Birational invariants, purity and the Gersten conjecture

Lectures at the 1992 AMS Summer School

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§ 0. Introduction

In the last ten years, classical fields such as the algebraic theory of quadratic forms and the theory of central simple algebras have witnessed an intrusion of the cohomological machinery of modern algebraic geometry. Merkur'ev's theorem on the e_3 invariant, one way or another, relates to étale cohomology; so does the general Merkur'ev/Suslin theorem on the norm residue symbol. One of the purposes of the 1992 Summer Institute was to introduce workers in the two mentioned fields to the tools of modern algebraic geometry.

The organizers asked me to lecture on étale cohomology. Excellent introductions to étale cohomology exist, notably the set of lectures by Deligne in [SGA4 1/2]. There was no point in trying to duplicate such lectures. I thus decided to centre my lectures on unramified cohomology. This notion, although not novel by itself, has attracted attention only in recent times; it lies half-way between the scheme-theoretic point of view and the birational point of view. The notion itself may be defined for other functors than cohomology (other functors of interest will be discussed in § 2.2).

Let k be a field and F a functor from the category of k-algebras to the category of abelian groups. Let X be an irreducible (reduced) algebraic variety X, and let $k(X)$ be its function field. Let $A \subset k(X)$ run through the rank one discrete valuation rings which contain k and whose field of fractions is $k(X)$. By definition, the unramified subgroup $F_{nr}(k(X)/k) \subset F(k(X))$ is the group of elements in $F(k(X))$ which lie in the image of $F(A)$ for each A as above.

In order to demonstrate the utility of this notion, in § 1, we shall start from scratch and establish the well-known fact that an elliptic curve is not birational to the projective line, the tool being unramified H^1 with $\mathbf{Z}/2$ coefficients, presented in an even more down to earth form. Copying this proof for unramified H^i for higher integers i leads, for $i = 2$, to another approach to the celebrated Artin-Mumford examples of nonrational unirational varieties [Ar/Mu72]. One may go further along this way (see § 4.2).

As one may observe immediately, the notion of unramified cohomology only depends on the function field $k(X)/k$ of the variety X. Indeed, one place where this notion has been extremely successful is Saltman's paper [Sa84], where the author shows that some function fields (invariant fields of a linear action of a finite group) are not purely transcendental over the ground field, thus settling Noether's problem over an algebraically closed field in the negative. Saltman's invariant is the unramified Brauer group $\mathrm{Br}_{nr}(k(X)/k)$, which may be shown to be equal to the (cohomological) Brauer group $\mathrm{Br}(X)$ of a smooth projective model X. In concrete cases, it is unclear how to construct such a model for a given function field. A key aspect of Saltman's paper is that the unramified point of view enables one to dispense with the construction of an explicit model, and even with the existence of such a model.

Various recent works have been devoted to extensions of these ideas, a number of them with the goal of detecting the nonrationality of some unirational varieties (see § 4.2.4 and § 4.2.5 below).

One aim of these lectures is to reconcile the birational point of view on unramified cohomology with the scheme-theoretic point of view. Among the reasons for doing so, let me mention :

– the study of the functorial behaviour of unramified cohomology with respect to arbitrary (not necessarily dominating) morphisms of varieties;

– the attempts to use unramified cohomology to introduce some equivalence relation on the rational points, or on the zero-cycles on a variety defined over a non-algebraically closed field (see e.g. [CT/Pa90]) ; the idea (a special case of the functoriality problem) is to try to evaluate an unramified cohomology class at an arbitrary point (some condition such as smoothness of the variety is essential at that point) ;

– the attempts to try to control the size of unramified cohomology groups over an algebraically closed ground field.

It will be demonstrated (§ 2) that good functorial behaviour of the unramified functor F_{nr} (and some other similar functors) attached to a functor F requires some basic properties, namely the specialization property, the injectivity property and the codimension one purity property (see Def. 2.1.4) for regular local rings.

The study of these properties, particularly the codimension one purity for étale cohomology, will lead us to a review of two fundamental properties of étale cohomology for varieties over a field (§ 3). The first property is cohomological purity, which is due to Artin ([SGA4]), and which is described in the standard textbooks on étale cohomology. The second property is the Gersten conjecture for étale cohomology, established by Bloch and Ogus [Bl/Og74], which is not studied in the standard textbooks. Although the circle of ideas around it is well-known to a few experts, to many people it still seems shrouded in mystery. In § 3, some motivation for this conjecture will be given together with its formal statement (§ 3.5). A guided tour through the literature will then be offered, but no proofs will be given. However, in § 5, an alternate approach to some results of Gersten type ([Oj80], [CT/Oj92]) will be described in some detail.

No survey paper would be satisfactory without a couple of new results, or at least a couple of results which are in the air. While preparing these lectures, I thus came across

– finiteness results for unramified H^3-cohomology (§ 4.3), some of which had already been obtained by L. Barbieri-Viale [BV92b] ;

– a rigidity theorem for unramified cohomology (§ 4.4), also considered by Jannsen (unpublished) ;

– a new proof of a purity theorem of Markus Rost (Theorem 5.3.1).

The present notes are the outcome of the five lectures I gave at Santa Barbara. I have made no attempt at writing a systematic treatise, and I hope that the written text retains some of the spontaneity of the oral lectures.

Acknowledgements

Section 5 dwells on a formalism due to M. Ojanguren in the Witt group context [Oj80] and further developed in our joint paper [CT/Oj92]. I had helpful discussions with my Indian colleagues, in particular Parimala and Sujatha, both during the preparation of the lectures and during the writing up of the notes. Some of these discussions were made possible by support from the Indo-French Centre for the Promotion of Advanced Research (IFCPAR). I thank Steve Landsburg for producing an on-the-spot preliminary write-up of the lectures.

Finally, I thank Bill Jacob and Alex Rosenberg for having given me an opportunity to reflect on some basic concepts, and for the superb setting of the conference.

Notation

Given an abelian group A and a positive integer n, we write ${}_nA$ for the subgroup of elements killed by n, and we write A/n for the quotient A/nA.

Given an integral domain A, we let $\mathrm{qf}(A)$ denote its fraction field.

In this paper, discrete valuation rings will be of rank one unless otherwise mentioned.

We let $\mathbf{A}^n_k$, resp. $\mathbf{P}^n_k$ denote n-dimensional affine, resp. projective, space over a field k.

If X is an irreducible variety, and p is a positive integer, we let X^p denote the set of all codimension p points of X, i.e. the set of points $M \in X$ whose local ring $\mathcal{O}_{X,M}$ is of dimension p.

§ 1. An exercise in elementary algebraic geometry

Let $\mathbf{C}(x)$ be the field of rational fractions in one variable over the complex field $\mathbf{C}$ and let $L = \mathbf{C}(x)(\sqrt{x(x-1)(x+1)})$ be the quadratic extension of $\mathbf{C}(x)$ obtained by adjoining the square root of $x(x-1)(x+1)$. This is a function field of transcendence degree one over $\mathbf{C}$, and it is none other than the field of rational functions of the elliptic curve E given by the affine equation

$$y^2 = x(x-1)(x+1).$$

Certainly one of the first statements in algebraic geometry is that E is not birational to the affine line $\mathbb{A}^1_{\mathbf{C}}$, in other words that L cannot be generated over $\mathbf{C}$ by a single element. There are several ways to see this. As a preparation to the notion of unramified cohomology, we give the following elementary proof, which only uses the notion of a discrete valuation (of rank one, with value group $\mathbf{Z}$).

Proposition 1.1. — *Let $L/\mathbf{C}$ be the field $L = \mathbf{C}(x)(\sqrt{x(x-1)(x+1)})$. Then L is not $\mathbf{C}$-isomorphic to the rational field $\mathbf{C}(t)$.*

Proof : a) Write $L = \mathbf{C}(x)(y)$ with

$$y^2 = x(x-1)(x+1) \in L.$$

Consider the element $x \in L$. We claim that for any discrete valuation v on the field L, the valuation $v(x)$ of x is *even.*

Case 1. If $v(x) = 0$, the result is clear.

Case 2. If $v(x) > 0$ then $v(x-1) = \inf(v(x), v(1)) = v(1) = 0$ and similarly $v(x+1) = 0$. From the identity

$$y^2 = x(x-1)(x+1)$$

in L, we deduce $2v(y) = v(x)$ and $v(x)$ is even.

Case 3. If $v(x) < 0$, then $v(x-1) = \inf(v(x), v(1)) = v(x)$ and similarly $v(x+1) = v(x)$ hence from the above identity we deduce $2v(y) = 3v(x)$ hence also $v(x)$ even.

b) We claim that x is not a square in L.

Given any quadratic field extension $K \subset L = K(\sqrt{a})$ with $\operatorname{char.}(K) \neq 2$, a straightforward computation yields a short exact sequence

$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow K^*/K^{*2} \longrightarrow L^*/L^{*2}$$

where the map $\mathbf{Z}/2 \longrightarrow K^*/K^{*2}$ sends 1 to the class of a. Applying this remark to the quadratic extension of fields $K = \mathbf{C}(x) \subset L = \mathbf{C}(x, \sqrt{x(x-1)(x+1)})$), we see that if x were a square in L, either x would be a square in $\mathbf{C}(x)$ or $x(x-1)(x+1)/x = (x-1)(x+1)$ would be a square. But for each of $x \in \mathbf{C}(x)$ and $(x-1)(x+1) \in \mathbf{C}(x)$ we may find a valuation of the field $\mathbf{C}(x)$ which takes the value 1 on either of these elements. Hence they are not squares, and x is not a square in L.

c) In the rational function field $\mathbf{C}(t)$ in one variable over $\mathbf{C}$, any non-zero element, all valuations of which are even, is a square. Indeed, any $z \in \mathbf{C}(t)^*$ after suitable multiplication by a square in $\mathbf{C}(t)$ may be rewritten as a polynomial in $\mathbf{C}[t]$ with simple roots $z = \prod_{i \in I}(t - e_i)$. If I is not empty, i.e. if $z \neq 1$, then for any $i \in I$, we may consider the valuation v_i associated to the prime ideal $(t - e_i)$, and $v_i(z) = 1$ is not even.

d) Putting a), b), c) together, we find that the field L is not $\mathbf{C}$-isomorphic to the field $\mathbf{C}(t)$. $\square$

Exercise 1.1.1 : Let p and q be two coprime integers, and let $e_i, i = 1, \ldots, q$ be distinct elements of $\mathbf{C}$. Let L be the function field $L = \mathbf{C}(x)(y)$, with $y^p = \prod_{i=i=1,\ldots,q}(x - e_i)$. Arguing with elements all valuations of which are divisible by p, show that L is not purely transcendental over $\mathbf{C}$.

Let us slightly formalize the proof of Proposition 1.1. Given a field k, $\mathrm{char}(k) \neq 2$, and a function field F/k (by function field F/k we mean a field F finitely generated over the ground field k) we consider the group

$$Q(F/k) = \{\alpha \in F^*, \forall v \text{ trivial on } k, \ \ v(\alpha) \in 2\mathbf{Z}\}/F^{*2}.$$

Let Ω be the set of discrete valuation rings A (of rank one) with $k \subset A$ and with fraction field K. The group $Q(F/k)$ may also be defined as :

$$Q(F/k) = \{\alpha \in F^*/F^{*2}, \forall A \in \Omega, \ \alpha \in \mathrm{Im}(A^*/A^{*2}).\}$$

Our proof of the non-rationality of $L = \mathbf{C}(x)(y)$ fell into the following parts :

a) We showed that if $F = \mathbf{C}(t)$, then $Q(F) = 0$. As the reader will easily check, this result may be extended. Namely, if $t_1, \ldots, t_n$ are independent variables, the natural map $k^*/k^{*2} \to Q(k(t_1, \ldots, t_n)/k)$ is an isomorphism. Even more generally, if K/k is a function field, the inclusion $K \subset K(t_1, \ldots, t_n)$ induces an isomorphism $Q(K/k) \simeq Q(K(t_1, \ldots, t_n)/k)$.

b) We produce a non-trivial element in $Q(L)$.

In the next section, we shall generalize the formalism above to functors F other than $F(A) = A^*/A^{*2}$. In most cases, the analogue of a) will be easy to check. However the computation of the analogue of $Q(L)$ will often turn out to be tricky. Before getting to this, we might comfort ourselves with the remark that in the case at hand, a few more elementary arguments enable one to compute the exact value of $Q(L)$.

If k is algebraically closed, the group k^* is divisible, hence any discrete valuation on F is trivial on k. In that case, we simply write $Q(F) = Q(F/k)$. In the above proof, we showed $Q(\mathbf{C}(t)) = 0$ and $Q(L) \neq 0$ for $L = \mathbf{C}(x, \sqrt{x(x-1)(x+1)})$. We shall go further and actually compute $Q(L)$.

First note that given any extension L/K of function fields over a field k, there is an induced map $Q(K/k) \to Q(L/k)$ (for any discrete valuation v on L, either v is trivial on K, or it induces a discrete valuation on K). When the extension L/K is finite and separable, there is a map the other way round :

LEMMA 1.2. — *Let L/K be a finite extension of function fields over k. Assume that L/K is separable. Then the norm map $N_{L/K} : L^*/L^{*2} \to K^*/K^{*2}$ induces a homomorphism $N_{L/K} : Q(L/k) \to Q(K/k)$.*

Proof : Given any discrete valuation ring $A \subset K$ with $k \subset A$ and $qf(A) = K$, the integral closure B of A in L is a semi-local Dedekind ring ([Se68], I, § 4). We thus have an exact sequence

$$0 \longrightarrow B^* \longrightarrow L^* \longrightarrow \bigoplus_{i=1,\ldots,m} \mathbf{Z} \longrightarrow 0$$

where the right hand side map is given by the valuations $v_i, i = 1, \ldots, m$ at the finitely many maximal ideals of B. If all valuations $v_i(\beta)$ are even, then this sequence shows that β may actually be written $\beta = \gamma.\delta^2$ with $\gamma \in B^*$ and $\delta \in L^*$. Now the norm map $N_{L/K} : L^* \to K^*$ sends B^* into A^*, and we conclude that $N_{L/K}(\beta)$ is the product of a unit in A^* by a square in K^*, i.e. that $v_A(N_{L/K}(\beta))$ is even. □

PROPOSITION 1.3. — *With notation as in Proposition 1.1, we have $Q(L) = \mathbf{Z}/2 \oplus \mathbf{Z}/2$.*

Proof : Given a quadratic field extension $K \subset L = K(\sqrt{a})$ with char.$(K) \neq 2$, the exact sequence mentioned in the proof of Prop. 1.1 may be extended to

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow K^*/K^{*2} \longrightarrow L^*/L^{*2} \xrightarrow{N} K^*/K^{*2}$$

To prove this, use Hilbert's theorem 90 for the cyclic extension L/K. (As a matter of fact, this sequence is part of a well-known infinite exact sequence in Galois cohomology.) Let us now take $K = \mathbf{C}(x)$ and $L = \mathbf{C}(x, \sqrt{x(x-1)(x+1)})$. From Lemma 1.2, the norm map induces a map $Q(L) \to Q(K)$. Now we have $Q(K) = Q(\mathbf{C}(x)) = 0$, as already mentioned. From the above sequence we conclude that any element of $Q(L)$ comes from K^*/K^{*2} via the natural map $K^*/K^{*2} \longrightarrow L^*/L^{*2}$ induced by $K \subset L$. We now need to determine which classes in K^*/K^{*2} have image in L^*/L^{*2} contained in $Q(L)$. Suppose that $\alpha \in K^*$ defines such a class. Consider the ring extension $\mathbf{C}[x,y]/(y^2 - x(x-1)(x+1))/\mathbf{C}[x]$. This is a finite extension of Dedekind rings and $\mathbf{C}[x,y]/(y^2 - x(x-1)(x+1))$ is the integral closure of $K = \mathbf{C}[x]$ in L (indeed, this $\mathbf{C}$-algebra is smooth over $\mathbf{C}$, hence is a regular ring), and one immediately checks that over any maximal ideal of $\mathbf{C}[x]$ corresponding to a point of the affine line different from $0, 1, -1$, there lie two disctinct prime ideals of $\mathbf{C}[x,y]/(y^2 - x(x-1)(x+1))$. For any such point, taking valuations at that point and at the two points lying above it yields a commutative diagram

$$\begin{array}{ccc} \mathbf{C}(X, \sqrt{X(X-1)(X+1)})^* & \longrightarrow & \mathbf{Z} \oplus \mathbf{Z} \\ \uparrow & & \uparrow \\ \mathbf{C}(X)^* & \longrightarrow & \mathbf{Z} \end{array}$$

where the right hand side vertical map is $1 \to (1,1)$. Let $P(x) \in \mathbf{C}(x)$, and suppose that its image in L has all its valuations even; then as an element of $\mathbf{C}(x)$, at all primes of the affine line different from $(x), (x-1), (x+1)$, $P(x)$ must have even valuation. Any element in K^*/K^{*2} may be represented by a square free polynomial in $\mathbf{C}[x]$. We conclude that α, up to a square in K^*, may be represented by a polynomial $x^a.(x-1)^b.(x+1)^c$, where a, b, c are 0 or 1. Now each of $x, x-1, x+1$ actually gives rise to an unramified element in L^*/L^{*2}, by the same argument that was used for x in Proposition 2.1. One easily checks that the classes of $x, x-1, x+1$ are linearly independent over $\mathbf{Z}/2$ in K^*/K^{*2}. They thus span a group $(\mathbf{Z}/2)^3 \subset K^*/K^{*2}$. On the other hand, the map $(\mathbf{Z}/2)^3 \longrightarrow L^*/L^{*2}$ which sends a, b, c to $x^a.(x-1)^b.(x+1)^c \in L^*/L^{*2}$ has kernel exactly equal to the subgroup $\mathbf{Z}/2$ spanned by $(1,1,1)$, in view of the exact sequence

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow K^*/K^{*2} \longrightarrow L^*/L^{*2}.$$

Thus $Q(L) \simeq (\mathbf{Z}/2)^2$, with the two generators given by the class of x and of $x-1$ in L^*/L^{*2}. $\square$

Exercise 1.3.1 : Let $e_i, i = 1, \ldots, 2g+1$ be distinct elements of $\mathbf{C}$, and let L be the the function field $L = \mathbf{C}(x, \sqrt{\prod_{i=1}^{2g+1}(x-e_i)})$. Using the same arguments as above, show $Q(L) = (\mathbf{Z}/2)^{2g}$. In particular, two such function fields with different values of g are not $\mathbf{C}$-isomorphic.

Remark 1.3.2 : The learned reader will have long recognized that the functor $L \longmapsto Q(L)$ which we have been using in the above example is a familiar invariant in algebraic geometry. Namely, if $\mathbf{C}(X)$ is the the function field L of a smooth, projective, connected variety $X/\mathbf{C}$, the group $Q(\mathbf{C}(X))$ is isomorphic to the 2-torsion subgroup of the Picard group $\mathrm{Pic}(X)$. For a curve of genus g, that group is well-known to be isomorphic to $(\mathbf{Z}/2)^{2g}$. This is what the above naive computation gives us for the function field of the hyperelliptic curve

$$y^2 = \prod_{i=1,\ldots,2g+1}(x-e_i),$$

where the e_i's are distinct.

Note that the birational invariant given by the torsion in $\mathrm{Pic}(X)$ is not a very subtle invariant. For instance, Serre [Se59] proved that if a smooth projective variety X over $\mathbf{C}$ is dominated by a projective space (i.e. is unirational), then $\mathrm{Pic}(X)$ has no torsion (he actually proved more, namely that the fundamental group of X is trivial). Subtler functors than $A \longmapsto A^*/A^{*2}$ are required to tell us that some unirational varieties are not rational.

§ 2. Unramified elements

§ 2.1 Injectivity, codimension one purity, homotopy invariance : a general formalism

Let $\mathcal{R}$ be the category of commutative rings with unit, and let $\mathcal{C} \subset \mathcal{R}$ be one of the following subcategories.

1) The category $\mathcal{R}$ of all commutative rings with unit, morphisms being arbitrary ring homomorphisms.

2) The category $\mathcal{R}_{fl}$ of all commutative rings with unit, morphisms being flat ring homomorphisms.

3) The category $k - Alg$ of all k-algebras (not necessarily of finite type) over a fixed field k, morphisms being k-homomorphisms of k-algebras.

4) The category $k - Alg_{fl}$ of k-algebras (not necessarily of finite type) over a fixed field k, morphisms being flat k-homomorphisms of k-algebras.

Let Ab be the category of abelian groups. We shall be interested in (covariant) functors from $\mathcal{C}$ to Ab. Although we want to keep this section at a formal level, it might help the reader if we already revealed which functors we have in mind. Let us here quote :

(i) the (Azumaya or cohomological) Brauer group [Au/Go60], [Gr68];

(ii) the Witt group [Kn77];

(iii) various étale cohomology groups with values in commutative group schemes [SGA4];

(iv) K-theory [Qu73] and K-theory with coefficients.

We may also mention more exotic functors on the category of k-algebras, such as :

(v) the functor $A \longmapsto A^*/\mathrm{Nrd}((D \otimes_k A)^*)$, where D is a central simple algebra over the field k and Nrd denotes the associated reduced norm;

(vi) the functor $A \longmapsto A^*/\Phi(A)$ where Φ is a Pfister form over k ($\mathrm{char}(k) \neq 2$) and $\Phi(A)$ denotes the subgroup of elements of A^* represented by Φ over A [CT78];

(vii) finally, functors with values in the category of pointed sets such as the functor $A \longmapsto H^1(A, G)$ where G is a linear algebraic group over k and $H^1(A, G)$ is the set of isomorphism classes of principal homogeneous spaces over $\mathrm{Spec}(A)$ under G.

