Rational points on conic bundle surfaces via additive combinatorics

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Joint with T. Browning and A. Skorobogatov

Conic bundle surface over number field k:
projective, non-singular surface with dominant k-morphism

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s.t. all fibres are conics.

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Conjecture (Colliot-Thélène & Sansuc)

Brauer–Manin obstruction is the only obstruction to HP and WA for conic bundle surfaces.

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Unconditional results

Let $r = \#$ degenerate geometric fibres of X.

The conjecture holds when $0 \leqslant r \leqslant 5$ and in special cases of $r = 6$. (Colliot-Thélène, Salberger, Sansuc, Skorobogatov, and Swinnerton-Dyer)

Main theorem

Theorem (Browning–M–Skorobogatov)

Let X be a conic bundle surface over $\mathbb{Q}$, assume that it has degenerate geometric fibres and that they are all defined over $\mathbb{Q}$. Then

- *$X(\mathbb{Q})$ is Zariski dense in X, and*
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where C conic over $\mathbb{Q}$, and $\mathcal{W}_{\boldsymbol{\lambda}} \subset \mathbb{A}^{2r+2}_{\mathbb{Q}}$ defined via

$$\{0 \neq u - e_i v = \lambda_i (x_i^2 - a_i y_i^2) : i = 1, \ldots, r\}$$

for suitable $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in (\mathbb{Q}^*)^r$.

Proof

$\mathcal{V} \subset \mathbb{A}_{\mathbb{Q}}^{2r+s}$ defined via

$$\{0 \neq x_i^2 - a_i y_i^2 = f_i(u_1, \ldots, u_s) : i = 1, \ldots, r\}$$

for homogeneous linear polynomials $f_i \in \mathbb{Z}[u_1, \ldots, u_s]$ s.t. $f_i \neq \alpha f_j$ whenever $i \neq j$, $\alpha \in \mathbb{Q}$.

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$$\sum_{\mathbf{u} \in \mathbb{Z}^s \cap N\mathcal{K}} \prod_{i=1}^{r} r_i(f_i(\mathbf{u}))$$

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Generalised von Neumann theorem

Replace r_i by general $h_i : \mathbb{Z} \to \mathbb{C}$ and consider

$$N^{-s} \sum_{\mathbf{u} \in (\mathbb{Z}/N\mathbb{Z})^s} \prod_{i=1}^{r} h_i(f_i(\mathbf{u}))$$

Generalised von Neumann theorem

If $\|h_i\|_\infty \leqslant 1$

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If $\|h_i\|_\infty \leqslant 1$ then Gowers' work on Szemerédi's theorem shows: If

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$$\max_i \|h_i - \delta_i\|_{U^{r-1}} = o(1) \quad \Longrightarrow$$

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Gowers uniformity norms

$$\|h\|^4_{U^2(\mathbb{Z}/N\mathbb{Z})} = N^{-3} \sum_{n,d_1,d_2 \in \mathbb{Z}/N\mathbb{Z}} h(n)\overline{h(n+d_1)h(n+d_2)}h(n+d_1+d_2)$$

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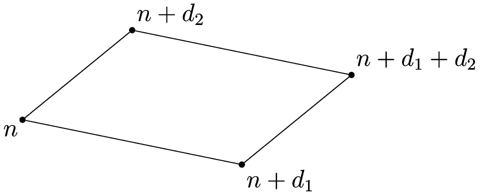

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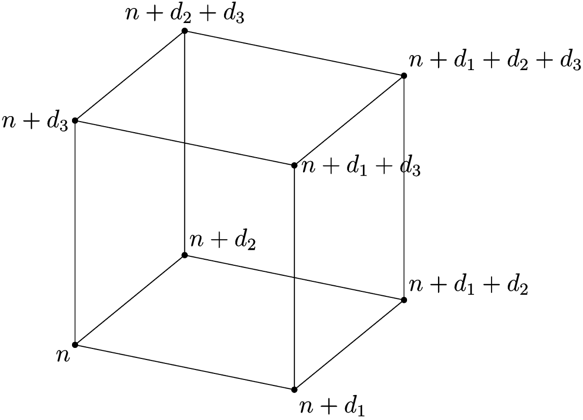