Interesting functors are also obtained by replacing the functor $A \longmapsto F(A)$ by the functor $A \longmapsto F(A \otimes_k B)$ where B is some fixed k-algebra.

Definition 2.1.1. — *Let F be a functor from $\mathcal{C}$ to Ab. Given any fields K and L in $\mathcal{C}$, with $K \subset L$, we say that $\alpha \in F(L)$ is unramified over K if for each rank one discrete valuation ring A with $K \subset A$ and quotient field* $\mathrm{qf}(A) = L$*, the element α belongs to the image of $F(A) \to F(L)$. We shall denote by $F_{nr}(L/K) \subset F(L)$ the group of all such elements, and refer to it as the unramified subgroup of $F(L)$.*

Remark 2.1.2 : Suppose that F is a functor to the category of pointed sets. Then there is a similar definition of the unramified subset $F_{nr}(L/K) \subset F(L)$. Most of the statements below have analogues in this context.

Lemma 2.1.3. — *a) Given fields $K \subset L$ in $\mathcal{C}$, the natural map $F(K) \longrightarrow F(L)$ induces a map $F(K) \longrightarrow F_{nr}(L/K)$.*

b) Let $E \subset K \subset L$ be fields in $\mathcal{C}$. The natural map $F(K) \to F(L)$ induces a map $F_{nr}(K/E) \to F_{nr}(L/E)$.

Proof : The first statement is clear. If $B \subset L$ is a discrete valuation ring with $E \subset B$ and $L = \mathrm{qf}(B)$, then either $K \subset B$ or $A = B \cap K$ is a discrete valuation ring of K. In either case, the image in L of any element of $F_{nr}(K/E) \subset K$ clearly lies in B. (Note that a) is just a special case of b).) $\square$

Let F be a functor from $\mathcal{C}$ to *Ab*. We shall be interested in various properties of such a functor.

Definiton 2.1.4. — *a)* **Injectivity property for a regular local ring A.** *Let A be a regular local ring A in $\mathcal{C}$, with field of fractions K. Then the map $F(A) \to F(K)$ has trivial kernel.*

b) **Codimension one purity property for a regular local ring A.** *Let A be a regular local ring A in $\mathcal{C}$, with field of fractions K. Then*

$$\mathrm{Im}(F(A) \to F(K)) = \bigcap_{p \text{ height one}} \mathrm{Im}(F(A_p) \longrightarrow F(K)).$$

c) **Specialization property for a regular local ring A.** *Suppose that $\mathcal{C}$ is either $\mathcal{R}$ or $k - Alg$. Let A be a regular local ring in $\mathcal{C}$, let K be its fraction field and κ its residue field. Then the kernel of $F(A) \to F(K)$ lies in the kernel of $F(A) \to F(\kappa)$.*

The specialization property is of course much weaker than the injectivity property.

Lemma 2.1.5. — *Let F be a covariant functor from the category $\mathcal{C}$ to Ab.*

a) If F satisfies the specialization property for all complete discrete valuation rings A, then it satisfies the specialization property for arbitrary regular local rings, in particular for arbitrary discrete valuation rings.

b) If F satisfies the injectivity property for all complete discrete valuation rings, then it satisfies the specialization property for arbitrary regular local rings.

c) Let k be a field and let A be a k-algebra which is a regular local ring, with fraction field K and residue field κ ; assume that the composite map $k \to A \to \kappa$ is an isomorphism. If F satisfies the specialization property, then the map $F(k) \to F(K)$ is injective.

Proof : Statement c) is obvious, and statement a) implies b). For the proof of a), which uses a well-known induction argument on the dimension of a regular local ring A, we refer the reader to [CT78], Prop. 2.1, [CT80], Lemme 1.1, [CT/Sa79], 6.6.1. $\square$

Definition 2.1.6. — *Let F be a covariant functor from the category $\mathcal{C}$ to Ab. We shall say that F satisfies* **field homotopy invariance** *if whenever K is a field in $\mathcal{C}$ and $K(t)$ is the field of rational functions in one variable over K, the induced map $F(K) \to F_{nr}(K(t)/K)$ is an isomorphism.*

Remark 2.1.7 : Note that there is another kind of homotopy invariance of the functor F, which is of a global nature, and may be taken into consideration. We shall say that the functor F from $\mathcal{C}$ to Ab satisfies **ring homotopy invariance** if for any commutative ring A in $\mathcal{C}$, if $A[t]$ denotes the polynomial ring in one variable over A, the map $F(A) \longrightarrow F(A[t])$ is an isomorphism. If $A[t_1, \ldots, t_n]$ denotes the polynomial ring in n variables over A, **ring homotopy invariance** immediately implies that the map $F(A) \longrightarrow F(A[t_1, \ldots, t_n])$ is an isomorphism.

Let F be a functor from $k - Alg$ to Ab. To any integral variety X/k, with field of functions $k(X)$, let us associate the following subgroups of $F(k(X))$:

$$F_1(X) = \{\alpha \in F(k(X)) \mid \forall P \in X^1\,,\ \alpha \in \operatorname{Im} F(\mathcal{O}_{X,P})\}$$

$$F_{loc}(X) = \{\alpha \in F(k(X)) \mid \forall P \in X\,,\ \alpha \in \operatorname{Im} F(\mathcal{O}_{X,P})\}$$

$$F_{nr}(k(X)/k) = \{\alpha \in F(k(X)) \mid \forall k \subset A \subset k(X)\,,\ A \text{ discrete valuation ring}\,, \\ \mathrm{qf}(A) = k(X)\,, \alpha \in \operatorname{Im} F(A)\}$$

$$F_{val}(k(X)/k) = \{\alpha \in F(k(X)) \mid \forall k \subset A \subset k(X)\,,\ A \ \text{ valuation ring}\,,\ \ \mathrm{qf}(A) = k(X)\,, \\ \alpha \in \operatorname{Im} F(A)\}$$

(In the last definition, A runs through all Krull valuation rings.)

Proposition 2.1.8. — *Let F be a functor from $k - Alg$ to Ab. Let X/k be an integral variety, with function field $k(X)$.*

a) *We have $F_{val}(k(X)/k) \subset F_{nr}(k(X)/k)$ and $F_{loc}(X) \subset F_1(X)$.*

b) *If X is normal, then we have $F_{nr}(k(X)/k) \subset F_1(X)$.*

c) *If X/k is proper, then we have $F_{loc}(X) \subset F_{val}(k(X)/k)$.*

d) *If X/k is smooth, and the functor F satisfies the (codimension one) purity property for regular local rings, then $F_{loc}(X) = F_1(X)$.*

e) *If X/k is smooth and proper, and F satisfies the (codimension one) purity property for regular local rings, then the four subgroups $F_1(X)$, $F_{loc}(X)$, $F_{nr}(k(X)/k)$ and $F_{val}(k(X)/k)$ of $F(k(X))$ coincide, and they are all k-birational invariants of smooth, proper, integral k-varieties.*

Proof : Statement a) is clear. As for b), if X is normal, then local rings at points of codimension 1 are discrete valuation rings, hence we have $F_{nr}(k(X)/k) \subset F_1(X)$. Assume that X/k is proper. Then the inclusion $\mathrm{Spec}(k(X)) \to X$ of the generic point extends to a morphism $\mathrm{Spec}(A) \to X$. We thus get inclusions $\mathcal{O}_{X,P} \subset A \subset k(X)$. Now if $\alpha \in F(k(X))$ belongs to $F_{loc}(X)$, it is in the image of $F(\mathcal{O}_{X,P})$; thus α comes from $F(A)$. This proves c). Let us assume that X/k is smooth and that F satisfies the (codimension one) purity property for regular local rings. Then X is normal, hence $F_{loc}(X) \subset F_1(X)$ as noted above. Conversely, given $\alpha \in F_1(X)$ and $P \in X$, apply the codimension one purity property to $A = \mathcal{O}_{X,P}$ to get $\alpha \in \operatorname{Im} F(\mathcal{O}_{X,P})$. This proves d). Statement e) gathers the previous results. The k-birational invariance of $F_{nr}(k(X)/k)$ is clear, hence also that of the other groups $\square$

Proposition 2.1.9. — *Let k be a field and let F be a functor from the category $\mathcal{C}$ of commutative k-algebras to abelian groups. Assume that F satisfies the specialization property for discrete valuation rings and the codimension one purity property for regular local rings A of dimension 2. Assume that for all fields $K \supset k$, the natural map $F(K) \to F_{nr}(K(t)/K)$ is a bijection, where $K(t)$ denotes the function field in one variable over K. Then for all fields $K \supset k$ and all positive integers n, the natural map $F(K) \to F_{nr}(K(t_1, \ldots, t_n)/K)$ is a bijection.*

Proof : First note that the assumption implies that for any field $K \supset k$, the map $F(K) \to F(K(t))$ is injective. Induction then shows that for any such field K, and any positive integer n, the map $F(K) \to F(K(t_1, \ldots, t_n))$ is injective (injectivity also follows from the specialization property : simply use a local ring at a K-rational point).

We shall prove the theorem by induction on n. The case $n = 1$ holds by assumption. Suppose we have proved the theorem for n. Consider the projection p of $\mathbf{P}^1_K \times_K \mathbf{P}^n_K$ onto the second factor $\mathbf{P}^n_K$. Note that these varieties are smooth over k, hence their local rings are regular. On function fields, the map p induces an inclusion $E = K(t_1, \cdots, t_n) \subset L = K(t_1, \cdots, t_{n+1})$.

Let $A \in \mathbf{P}^1_K(K)$ be a fixed K-rational point. The map $x \to (A, x)$ defines a section σ of the projection p (i.e. $p \circ \sigma = \mathrm{id}_{\mathbf{P}^n_K}$). Let η denote the generic point of $\mathbf{P}^n_K$.

Let now $\alpha \in F(K(t_1, \cdots, t_{n+1}))$ be an element of $F_{nr}(K(t_1, \cdots, t_{n+1})/K)$. We may view α as an element of $F(L) = F(E(\mathbf{P}^1_E))$. It certainly belongs to $F_{nr}(E(\mathbf{P}^1_E)/E)$. From the hypothesis, we conclude that $\alpha \in F(L)$ is the image of a unique $\beta \in F(E)$ under the inclusion $p^* : F(E) \hookrightarrow F(L)$.

Let $x \in \mathbf{P}^n_K$ be an arbitrary codimension one point. Let $y = \sigma(x) \in \mathbf{P}^1_K \times_K \mathbf{P}^n_K$. This is a codimension 2 point on $\mathbf{P}^1_K \times_K \mathbf{P}^n_K$, which is a specialization of the codimension one point $\omega = \sigma(\eta)$. Let A be the (dimension one) local ring at x, B the (dimension two) local ring at y and C the (dimension one) local ring at ω. The local ring at η is the field $E = K(t_1, \cdots, t_n)$. We have inclusions $A \subset E$, $B \subset C \subset L$. We also have compatible maps $\sigma^* : F(B) \to F(A)$ and $\sigma^* : F(C) \to F(E)$

Since F satisfies the codimension one purity assumption for regular local rings of dimension two, and α belongs to $F_{nr}(K(\mathbf{P}^1_K \times_K \mathbf{P}^n_K)/K)$, there exists an element $\gamma \in F(B)$ whose image in $F(L)$ is α. Now the image γ_1 of γ in $F(C)$ under $F(B) \to F(C)$ and the image β_1 of β in $F(C)$ under the map $F(E) \to F(C)$ both restrict to α in $F(L)$. The ring C is a discrete valuation ring, the natural map $C \to E$ from C to its residue field E being given by σ^*.

By the specialization property, we conclude that $\sigma^*(\beta_1) = \sigma^*(\gamma_1) \in F(E)$. But $\sigma^*(\beta_1) = \sigma^*(p^*(\beta)) = \beta$ and $\sigma^*(\gamma_1)$ is the restriction to $F(E)$ of $\sigma^*(\gamma) \in F(A)$. We therefore conclude that $\beta \in F(E)$ lies in the image of $F(A)$. Since A was the local ring of $\mathbf{P}^n_K$ at an arbitrary codimension one point, we conclude that β lies in

$F_{nr}(E/K) = F_{nr}(K(t_1, \cdots, t_n)/K)$, hence by induction comes from $F(K)$. ∎

Our reason for phrasing the proposition as we have done is that purity in codimension one, for most functors of interest, is a property which is not easy to check. As a matter of fact, for some functors of interest, such as the Witt group, it is not yet known to hold for local rings of smooth varieties of arbitrary dimension over a field. For regular local rings of dimension two, there are specific arguments – many of them relying on the basic fact that a reflexive module over a regular local ring of dimension two is free. A well-known case, due to Auslander and Goldman [Au/Go60] is that of the (Azumaya) Brauer group. This was extended by Grothendieck to the cohomological Brauer group ([Gr68]. Another case is that of the set of isomorphism classes of principal homogeneous spaces under a reductive group (see [CT/Sa79], § 6), which in turn yields a similar result for the Witt group (cf. [CT/Sa79], § 2).

Suppose we know the specialization property for our given functor F, but that we do not know the purity property in codimension one for regular local rings of dimension 2. In the literature, one finds two other methods to use unramified elements, or some variant, in showing that some function fields are not purely transcendental over their ground field.

One of them uses residue maps and their functorial behaviour. We shall not formalize this proof here. The reader will see it at work in the context of étale cohomology in [CT/Oj89] (see also Prop. 3.3.1, Theorem 4.1.5 and § 4.2.4 and 4.2.5 below)). It is also used in the Witt group context in [CT/Oj89] and [Oj90].

The other method ([CT78], [CT80] does not use unramified elements directly. It uses the functor $F_{loc}(X)$ associated to a smooth integral variety X over a field k.

Proposition 2.1.10 ([CT80], Prop. 1.2). — *Let k be a field and let F be a functor from the category $\mathcal{C}$ of commutative k-algebras to abelian groups. If the functor F satisfies the specialization property for discrete valuation rings, hence for regular local rings, then associating to any integral smooth k-variety X the group $F_{loc}(X)$ defines a contravariant functor on the category of all smooth integral k-varieties, with morphisms arbitrary k-morphisms.*

Proof : Let $f : X \to Y$ be a k-morphism of integral k-varieties, with Y smooth. Let ξ be the generic point of X, let $P = f(\xi) \in Y$. The fraction field of the local ring $\mathcal{O}_{Y,P}$ is $k(Y)$. Let κ_P be its residue field. Let $\alpha \in F_{loc}(Y)$. Then α comes from an element α_P of $F(\mathcal{O}_{Y,P})$. Let $\beta = f^*(\alpha_p) \in F(k(X))$. Since F satisfies the specialization property, the element β is well-defined. Indeed, even though $\alpha_P \in F(\mathcal{O}_{Y,P})$ may not be uniquely defined, its image γ_P in $F(\kappa_P)$ is uniquely defined by the specialization property for the regular local ring $\mathcal{O}_{Y,P}$, and β is obtained as the image of α_P under the composite map $F(\mathcal{O}_{Y,P}) \to F(\kappa_P) \to F(k(X))$.

Let us show that β belongs to $F_{loc}(X)$. Let $x \in X$ be an arbitrary point. We then have the commutative diagram of local homomorphisms of local rings, where the horizontal maps are induced by f^* :

$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,P} & \longrightarrow & \mathcal{O}_{X,\xi} = k(X) \end{array}$$

Since α belongs to $F_{loc}(Y)$, there exists an element $\alpha_{f(x)} \in F(\mathcal{O}_{Y,f(x)})$ with image $\alpha \in F(k(Y))$. The image of that element in $F(\mathcal{O}_{Y,P})$ may differ from α_P, but their images in $F(k(Y))$ coincide. Arguing as above, we find that β is the image of $\alpha_{f(x)}$ in $F(k(Y))$ under the composite map $F(\mathcal{O}_{Y,f(x)}) \to F(\mathcal{O}_{Y,P}) \to F(k(X))$. From the above diagram we conclude that β is the image of $\alpha_{f(x)}$ under the composite map $F(\mathcal{O}_{Y,f(x)}) \to F(\mathcal{O}_{X,x}) \to F(k(X))$, hence that β comes from $F(\mathcal{O}_{X,x})$. Since x was arbitrary, we conclude that β belongs to $F_{loc}(X)$. We thus have a map $f^* : F_{loc}(Y) \longrightarrow F_{loc}(X)$, under the sole assumption that X is integral and that Y is integral and that the specialization property holds for F and the local rings of Y.

It remains to show contravariance, i.e. given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ with Y and Z regular, we need to show that the two maps $(g \circ f)^*$ and $f^* \circ g^*$ from $F_{loc}(Z)$ to $F_{loc}(X)$ coincide. To check this, let $\xi \in X$, resp. $\eta \in Y$ be the generic points of X, resp. Y. Let $P = f(\xi) \in Y$, $Q = g(P) \in Z$, $R = g(\eta) \in Z$. We then have the commutative diagram of local homomorphisms of local rings :

$$\begin{array}{ccccc} \mathcal{O}_{X,\xi} & \longleftarrow & \mathcal{O}_{Y,P} & \longleftarrow & \mathcal{O}_{Z,Q} \\ & & \downarrow & & \downarrow \\ & & \mathcal{O}_{Y,\eta} & \longleftarrow & \mathcal{O}_{Z,R} \end{array}$$

Given $\alpha \in F_{loc}(Z) \subset F(k(Z))$, we may represent it as an element $\alpha_Q \in F(\mathcal{O}_{Z,Q})$, and this element restricts to an element $\alpha_R \in F(\mathcal{O}_{Z,R})$ with image α in $F(k(Z))$. Applying functoriality of F on rings to the above commutative diagram, we get :

$$\begin{array}{ccccc} \beta & \longleftarrow\!\shortmid & \alpha_P & \longleftarrow\!\shortmid & \alpha_Q \\ & & \downarrow & & \downarrow \\ & & \alpha_i & \longleftarrow\!\shortmid & \alpha_R. \end{array}$$

Now $(g \circ f)^*(\alpha) \in F_{loc}(X)$, by definition, is none but $(g \circ f)^*(\alpha_Q) \in F(k(X))$, which by functoriality of F on rings coincides with $f^*(g^*(\alpha_Q)) = f^*(\alpha_P)$. On the other hand, $g^*(\alpha) \in F_{loc}(Y)$, by definition, is $\alpha_\eta = g^*(\alpha_R) \in F(k(Y))$. Now to compute $f^*(g^*(\alpha))$, we must take a representative of $g^*(\alpha)$ in $F(\mathcal{O}_{Y,P})$ and take its image in $F(\mathcal{O}_{X,\xi}) = F(k(X))$ under f^*. But such a representative is given by α_P, which completes the proof of the equality

$$(g \circ f)^*(\alpha) = f^*(g^*(\alpha)) \in F_{loc}(Z).$$

∎

Remark 2.1.11 : As the reader will check, all that is needed in the definition of the functor F_{loc}, and in the above proof, is a (covariant) functor F on the category of local rings, where morphisms are local homomorphisms.

Proposition 2.1.12. — *Let k be a field and let X/k be a smooth, proper, integral variety. Assume that the functor F satisfies the specialization property for discrete valuation rings containing k and that for all fields $K \supset k$, the projection $\mathbb{A}^1_K = \mathrm{Spec}(K[t]) \to \mathrm{Spec}(K)$ induces a bijection $F(K) \xrightarrow{\sim} F_1(\mathbb{A}^1_K) = F_{loc}(\mathbb{A}^1_K) \subset F(K(t))$. If X is k-birational to projective space, i.e. if the function field $k(X)$ is purely transcendental over k, then the natural map $F(k) \to F_{loc}(X)$ is an isomorphism.*

Proof : See [CT80], Théorème 1.5. ∎

Remark 2.1.13 : If X/k is smooth, Proposition 2.1.10 enables us to evaluate elements of $F_{loc}(X)$ at k-points (rational points) of X. Let X/k be a smooth, proper integral variety. If we find $\alpha \in F_{loc}(X)$ and two k-points A and B such that $\alpha(A) \neq \alpha(B)$ in $F(k)$, then α does not come from $F(k)$, hence X is not k-birational to affine space by Proposition 2.1.12. The reader will find concrete examples in [CT78], § 5, and [CT80], § 2.5.2.

The two propositions above may be applied in various contexts ([CT80], § 2). However the reader should keep in mind that application of this technique for a given proper variety X requires checking that a given element of $F(k(X))$ comes from $F(\mathcal{O}_{X,P})$ at every point $P \in X$. This may be much harder than checking such a condition on discrete valuation rings (and it might require working on an explicit smooth proper model...). Of course, if X is smooth and codimension one purity holds, these conditions are equivalent, but purity is also hard to establish.

§ 2.2 A survey of various functors

In this subsection, we shall briefly survey various functors which may be considered for the formalism above. For each of these functors, we shall mention the known results and open questions regarding the various properties mentioned above : specialization property, injectivity property, (codimension one) purity.

§ 2.2.1 Etale cohomology with coefficients $\mu_n^{\otimes j}$

This case will be discussed in full detail in § 3 and § 4. We let A be a ring, n be a positive integer invertible on A, and we consider the cohomology groups $H^i_{\text{ét}}(A, \mu_n^{\otimes j}) = H^i_{\text{ét}}(\mathrm{Spec}(A), \mu_n^{\otimes j})$ (see § 3 for definitions). The injectivity property is known for discrete valuation rings (see § 3.6 below), hence also the specialization property.

The injectivity property is known for local rings of smooth varieties over a field k ([Bl/Og74], see § 3.8 below). The injectivity property for arbitrary regular local rings is known for $H^1_{\text{ét}}(A, \mu_n^{\otimes j})$ (it actually holds for any noetherian normal domain, cf. [CT/Sa79], Lemma 2.1) and for $H^2_{\text{ét}}(A, \mu_n)$ (it is a consequence of the similar result for the Brauer group, see (3.2) and 2.2.2 below). For $i \geq 3$ and arbitrary regular local rings it is an open question, even when $\dim(A) = 2$.