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$$\|h\|_{U^k(\mathbb{Z}/N\mathbb{Z})}^{2^k} = N^{-(k+1)} \sum_{\substack{n,d_1,\ldots,d_k \\ \in \mathbb{Z}/N\mathbb{Z}}} \prod_{\boldsymbol{\omega} \in \{0,1\}^k} \mathcal{C}^{|\boldsymbol{\omega}|} h(n + \boldsymbol{\omega} \cdot \mathbf{d})$$

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Important fact: $\|h\|_{U^2(\mathbb{Z}/N\mathbb{Z})} = \|\hat{h}\|_{\ell^4}.$

Thus, $\|h\|_{U^2(\mathbb{Z}/N\mathbb{Z})} \geqslant \delta$ iff h has a large Fourier coefficient:

$$\left| \frac{1}{N} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} h(n) e(\theta n) \right| \geqslant 2\delta^2$$

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Example:

Let $A \subset \{1, \ldots, N\}$, $|A| = \alpha N$. Then

$$\#\{\text{3-term AP's in A}\} = \sum_{n,d:1\leqslant n,n+2d\leqslant N} 1_A(n)1_A(n+d)1_A(n+2d)$$
$$\sim \alpha^3 N^2/2$$

iff $1_A - \alpha$ has no large Fourier coefficient.

If $\frac{1}{x}\sum_{n\leqslant x} h_i(n) = \delta_i + o(1)$, then

$$\max_i \|h_i - \delta_i\|_{U^{r-1}} = o(1) \qquad \Longrightarrow$$

$$\sum_{\mathbf{u}\in\mathbb{Z}^s\cap K}\prod_{i=1}^{r} h_i(f_i(\mathbf{u})) = \operatorname{vol} K\,\delta_1\ldots\delta_r + o(N^s)$$

for convex $K \subseteq [-N,N]^s$.

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Thus, Fourier analysis does not capture the U^3 norm. Indeed

Theorem (Inverse theorem for U^3, Green–Tao)

If $h : \{1, \ldots, N\} \mapsto \mathbb{C}$, $\|h\|_\infty \leqslant 1$ and $\|h\|_{U^3} \geqslant \delta$, then there is a generalised quadratic phase

$$\phi(n) = \sum_{r,s \leqslant C_1(\delta)} \beta_{rs}\{\theta_r n\}\{\theta_s n\} + \gamma_r\{\theta_r n\}$$

where $\beta_{rs}, \gamma_r, \theta_r \in \mathbb{R}$, s.t. $\quad \left|\frac{1}{N}\sum_{n \leqslant N} h(n) e(\phi(n))\right| \gg_\delta 1.$

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with Lipschitz function $F : G/\Gamma \to \mathbb{C}$, $\|F\|_\infty \leqslant 1$.

The general inverse theorem

Theorem (Green-Tao-Ziegler, 2012)

Let $k \geqslant 0$ be an integer, $0 < \delta \leqslant 1$. Then there is a finite collection $\mathcal{M}_{k,\delta}$ of k-step nilmanifolds $(G/\Gamma, d_{G/\Gamma})$ s.t.:
If $N \geqslant 1$, $h : \{1, \ldots, N\} \to \mathbb{C}$, $\|h\|_\infty \leqslant 1$ and

$$\|h\|_{U^{k+1}[N]} \geqslant \delta \,,$$

then there is a $G/\Gamma \in \mathcal{M}_{k,\delta}$, and a nilsequences $(F(g(n)\Gamma))_{n \leqslant N}$ with $\|F\|_{Lip} = O_{k,\delta}(1)$ s.t.

$$\Big|\frac{1}{N}\sum_{n \leqslant N} h(n)F(g(n)\Gamma)\Big| \gg_{k,\delta} 1 \,.$$

What have we gained?