Codimension 1 purity is known for local rings of smooth varieties over a field ([Bl/Og74], see § 3.8 below; see also Thm. 5.2.7). For arbitrary regular local rings it is known for $H^1_{\text{ét}}(A, \mu_n^{\otimes j})$ (it actually holds for any noetherian normal domain, cf. [CT/Sa79], Lemma 2.1). For regular local rings with $\dim(A) \leq 3$ it is known for $H^2_{\text{ét}}(A, \mu_n)$ (it is a consequence of the similar result for the Brauer group, see (3.2) and 2.2.2 below). For arbitrary regular local rings and $i \geq 3$ it is an open question, even when $\dim(A) = 2$.

The results valid for local rings of smooth varieties over a field also hold for local rings of schemes smooth over a discrete valuation ring (Gillet, unpublished).

§ 2.2.2 The Brauer group

Let $\mathrm{Br}(A) = H^2_{\text{ét}}(A, \mathbb{G}_m)$ be the étale cohomological Brauer group. For an arbitrary regular domain A, with field of fraction K, the map $\mathrm{Br}(A) \to \mathrm{Br}(K)$ is injective (Auslander-Goldman [Au/Go60], Grothendieck [Gr68]) (whence $\mathrm{Br}(A)$ is a torsion group). Thus both injectivity and specialization hold for arbitrary regular local rings.

As for codimension one purity, it is known to hold for $\dim(A) = 2$ (Auslander-Goldman [Au/Go60], [Gr68]) and $\dim(A) \leq 3$ (Gabber [Ga81a]). If A is a local ring of a smooth variety over a field k, purity holds for the prime-to-p torsion of the (torsion group) $\mathrm{Br}(A)$ (see § 3.8). Gabber has very recently announced a proof of codimension one purity for arbitrary regular local rings.

§ 2.2.3 The Witt group

If A is a commutative ring with $2 \in A^*$, we let $W(A)$ be the Witt ring of A as defined in [Kn77]. If A is an arbitrary Dedekind domain, and K is its field of fractions, the map $W(A) \to W(K)$ is injective (see [Kn77]). Thus the specialization property holds. If A is a (not necessarily local) regular domain and K is its fraction field , the map $W(A) \to W(K)$ is known to be injective if $\dim(A) = 2$ (Ojanguren [Oj82a], Pardon [Pa82b]), $\dim(A) = 3$ (Ojanguren [Oj82a] in the local case, Pardon [Pa82b]), $\dim(A) = 4$ (Pardon [Pa82b]). For dimension at least 4, and A global regular, counterexamples are known (Knus, see [Oj82]). The injectivity property is an open question for general regular local rings in dimension at least 5. If A is a local ring of a smooth variety over a field, the injectivity property is known (Ojanguren [Oj80], see § 5.1 and § 5.2 below).

Purity in codimension one for the Witt group is known for regular local rings of dimension two ([CT/Sa79]). It is unknown in higher dimension, even in the geometric case (local ring of a smooth variety over a field). It is quite likely that an L-theory version of Quillen's work [Qu73], in particular of his proof of Gersten's conjecture in the geometric case, could be developed (see [Pa82a]).

§ 2.2.4 The K-theory groups $K_n(A)$

The situation here is rather amazing : the injectivity property for arbitrary discrete valuation rings and $K_n(A)$ for $n \geq 3$ is not known. For K_1, injectivity is trivial. For K_2, it is known for regular local rings of dimension two (Van der Kallen [vdK76]). We refer the reader to Sherman's article [Sh89] for a recent survey.

Merkur'ev points out that specialization holds. It is enough to prove it for a discrete valuation ring A. Let E be the field of fractions of A, let κ be the residue field, and let $t \in A$ be a uniformizing parameter. For any positive integer n, let us consider the composite map $K_n(A) \to K_n(E) \to K_{n+1}(E) \to K_n\kappa$, where the first map is the obvious map, the second one is multiplication by $t \in E^* = K_1(E)$ and where the last map is the residue map. Then this composite map is none other than the reduction map $K_n(A) \to K_n(\kappa)$ (see [Gi86]). It is then clear that any element in the kernel of $K_n(A) \to K_n(E)$ lies in the kernel of $K_n(A) \to K_n(\kappa)$. The argument may be applied to other functors (the Witt group, K-theory with finite coefficients, étale cohomology).

For K-theory, both injectivity and codimension one purity are known for local rings of smooth varieties over a field, as follows from Quillen's proof of the Gersten conjecture ([Qu73]). A different proof of injectivity appears in [CT/Oj92], see § 5.2 below.

§ 2.2.5 The K-theory groups with coefficients $K_n(A, \mathbf{Z}/m)$

The situation here is much better. The injectivity property for discrete valuation rings is known (Gillet [Gi86]). Thus the specialization property certainly holds (alternatively, we could use the same argument as above).

In the geometric case, both injectivity and codimension one purity hold – one only needs to mimic Quillen's argument for K-theory.

§ 2.2.6 Isomorphism classes of principal homogeneous spaces

Given X a scheme and G/X a smooth affine group scheme, the étale Čech cohomology set $\check{\mathrm{H}}^1_{\text{ét}}(X, G)$, denoted $H^1(X, G)$ further below, classifies the set of isomorphism classes of principal homogeneous spaces (torsors) over Y/X under G (cf. [Mi80], chap. III). This is a pointed set, with distinguished element the class of the trivial torsor G/X.

If A is a regular local ring with fraction field K, and G/A is a reductive group scheme, one may ask whether the map $H^1(A, G) \to H^1(K, G)$ has trivial kernel (injectivity property), i.e. whether a torsor which is rationally trivial is trivial over the local ring A.

One may ask whether an element of $H^1(K,G)$ which at each prime p of height one of A comes from an element of $H^1(A_p,G)$ actually comes from an element of $H^1(A,G)$.

The injectivity question was already raised by Serre and Grothendieck in the early days of étale cohomology (1958) (cf. [CT79]).

For arbitrary discrete valuation rings, injectivity was proved by Nisnevich [Ni84]. For split reductive groups, Nisnevich [Ni89] also proved injectivity over a regular local ring of dimension two. For local rings of smooth varieties over an infinite perfect field k, and for a reductive group G defined over k, injectivity was established in [CT/Oj92] (see § 5 below). The case where k is infinite but not necessarily perfect has since been handled by Raghunathan [Ra93].

Codimension one purity holds over a regular local ring of dimension two ([CT/Sa79]). In higher dimension, it is an open question, even for A a local ring of a smooth k-variety and G a reductive k-group scheme. One special case is known, namely codimension one purity for $H^1(A,\mathbb{SL}(D))$ for A a local ring of a smooth variety over a field k and D/k a central simple algebra. Various proofs are available ([CT/Pa/Sr89] in the square free index case, [Ro90], [CT/Oj92] ; another proof will be given in § 5.3).

§ 2.2.7 One more general question

To conclude this section, let us mention a question related to the injectivity property. Let A be a regular local ring, let K be its field of fractions. Let G/A be a reductive group scheme and let X/A be an A-scheme which is a homogeneous space of G. If the set $X(K)$ of K-rational points is nonempty, does it follow that the set of A-points $X(A)$ is nonempty ? If A is a discrete valuation ring, and X/A is proper, this is trivially so. But for $\dim(A) \geq 2$ and X/A proper, the answer is far from obvious. One known case is that of Severi-Brauer schemes ([Gr68]).

As a special example of the question, suppose $2 \in A^*$, let $a_i \in A^*$ $(i = 1, \ldots, n)$. Suppose that the quadratic form $\sum_{i=1,\ldots,n} a_i X_i^2$ has a nontrivial zero with coordinates in K^n. Does it have a zero $(\alpha_1, \ldots, \alpha_n) \in A^n$ with at least one of the α_i's a unit ? For more on this topic, see [CT79]. The case of a regular local ring of dimension two was handled by Ojanguren [Oj82b].

§ 3. Étale cohomology

The aim of this section is to recall some basic facts from étale cohomology, including cohomological purity, as proved in [SGA4], then to go on to the Gersten conjecture in the étale cohomological context, as stated and proved by Bloch and Ogus [Bl/Og74]. We shall try to motivate the Gersten conjecture (§ 3.5). In § 3.8 we shall see that the injectivity and codimension one purity theorems for local rings of smooth varieties over a field are immediate consequences of the Gersten conjecture.

§ 3.1 A few basic properties of étale cohomology

The reader is referred to [SGA4], to Deligne's introductory lectures ([SGA4 1/2], pp. 4-75) or to Milne's book [Mi80] for the definition of étale cohomology. Given a scheme X

and an étale sheaf $\mathcal{F}$ on X, the étale cohomology groups $H^i_{\text{ét}}(X,\mathcal{F})$ will often be denoted $H^i(X,\mathcal{F})$. Subscripts will be used when referring to other topologies, e.g. $H^i_{Zar}(X,\mathcal{F})$. In this section we shall only recall a few basic facts that will enable us to define and study unramified cohomology.

First recall that étale cohomology of an étale sheaf over the spectrum of a field k may be canonically identified with the Galois cohomology of the associated discrete module upon which the absolute Galois group of k acts continously; exact sequences of étale cohomology correspond to exact sequences in Galois cohomology.

Let X be a scheme. The group scheme $\mathbb{G}_m = \mathbb{G}_{m,X}$ which to any X-scheme Y associated the group $\mathbb{G}_m(Y)$ of units of Y defines a sheaf on X for the flat topology, hence in particular for the étale topology and the Zariski topology.

There is a natural isomorphism

$$\mathrm{Pic}(X) = H^1_{Zar}(X,\mathbb{G}_m) \simeq H^1_{\text{ét}}(X,\mathbb{G}_m)$$

(this is Grothendieck's version of Hilbert's theorem 90). Recall that given an integral noetherian scheme X one defines the group $\mathrm{Div}(X)$ of Weil divisors as the free group on points of codimension 1, and the Chow group $CH^1(X)$ as the quotient of the group $\mathrm{Div}(X)$ by the (Weil) divisors of rational functions on X. There is a natural map $\mathrm{Pic}(X) = H^1_{Zar}(X,\mathbb{G}_m) \to CH^1(X)$ which is an isomorphism if X is locally factorial, for instance if X is regular.

In these notes, by Brauer group of X we shall mean Grothendieck's Brauer group, namely

$$\mathrm{Br}(X) = H^2_{\text{ét}}(X,\mathbb{G}_m).$$

The torsion of this group is known to coincide with the Azumaya Brauer group in many cases (and O. Gabber has announced a proof that this is always the case). When X is regular, $\mathrm{Br}(X)$ is a torsion group, more precisely it is a subgroup of the Brauer group $\mathrm{Br}(k(X))$ of the function field of X.

Let n be a positive integer invertible on X. The X-group scheme of n-th roots of unity defines a sheaf μ_n for the étale topology on X. Given any positive integer j, we define the étale sheaf $\mu_n^{\otimes j} = \mu_n \otimes \ldots \otimes \mu_n$ as the tensor product of j copies of μ_n. We let $\mu_n^{\otimes 0}$ be the constant sheaf $\mathbf{Z}/n$ on X. If j is a negative integer, we let $\mu_n^{\otimes j} = \mathrm{Hom}(\mu_n^{\otimes(-j)}, \mathbf{Z}/n)$, where the Hom is taken in the category of étale sheaves on X. For *any* integers j and k, we have $\mu_n^{\otimes j} \otimes \mu_n^{\otimes k} = \mu_n^{\otimes(j+k)}$.

The étale cohomology groups which will be of interest to us here will be the étale cohomology groups $H^i_{\text{ét}}(X,\mu_n^{\otimes j})$, along with $\mathrm{Pic}(X)$ and $\mathrm{Br}(X)$.

One basic property of étale cohomology with torsion coefficients (torsion prime to the characteristic) is *homotopy invariance*, namely for any scheme X, any integer n invertible on X, any nonnegative integer i and any integer j, the natural map

$$H^i_{\text{ét}}(X,\mu_n^{\otimes j}) \longrightarrow H^i_{\text{ét}}(\mathbb{A}^m_X,\mu_n^{\otimes j})$$

induced by the projection of affine m-space $\mathbb{A}^m_X$ onto X is an isomorphism. The reader is referred to [SGA4], XV, Cor. 2.2, or to [Mi80] , VI. Cor. 4.20 p. 240 (for the definition of acyclicity, see [Mi80] p. 232).

For X a scheme and n a positive integer invertible on X, there is a basic exact sequence of étale sheaves on X, the Kummer sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 1\,.$$

where the map $\mathbb{G}_m \to \mathbb{G}_m$ send x to x^n. This sequence gives rise to the short exact sequences

$$0 \longrightarrow H^0(X,\mathbb{G}_m)/H^0(X,\mathbb{G}_m)^n \longrightarrow H^1_{\text{ét}}(X,\mu_n) \longrightarrow {}_n\operatorname{Pic}(X) \longrightarrow 0 \tag{3.1}$$

and

$$0 \longrightarrow \operatorname{Pic}(X)/n\operatorname{Pic}(X) \longrightarrow H^2_{\text{ét}}(X,\mu_n) \longrightarrow {}_n\operatorname{Br}(X) \longrightarrow 0. \tag{3.2}$$

Let X and n be as above, and j be any integer. Let Y be a closed subscheme of X and U be the complement of Y in X. There is a long exact sequence of étale cohomology groups

$$\cdots \to H^i(X,\mu_n^{\otimes j}) \to H^i(U,\mu_n^{\otimes j}) \to H^{i+1}_Y(X,\mu_n^{\otimes j}) \to H^{i+1}(X,\mu_n^{\otimes j}) \to \ldots \tag{3.3}$$

where $H^n_Y(X,.)$ denotes étale cohomology with support in the closed subset Y.

To any morphism $f : V \to X$ one associates the cohomology group $H^n_{f^{-1}(Y)}(V,\mu_n^{\otimes j})$. Letting f run through étale maps, we may sheafify this construction, thus giving rise to étale sheaves $\mathbf{H}^n_Y(\mu_n^{\otimes j})$ on X. These sheaves are actually concentrated on Y.

Another way to define them is as follows. To $\mathfrak{F}$, associate a new sheaf

$$\mathbf{H}^0_Y\mathfrak{F} = \operatorname{Ker}[\mathfrak{F} \longrightarrow j_*j^*\mathfrak{F}]$$

and take derived functors.

There is a spectral sequence

$$E_2^{pq} = H^p_{\text{ét}}(Y,\mathbf{H}^q_Y\mathfrak{F}) \Longrightarrow H^{p+q}_Y(X,\mathfrak{F}). \tag{3.4}$$

§ 3.2 The cohomological purity conjecture

Let X be a regular scheme, $Y \subset X$ a regular closed subscheme. Assume that $Y \subset X$ is everywhere of codimension $c > 0$. Let $n > 0$ be an integer invertible on X. There is a natural map given by the local fundamental class ([SGA4 1/2], Cycle 2.2) :

$$(\mathbf{Z}/n)_Y \longrightarrow \mathbf{H}^{2c}_Y(\mu_{n,X}^{\otimes c})$$

and similarly

$$\mu_{n,Y}^{\otimes(j-c)} \longrightarrow \mathbf{H}^{2c}_Y(\mu_{n,X}^{\otimes j}).$$

In [SGA4], chap. XVI and XIX, [SGA4 1/2], Cycle, and in [SGA5], I.3.1.4, there is a discussion of the (absolute)

Cohomological purity conjecture 3.2.1. — *For $i \neq 2c$ and any integer j, we have*

$$\mathbf{H}^i_Y((\mu_n^{\otimes j})_X) = 0$$

and for $i = 2c$ and any integer j, the map

$$\mu_{n,Y}^{\otimes(j-c)} \longrightarrow \mathbf{H}^{2c}_Y((\mu_n^{\otimes j})_X)$$

is an isomorphism.

Assume that cohomological purity holds for $Y \subset X$ as above. Then the local-to-global spectral sequence degenerates and we get isomorphisms :

$$H^i_Y(X, \mu_n^{\otimes j}) \simeq H^{i-2c}(Y, \mu_n^{\otimes(j-c)}).$$

Let U be the complement of Y in X. In view of spectral sequence (3.4), a consequence of purity would be an exact sequence, often referred to as the Gysin sequence

$$\text{(3.5)} \qquad \dots \to H^i(X, \mu_n^{\otimes j}) \to H^i(U, \mu_n^{\otimes j}) \to H^{i+1-2c}(Y, \mu_n^{\otimes(j-c)}) \to H^{i+1}(X, \mu_n^{\otimes j}) \to \dots$$

where the Gysin map $H^{i+1-2c}(Y, \mu_n^{\otimes(j-c)}) \to H^{i+1}(X, \mu_n^{\otimes j})$ goes "the wrong way".

Remark 3.2.2 : There is a similar discussion in [Gr68, GBIII], § 6. Note however that the varieties Y_i there should at least be assumed to be normal.

Remark 3.2.3 : Very important progress on the absolute cohomological purity conjecture is due to Thomason [Th84] (see his Corollary 3.7).

§ 3.3 Cohomological purity for discrete valuation rings ; the residue map

Let $X = \mathrm{Spec}(A)$, A a discrete valuation ring with residue field κ and quotient field K, let $Y = \mathrm{Spec}(\kappa) \subset X$ be the closed point and $\eta = \mathrm{Spec}(K) \subset X$ be the generic point. Let $n > 0$ be an integer invertible in A. The purity conjecture is known in that case and more generally for a Dedekind scheme. For the proof, see Grothendieck/Illusie, [SGA5], chap. I, § 5.

The Gysin exact sequence (3.5) here reads

$$\text{(3.6)} \quad \dots \to H^i_{\text{ét}}(A, \mu_n^{\otimes j}) \to H^i_{\text{ét}}(K, \mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}_{\text{ét}}(\kappa, \mu_n^{\otimes(j-1)}) \to H^{i+1}_{\text{ét}}(A, \mu_n^{\otimes j}) \to \cdots$$

In particular, we have

$$\mathrm{Ker}(\partial_A) = \mathrm{Im}\,[H^i(A, \mu_n^{\otimes j}) \to H^i(K, \mu_n^{\otimes j})].$$

The map ∂_A, which is called the *residue map* has a description in terms of Galois cohomology (see the discussion in [SGA5], chap. I, pp. 50-52). We may replace A by its completion and thus assume that A is a complete discrete valuation ring. Let $\overline{K}$ be a separable closure of K and let $K_{nr} \subset \overline{K}$ be the maximal unramified extension of K inside $\overline{K}$. Let $\mathfrak{G} = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group, $I = \mathrm{Gal}(\overline{K}/K_{nr})$ be the inertia group and $G = \mathrm{Gal}(K_{nr}/K) = Gal(\overline{\kappa}/\kappa)$. We then have the Hochschild-Serre spectral sequence for Galois cohomology

$$E_2^{pq} = H^p(G, H^q(I, \mu_n^{\otimes j})) \Longrightarrow H^{p+q}(\mathfrak{G}, \mu_{\mathbf{n}}^{\otimes \mathrm{j}}).$$

Let p be the characteristic of κ. Now we have an exact sequence

$$1 \longrightarrow I_p \longrightarrow I \longrightarrow \prod_{l \neq p} \mathbf{Z}_l(1) \longrightarrow 1$$

where $\mathbf{Z}_l(1)$ is the projective limit over m of the G-modules $\mu_{l^m} = \mu_{l^m}(\overline{K}) = \mu_{l^m}(K_{nr})$ (Here $I_p = 0$ if $p = 0$). For this, see [Se68]. Also, we have $H^q(I, \mu_n^{\otimes j}) = 0$ for $q > 2$, and $H^1(I, \mu_n^{\otimes j}) = \mu_n^{\otimes j-1}$, as a G-module. The above spectral sequence thus gives rise to a long exact sequence

$$\ldots \to H^i(G, \mu_n^{\otimes j}) \to H^i(\mathfrak{G}, \mu_n^{\otimes j}) \to H^{i-1}(G, \mu_n^{\otimes j-1}) \to H^{i+1}(G, \mu_n^{\otimes j}) \to \ldots$$

hence in particular to a map

$$\partial : H^i(K, \mu_n^{\otimes j}) \longrightarrow H^{i-1}(\kappa, \mu_n^{\otimes (j-1)}) \,.$$

and one may check that this map agrees (up to a sign) with the map ∂_A in (3.6) .

Equipped with the previous description of the map ∂_A, one proves :

Proposition 3.3.1. — *Let $A \subset B$ be an inclusion of discrete valuation rings with associated inclusion $K \subset L$ of their fields of fractions (which need not be a finite extension of fields). Let κ_A and κ_B be their respective residue fields. Let n be an integer invertible in A, hence in B, and let $e = e_{B/A}$ be the ramification index of B over A, i.e. the valuation in B of a uniformizing parameter of A. The following diagram commutes :*

$$\begin{array}{ccc} H^i(K, \mu_n^{\otimes j}) & \xrightarrow{\partial_A} & H^{i-1}(\kappa_A, \mu_n^{\otimes j-1}) \\ \downarrow \mathrm{Res}_{K,L} & & \downarrow \times e_{B/A}.\mathrm{Res}_{\kappa_A, \kappa_B} \\ H^i(L, \mu_n^{\otimes j}) & \xrightarrow{\partial_B} & H^{i-1}(\kappa_B, \mu_n^{\otimes j-1}). \end{array}$$

∎

Remark 3.3.2 : Let A be a discrete valuation ring with fraction field K and residue field κ, with $\mathrm{char}(\kappa) = 0$. There also exist residue maps

$$\partial_A : \mathrm{Br}(K) \to H^1(\kappa, \mathbf{Q}/\mathbf{Z}).$$

Such maps may be defined in one of two ways.

The first one is by applying the definition above, since $\mathrm{Br}(K)$ is the union of all $H^2(K, \mu_n)$ and $H^1(\kappa, \mathbf{Q}/\mathbf{Z})$ is the union of all $H^1(\kappa, \mathbf{Z}/n)$.

The other one, which has the advantage of being defined under the sole assumption that the residue field κ is perfect, proceeds as follows. First of all, we may assume that A is complete (henselian would be enough). Let K_{nr} be the maximal unramfied extension of K. One first shows $\mathrm{Br}(K_{nr}) = 0$, hence $\mathrm{Br}(K) = \mathrm{Ker}[\mathrm{Br}(K) \to \mathrm{Br}(K_{nr})] = H^2(G, K^*_{nr})$, where $G = \mathrm{Gal}(\kappa_s/\kappa) = \mathrm{Gal}(K_{nr}/K)$. One now has the map $H^2(G, K^*_{nr}) \to H^2(G, \mathbf{Z})$ given by the valuation map. Finally, the group $H^2(G, \mathbf{Z})$ is identified with $H^1(G, \mathbf{Q}/\mathbf{Z}) = H^1(\kappa, \mathbf{Q}/\mathbf{Z})$.

Serre (unpublished) has checked that these two residue maps are actually the opposite of each other.

There is another way to define the residue map

$$\partial_A : \mathrm{Br}(K) \to H^1(\kappa, \mathbf{Q}/\mathbf{Z})$$

when the residue field is perfect. This appears in Grothendieck's paper [GB III], § 2. Comparing these various residue maps might be cumbersome. In many cases, the only thing of interest is that the kernel of each residue map $\partial_A : \mathrm{Br}(K) \to H^1(\kappa, \mathbf{Q}/\mathbf{Z})$ coincides with the image of $\mathrm{Br}(A) \to \mathrm{Br}(K)$.

To complete the picture, it should be noted that ring theorists have a way of their own to define the residue map $\mathrm{Br}(K) \to H^1(\kappa, \mathbf{Q}/\mathbf{Z})$ (K being complete and κ perfect). For this, see [Dr/Kn80].

§ 3.4 Cohomological purity for smooth varieties over a field

For $Y \subset X$ smooth varieties over a field, purity is established by Artin in [SGA4], chap. XVI (see 3.6, 3.7, 3.8, 3.9, 3.10).

THEOREM 3.4.1. — *Let k be a field, let $Y \subset X$ be smooth k-varieties with Y closed in X of pure codimension c. Let $n > 0$ be prime to the characteristic of k. Then cohomological purity holds for $\mu_n^{\otimes j}$. In particular we have a long exact sequence*

$$\ldots \to H^i(X, \mu_n^{\otimes j}) \to H^i(X - Y, \mu_n^{\otimes j}) \to H^{i+1-2c}(Y, \mu_n^{\otimes (j-c)}) \to H^{i+1}(X, \mu_n^{\otimes j}) \to \ldots$$

This has the following corollary :

COROLLARY 3.4.2. — *Let X/k be smooth and irreducible over a field k. Let $F \subset X$ any closed subvariety (not necessarily smooth), $\mathrm{cod}_X F \geq c$. Then the restriction maps*

$$H^i_{\text{ét}}(X, \mu_n^{\otimes j}) \longrightarrow H^i_{\text{ét}}(X - F, \mu_n^{\otimes j})$$

are injective for $i < 2c$ and isomorphisms for $i < 2c - 1$.

Proof: Suppose first that k is a perfect field. Fix i. Let us prove that the restriction map is surjective for $c > (i+1)/2$. The proof is by descending induction on the codimension of F in X, subject to the condition $c > (i+1)/2$. Assume it has been proved for $c+1$. There exists a closed subset $F_1 \subset F$ of codimension at least $c+1$ in X such that $F - F_1$ is regular, hence smooth over the perfect field k, and of pure codimension c in $X - F_1$ (the closed set F_1 is built out of the components of F of dimension strictly smaller than the dimension of F and of the singular locus of F). From $i+1-2c < 0$ and the theorem, we conclude that the restriction map $H^i_{\text{ét}}(X - F_1, \mu_n^{\otimes j}) \longrightarrow H^i_{\text{ét}}(X - F, \mu_n^{\otimes j})$ is surjective. The induction assumption now implies that the restriction map $H^i_{\text{ét}}(X, \mu_n^{\otimes j}) \longrightarrow H^i_{\text{ét}}(X - F, \mu_n^{\otimes j})$ is surjective, hence $H^i_{\text{ét}}(X, \mu_n^{\otimes j}) \longrightarrow H^i_{\text{ét}}(X - F, \mu_n^{\otimes j})$ is surjective. That the restriction map is injective for $c > i/2$ is proved in a similar manner. The proof will be left to the reader. For the induction to work we need to be over a perfect field, to ensure that regular schemes are smooth. But since étale cohomology is invariant under purely inseparable extensions ([SGA4] VIII 1.1, [Mi80] Rem 3.17 p. 77), the corollary holds over all fields. $\square$

Let X be a smooth integral k-variety. For $c = 1$, Corollary 3.4.2 says that for a nonempty open set $U \subset X$, the restriction maps on sections $H^0_{\text{ét}}(X, \mu_n^{\otimes j}) \longrightarrow H^0_{\text{ét}}(U, \mu_n^{\otimes j})$ are isomorphisms, which is nearly obvious, and that the maps $H^1_{\text{ét}}(X, \mu_n^{\otimes j}) \longrightarrow H^1_{\text{ét}}(U, \mu_n^{\otimes j})$ are injective.

For $c = 2$, Corollary 3.4.2 says that if F is a closed subset of codimension at least 2, the restriction maps $H^i_{\text{ét}}(X, \mu_n^{\otimes j}) \longrightarrow H^i_{\text{ét}}(X - F, \mu_n^{\otimes j})$ are injective for $i \leq 3$ and bijective for $i \leq 2$. Using purity for discrete valuation rings, commutativity of étale cohomology with direct limits ([Mi80], III 3.17 p. 119), and the Mayer-Vietoris sequence ([Mi80], III.2.24 p. 110), from the result just proved one may deduce the existence of exact sequences

$$0 \longrightarrow H^1(X, \mu_n^{\otimes j}) \longrightarrow H^1(k(X), \mu_n^{\otimes j}) \longrightarrow \bigoplus_{x \in X^{(1)}} H^0(k(x), \mu_n^{\otimes (j-1)}) \tag{3.7}$$

and

$$H^2(X, \mu_n^{\otimes j}) \longrightarrow H^2(k(X), \mu_n^{\otimes j}) \longrightarrow \bigoplus_{x \in X^{(1)}} H^1(k(x), \mu_n^{\otimes (j-1)}). \tag{3.8}$$

(A similar argument, with more details, will appear in the proof of Theorem 3.8.2 below.)

From the Kummer sequence we get a functorial surjection $H^2(X, \mu_n) \to {}_n\mathrm{Br}(X)$ (see (3.2)). The map $H^2_{\text{ét}}(X, \mu_n) \longrightarrow H^2_{\text{ét}}(X - F, \mu_n)$ is bijective by Corollary 3.4.2. Hence if X/k is a smooth integral variety over a field k of characteristic zero, and F is a closed subset of codimension at least 2, the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(X - F)$ is surjective. It is therefore an isomorphism, since over any regular integral scheme X, for any nonempty open set U, the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective ([GB II] 1.10). All in all, for X a smooth variety over a field k and $\mathrm{char}(k) = 0$, there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(k(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} H^1(k(x), \mathbf{Q}/\mathbf{Z}). \tag{3.9}$$

If $\operatorname{char}(k) = p \neq 0$, then for any prime $l \neq p$, we have a similar exact sequence for the l-primary torsion subgroups of the groups above.

For applications of purity to the Brauer group, see [Fo89] and [Fo92].

§ 3.5 The Gersten conjecture

In general, for $\operatorname{codim}(F) \geq 2$, the restriction map $H^3_{\text{ét}}(X, \mu_n^{\otimes j}) \longrightarrow H^3_{\text{ét}}(X - F, \mu_n^{\otimes j})$ need not be surjective. Here is the simplest example. Let $k = \mathbf{C}$, let X be a smooth, irreducible, affine surface, and let P be a closed point of X. Since X is affine over an algebraically closed field, one basic theorem of étale cohomology ([Mi80], VI.7.2 p. 253) ensures that $H^i_{\text{ét}}(X, \mu_n^{\otimes j}) = 0$ for $i > 2 = \dim(X)$. The Gysin sequence (3.5) for the pair (X, P) and $j = 2$ simply reads

$$0 = H^3(X, \mu_n^{\otimes 2}) \to H^3(X - P, \mu_n^{\otimes 2}) \to H^0(P, \mathbf{Z}/n) \to H^4(X, \mu_n^{\otimes 2}) = 0$$

and since $H^0(P, \mathbf{Z}/n) = \mathbf{Z}/n$, clearly the map $H^3(X, \mu_n^{\otimes 2}) \to H^3(X - P, \mu_n^{\otimes 2})$ is not surjective. The same example could be given with $X = \operatorname{Spec}(A)$ and A the local ring of X at P.

Let k be an arbitrary field, let n be prime to the characteristic of k. Let X/k be a smooth irreducible surface over k, let P be a k-point, and let $Y \subset X$ be a smooth irreducible curve going through P. We then have the following diagram

$$\begin{tikzcd}
 & & & \mathbf{Z}/n \arrow[dl, equal] \\
H^3(X, \mu_n^{\otimes 2}) \arrow[r] \arrow[d, equal] & H^3(X - P, \mu_n^{\otimes 2}) \arrow[r] \arrow[d] & H^4_P(X, \mu_n^{\otimes 2}) \arrow[d] & \\
H^3(X, \mu_n^{\otimes 2}) \arrow[r] & H^3(X - Y, \mu_n^{\otimes 2}) \arrow[r] & H^4_Y(X, \mu_n^{\otimes 2}) \arrow[dr, equal] & \\
 & & & H^2(Y, \mu_n) \\
 & & & \operatorname{Pic} Y / n \operatorname{Pic} Y \arrow[u, "\text{map from Kummer sequence}"']
\end{tikzcd}$$

Let $\alpha \in H^2_{\text{ét}}(Y, \mu_n)$ be the image of $1 \in \mathbf{Z}/n$ under the composite map

$$\mathbf{Z}/n \simeq H^4_P(X, \mu_n^{\otimes 2}) \to H^4_Y(X, \mu_n^{\otimes 2}) \simeq H^2_{\text{ét}}(Y, \mu_n).$$

The point P defines a divisor on the curve Y. Let $[P] \in \mathrm{Pic}(Y)$ be its class. Checking through the definitions ([SGA4 1/2], Cycle) one sees that the image of $[P]$ under the map $\mathrm{Pic}(Y) \to H^2_{\text{ét}}(Y, \mu_n)$ deduced from the Kummer sequence is none other than α.

On the smooth irreducible curve Y, we may find a rational function $f \in k(Y)^*$ with divisor $\mathrm{div}_Y(f) = P + \sum_{i=1,\dots,n} m_i P_i$ where the P_i are closed points of Y, distinct from P and the m_i are integers. If we now define X_1 to be X with the closed points P_i deleted, and Y_1 to be Y with these closed points deleted, we find that the map

$$H^4_P(X, \mu_n^{\otimes 2}) = H^4_P(X_1, \mu_n^{\otimes 2}) \to H^4_{Y_1}(X_1, \mu_n^{\otimes 2})$$

is zero – the first equality follows from the excision property for cohomology with support ([Mi80] III 1.27 p.92 or [CT/Oj92], Prop. 4.4).

In particular, this proves that any element of $H^3(X - P, \mu_n^{\otimes 2})$, once restricted to $H^3(X_1 - Y, \mu_n^{\otimes 2})$, comes from $H^3(X_1, \mu_n^{\otimes 2})$.

Passing over to the cohomology of the local ring of X at P, one gets the following perhaps more striking statement. Let $X = \mathrm{Spec}(A)$ be the local ring of a k-surface at a smooth k-point P. Let $Y = \mathrm{Spec}(A/f) \subset X$ be defined by a regular parameter $f \in A$. An element $\gamma \in H^3(X - P, \mu_n^{\otimes 2})$ need not lift to an element in $H^3(X, \mu_n^{\otimes 2})$. However, its image in $H^3(X - Y, \mu_n^{\otimes 2})$ does lift to an element of $H^3(X, \mu_n^{\otimes 2})$.

What the Gersten conjecture postulates is that what we have just observed is a general phenomenon.

Let X be a regular noetherian scheme of finite Krull dimension. Following Bloch and Ogus [Bl/Og74], we shall say that the Gersten conjecture holds for étale cohomology over X if the following key property holds :

Local acyclicity (Gersten's conjecture for étale cohomology). — *Let $n > 0$ be an integer invertible on X and let j be an integer. Let S be a finite set of points in an affine open set of X. Let $Y \subset X$ be a closed subset of codimension at least $p+1$ in X. Then there exists a closed subset $Z \subset X$, with $Y \subset Z$ and $\mathrm{codim}(Z) = p$ and an open set $U \subset X$ containing S such that the composite map*

$$H^i_Y(X, \mu_n^{\otimes j}) \to H^i_Z(X, \mu_n^{\otimes j}) \to H^i_{Z \cap U}(U, \mu_n^{\otimes j})$$

is zero.

§ 3.6 The Gersten conjecture for étale cohomology : discrete valuation rings

Let $X = \mathrm{Spec}(A)$, A a discrete valuation ring with residue field κ and quotient field K, let $Y = \mathrm{Spec}(\kappa) \subset X$ be the closed point and $\eta = \mathrm{Spec}(K) \subset X$ be the generic point. Let $n > 0$ be an integer invertible in A.

The Gersten conjecture in that case simply asserts that all maps $H^i_Y(X,\mu_n^{\otimes j}) \to H^i_{\text{ét}}(X,\mu_n^{\otimes j})$ are zero, in other words, in view of purity, that the maps $H^{i-1}_{\text{ét}}(\kappa,\mu_n^{\otimes(j-1)}) \to H^{i+1}_{\text{ét}}(A,\mu_n^{\otimes j})$ in the long exact sequence (3.6) are all zero, which in turn is equivalent to saying that this long exact sequence breaks up into short exact sequences

$$(3.10) \qquad 0 \to H^i_{\text{ét}}(A,\mu_n^{\otimes j}) \to H^i_{\text{ét}}(K,\mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}_{\text{ét}}(\kappa,\mu_n^{\otimes(j-1)}) \to 0.$$

This conjecture is known. Although it does not seem to be written explicitly in the literature, the reader will have no difficulty to write down a proof by copying the proof given by Gillet [Gi86] for the analogous statement for K-theory with coefficients. The two key ingredients are the *cup-product* and the *norm map* (also called transfer, or corestriction) on cohomology for finite flat extensions of schemes.

On the discrete valuation ring A, the Kummer sequence gives rise to isomorphisms $H^2(A,\mu_n) \simeq {}_n\mathrm{Br}(A)$ and $H^2(K,\mu_n) \simeq {}_n\mathrm{Br}(K)$. Provided we restrict attention to torsion prime to the residue characteristic (denoted by a dash), we thus have an exact sequence

$$(3.11) \qquad 0 \longrightarrow \mathrm{Br}(A)' \longrightarrow \mathrm{Br}(K)' \longrightarrow H^1(\kappa,\mathbf{Q}/\mathbf{Z})' \longrightarrow 0.$$

This exact sequence was first discussed by Auslander and Brumer [Au/Br68].

§ 3.7 The Gersten conjecture for smooth varieties over a field : results of Bloch and Ogus and some recent developments

In 1974, inspired by the work of Gersten and Quillen ([Qu73]) in algebraic K-theory, Bloch and Ogus stated the analogue of the Gersten conjecture – initially stated by Gersten in the K-theory context – for étale cohomology of smooth varieties over a field. They proved the local acyclicity result, as stated at the end of § 3.5, for varieties X smooth over an infinite field ([Bl/Og74], § 5).

They actually worked in the broader set-up of a Poincaré duality theory with support. The proofs in [Bl/Og74] are by no means easy to follow, if only because they rely on the rather formidable machinery of Grothendieck's duality theory, and on étale homology theory.

As far as the geometry is concerned, their proof relied on Quillen's (semi-local) presentation of hypersurfaces of smooth varieties over an infinite field ([Qu73]). Let us recall here what this presentation is. Given $X = \mathrm{Spec}(A)$ a smooth, integral, affine variety of dimension d over a field k, a finite set $S \subset X$ of points of X and $Y = Spec(A/f) \subset X$ a hypersurface in X, after suitable shrinking of X around S, Quillen produces a projection $X \to \mathbb{A}^{d-1}_k$ such that the induced map $Y \to \mathbb{A}^{d-1}_k$ is finite and étale at the points of $S \cap Y$. As in Quillen's proof, Bloch and Ogus then consider the fibre product of $X \to \mathbb{A}^{d-1}_k$ and $Y \to \mathbb{A}^{d-1}_k$ over $\mathbb{A}^{d-1}_k$.

Since 1974, there have been various efforts to obtain a better understanding of the Gersten conjecture in the étale cohomological context and in other contexts. Let us here mention the paper of Gabber [Ga81b] and various talks by Gillet around 1985. Gillet advocated an approach that would only use the basic properties of étale cohomology with support (functoriality, excision, purity), but would not rely on Grothendieck's duality theory nor on étale homology. This approach was taken anew by Srinivas in a lecture at the Tata Institute in 1992. Gillet and Srinivas both used Quillen's presentation of hypersurfaces of smooth varieties over an infinite field.

On the other hand, a different geometric local presentation of subvarieties of smooth varieties over an infinite field was introduced around 1980.

Motivated by a quadratic version of the Gersten conjecture in K-theory, Ojanguren ([Oj80], [Kn91] proved the following geometric lemma. Given $X = \mathrm{Spec}(A)$ a smooth, integral, affine variety of dimension d over an infinite field k, a closed point $P \in X$ and $Y = Spec(A/f) \subset X$ a hypersurface in X, after suitable shrinking of X around P, one may find an étale map $p : \mathrm{Spec}(A) \to \mathbb{A}^d_k$ and a nonzero element $g \in k[t_1, \ldots, t_d]$ such that the induced map $k[t_1, \ldots, t_d]/g \to A/f$ is an isomorphism after localization at $Q = f(P)$. That approach was again used in [CT/Oj92] to get some analogues of the Gersten conjecture in a noncommutative context (principal homogeneous spaces under linear algebraic groups, over a base which is smooth over a field).

But let us go back to the early eighties. Independently of Ojanguren, in his I.H.E.S. preprint [Ga81b], Gabber proved a preparation lemma (op. cit., Lemma 3.1) which covered Ojanguren's result. This preparation lemma was used again by Gros and Suwa ([Gr/Su88],§ 2). In his talk at Santa Barbara in July 1992, Hoobler sketched a proof of Gersten's conjecture in the geometric case (the Bloch-Ogus result) based on this presentation of geometric semi-local rings on the one hand, on cohomology with support (and such basic property as homotopy invariance) on the other hand. This approach seems to be the nicest one to date, and it gives some additional properties (universal exactness, see already [Ga81b]) which might not have been so easy to prove along the original lines of [Bl/Og74]. The details of this approach are presently being written down ([CT/Ho/Ka93]). Some simple cases of this approach will be discussed in § 5 below.

An amusing aspect of this historical sketch is that not until June 1993 did I realize that Ojanguren's presentation lemma and Gabber's presentation were so close to each other.

To close this discussion of the Gersten conjecture, let us point out that in his 1985 talks, Gillet had announced a proof of the Gersten conjecture for étale cohomology (local acyclicity lemma) for schemes smooth over a Dedekind ring.

§ 3.8 Injectivity property and codimension one purity property

THEOREM 3.8.1 (INJECTIVITY PROPERTY). — *Let $i > 0$, $n > 0$ and j be integers. Let k be a field of characteristic prime to n. If A is a semi-local ring of a smooth integral k-variety X, and $K = k(X)$ is the fraction field of A, then the natural map*

$$H^i(A, \mu_n^{\otimes j}) \to H^i(K, \mu_n^{\otimes j})$$

is injective.

Proof : Let us first assume that k is infinite. We may assume that X is affine. Let S be the finite set of points of X with associated semi-local ring A. Let $\alpha \in H^i(A, \mu_n^{\otimes j})$ be in the kernel of the above map. Using the commutativity of étale cohomology with direct limit of commutative rings with affine flat transition homomorphisms, after shrinking X we may assume that α is represented by a class $\beta \in H^i(X, \mu_n^{\otimes j})$ which vanishes in $H^i(U, \mu_n^{\otimes j})$ for $U \subset X$ a suitable nonempty open set. Let Y be the closed set which is the complement of U in X. We have the exact sequence of cohomology with support

$$H^i_Y(X, \mu_n^{\otimes j}) \to H^i(X, \mu_n^{\otimes j}) \to H^i(U, \mu_n^{\otimes j})$$

hence β comes from $\gamma \in H^i_Y(X, \mu_n^{\otimes j})$. Now the local acyclicity theorem ensures that there exists a closed set Z containing Y and a nonempty open set $V \subset X$ containing S such that the composite map

$$H^i_Y(X, \mu_n^{\otimes j}) \to H^i_Z(X, \mu_n^{\otimes j}) \to H^i_{Z \cap V}(V, \mu_n^{\otimes j})$$

is zero. By functoriality, this implies that the image of γ, hence also of β in $H^i(V, \mu_n^{\otimes j})$ vanishes. Thus $\alpha = 0$. The proof is therefore complete when k is infinite. If k is finite, one may find two infinite extensions of k of coprime pro-order. If $\alpha \in H^i(A, \mu_n^{\otimes j})$ is as above, then by the commuting property of étale cohomology just mentioned, one finds two finite extensions k_1/k and k_2/k of coprime degrees, such that α vanishes in $H^i(A \otimes_k k_r, \mu_n^{\otimes j})$ for $r = 1, 2$. A standard argument using traces (transfers) then shows $\alpha = 0 \in H^i(A, \mu_n^{\otimes j})$. □

Theorem 3.8.2 (CODIMENSION ONE PURITY THEOREM). — *Let $i > 0$, $n > 0$ and j be integers. Let k be a field of characteristic prime to n. Let A be a semi-local ring of a smooth integral k-variety X, with fraction field $K = k(X)$, and let α be an element of $H^i(K, \mu_n^{\otimes j})$. If for each height one prime p of A, α belongs to the image of $H^i(A_p, \mu_n^{\otimes j})$, then α comes from a (unique) element of $H^i(A, \mu_n^{\otimes j})$.*

Proof : Unicity of the lift in $H^i(A, \mu_n^{\otimes j})$ follows from the previous theorem. Let us first assume that k is infinite. Let $S \subset X$ be the finite set of points corresponding to the maximal ideals of A.