To obtain an asymptotic formula for

$$\sum_{\mathbf{u}\in(\mathbb{Z}/N\mathbb{Z})^s}\prod_{i=1}^{r}h_i(f_i(\mathbf{u})),$$

f_i linear polynomials, no two proportional

we may, instead of proving that $\|h_i-\delta_i\|_{U^{r-1}}$ are small, attempt to show that

$$\Big|\frac{1}{N}\sum_{n\leqslant N}(h_i(n)-\delta_i)F(g(n)\Gamma)\Big| = o_{G/\Gamma,r}(1)$$

for each $i=1,\ldots,r$.

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Problem:
The h_i have to be bounded!

The transference principle

All mentioned results continue to hold for functions $h_i : \{1, \ldots, N\} \to \mathbb{C}$ s.t. $\sum_{n \leqslant x} h_i(n) = \delta_i x + o(x)$ that, instead of

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Green-Tao transference result

If $h : \{1, \ldots, N\} \to \mathbb{C}$ has a pseudo-random majorant, then

$$h = h_1 + h_2$$

where h_1 bounded, and h_2 Gowers uniform.

Summary

In order to apply these methods to the representation functions r_i of irreducible binary quadratic forms, we have to

1. construct a pseudo-random majorant
2. show that the r_i do not correlate with nilsequences

Further results

Theorem (Browning–M–Skorobogatov)

Let X be a conic bundle surface over $\mathbb{Q}$ and assume that all degenerate geometric fibres are defined over $\mathbb{Q}$. Then

- $X(\mathbb{Q})$ *is Zariski dense in* X*, and*
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Theorem (BMS)

If all degenerate geometric fibres are defined over $\mathbb{Q}$, then the Brauer–Manin obstruction is the only obstruction to WA on any smooth and projective model of X.

Further results

Higher dimensional varieties

$$X = \left\{ \ f_1(\mathbf{t})X_1^2 + \cdots + f_n(\mathbf{t})X_n^2 = 0 \ \right\}$$

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Theorem (BMS)

Brauer–Manin is the only obstruction to WA on smooth and projective models of X, provided $f_1, \ldots, f_n$ are products of linear poly's over $\mathbb{Q}$.

Further results

$$X = X_1 \times_{\mathbb{P}^1_{\mathbb{Q}}} X_2 \times_{\mathbb{P}^1_{\mathbb{Q}}} \cdots \times_{\mathbb{P}^1_{\mathbb{Q}}} X_n$$

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Applies to intersections of quadrics:

$$\left\{(u - e_{2i-1}v)(u - e_{2i}v) = c_i(x_i^2 - a_i y_i^2), \quad i = 1, \ldots, n\right\} \subseteq \mathbb{P}^{2n+1}_{\mathbb{Q}}$$

$a_i \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}, \quad c_i \in \mathbb{Q}^*, \quad$ pairwise distinct $e_1, \ldots, e_{2n} \in \mathbb{Q}$.

Surveys on Green-Tao/Green-Tao-Ziegler material

[1] B.J. Green, *Generalising the Hardy-Littlewood method for primes*, International Congress of Mathematicians. Vol. II, 373–399, Eur. Math. Soc., Zurich, 2006.

[2] B.J. Green, T. Tao and T. Ziegler, *An inverse theorem for the Gowers U^{s+1}-norm (announcement)*, Electron. Res. Annouce. Math. Sci. 18 (2011), 69–90.

Pseudo-random majorant

The total mass is roughly 1:

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The D-linear forms condition

For all integers $0 < t, d \leqslant D$, we have

$$\frac{1}{N^d}\sum_{\mathbf{u}\in(\mathbb{Z}/N\mathbb{Z})^d}\nu(f_1'(\mathbf{u}))\dots\nu(f_t'(\mathbf{u})) = 1 + o(1)$$

for $f_1',\dots,f_t' : \mathbb{Z}^d \to \mathbb{Z}^t$ linear poly's with bounded coefficients s.t.

$$f_i' \neq \alpha f_j', \quad i \neq j, \alpha \in \mathbb{Q}.$$