We first claim that there exists an open set U of X which contains the generic points of all the codimension one irreducible closed subvarieties of X going through a point of S and is such that the class α comes from $\beta \in H^i(U, \mu_n^{\otimes j})$. Since étale cohomology commutes with filtering projective limits of schemes with flat affine transition morphisms, we may find an open set U of X such that $\alpha \in H^i(K, \mu_n^{\otimes j})$ comes from $\alpha_U \in H^i(U, \mu_n^{\otimes j})$. Assume that U does not contain the generic point P of a codimension one irreducible closed subvariety of X going through a point of S. By assumption α comes from some class $\alpha_P \in H^i(A_p, \mu_n^{\otimes j})$, which we may extend to a class in $H^i(V, \mu_n^{\otimes j})$ for V some open set of X containing P. We would like α_U and α_V to agree on the overlap $V \cap U$. But (P being fixed) $\lim\limits_{P \in V} (U \cap V) = \operatorname{Spec} k(X)$ and α_U and α_V agree when restricted to Spec $k(X)$, hence they agree on some $U \cap V$ with V small enough. From the Mayer-Vietoris sequence in étale cohomology ([Mi80], III 2.24 p. 110), we conclude that α comes from $H^i(U \cup V, \mu_n^{\otimes j})$.

Shrinking X around S, we may further assume that the closed set $F = X - U$ is of codimension at least 2 in X. In the case $i \leq 2$, cohomological purity ensures that the restriction map $H^i(X, \mu_n^{\otimes j}) \to H^i(U, \mu_n^{\otimes j})$ is surjective (§ 3.4), and the theorem follows. Suppose $i \geq 3$. By the local acyclicity theorem (§ 3.6 and § 3.7) there exists a closed set F_1 containing F, with $\mathrm{codim}(F) \geq 1$, and an open set $V \subset X$ containing S such that the composite vertical map on the right hand side of the following diagram is zero :

$$\begin{array}{ccccc}
H^i(X, \mu_n^{\otimes j}) & \longrightarrow & H^i(X - F, \mu_n^{\otimes j}) & \longrightarrow & H^i_F(X, \mu_n^{\otimes j}) \\
\downarrow & & \downarrow & & \downarrow \\
H^i(X, \mu_n^{\otimes j}) & \longrightarrow & H^i(X - F_1, \mu_n^{\otimes j}) & \longrightarrow & H^i_{F_1}(X, \mu_n^{\otimes j}) \\
\downarrow & & \downarrow & & \downarrow \\
H^i(V, \mu_n^{\otimes j}) & \longrightarrow & H^i(V - F_1, \mu_n^{\otimes j}) & \longrightarrow & H^i_{F_1 \cap V}(V, \mu_n^{\otimes j})
\end{array}$$

Now the image of β in $H^i(V - F_1, \mu_n^{\otimes j})$ comes from $H^i(V, \mu_n^{\otimes j})$, and the theorem is proved when k is infinite. For k, the result follows as in 3.8.1. $\square$

There are Brauer group versions of the above results, which we gather in one result.

THEOREM 3.8.3. — *Let k be a field of characteristic zero. Let A be a semi-local ring of a smooth integral k-variety X with fraction field $K = k(X)$, and let α be an element of* $\mathrm{Br}(K)$. *If for each prime p of height one of A, the element α belongs to the image of* $\mathrm{Br}(A_p)$*, then α comes from a (unique) element of* $\mathrm{Br}(A)$. *There is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(A) \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{p \in A^{(1)}} H^1(\kappa_p, \mathbf{Q}/\mathbf{Z})$$

where the last map is the direct sum of the residue maps $\partial_{A_p} : \mathrm{Br}(K) \longrightarrow H^1(\kappa_p, \mathbf{Q}/\mathbf{Z})$.

Proof : It is enough to note that for any positive n (here n prime to the characteristic of k would be enough), and A as above, in particular local, the Kummer exact sequence gives rise to compatible isomorphisms $H^2(A, \mu_n) \simeq {}_n\mathrm{Br}(A)$ and $H^2(K, \mu_n) \simeq {}_n\mathrm{Br}(K)$ (see (3.2)). The first result now follows from the previous results, and the exact sequence then follows from an application of (3.6) to each discrete valuation ring A_p. $\square$

Remark 3.8.4 : Except for the injectivity of the Brauer group of A into the Brauer group of K – which actually holds for any regular integral domain – the previous theorem is nothing but the semi-local version of sequence (3.9). Indeed, at the level of H^2, only the purity theorem comes into play. It is only from H^3 onwards that one must appeal to the local acyclicity theorem (Gersten's conjecture).

§ 4. Unramified cohomology

§ 4.1 Equivalent definitions of unramified cohomology, homotopy invariance

THEOREM 4.1.1. — *Let k be a field, X/k a smooth integral k-variety and $n > 0$ be an integer prime to* char*(k). Let $k(X)$ be the function field of X. Let i and j be integers. The following subgroups of $H^i(k(X), \mu_n^{\otimes j})$ coincide :*

a) *the group of elements $\alpha \in H^i(k(X), \mu_n^{\otimes j})$ which, at any point P of codimension 1 in X with local ring $\mathcal{O}_{X,P}$ and residue field κ_P, come from a class in $H^i(\mathcal{O}_{X,P}, \mu_n^{\otimes j})$, or equivalently which satisfy $\partial_{\mathcal{O}_{X,P}}(\alpha) = 0 \in H^{i-1}(\kappa_P, \mu_n^{\otimes(j-1)})$;*

b) *the group of elements of $H^i(k(X), \mu_n^{\otimes j})$ which at any point $P \in X$ come from a class in $H^i(\mathcal{O}_{X,P}, \mu_n^{\otimes j})$;*

c) *the group $H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ of global sections of the Zariski sheaf $\mathcal{H}^i(\mu_n^{\otimes j})$, which is the sheaf associated to the Zariski presheaf $U \to H^i_{\text{ét}}(U, \mu_n^{\otimes j})$. ;*

d) *if X/k is complete, the group $H^i_{nr}(k(X)/k, \mu_n^{\otimes j})$ consisting of elements α of $H^i(k(X), \mu_n^{\otimes j})$ such that for any discrete valuation ring $A \subset k(X)$ with $k \subset A$ and with field of fractions $k(X)$, the element α comes from $H^i(A, \mu_n^{\otimes j})$, or equivalently the residue $\partial_A(\alpha) = 0 \in H^{i-1}(\kappa_A, \mu_n^{\otimes(j-1)})$;*

e) *if X/k is complete, the group of elements $\alpha \in H^i(k(X), \mu_n^{\otimes j})$ such that for any valuation ring (not necessarily discrete) A of $k(X)$ containing k, α comes from $H^i(A, \mu_n^{\otimes j})$.*

Proof : The proof follows from the codimension one purity property for local rings of X (Thm. 3.8.2) and from the general argument given in Proposition 2.1.8. (Proposition 2.1.8 assumed the codimension one purity property for all regular local rings, but only used it for those of geometric origin.) The characterisation of "unramified classes" by means of residues at discrete valuation rings comes from cohomological purity for such rings (§ 3.3).

Remark 4.1.2 : In the notation of § 2, the group appearing in b), is none other than the group $F_{loc}(X)$ associated to the functor $F(A) = H^i(A, \mu_n^{\otimes j})$. According to the above theorem, the group in b) coincides with the group $H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ in c). The functor $X \longmapsto H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ is a natural contravariant functor on the category of all k-varieties. Restricting ourselves to the category smooth integral k-varieties, we thus get another proof of Prop. 2.1.10 for the functor $F(A) = H^i(A, \mu_n^{\otimes j})$. Note however that the proof of Prop. 2.1.10 only used the specialization property, whereas the present one uses the codimension one purity theorem, hence the Gersten conjecture.

Note that the functor $X \longmapsto H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ is a natural contravariant functor on the category of all k-varieties. If we restrict this functor to the category of smooth integral k-varieties, and we replace $H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ by the group appearing in b), which is in the notation of § 2 is none other than $F_{loc}(X)$ for the functor $F(A) = H^i(A, \mu_n^{\otimes j})$, we get another proof of Prop. 2.1.10. Note however that the proof of Prop. 2.1.10 only used the specialization property, whereas the present one uses the codimension one purity theorem, hence the Gersten conjecture.

Remark 4.1.3 : The k-birational invariance of the groups $H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ on smooth, proper, integral varieties, was also observed (in zero characteristic) by Barbieri-Viale [BV92a]. M. Rost tells me that the k-birational invariance of the group of elements in Theorem 4.1.1.a can be proved in a "field-theoretic" fashion, without using the Gersten conjecture (see [Ro93]).

Proposition 4.1.4. — *Let k be a field and $n > 0$ be an integer prime to* char.(k). *Let i, j and $m > 0$ be integers. Let $K = k(t)$ be the rational field in one variable over k. Then the natural map $H^i(k, \mu_n^{\otimes j}) \to H^i(k(t), \mu_n^{\otimes j})$ induces a bijection $H^i(k, \mu_n^{\otimes j}) \simeq H^i_{nr}(k(t)/k, \mu_n^{\otimes j})$.*

Proof : First assume that k is perfect. For the affine line $\mathbb{A}^1_k$, and for a closed, reduced, proper subset $F \subset \mathbb{A}^1_k$, we may write the long exact sequence of cohomology with support and use purity (§ 3.3 or § 3.4) to translate it as

$$\ldots \to H^i(\mathbb{A}^1_k, \mu_n^{\otimes j}) \to H^i(\mathbb{A}^1_k - F, \mu_n^{\otimes j}) \to \bigoplus_{P \in F} H^{i-1}(k(P), \mu_n^{\otimes j-1}) \to \ldots$$

Now homotopy invariance for étale cohomology (see § 3.1) guarantees that the pull-back map $H^i(k, \mu_n^{\otimes j}) \to H^i(\mathbb{A}^1_k, \mu_n^{\otimes j})$ induced by the structural morphism is an isomorphism. For any F, the map $H^i(k, \mu_n^{\otimes j}) \to H^i(\mathbb{A}^1_k - F, \mu_n^{\otimes j})$ is injective, as may be seen by specializing to a k-rational point if k is infinite, or by using a 0-cycle of degree one and a norm argument if k is finite (as a matter of fact, $H^i(k, \mu_n^{\otimes j}) = 0$ for k finite and $i > 1$). We thus get short exact sequences

$$0 \to H^i(k, \mu_n^{\otimes j}) \to H^i(\mathbb{A}^1_k - F, \mu_n^{\otimes j}) \to \bigoplus_{P \in F} H^{i-1}(k(P), \mu_n^{\otimes j-1}) \to 0.$$

We may now let F be bigger and bigger, and we ultimately get the exact sequence

$$0 \to H^i(k, \mu_n^{\otimes j}) \to H^i(k(t), \mu_n^{\otimes j}) \to \bigoplus_{P \in \mathbb{A}^{1(1)}_k} H^{i-1}(k(P), \mu_n^{\otimes j-1}) \to 0 \tag{4.1}$$

from which the proposition follows. (If k is not perfect, simply observe that étale cohomology does not change by purely inseparable extensions.) ∎

Theorem 4.1.5. — *Let k be a field and $n > 0$ be an integer prime to* $\mathrm{char}(k)$*. Let E be a function field over k. Let i, j and $m > 0$ be integers. Let $K = E(t_1, \ldots, t_m)$ be the rational field in m variables over E. Then the natural map $H^i(E, \mu_n^{\otimes j}) \to H^i(K, \mu_n^{\otimes j})$ induces a bijection $H^i_{nr}(E/k, \mu_n^{\otimes j}) \simeq H^i_{nr}(K/k, \mu_n^{\otimes j})$. In particular, the natural map $H^i(k, \mu_n^{\otimes j}) \to H^i(k(t_1, \ldots, t_m), \mu_n^{\otimes j})$ induces a bijection $H^i(k, \mu_n^{\otimes j}) \to H^i_{nr}(k(t_1, \ldots, t_m)/k, \mu_n^{\otimes j})$.*

Proof : Let be t a variable, and let E be a function field over k. It is enough to show that the map $H^i(E, \mu_n^{\otimes j}) \to H^i(E(t), \mu_n^{\otimes j})$ induces an isomorphism

$$H^i_{nr}(E/k, \mu_n^{\otimes j}) \to H^i_{nr}(E(t)/k, \mu_n^{\otimes j}).$$

First note that the map $H^i(E, \mu_n^{\otimes j}) \to H^i(E(t), \mu_n^{\otimes j})$ is injective (any class $\alpha \in H^i(E, \mu_n^{\otimes j})$ which vanishes in $H^i(E(t), \mu_n^{\otimes j})$ actually vanishes in $H^i(U, \mu_n^{\otimes j})$ for some open set U of the affine line over E; if E is infinite, evaluation at an E-point shows that $\alpha = 0$, if E is finite, see above). We have a natural embedding (Lemma 2.1.3)

$$H^i_{nr}(E/k, \mu_n^{\otimes j}) \subset H^i_{nr}(E(t)/k, \mu_n^{\otimes j}).$$

Take $\beta \in H^i_{nr}(E(t)/k, \mu_n^{\otimes j})$. Restricting attention to discrete valuation rings of $E(t)$ which contain E, and using Prop. 4.1.4, one sees that β comes from a class γ in $H^i(E, \mu_n^{\otimes j})$. Let $A \subset E$ be a discrete valuation ring with $k \subset A$ and $E = \mathrm{qf}(A)$. Let $\pi \in A$ be a uniformizing parameter of A and let $B \subset E(t)$ be the discrete valuation ring which is the local ring of $A[t]$ at the ideal of height one defined by π in $A[t]$. If κ_A is the residue field of A, the residue field κ_B of B is none other than the rational function field $\kappa_A(t)$. The discrete valuation ring B is unramified over A, in other words $e_{B/A} = 1$. By the functorial behaviour of the residue map (Prop. 3.3.1), we have a commutative diagram :

$$\begin{array}{ccc}
H^i(E, \mu_n^{\otimes j}) & \xrightarrow{\partial_A} & H^{i-1}(\kappa_A, \mu_n^{\otimes j-1}) \\
\downarrow \mathrm{Res}_{E,E(t)} & & \downarrow \mathrm{Res}_{\kappa_A,\kappa_A(t)} \\
H^i(E(t), \mu_n^{\otimes j}) & \xrightarrow{\partial_B} & H^{i-1}(\kappa_A(t), \mu_n^{\otimes j-1}).
\end{array}$$

Now the map $\mathrm{Res}_{\kappa_A,\kappa_B} : H^{i-1}(\kappa_A, \mu_n^{\otimes j-1}) \to H^{i-1}(\kappa_B, \mu_n^{\otimes j-1})$ is *injective* (same arguments as above), hence $\partial_B(\beta) = 0$ implies $\partial_A(\gamma) = 0$. Since A was arbitrary, we conclude that γ belongs to $H^i_{nr}(E/k, \mu_n^{\otimes j})$. □

Remark 4.1.6 : In the context of the Witt group, a similar proof may be given (see [CT/Oj89] and [Oj90], § 7). This is also the case in the context of the Brauer group (see [Sa85], [CT/S93]). Another proof could be given along the lines of Proposition 2.1.9; however such a proof relies on the (known, but difficult) codimension one purity theorem 3.8.2 for a two dimensional regular local ring of a smooth variety over a field.

§ 4.2 Computing unramified cohomology

In this section, we list cases where unramified cohomology has been computed, or at least where methods have been devised to detect nontrivial elements in unramified cohomology groups.

Let us first discuss unramified H^1.

PROPOSITION 4.2.1. — *Let X be a smooth, complete, connected variety over a field k. Let n be a positive integer prime to* $\mathrm{char}(k)$.

(a) For any $j \in \mathbf{Z}$, the natural map $H^1(X, \mu_n)^{\otimes j}) \to H^1(k(X), \mu_n^{\otimes j})$ induces an isomorphism between $H^1(X, \mu_n^{\otimes j})$ and $H^1_{nr}(k(X)/k, \mu_n^{\otimes j})$.

(b) If k is algebraically closed, this map induces an isomorphism of finite groups ${}_n\mathrm{Pic}(X) \simeq H^1_{nr}(k(X)/k, \mu_n)$.

(c) If k is algebraically closed and $\mathrm{char}(k) = 0$*, then there is a (non-canonical) isomorphism $H^1_{nr}(k(X)/k, \mu_n^{\otimes j}) \simeq (\mathbf{Z}/n)^{2q} \oplus {}_n\mathrm{NS}(X)$, where q denotes the dimension of the Picard variety of X, also equal to the dimension of the coherent cohomology group $H^1(X, \mathcal{O}_X)$, and where* $\mathrm{NS}(X)$ *denotes the Néron-Severi group of X, which is a finitely generated abelian group.*

Proof : After Theorem 4.1.1, Statement (a) is just a reinterpretation of exact sequence (3.7) (as a matter of fact, the special case of Purity required for (a) is easy to prove and holds under quite general assumtions).

Statement (b) then follows from exact sequence (3.1), since for X/k as above, $H^0(X, \mathbb{G}_m) = k^*$ and $k^*/k^{*n} = 1$.

For X/k as in (c), the Néron-Severi group $\mathrm{NS}(X)$ of classes of divisors modulo algebraic equivalence is well-known to be a finitely generated group. The kernel of the natural surjective map $\mathrm{Pic}(X) \to \mathrm{NS}(X)$ is the group $A(k)$ of k-points of an abelian variety A, the Picard variety of X. As such, it is a divisible group. Thus, as abelian groups, and in a non-canonical way, we have $\mathrm{Pic}(X) \simeq A(k) \oplus \mathrm{NS}(X)$. Since $\mathrm{char}(k) = 0$, the dimension of the Picard variety is equal to $q = \dim H^1(X, \mathcal{O}_X)$. By the theory of abelian varieties, this implies that ${}_nA(k) \simeq (\mathbf{Z}/n)^{2q}$ for all $n > 0$ (here again, use is made of the $\mathrm{char}(k) = 0$ hypothesis). ∎

Remark 4.2.2 : Let k be a field of characteristic prime to the positive integer n. Let A be a discrete valuation ring with $k \subset A$ and let K be the fraction field of A. By means of the Kummer sequence one may identify the natural map $A^*/A^{*n} \to K^*/K^{*n}$ with the map $H^1(A, \mu_n) \to H^1(K, \mu_n)$. Thus for any smooth, complete, integral variety X/k, the group $H^1_{nr}(k(X)/k, \mu_n)$ coincides with the group of unramified elements in $k(X)$ associated to the functor $R \to R^*/R^{*n}$, in the sense of Definition 2.1 For any A as above, there is an obvious exact sequence

$$1 \to A^*/A^{*n} \to K^*/K^{*n} \to \mathbf{Z}/n \to 0.$$

We may now interpret the computations of § 1 as a computation of unramified H^1 with coefficients $\mathbf{Z}/2$ for hyperelliptic curves over the complex field, that is as a computation in the 2-torsion subgroup of the Picard group of such curves.

Let us now discuss unramified H^2. The following proposition is just a reformulation of some results of Grothendieck ([Gr68]).

PROPOSITION 4.2.3. — *Let X be a smooth, projective, connected variety over a field k.*
(a) Let n be a positive integer prime to $\mathrm{char}(k)$. *The isomorphism $H^2(k(X),\mu_n) \simeq {}_n\mathrm{Br}(k(X))$ coming from the Kummer exact sequence induces an isomorphism between $H^2_{nr}(k(X)/k,\mu_n)$ and ${}_n\mathrm{Br}(X)$.*
(b) If k is algebraically closed and n is a power of a prime number $l \neq \mathrm{char}(k)$, then $H^2_{nr}(k(X)/k,\mu_n) \simeq (\mathbf{Z}/n)^{B_2-\rho} \oplus {}_nH^3(X,\mathbf{Z}_l)$, where $B_2-\rho$ is the "number of transcendental cycles, i.e. the difference between the rank of the étale cohomology group $H^2(X,\mathbf{Q}_l)$ and the rank of $NS(X)\otimes\mathbf{Q}_l$, and where $H^3(X,\mathbf{Z}_l)$ denotes the third étale cohomology group with $\mathbf{Z}_l$-coefficients.

Proof : After Theorem 4.1.1, statement (a) is just a reformulation of the discussion at the end of § 3.4 (see the exact sequences (3.7) and (3.8)). As for b), Grothendieck explains in [Gr68] how starting from exact sequence (3.2), for any prime $l \neq \mathrm{char}(k)$, one gets the exact sequence

$$0 \to \mathrm{NS}(X)\otimes\mathbf{Q}_l/\mathbf{Z}_l \to H^2(X,\mathbf{Q}_l/\mathbf{Z}_l) \to \mathrm{Br}(X)\{l\} \to 0$$

where $\mathrm{Br}(X)\{l\}$ denotes the l-primary subgroup of the torsion group $\mathrm{Br}(X)$ (twisting by the roots of unity has been ignored). Statement (b) easily follows. □

Let us assume that k is algebraically closed of characteristic zero. If the variety X is unirational, then it may be shown that $B_2-\rho=0$. Thus the only possible nontrivial part in the Brauer group (i.e. in $H^2_{nr}(k(X),\mu_n)$ for some n) comes from the torsion subgroup of $H^3(X,\mathbf{Z}_l)$ for some prime l. It may be shown by comparison with classical cohomology that this torsion subgroup is zero for almost all l.

4.2.4. Literature on unramified H^2 : The invariant appearing in Proposition 4.2.3 has been much studied. The first computation in this direction is due to Artin and Mumford ([Ar/Mu72]). They produced one of the first examples of unirational varieties that are not rational. Their example is a threefold, a conic bundle over the complex plane. Their proof used an explicit smooth projective variety X and a direct computation of the torsion subgroup in the third integral cohomology group of this variety.

That there was something to be gained in apprehending the Brauer group as an unramified Brauer group was revealed by Saltman [Sa84], when he exhibited fields $k(V)^G$ of invariants for a faithful linear action of a finite group G on a vector space V, over an algebraically closed field k of characteristic zero, that were not rational because the unramified Brauer group is nonzero. In this case, one would have been at a loss to construct a smooth projective model.

The unramified point of view was further developed by Saltman ([Sa85], [Sa88],[Sa90]) and Bogomolov ([Bo87], [Bo89]). Their work is discussed in the survey [CT/Sa88].

Bogomolov [Bo87] (see also [CT/Sa88]) gave a formula for $\mathrm{Br}_{nr}(k(V)^G)$ for G finite and V/k as above, namely he showed that this group may be identified with the subgroup of the integral cohomology group $H^2(G,\mathbf{Q}/\mathbf{Z}) \simeq H^3(G,\mathbf{Z})$ consisting of classes whose restrictions to all abelian subgroups of G vanish. This condition is equivalent to the vanishing of restrictions to all abelian subgroups with at most two generators.

Bogomolov's formula in turn is subsumed in Saltman's formula [Sa90] for the unramified Brauer group of twisted multiplicative field invariants.

Let G be a connected, reductive group over an algebraically closed field k of characteristic zero. Let $G \subset GL_n$ be an injective homomorphism. One may then consider the quotient variety $X = GL_n/G$, which is an affine variety since G is reductive. It is an open question whether the (obviously unirational) variety X is a rational variety. Saltman ([Sa85], [Sa88]) and Bogomolov [Bo89] have studied this problem. For $G = PGL_r$, Saltman [Sa85] (see [CT/Sa87] for a different proof) and Bogomolov for G arbitrary ([Bo89], see also [CT/Sa88]) showed that the unramified Brauer group of $k(GL_n/G)$ is always zero.

Taking the point of view of unramified cohomology, one may handle the Artin-Mumford example quoted above, as well as other examples, in a manner totally parallel to that of § 1. Indeed, in § 1, the function field $L = \mathbf{C}(X)$ is a quadratic field extension of the rational function field in one variable $K = \mathbf{C}(t)$, that is it is the function field of a 0-dimensional conic over $\mathbf{C}(t)$. A key point in the proof we gave in § 1 is the exact sequence

$$1 \to \mathbf{Z}/2 \to K^*/K^{*2} \to L^*/L^{*2}$$

that is

$$1 \to \mathbf{Z}/2 \to H^1(K,\mathbf{Z}/2) \to H^1(L,\mathbf{Z}/2)$$

associated to the quadratic field extension L/K., which gives control on the kernel of the map $K^*/K^{*2} \to L^*/L^{*2}$. Let $\alpha \in H^1(K,\mathbf{Z}/2)$ be the generator of the above kernel. Note that α is a ramified element in K^*/K^{*2}. Nontrivial unramified elements in L^*/L^{*2} are then produced in the following way. Let $\beta \neq 1,\alpha$ be a ramified element in K^*/K^{*2}. Suppose that all ramification of β is "contained" in the ramification of α. Since α dies in L^*/L^{*2}, so does its ramification, hence so does the ramification of β when going over to L. However, because β does not lie in the the kernel of $K^*/K^{*2} \to L^*/L^{*2}$, β itself does not die when going up to L : this produces a nontrivial unramified element in L^*/L^{*2}.

Now let L be the function field of a 1-dimensional conic over the rational function field in two variables $K = \mathbf{C}(t_1,t_2)$. If L is the function field of a smooth projective conic over a field K, it is a special case, already known to Witt ([Wi35] p.465), of a result of Amitsur that the kernel of the map $H^2(K,\mathbf{Z}/2) \to H^2(L,\mathbf{Z}/2)$ is either 0 or $\mathbf{Z}/2$, depending on whether the conic has a K-rational point or not. In the second case, the nontrivial element in $H^2(K,\mathbf{Z}/2) \simeq {}_2\mathrm{Br}(K)$ is the class of the quaternion algebra α associated to the conic. Now this is an exact analogue of the situation above. To produce an unramified element in $H^2(L,\mathbf{Z}/2)$ one should therefore look for an element $\beta \in H^2(K,\mathbf{Z}/2)$ which is neither 0 nor α, but whose ramification (as an element of $H^2(K,\mathbf{Z}/2)$) is "somehow" dominated by the ramification of α, so that when going over to L, all ramification of $\beta_L \in H^2(L,\mathbf{Z}/2)$ vanishes – although β itself does not vanish.

This is precisely what happens in the Artin-Mumford counterexample. More examples of the kind may be produced along these lines. For more details, see [CT/Oj89].

4.2.5 Beyond H^2 :

In the following discussion, unless otherwise specified, we assume that the ground field k is algebraically closed of characteristic zero. Since algebraic geometry often produces unirational varieties whose rationality is an open question (such as $X = GL_n/G$ for G a reductive subgroup of GL_n, see above), it would be very useful to be able to compute the higher unramified cohomology groups. $H^i_{nr}(k(X), \mathbf{Z}/n)$ for such varieties. Also, for some of these varieties the lower unramified cohomology groups might be trivial. This is the case for unramified H^2 of $k(X)$ for $X = GL_n/G$ with G as above and connected (Bogomolov, see comments above), and it may be the case for some finite groups $G \subset GL_n$. However, for higher unramified cohomology groups, there is no ready-made formula available, such as those appearing in Propositions 4.2.1 and 4.2.3 (see however the exact sequence (4.2) below). As a matter of fact, one does not even know whether the groups $H^i_{nr}(k(X), \mathbf{Z}/n)$ for $i \geq 3$ are finite. For some special varieties for which finiteness of $H^3_{nr}(k(X), \mathbf{Z}/n)$ is known, see § 4.3.

The first task when dealing with higher unramified cohomology has been to produce examples where it is nontrivial. In [CT/Oj89], Ojanguren and I produced an example of a unirational variety (of dimension 6) for which none of the previously used birational invariants could be used, but for which unramified H^3 could detect non-rationality. The construction was inspired by the Artin-Mumford example, as reexamined above. As a substitute for the Witt/Amitsur result, we used a result of Arason [Ar75] computing the kernel of $H^3(K, \mathbf{Z}/2) \to H^3(F, \mathbf{Z}/2)$ when F is the function field of a quadric defined by a 3-fold Pfister form $< 1, -a > \otimes < 1, -b > \otimes < 1, -c >$ $(a, b, c \in K^*)$. That kernel is spanned by the cup product $(a) \cup (b) \cup (c)$, where $(a), (b), (c) \in H^1(K, \mathbf{Z}/2) \simeq K^*/K^{*2}$ are the cohomology classes associated to the respective classes of a, b, c in K^*/K^{*2}.

(Similar computations may be done at the level of the Witt group. The analogue of Arason's result in that case is an earlier and simpler result of Arason and Pifster. This was briefly noticed in [CT/Oj89]. For a pleasant approach to this purely quadratic form theoretic point of view, see Ojanguren's book [Oj90].)

Arason's theorem computes the kernel of $H^3(K, \mathbf{Z}/2) \to H^3(F, \mathbf{Z}/2)$ for F the function field of the affine quadric

$$X^2 - aY^2 - bZ^2 + abT^2 = c$$

$(a, b, c \in K^*)$. Now such an equation also reads :

$$\mathrm{Nrd}_{D/k}(\Xi) = c$$

where D denotes the quaternion algebra (a, b) over K,

$$\Xi = X + iY + jZ + kT$$

(with the usual notation for quaternions) and Nrd denotes the reduced norm. Using hard techniques of algebraic K-theory, Suslin [Su91] has generalized the latter result. Let D/K

be a central simple algebra of prime index p over a field K ($\mathrm{char}(K) \neq p$). Let X/K be the affine (p^2-1)-dimensional affine variety given by the equation

$$\mathrm{Nrd}_{D/k}(\Xi) = c$$

for some $c \in K^*$. Then (Suslin, loc. cit.) the kernel of the map $H^3(K, \mu_p^{\otimes 2}) \to H^3(K(X), \mu_p^{\otimes 2})$ is spanned by the class $(D) \cup (c)$ where $(c) \in H^1(K, \mu_p)$ is the class of c in K^*/K^{*p} (under the Kummer identification) and $(D) \in H^2(K, \mu_p) \simeq {}_p\mathrm{Br}(K)$ is the class of the central simple algebra D in the Brauer group of K.

On the basis of this result, and with some inspiration from Bogomolov's extension [Bo87] of Saltman's results [Sa84], for arbitrary prime p, E. Peyre [Pe93] managed to produce many examples of unirational varieties X over an algebraically closed field k for which the unramified cohomology group $H^3(k(X), \mathbf{Z}/p)$ does not vanish, hence which are not rational, even though the whole unramified Brauer group of $k(X)$ vanishes.

Let us come back to the case $p = 2$. In this case, Amitsur's H^2 result and Arason's H^3 result have been extended by Jacob and Rost [Ja/Ro89] to H^4. Let K be a field, $\mathrm{char}(K) \neq 2$, let a, b, c, d be elements of K^* and let X be the smooth projective quadric associated to the 4-fold Pfister form $< 1, -a > \otimes < 1, -b > \otimes < 1, -c > \otimes < 1, -d >$ ($a, b, c, d \in K^*$). Then, with notation as above, the kernel of the map $H^4(K, \mathbf{Z}/2) \to H^4(K(X), \mathbf{Z}/2)$ is spanned by the class of the cup product $(a) \cup (b) \cup (c) \cup (d)$. Here again, Peyre was able to use this result to produce unirational varieties whose non-rationality is detected by unramified H^4 (with coefficients $\mathbf{Z}/2$).

When the ground field is not algebraically closed, unramified cohomology may still provide some information (such is already the case for the Brauer group). For instance, it may detect whether some varieties over k, even though they are rational over the algebraic closure of the ground field, are not rational over the ground field.

I shall here mention two general cases where some higher cohomology groups for varieties over a non algebraically closed field have been computed.

Suppose that k is the field $\mathbf{R}$ of real numbers. Let $X/\mathbf{R}$ be a smooth variety over $\mathbf{R}$. Then for any integer $n > \dim(X)$, the group $H^0(X, \mathcal{H}^n(\mathbf{Z}/2))$ is isomorphic to $(\mathbf{Z}/2)^s$, where s denotes the number of real components of the topological space $X(\mathbf{R})$ ([CT/Pa90]; that result has since been proved for arbitrary separated varieties). In particular, if $X/\mathbf{R}$ is a smooth, proper, integral variety over the reals, for any $n > \dim(X)$, the unramified cohomology group $H^n_{nr}(\mathbf{R}(X)/\mathbf{R}, \mathbf{Z}/2)$ is isomorphic to $(\mathbf{Z}/2)^s$, where s denotes the number of connected components of $X(\mathbf{R})$ (well-known not to depend on the particular smooth complete model). These results are proved in a joint paper with Parimala [CT/Pa90].

Suppose that $k = \mathbf{F}$ is a finite field of characteristic p. Let $X/\mathbf{F}$ be a smooth, projective, integral variety of dimension d. Let n be a positive integer prime to p. According to a conjecture of Kato [Ka86], the unramified cohomology groups $H^{d+1}_{nr}(\mathbf{F}(X)/\mathbf{F}, \mu_n^{\otimes d})$ should be trivial. For a curve, this is just a reformulation of the classical result $\mathrm{Br}(X) = 0$. For surfaces, this was proved by Sansuc, Soulé and the author in 1983, and independently by K. Kato (see [Ka86]). For threefolds, the limit result $H^4_{nr}(\mathbf{F}(X)/\mathbf{F}, \mathbf{Q}_l/\mathbf{Z}_l(3)) = 0$ (here $l \neq p$ is a prime and $\mathbf{Q}_l/\mathbf{Z}_l(3)$ denotes the direct limit of all $\mu_{l^m}^{\otimes 3}$ for m tending to infinity) has recently been given two independent proofs (S. Saito [Sa92], the author [CT92]).

There are many open problems in this area. Here are three of them.

(i) Let k be field, $\mathrm{char}(k) = 0$. Let G be a finite group, and let V be a faithful finite dimensional representation of G over k. Let $k(V)$ be the fraction field of the symmetric algebra on V, and let $K = k(V)^G$ be the field of G-invariants.

For any integer i and any positive integer n, give a formula for $H^i_{nr}(K/k, \mu_n^{\otimes(i-1)})$.

(One reason for expecting a nicer formula for the $(i-1)$ twist is that one expects – and one knows for $i \leq 3$ – that for n dividing m the natural map from $H^i_{nr}(K/k, \mu_n^{\otimes(i-1)})$ to $H^i_{nr}(K/k, \mu_m^{\otimes(i-1)})$ is an injection.)

It is worth observing that, thanks to the “no-name lemma” (see [CT/Sa88]), for fixed i, j and n, the group $H^i_{nr}(k(V)^G/k, \mu_n^{\otimes j})$ does not depend on the particular faithful representation V of G. For k algebraically closed and $i = 2$, such a formula, purely in terms of the cohomology of G and of its subgroups, was given by Bogomolov [Bo87] (see above) (see also [Sa90] and [CT/Sa88]).

(ii) Let k be a (non algebraically closed!) field and let T/k be an algebraic k-torus (i.e. an algebraic group over k, which over an algebraic closure of k becomes isomorphic to a product of multiplicative groups $\mathbf{G}_m$). Let $k(T)$ be the function field of T.

For any integer i and any positive integer n prime to the characteristic of k, give a formula for the group $H^i_{nr}(k(T)/k, \mu_n^{\otimes(i-1)})$.

Here the k-torus T is entirely determined by its character group $\hat{T}$, which is a lattice with an action of the absolute Galois group $\mathcal{G}$ of k. For $i = 2$, there is such a formula, purely in terms of the cohomology of the $\mathcal{G}$-module $\hat{T}$ ([CT/Sa87]). Namely, if $G = \mathrm{Gal}(K/k)$ is the Galois group of a finite splitting extension of the torus T, the group $\mathrm{Br}_{nr}(k(X)/k)$ coincides with the group of elements in $H^2(G, \hat{T})$ which vanish by restriction to all cyclic subgroups of G. The group $H^2_{nr}(k(T)/k, \mu_n)$ coincides with the n-torsion subgroup of $\mathrm{Br}_{nr}(k(X)/k)$.

(iii) Let k be an algebraically closed field, $G \subset GL_n$ be a connected reductive subgroup of GL_n and $X = GL_n/G$.

Compute the unramified cohomology groups $H^i_{nr}(k(X), \mathbf{Z}/n)$.

For $i = 3$ and $G = PGL_r$, this is the problem addressed by Saltman [Sa93] in these Proceedings.

§ 4.3 Finiteness results for unramified H^3 and for $CH^2(X)/n$

Given a smooth variety X over a separably closed field k, an integer $n > 0$ prime to the characteristic of k, and i and j integers, the cohomology groups $H^i(X, \mu_n^{\otimes j})$ are finite ([SGA4], XVI, 5.2; [Mi80], VI 5.5, restricted to separably closed fields). In characteristic zero, this is known to hold for arbitrary varieties ([SGA4], XVI, 5.1).

There is a natural map $H^i(X, \mu_n^{\otimes j}) \to H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$. If X is smooth, and $i \leq 2$, this map is surjective by Purity (see § 3.4), hence the groups $H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ are finite for $i \leq 2$. However, for $i \geq 3$, the above map need not be surjective – however the surjectivity for smooth, projective, varieties is an open question.

One may wonder whether for X/k smooth the groups $H^0(X, \mathcal{H}^i(\mu_n^{\otimes j}))$ are finite. In this section, for $i = 3$, we shall give sufficient conditions on X for this to be the case. We shall also be interested in non algebraically closed ground fields. First, we recall various tools which will be used in the proof of each of our finiteness theorems.

Quite generally, for any variety X over a field k, and $n > 0$ prime to char(k), there is a spectral sequence

$$E_2^{pq} = H^p_{Zar}(X, \mathcal{H}^q(\mu_n^{\otimes j})) \Longrightarrow H^{p+q}_{\text{ét}}(X, \mu_n^{\otimes j}) \,.$$

The terms E_2^{pq} are zero for $p > d = \dim(X)$. If X is smooth, the terms E_2^{pq} are zero for $p > q$. This is a consequence of the Gersten conjecture for étale cohomology, as proved by Bloch and Ogus [Bl/Og74]. In particular, for X/k smooth (not necessarily proper), there is a short exact sequence

$$H^3(X, \mu_n^{\otimes j}) \to H^0(X, \mathcal{H}^3(\mu_n^{\otimes j})) \to H^2(X, \mathcal{H}^2(\mu_n^{\otimes j})) \to H^4(X, \mu_n^{\otimes j})$$

which for $j = 2$ may be rewritten

$$H^3(X, \mu_n^{\otimes 2}) \to H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})) \to CH^2(X)/n \to H^4(X, \mu_n^{\otimes 2}). \tag{4.2}$$

Recall that $CH^i(X)$, resp. $CH_i(X)$, denotes the i-th Chow group of codimension i cycles, resp. dimension i cycles on X, modulo rational equivalence (see [Fu84] for the theory of Chow groups). That $H^2(X, \mathcal{H}^2(\mu_n^{\otimes 2}))$ may be identified with $CH^2(X)$ on a smooth variety (over an arbitray field k of characteristic prime to n) also follows from the work of Bloch/Ogus (see [Bl/Og74], [Bl80], [Bl81] and [CT93]).

Let us recall the localization sequence. Given X/k an arbitrary variety, $U \subset X$ an open set and F the complement of U in X, for any integer i, there is an exact sequence, the localization sequence (see [Fu84], I. Prop. 1.8 p. 21) :

$$CH_i(F) \longrightarrow CH_i(X) \longrightarrow CH_i(U) \longrightarrow 0 \tag{4.3}$$

Given X and Y two k-varieties and $f : X \to Y$ a finite and flat map of constant rank d, given any integer i, the composite map $p_* \circ p^*$ of the flat pull-back p^* with the proper push-forward map p_* is multiplication by d on $CH_i(X)$ ([Fu84], I, Example 1.7.4 p.20).

We shall make use of the following fact, which is a special case of Theorem 4.1.1. If X/k is smooth and integral, there is a natural embedding

$$H^i_{nr}(k(X)/k, \mu_n^{\otimes j}) \hookrightarrow H^0(X, \mathcal{H}^i(\mu_n^{\otimes j})). \tag{4.4}$$

Finally, we shall also use the following consequence, foreseen by Bloch [Bl81], of the Merkur'ev-Suslin theorem [Me/Su82]. If X/k is a smooth variety over the field k, for any integer $m > 0$ prime to char(k), the m-torsion subgroup ${}_mCH^2(X)$ is a subquotient of the étale cohomology group $H^3(X, \mu_m^{\otimes 2})$ (see [CT93], 3.3.2).

Now we are prepared to state and prove some finiteness theorems. The first explicit appearance of such theorems is in Parimala's paper [Pa88] (with application to finiteness theorems for the Witt group of unirational threefolds over the reals). Over **C**, the first theorem below was stated and proved independently by L. Barbieri-Viale ([BV92]). The theorem applies to unirational varieties; it also applies to threefolds which are conic bundles over arbitrary surfaces. Concerning these last varieties, it may actually be shown that for such a threefold X over an algebraically closed field k of characteristic prime to n, all the groups $H^3_{nr}(k(X)/k, \mu_n^{\otimes j})$ vanish (see [Pa89], appendix).

Theorem 4.3.1. — *Let k be an algebraically closed field of characteristic zero, X a smooth integral variety of dimension d over k. Assume that there exists a dominant rational map of degree N from $\mathbb{A}^{d-2} \times S$ to X for some integral surface S. Then the groups $CH^2(X)/n$ and $H^3_{nr}(k(X)/k, \mathbb{Z}/n)$ are finite, and this last group is zero if n is prime to N.*

Proof : The assumption may also be phrased as follows : there is a function field K/k of transcendence degree 2, such that the rational function field $K(t_1, \ldots, t_{d-2})$ is a field extension of degree N of the function field $k(X)$.

We may and will assume that S is smooth and affine. First consider the maps

$$H^i_{nr}(k(X)/k, \mathbb{Z}/n) \to H^i_{nr}(k(S)(t_1, \ldots, t_{d-2})/k, \mathbb{Z}/n) \xrightarrow{\text{Cores}} H^i(k(X)/k, \mathbb{Z}/n)$$

where the composition is multiplication by N (as a matter of fact, the image of the last map lies in $H^i_{nr}(k(X)/k, \mathbb{Z}/n)$, but this is not needed for the proof). The central term is isomorphic to $H^i_{nr}(k(S)/k, \mathbb{Z}/n)$ by Theorem 4.1.5, hence is a subgroup of $H^i(k(S)/k, \mathbb{Z}/n)$ which is zero for $i \geq 3$ ([Se65]). This shows that N kills $H^3_{nr}(k(X)/k, \mathbb{Z}/n)$, hence that this group is zero for all integers n prime to N.

We may find smooth open sets $V \subset \mathbb{A}^{d-2} \times S$ and $U \subset X$ and a finite and flat morphism $p : V \to U$ of constant rank N such that the composite morphism $V \to U \to X$ defines the given rational map from $\mathbb{A}^{d-2} \times S$ to X.

Projection $f : \mathbb{A}^{d-2} \times S \longrightarrow S$ induces an isomorphism

$$f^* : CH_0(S) \to CH_{d-2}(\mathbb{A}^{d-2} \times S)$$

([Fu84] Thm. 3.3 p. 64). The restriction map $CH_{d-2}(\mathbb{A}^{d-2} \times S) \longrightarrow CH_{d-2}(V)$ is trivially surjective. Now since S is a connected affine variety, the group $CH_0(S)$ is divisible. Indeed, to prove this, it is enough to do it for an affine connected curve. By normalization, this last case reduces to that of a smooth connected affine curve C. Let J be the jacobian of the smooth projective completion of C. The divisibility of $CH_0(C)$ follows from that of the group $J(k)$. Thus for our surface S we have $CH_0(S)/n = 0$, from which we deduce $CH_{d-2}(\mathbb{A}^{d-2} \times S)/n = 0$, hence $CH_{d-2}(V)/n = 0$ for all $n > 0$.

Since the open set $U \subset X$ is smooth, as recalled above we have an embedding

$$H^3_{nr}(k(X)/k, \mathbb{Z}/n) \hookrightarrow H^0(U, \mathcal{H}^3(\mathbb{Z}/n)).$$

Thus to show that $H^3_{nr}(k(X)/k, \mathbb{Z}/n)$ is finite, it suffices to show that $H^0(U, \mathcal{H}^3(\mathbb{Z}/n))$ is finite. Since, as recalled at the beginning of this section, the group $H^3(U, \mu_n^{\otimes 2})$ is finite (because k is algebraically closed), by (4.2), finiteness of $H^0(U, \mathcal{H}^3(\mathbb{Z}/n))$ is equivalent to finiteness of $CH^2(U)/nCH^2(U)$ (note that $\mathbb{Z}/n \simeq \mu_n^{\otimes 2}$ since k is algebraically closed).

The finite and flat morphism p gives rise to morphisms $p^* : CH_{d-2}(U) \to CH_{d-2}(V)$ and $p_* : CH_{d-2}(V) \to CH_{d-2}(U)$. The composite map

$$p_* \circ p^* : CH_{d-2}(U) \to CH_{d-2}(V) \to CH_{d-2}(U)$$

is multiplication by N.

The group ${}_mCH^2(U)$ is a subquotient of the group $H^3(U, \mu_m^{\otimes 2})$, as recalled above (the Merkur'ev/Suslin result is used here), and this last group is finite since k is algebraically closed.

Letting $A = CH_{d-2}(U)$ and $B = CH_{d-2}(V)$, and bearing in mind the vanishing, hence finiteness of $CH_{d-2}(V)/n = 0$ for all $n > 0$, the finiteness of $CH^2(U)/nCH^2(U)$ now follows from the purely formal lemma :

Lemma 4.3.2. — *Let $f : A \to B$ and $g : B \to A$ be homomorphisms of abelian groups. Assume that $g \circ f$ is multiplication by the positive integer $N > 0$. Assume that ${}_NA$ and B/NB are finite. Let $n \geq 0$. Then, if B/nB is finite, so is A/nA.*

Proof : Let $b_i \in B,\ i \in I$ be representatives of the finite set $I = B/NB$. For each $i \in I$ such that $g(b_i)$ belongs to NA, fix an element $a_i \in A$ with $g(b_i) = Na_i$. For any $a \in A$ we may find $b \in B$ and $i \in I$ such that $f(a) = Nb + b_i$. Now $Na = g \circ f(a) = Ng(b) + g(b_i)$. Hence $a - g(b) - a_i$ belongs to the the N-torsion subgroup ${}_NA$, which is finite. We conclude that the quotient $C = A/g(B)$ is a finitely generated abelian group. For any positive integer n, we have an exact sequence

$$B/nB \longrightarrow A/nA \longrightarrow C/nC \longrightarrow 0$$

hence A/nA is finite if B/nB is finite. $\square$

It remains to prove the finiteness of $CH^2(X)/n$. The argument given above shows that there exists a nonempty open set $U \subset X$ such that $CH^2(U)/n$ is finite. Let $F \subset X$ be the complement of U in X. We have the localization sequence

$$CH_{d-2}(F) \to CH_{d-2}(X) \to CH_{d-2}(U) \to 0.$$

Since $\dim(F) \leq d-1$, the finiteness of $CH_{d-2}(X)/n$ follows from the lemma :

Lemma 4.3.3. — *Let Z/k be a variety of dimension $\leq d$ over the algebraically closed field k. Then for any $n > 0$, the group $CH_{d-1}(Z)/n$ is finite.*

Proof : If $\dim(Z) < d-1$, then $CH_{d-1}(Z) = 0$ and if $\dim(Z) = d-1$ then $CH_{d-1}(Z)$ is finitely generated (one generator for each irreducible component of Z). Let us assume $\dim(Z) = d$. We may also assume that all components of Z have dimension d. Let Z_{sing} be the singular locus of Z and let $U \subset Z$ be the complement. We have the localization sequence

$$CH_{d-1}(Z_{\text{sing}}) \to CH_{d-1}(Z) \to CH_{d-1}(U) \to 0.$$

By the argument above, the abelian group $CH_{d-1}(Z_{\text{sing}})$ is finitely generated. We have $CH_{d-1}(U) = CH^1(U) \simeq \mathrm{Pic}(U)$, hence for any $n > 0$, we have $CH_{d-1}(U)/n \simeq \mathrm{Pic}(U)/n \hookrightarrow H^2(U, \mu_n)$ and this last group is finite since k is algebraically closed. □

Remark 4.3.4 : As indicated above, over $\mathbb{C}$, essentially the same theorem was proved by L. Barbieri-Viale ([BV92b]). His proof, which is shorter, uses a result of Bloch/Srinivas [Bl/Sr83], namely the fact that for a smooth variety $X/\mathbf{C}$, the Zariski sheaf $\mathcal{H}^3(\mathbf{Z})$ associated to the Zariski presheaf $U \longmapsto H^3_{Betti}(U(\mathbf{C}), \mathbf{Z})$ has no torsion (that result in its turn is a consequence of the Merkur'ev-Suslin theorem). Assume that $X/\mathbf{C}$ is smooth and projective. Let $NS^2(X)$ be the quotient of $CH^2(X)$ by the subgroup $A^2(X)$ of classes algebraically equivalent to zero. From the geometric assumption in Theorem 4.3.1, one deduces that the group $H^0(X, \mathcal{H}^3(\mathbf{Z}))$ is torsion, hence that it is zero by the Bloch/Srinivas result. Now from [Bl/Og74] we have an exact sequence

$$H^0(X, \mathcal{H}^3(\mathbf{Z})) \longrightarrow NS^2(X) \longrightarrow H^4_{Betti}(X, \mathbf{Z})$$

and we conclude that the group $NS^2(X)$ is a finitely generated group. Since the group $A^2(X)$ is divisible, for any positive integer n, the induced map $CH^2(X)/n \longrightarrow NS^2(X)/n$ is an isomorphism, hence $CH^2(X)/n$ is finite.

Remark 4.3.5 : The same result can also be obtained under any of the following weaker (and related) hypotheses (cf. [Bl/Sr83]) :

(i) There exists a morphism f from a smooth projective surface $S/\mathbf{C}$ to the smooth projective variety $X/\mathbf{C}$ such that the induced map $f_* : CH_0(S) \to CH_0(X)$ is surjective.

(ii) Let $\Delta \subset X \times X$ be the diagonal. There exists a positive integer m such that the cycle $m\Delta$ is rationally equivalent to a sum of codimension two cycles $Z_1 + Z_2$, the support of Z_1 being included in $S \times X$ and the support of Z_2 in $X \times Y$, where $S \subset X$ is a surface and $Y \subset X$ is of codimension 1.

Indeed the correspondence formalism ([Bl/Sr83]) and the torsion-freeness of $\mathcal{H}^3(\mathbf{Z})$ (see above) yield $H^0(X, \mathcal{H}^3(\mathbf{Z})) = 0$ and we may conclude just as above.

The more complicated proof of Theorem 4.3.1 given above is however better suited for extensions of this finiteness theorem to varieties over non algebraically closed fields, as we shall see in the balance of § 4.3.

For $S = \mathbb{A}^2$, and $d = 3$, statement b) in the following theorem is due to Parimala [Pa88].

THEOREM 4.3.6. — *Let k be a real closed field, and let X be a smooth, geometrically integral variety of dimension d over k. Assume that there exists a dominant rational map of degree N from $\mathbb{A}^{d-2} \times_{\overline{k}} S$ to $\overline{X}$, for some integral surface $S/\overline{k}$. Let $n > 0$ be an integer. Then :*

a) The group $H^3_{nr}(k(X)/k, \mu_n^{\otimes 2})$ is finite, and it is zero for all integers n prime to $2N$.

b) If X/k is smooth, the group $CH^2(X)/n$ is finite.

Proof : One may assume that $S/\overline{k}$ is smooth and affine and find a finite and flat map of constant degree $2N$ from a nonempty open set V of $\mathbb{A}^{d-2} \times_{\overline{k}} S$ to an open set U of X. The proof of the theorem is then essentially identical to the proof of Theorem 4.3.1, in view of the following facts :

(i) For any variety U/k with k real closed, the étale cohomology groups $H^i(U, \mu_n^{\otimes j})$ are finite. This follows from the Hochschild-Serre spectral sequence for the extension $\overline{k}/k$ together with the finiteness of étale cohomology of varieties over an algebraically closed field and finiteness of the Galois cohomology of finite $\mathrm{Gal}(\overline{k}/k)$-modules.

(ii) For any smooth variety X over a real closed field k, and any positive integer n, the group ${}_nCH^2(X)$ is finite. Indeed, it is a subquotient of $H^3(X, \mu_n^{\otimes 2})$ which we have just seen is finite.

(iii) $H^3(k, \mu_n^{\otimes 2}) = 0$ for n odd.

(iv) Lemma 4.3.3 still holds, with the same proof, when k is real closed. □

THEOREM 4.3.7. — *Let k be a p-adic field (finite extension of $\mathbb{Q}_p$), let X a smooth geometrically integral variety of dimension d over k.*

a) Assume that $\overline{X} = X \times_k \overline{k}$ is rationally dominated by the product of an integral curve and an affine space of dimension $d-1$; then for any integer $n > 0$, the groups $CH^2(X)/n$ and $H^3_{nr}(k(X)/k, \mu_n^{\otimes 2})$ are finite.

b) If n is prime to p, then the same finiteness results hold if $\overline{X}$ is rationally dominated by the product of an integral surface and an affine space of dimension $d-2$.

Proof : The proof of the first statement is essentially identical to the proof of Theorems 4.3.1 and 4.3.6. One chooses a finite extension K/k of fields, a smooth integral curve C/K, resp. surface S/K, and a finite, flat morphism of constant degree N from a nonempty open set of $C \times_K \mathbb{A}^{d-1}_K$ to an open set of X. One then combines the following facts :

(i) For any variety U/k with k p-adic, the étale cohomology groups $H^i(U, \mu_n^{\otimes j})$ are finite. This follows from the Hochschild-Serre spectral sequence for the extension $\overline{k}/k$ together with the finiteness of étale cohomology of varieties over an algebraically closed field and finiteness of the Galois cohomology of finite $\mathrm{Gal}(\overline{k}/k)$-modules ([Se65]). This implies that for any smooth variety U over k p-adic, and any positive integer n, the group ${}_nCH^2(U)$ is finite.

(ii) For the curve C, we obviously have $CH^2(C \times_K \mathbb{A}^{d-1}_K) = CH^2(C) = 0$. For the smooth surface S, if n is prime to p, the group $CH^2(S)/n$ is finite. It is enough to prove this for a smooth projective surface S over the p-adic field K. In that case, Saito and Sujatha [Sa/Su93] have proved that $H^0(X, \mathcal{H}^3(S, \mu_n^{\otimes 2}))$ is finite . The result then follows from exact sequence (4.2) together with the finiteness results for étale cohomology in (i) above.

(iii) Lemma 4.3.3 still holds, with the same proof, when k is a p-adic field (indeed, with notation as in that Lemma, $H^2(U, \mu_n)$ is finite according to (i) above). □

The same line of investigation as above, combined with results of [CT/Ra91], [Sal93] and [Sa91], enables one to prove theorems which may be regarded as weak Mordell-Weil theorems for codimension two cycles. Whether such theorems hold for arbitrary smooth varieties is a big open problem.

Theorem 4.3.8. — *Let X/k be a smooth geometrically integral variety of dimension d over a number field k. If $\overline{X} = X \times_k \overline{k}$ is rationally dominated by the product of a smooth integral curve C and an affine space of dimension $d-1$, then $CH^2(X)/n$ is a finite group.*

Proof: Given an m-dimensional variety Z over a number field k, the group $CH_{m-1}(Z)$ is a finitely generated abelian group : this is a known consequence of the Mordell-Weil theorem and of the Néron-Severi theorem. This holds more generally if k is finitely generated over **Q**. Let n be a positive integer. So Lemma 4.3.3 still holds over such ground fields. Using the localization sequence, we see that if Y is smooth and k-birational to X, then $CH^2(Y)/n$ is finite if and only if $CH^2(X)/n$ is finite. Similarly, we see that ${}_nCH^2(Y)$ is finite if and only if ${}_nCH^2(X)$ is finite.

Let Y/k be a smooth projective model of X. Since $\overline{Y}$ is dominated by the product of a curve C and a projective space, we conclude that the coherent cohomology group $H^2(Y, \mathcal{O}_Y)$ vanishes. Here is one way to see this. Let Ω be a universal domain containing k. Since the Chow group of zero-cycles of smooth projective varieties, the hypothesis implies that the Chow group $CH_0(Y_\Omega)$ of 0-dimensional cycles is representable. An argument of Roitman (see [Ja90] p.157) then implies that all $H^i(Y, \mathcal{O}_Y)$ vanish for $i \geq 2$. According to Raskind, the author, and Salberger ([CT/Ra91], [Sal93]), this implies that for any integer $n > 0$, the group ${}_nCH^2(Y)$ is finite. Hence ${}_nCH^2(X)$ and ${}_nCH^2(U)$ for any open set of X are also finite groups.

We may now proceed as in the proof of Theorem 4.3.7. We find a finite extension K/k of fields, a smooth integral curve C/K and a finite, flat morphism of constant degree N from a nonempty open set V of $C \times_K \mathbb{A}_K^{d-1}$ to an open set U of X. We have $CH^2(C \times_K \mathbb{A}_K^{d-1}) = CH^2(C) = 0$, hence $CH^2(V) = 0$. Since ${}_nCH^2(U)$ is finite, Lemma 4.3.2 now implies that $CH^2(U)/n$ is finite for any $n > 0$, hence also $CH^2(X)/n$. □

Theorem 4.3.9. — *Let X/k be a smooth integral variety of dimension d over a field k finitely generated over $\mathbb{Q}$. If $\overline{X} = X \times_k \overline{k}$ is rationally dominated by d-dimensional projective space, and if X has a rational k-point, then $CH^2(X)/n$ is a finite group.*

Proof : The proof differs from the proof of the previous theorem only in two points. The assumption that $\overline{X} = X \times_k \overline{k}$ is rationally dominated by d-dimensional projective space is used to ensure that both $H^2(Y, \mathcal{O}_Y)$ and $H^1(Y, \mathcal{O}_Y)$ vanish for Y a smooth projective model of X (use [Ja90] as above). Together with the assumption that X has a k-rational point, one may then use a result of Saito ([Sa91], see also [CT93], § 7, Thm. 7.6) ensuring that the torsion subgroup of $CH^2(Y)$ is finite. $\square$

Versions of the following theorem, which combines many of the previous results, were brought to my attention by N. Suwa and L. Barbieri-Viale (independently).

THEOREM 4.3.10. — *Let X/k be a smooth, projective, geometrically integral variety of dimension d over a field $k \subset \mathbf{C}$. Under any of the two sets of hypotheses :*

(i) the field k is a number field and $CH_0(X_{\mathbf{C}})$ is represented by a curve ;

(ii) the field k is finitely generated over $\mathbf{Q}$, there is a k-rational point on X and the degree map $CH_0(X_{\mathbf{C}}) \longrightarrow \mathbf{Z}$ is an isomorphism ;

the group $CH^2(X)$ is a finitely generated abelian group.

Remarks : (a) The geometric assumption in (i) is that there exists a curve $C/\mathbf{C}$ a $\mathbf{C}$-morphism from C to $X_{\mathbf{C}}$ such that the induced map $CH_0(C_{\mathbf{C}}) \longrightarrow CH_0(X_{\mathbf{C}})$ is surjective. Concrete examples are provides by varieties dominated by the product of a curve and and an affine space. A special case is that of quadric bundles (of relative dimension at least one) over a curve.

(b) The geometric assumption in (ii) is satisfied by unirational varieties, by quadric bundles (of relative dimension at least one) over rational varieties, and also by Fano varieties (Miyaoka, Campana [Ca92]).

Proof of the theorem :

Let $\overline{k} \subset \mathbf{C}$ be a fixed algebraic closure of k, let $G = \mathrm{Gal}(\overline{k}/k)$, let $\overline{X} = X \times_k \overline{k}$ and $X_{\mathbf{C}} = X \times_k \mathbf{C}$. There is a natural filtration on the Chow group $CH^2(\overline{X})$:

$$CH^2(\overline{X})_{alg} \subset CH^2(\overline{X})_{hom} \subset CH^2(\overline{X}).$$

The smallest subgroup is that of cycles algebraically equivalent to zero, the middle subgroup is that of cycles homologically equivalent to zero, that is those cycles which are in the kernel of the composite map

$$CH^2(X \times_k \overline{k}) \longrightarrow CH^2(X_{\mathbf{C}}) \longrightarrow H^4_{Betti}(X(\mathbf{C}), \mathbf{Z}).$$

A standard specialization argument shows that the map

$$CH^2(X \times_k \overline{k}) \longrightarrow CH^2(X_{\mathbf{C}})$$

is injective. On the other hand, the group $H^4_{Betti}(X(\mathbf{C}), \mathbf{Z})$ is a finitely generated abelian group. Thus the quotient $CH^2(\overline{X})/CH^2(\overline{X})_{hom}$ is a finitely generated group. By the same specialization argument, the quotient $CH^2(\overline{X})_{hom}/CH^2(\overline{X})_{alg}$ is a subgroup of the classical Griffiths group $CH^2(X_{\mathbf{C}})_{hom}/CH^2(X_{\mathbf{C}})_{alg}$.

Under any of the two assumptions in the theorem, and even under the weaker assumption that $CH_0(X_{\mathbf{C}})$ is representable by a surface, the group $CH^2(X_{\mathbf{C}})_{hom}/CH^2(X_{\mathbf{C}})_{alg}$

vanishes : this is a result of Bloch and Srinivas [Bl/Sr83], whose proof relies on the Merkur'ev-Suslin theorem.

Thus the group $CH^2(\overline{X})/CH^2(\overline{X})_{hom}$ is an abelian group of finite type.

According to standard usage, let us write $A^2(\overline{X}) = CH^2(\overline{X})_{hom}$. Murre [Mu83], relying on work of H. Saito [Sa79] and Merkur'ev-Suslin [Me/Su82], has shown that there exists a universal regular map $\rho : A^2(\overline{X}) \longrightarrow A(\overline{k})$ where A is a certain abelian variety defined over $\overline{k}$. Under the assumption that $CH_0(X_{\mathbf{C}})$ is represented by a curve, Bloch and Srinivas (loc. cit.) show that the map ρ is an isomorphism. There exists an abelian variety B over $\overline{k}$ and a cycle $Z \in CH^2(B \times \overline{X})$ which induces maps $B(\overline{k}) \longrightarrow A^2(\overline{X}) \longrightarrow A(\overline{k})$, the composite map being induced by a $\overline{k}$-morphism of abelian varieties $\varphi : B \longrightarrow A$ (see [Sa79], Prop. 1.2 (ii)). Now the varieties A, B, the morphism φ and the cycle Z are all defined over a finite field extension L of k. Let $H = \mathrm{Gal}(\overline{k}/L)$. It then follows that the isomorphism $\rho : A^2(\overline{X}) \longrightarrow A(\overline{k})$ is H-equivariant. Thus the group $(A^2(\overline{X}))^G \subset (A^2(\overline{X}))^H$ is a subgroup of $A(L)$, and for k, hence L, finitely generated over $\mathbf{Q}$, this last abelian group is finitely generated by the Mordell-Weil-Néron theorem. Hence $(A^2(\overline{X}))^G$ is a finitely generated abelian group. To conclude the proof, it only remains to show that under the assumptions of (i) and (ii), the group

$$\mathrm{Ker}[CH^2(X) \longrightarrow CH^2(\overline{X}]$$

is a finite group. In case (i), observe that the geometric assumption on 0-cycles implies the vanishing of $H^2(X, \mathcal{O}_X)$ (see [Ja90] p.157). The required finiteness now follows from [CT/Ra91] and [Sal93]. In case (ii), the geometric assumption on 0-cycles implies the vanishing of both $H^2(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$. The required finiteness result is then due to S. Saito ([Sa91],[CT93]). □

§ 4.4 Rigidity for unramified cohomology

The following theorem is an adaptation to étale cohomology of Suslin's celebrated rigidity theorem for K-theory with coefficients ([Su83] [Su88], see also Lecomte's paper [Le86] for interesting variants in the Chow group context) :

Theorem 4.4.1. — *Let $k \subset K$ be separably closed fields, let X/k be a smooth, integral, proper k-variety. Let $k(X)$ be the function field of X and $K(X)$ the function field of $X_K = X \times_k K$. Let $n > 0$ be an integer prime to* char*(k). Let $i > 0$ and j be integers. Then the natural map $H^i(k(X), \mu_n^{\otimes j}) \to H^i(K(X), \mu_n^{\otimes j})$ induces an isomorphism*

$$H^i_{nr}(k(X)/k, \mu_n^{\otimes j}) \simeq H^i_{nr}(K(X)/K, \mu_n^{\otimes j})$$

i.e. an isomorphism

$$H^0(X, \mathcal{H}^i(\mu_n^{\otimes j})) \simeq H^0(X_K, \mathcal{H}^i(\mu_n^{\otimes j})).$$

In other words, unramified cohomology is rigid : it does not change by extension of separably closed field.

Proof : 1) Since étale cohomology does not change under inseparable extensions, we may assume that k and K are algebraically closed.

2) The functor $V \to H^0(V, \mathcal{H}^i(\mu_n^{\otimes j}))$ is functorial contravariant on the category of all k-schemes. Indeed, étale cohomology is functorial contravariant under arbitrary morphisms. If $f : V \to W$ is any k-morphism of k-varieties, we have an induced map of sheaves on $W : \mathcal{H}^i_W(\mu_n^{\otimes j}) \to f_*(\mathcal{H}^i_V(\mu_n^{\otimes j}))$ hence a map

$$f^* : H^0(W, \mathcal{H}^i_W(\mu_n^{\otimes j})) \to H^0(V, \mathcal{H}^i_V(\mu_n^{\otimes j}))$$

and one easily checks that this map is functorial, i.e. respects composition of morphisms.

3) Given a finite field extension $K \subset L$ with $k \subset K$, an excellent discrete valuation ring A with $k \subset A$, with residue field κ and with field of fractions K, let B be the integral closure of A in L, which we assume to be of finite type over A. Let q_α be the finitely many maximal ideals of B, and let κ_α be the corresponding residue field. There is a commutative diagram

$$\begin{array}{ccc}
H^i(L, \mu_n^{\otimes j}) & \xrightarrow{(\partial_\alpha)} & \bigoplus_\alpha H^{i-1}(\kappa_\alpha, \mu_n^{\otimes j-1}) \\
\downarrow \mathrm{Cores}_{L,K} & & \downarrow \sum_\alpha \mathrm{Cores}_{\kappa_\alpha,\kappa} \\
H^i(K, \mu_n^{\otimes j}) & \xrightarrow{\partial_A} & H^{i-1}(\kappa, \mu_n^{\otimes j-1}).
\end{array}$$

This is most easily proved by means of the Galois cohomological description of the residue map.

If now $f : V \to W$ is a finite flat morphism of smooth integral k-varieties, we deduce that there is an induced norm map

$$f_* : H^0(V, \mathcal{H}^i_V(\mu_n^{\otimes j})) \to H^0(W, \mathcal{H}^i_W(\mu_n^{\otimes j})).$$

In particular given a smooth, proper, integral k-variety X and $f : V \to W$ a finite flat morphism of smooth integral k-curves, there is an induced norm map

$$f_* : H^0(X \times V, \mathcal{H}^i_V(\mu_n^{\otimes j})) \to H^0(X \times W, \mathcal{H}^i_W(\mu_n^{\otimes j})).$$

Suppose that V is a smooth affine curve and that $W = \mathbf{A}^1_k$. Given any point $x \in \mathbf{A}^1(k)$, we have a commutative diagram

$$\begin{array}{ccc}
H^0(X \times C, \mathcal{H}^i) & \longrightarrow & H^0(X, \mathcal{H}^i) \\
\mathrm{Cores} \downarrow & & id \downarrow \\
H^0(X \times \mathbf{A}^1_k, \mathcal{H}^i) & \longrightarrow & H^0(X, \mathcal{H}^i).
\end{array}$$

Here the top row is the sum $\sum_{i\in I} e_i\rho_i$, where I denotes the set of points $y_i \in C(k)$ above x, where e_i denotes the multiplicity of y_i in the fibre $f^{-1}(x) \subset C$, and where ρ_i denotes evaluation at y_i. The right vertical arrow is the identity. The left vertical map is the norm map described above. The bottom horizontal map is evaluation at x.

In order to show that this diagram actually commutes, one may restrict $\alpha \in H^0(X \times C, \mathcal{H}^i)$ to the semi-local ring S of $X \times C$ defined by the points above the generic point of $(X \times x) \subset X \times C$, whose local ring on $X \times C$ will be denoted by R. We now use Gersten's conjecture for semi-local rings of smooth varieties over a field (see § 3.7 and § 3.8). In particular, for such a semi-local ring S, we have $H^i(S, \mu_n^{\otimes j}) = H^0(\mathrm{Spec}(S), \mathcal{H}^i)$. The commutativity to be proved now boils down to a functoriality for the evaluation of the norm map on étale cohomology, for which we refer to [SGA4], Exposé XVII, Thm. 6.2.3 p. 422.

4) We are now ready to use the machinery described by Suslin in his ICM86 address ([Su88], § 2) (and which is an adaptation by Gabber, Gillet and Thomason of an older argument of Suslin). We will not repeat the arguments here, but we shall give all the ingredients which make the machinery work in the case in point. We consider the functor V which to a k-scheme Y associates the torsion group $V(Y) = H^0(X \times Y, \mathcal{H}^i)$.

5) Arguing as in Theorem 4.1.5 above, one proves that the map $H^0(X, \mathcal{H}^i) \to H^0(X \times \mathbf{A}^1_k, \mathcal{H}^i)$ is an isomorphism for all smooth k-varieties X. (This is the homotopy invariance for the functor V.)

6) Using the various functorialities described above, and the divisibility of generalized jacobians, together with the fact that the unramified cohomology groups are torsion groups, one finds that for any two points $x, y \in C(k)$, where C is a smooth connected affine curve, and any $\alpha \in H^0(X \times C, \mathcal{H}^i)$, the evaluations of α at x and y give the same element in $H^0(X, \mathcal{H}^i)$.

7) In order to complete the proof of the theorem, one needs to show that any class in $H^0(X_K, \mathcal{H}^i)$ comes from some class in $H^0(X_A, \mathcal{H}^i)$ for a suitable smooth k-algebra A, $A \subset K$ (cf. [Su88], Cor. 2.3.3)).

We first note that K is the direct limit of all (finitely generated) smooth k-algebras $A_\alpha \subset K$. Given a class $\xi \in H^0(X_K, \mathcal{H}^i)$, one easily produces a finite covering $X_K = \bigcup_{r\in R} U_r$, finite coverings $U_r \cap U_s = \bigcup_{t\in I_{r,s}} W_t$ for each $r, s \in R$, elements $\xi_r \in H^i(U_r, \mu_n^{\otimes j})$ such that each ξ_r restricts to $\xi \in H^0(X_K, \mathcal{H}^i) \subset H^i(K(X), \mu_n^{\otimes j})$ and such that ξ_r and ξ_s have equal restrictions in $H^i(U_r \cap U_s, \mu_n^{\otimes j})$. Now the open sets U_r, W_t are defined by finitely many equations. Hence they are already defined at the level of some A_α, i.e. $X_{A_\alpha} = \bigcup_{r\in R} U_{\alpha,r}$ with $U_{\alpha,r} \times_{A_\alpha} K = U_r$ and similarly $W_t = W_{\alpha,t} \times_{A_\alpha} K = W_t$. Because cohomology commmutes with filtering limits, after replacing A_α by a bigger smooth k-algebra inside K, which we will still call A_α, we may ensure that ξ_r comes from $\xi_{\alpha,r} \in H^i(U_{\alpha,r}, \mu_n^{\otimes j})$, and that the restrictions of $\xi_{\alpha,r}$ and $\xi_{\alpha,s}$ to each $W_{\alpha,t}, t \in I_{r,s}$ coincide. The elements ξ_r thus give rise to an element $\xi_\alpha \in H^0(X_{A_\alpha}, \mathcal{H}^i)$ whose restriction to $H^0(X_K, \mathcal{H}^i)$ is ξ.

Remark 4.4.2 : As pointed out to me by Jannsen, analogous arguments should give a similar rigidity theorem for all groups $H^r_{\mathrm{Zar}}(X, \mathcal{H}^i(\mu_n^{\otimes j}))$.

§ 5 Back to the Gersten conjecture

As already mentioned in § 2.2.3 and § 3.7, in 1980, Ojanguren [Oj80] proved that the Witt group of a local ring of a smooth variety over a field injects into the Witt group of its field of fractions. In 1989, Ojanguren and I [CT/Oj92] axiomatized Ojanguren's method. We were thus able to prove injectivity, in the sense of § 2.1, for various functors. In this section I will describe the method (§ 5.1) and give a few more injectivity results (§ 5.2). We shall actually prove injectivity results "with parameters", i.e. for functors on the category of k-algebras, of the shape $A \longmapsto F(Z \times_k A)$, where Z/k is a fixed k-variety. Based on one such an injectivity result for the Chow groups, in § 5.3 we shall give a new proof of a codimension one purity theorem due to M. Rost.

§ 5.1 A general formalism

Let k be a field and F be a covariant functor $A \longmapsto F(A)$ from the category of noetherian k-algebras (not necessarily of finite type), with morphisms the flat homomorphisms of rings, to the category of pointed sets, i.e. sets equipped with a distinguished element. The distinguished element in $F(A)$ shall be denoted $\mathbf{1}_A$, and often simply $\mathbf{1}$. Given $A \to B$, the kernel of $F(A) \to F(B)$ is the set of elements of $F(A)$ whose image is $\mathbf{1}_B$. Consider the following properties.

A1. F commutes with filtering direct limits of rings (with flat transition homomorphisms).

A2. Weak homotopy : for all fields L containing k, and for all $n \geq 0$, the map

$$F(L[t_1, \ldots, t_n]) \longrightarrow F(L(t_1, \ldots, t_n))$$

has trivial kernel. (i.e. kernel reduced to 1).

A3. Patching : Given an étale inclusion of integral k–algebras $A \to B$ and given a non-zero element $f \in A$ such that the induced map $A/f \longrightarrow B/f$ is an isomorphism, the induced map on kernels

$$\mathrm{Ker}[F(A) \to F(A_f)] \longrightarrow \mathrm{Ker}[F(B) \to F(B_f)]$$

is onto.

Theorem 5.1.1 ([CT/Oj92]). — *Let k be an infinite field. Assume that F satisfies* **A1**, **A2** *and* **A3**. *If $L \supset k$ and A is a local ring of a smooth L-variety, with fraction field K, then*

$$\ker[F(A) \longrightarrow F(K_A)] = 1\,.$$

Proof :

1. One starts from the presentation for principal hypersurfaces in a smooth variety over an infinite field already mentioned in § 3.7. This presentation, which differs from Quillen's, will henceforth be referred to as Ojanguren's presentation. Let us repeat the description here.

Let $\mathrm{Spec}(B)/\mathrm{Spec}(k)$ be smooth and integral of dimension d, let $f \in B$ be non-zero and let P be a closed point on the zero set of f. Then, up to shrinking $\mathrm{Spec}(B)$ around P, there is an étale map $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(R)$ where $\mathrm{Spec}(R)$ is an open set in affine space $\mathbb{A}^d_k$ and a $g \in R$ such that $g \longmapsto f$ and $R/g \xrightarrow{\approx} A/f$. That is : *"A germ of a hypersurface on a smooth variety is analytically isomorphic to a germ of a hypersurface in affine space of the same dimension"*.

2. One then uses a patching argument to reduce the problem to the case of a local ring of an affine space.

Start with A a local ring of a smooth variety X/k at a closed point P (one easily reduces to the case of a closed point). Let K be the fraction field of A. Assume $\alpha \in F(A)$ is sent to $1 \in F(K)$. By Axiom **A1**, α comes from $\alpha_B \in F(B)$ for some B of finite type and smooth over k, and $\alpha \longmapsto 1 \in F(B_g)$ for some $g \in B$. By Ojanguren's presentation, shrinking $\mathrm{Spec}(B)$ somewhat around the closed point P, we find an étale map $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(R)$ where $\mathrm{Spec}(R)$ is an open set in affine space $\mathbb{A}^d_k$ and a $g \in R$ such that $g \longmapsto f$ and $R/g \xrightarrow{\approx} A/f$. By Axiom **A3**, α_B lifts to $\alpha_R \in F(R)$ such that $\alpha_R \longmapsto 1 \in F(R_f)$. This reduces us to the case of a local ring of a closed point in affine space $\mathbb{A}^n_k$.

3. Let A be the local ring at a closed point M of $\mathbb{A}^n_k$, let K be the fraction field of A and let $\alpha \in F(A)$ have image $\mathbf{1} \in F(K)$. Rather than giving the complete argument, for which we refer to [CT/Oj92], we do the case $n = 1$ and sketch the case $n = 2$. Let $\mathfrak{m}$ be the maximal ideal of $k[t_1, \ldots, t_n]$ corresponding to M.

Case $n = 1$. Let $k[t]$ be the polynomial ring in one variable By axiom **A1**, there exists an $f \in k[t]$, $f \notin \mathfrak{m}$ such that α comes from some $\alpha_1 \in F(k[t]_f)$. By **A1** again, there exists a $g \in k[t]$ such that α_1 has image $\mathbf{1} \in F(k[t]_{fg})$. We may change $g(t)$ so that it becomes coprime with $f(t)$. The map $k[t] \to k[t]_f$ then induces an isomorphism $k[t]/g \simeq (k[t]_f)/g$. (This uses one-dimensionality : if the two elements f and g in the ring $k[t]$ have no common divisor, then they span the whole of $k[t]$.) From the patching axiom **A3** we conclude that there exists $\alpha_2 \in F(k[t])$ with image α_1 in $F(k[t]_f)$, and with image $\mathbf{1}$ in $F(k[t]_{fg}$, hence also in $F(k(t))$. Now the weak homotopy axiom **A2** implies $\alpha_2 = \mathbf{1} \in F(k[t])$, hence $\alpha_1 = \mathbf{1}$ and $\alpha = \mathbf{1} \in F(A)$.

$n = 2$ (detailed sketch) : By axiom **A1**, α comes from a class $\alpha_1 \in F(k[x,y]_f)$ for some $f \notin \mathfrak{m}$.Then α_1 maps to $1 \in F(k[x,y])_{fg})$ for some $g \in k[x,y]$. We want to patch α to $\mathbf{1}$ as before. Since $k[x,y]$ is a unique factorization domain, we may still change g so that f and g have no common divisor. But now the closed set defined by $f = g = 0$ may be nonempty and the map $k[x,y]/g \to (k[x,y]_f)/g$ need not be an isomorphism!

To overcome this problem, one makes a general position argument – here again we use the fact that k is infinite. One thus reduces to the case where :

a) the projection of M on $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$ is the point O defined by $x = 0$,

b) f and g monic in the second variable y,

c) $f(0,y)$ and $g(0,y)$ are coprime, i.e. $f = 0$ and $g = 0$ do not meet above O as shown below.

M

O $\mathbb{A}^1_k$

Over some interval around $O \in \mathbb{A}^1_k$, say on $\mathrm{Spec}(k[X]_{h(X)}[Y])$, $f = 0$ and $g = 0$ do not meet and we can use the patching axiom **A3** to produce $\beta \in \mathrm{Spec}(k[X]_{h(X)}[Y])$ which lifts $\alpha \in F(A)$. Now β becomes $\mathbf{1} \in F(k(X,Y))$ and so by the homotopy axiom **A2** (with the ground field $k(X)$) it becomes $\mathbf{1}$ already in $F(k(X)[Y])$. Thus by axiom **A1** there exists a polynomial $r(X)$ such that β becomes $\mathbf{1}$ in $F(k[X]_{h(X)r(X)}[Y])$. We may change r to ensure that h and r are coprime. Now the open sets $r \neq 0$ and $h \neq 0$ cover the entire line $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$, hence their inverse images cover the whole plane $\mathbb{A}^2_k$ and we may use axiom **A3** to produce a class $\gamma \in F(k[x,y])$ with image $\beta \in \mathrm{Spec}(k[X]_{h(X)}[Y])$ and with trivial image in $F(k(x,y))$. From the weak homotopy axiom, we conclude $\gamma = \mathbf{1} \in F(k[x,y])$, hence also $\beta = \mathbf{1}$, hence finally $\alpha = \mathbf{1}$.

This completes the proof in the two-dimensional case. The higher dimension case can be done by a more intricate version of the same argument ([CT/Oj92]). $\square$

§ 5.2 The injectivity property

Let $W(R)$ denote the Witt group of a ring R with $2 \in R^*$. When $B = k$, the following result is Ojanguren's initial result [Oj80].

Theorem 5.2.1. — *Let k be a field, and let A be a local ring of a smooth k-variety. Let*

K be the fraction field of A. Assume $2 \in A^*$. Let B be a k-algebra of finite type. Then the map $W(B \otimes_k A) \to W(B \otimes_k K)$ on Witt groups is an injection.

Proof : Axiom **A1** is easy. Axiom **A2** here is just a special case of a result of Karoubi [Ka73] [Ka75] [Oj84] : for any commutative ring R with $2 \in R^*$, the map $W(R) \to W(R[t])$ is an isomorphism. To show that axiom **A3** (patching) holds one reduces to a patching property for quadratic spaces. That property is Theorem 1 of [Oj80]. See also [CT/Oj92] Prop. 2.6. $\square$

Theorem 5.2.2. — *Let k be an infinite field. Let Z/k be an arbitrary variety. If A is a local ring of a smooth variety over k, and K is its fraction field, then for any integer $n \geq 0$*

1) The maps on G-theory groups

$$G_n(Z \times_k A) \longrightarrow G_n(Z \times_k K)$$

are injective.

2) If Z/k is smooth, the maps on K-theory groups

$$K_n(Z \times_k A) \longrightarrow K_n(Z \times_k K)$$

are injective.

Proof : By $Z \times_k A$ we actually mean $Z \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$. The groups $G_n(Y)$, also denoted $K'_n(Y)$, are the groups associated by Quillen to the exact category of coherent modules on a scheme Y and the groups $K_n(Y)$ are the groups associated to the exact category of locally free modules on Y. For each n there is a natural map $K_n(Y) \to G_n(Y)$ and this maps is an isomorphism if Y is a regular scheme ([Qu73], § 4, Cor. 2 p. 26). The second statement therefore follows from the first one.

We only have to check axioms **A1**, **A2** and **A3** for the covariant functor from the category of k-algebras (with morphisms the flat k-homomorphisms) to abelian groups given by $F(A) = G_n(Z \times_k A)$. For axiom **A1**, see [Qu73], § 2 (9) p.20 and [Qu73], § 7, Prop. 2.2 p.41. Much more than axiom **A2** holds in the present context. Namely, for A noetherian, the maps $G_n(Z \times_k A) \to G_n(Z \times_k A[t_1, \ldots, t_m])$ are isomorphisms ([Qu73], § 6, Theorem 8 p.38). Thus if $L \supset k$ is a field and α is in the kernel of $G_n(Z \times_k L[t_1, \ldots, t_m]) \to G_n(Z \times_k L(t_1, \ldots, t_m))$, then α comes from an element $\beta \in G_n(Z \times_k L)$ which vanishes in $G_n(Z \times_k L(t_1, \ldots, t_m))$, hence by axiom **A1** vanishes in $G_n(Z \times_k L[t_1, \ldots, t_m]_f)$ for some non-zero $f \in L[t_1, \ldots, t_m]$. Now, since k and L are infinite, we may specialize to an L-rational point of $\mathrm{Spec}(L[t_1, \ldots, t_m]_f)$ and we conclude $\alpha = 0$. As for the patching axiom **A3**, it follows from the localization sequence for G-theory ([Qu73], § 7, Prop. 3.2 and Remark 3.4 p.44; see [CT/Oj92] p. 112, where the whole argument is developed in the special case $Z = \mathrm{Spec}(k)$). $\square$

Remark 5.2.3 : When $Z = \mathrm{Spec}(k)$, the above result is just a special case of the Gersten conjecture for K-theory of smooth k-varieties, as proved by Quillen ([Qu73]). It is likely that Quillen's approach could also yield the more general result given above.

Remark 5.2.4 : For A as above, the injection $G_n(A) \to G_n(K)$ is part of the long exact sequence

$$0 \to G_n(A) \to G_n(K) \to \bigoplus_{x \in A^{(1)}} G_{n-1}(k(x)) \to \dots,$$

the Gersten sequence. In my Santa Barbara lectures, I mentioned that the theorem above makes it likely that for an arbitrary variety Z/k the complex

$$0 \to G_n(Z \times_k A) \to G_n(Z \times_k K) \to \bigoplus_{x \in A^{(1)}} G_{n-1}(Z \times_k k(x)) \to \dots$$

is exact. This has since been checked (see [CT/Ho/Ka93]).

THEOREM 5.2.5. — *Let k be an infinite field. Let Z/k be an arbitrary variety. If A is a local ring of a smooth variety over k, and K is its fraction field, then for any integer $n \geq 0$ the map*

$$CH^n(Z \times_k A) \longrightarrow CH^n(Z \times_k K)$$

is injective. In particular, the group $CH^n(Z \times_k A)$ is zero for $n > \dim Z$.

Proof : Rather than invoking the general formalism, let us go through a variant of it. First of all, Chow groups are generally defined only for varieties over a field ([Fu84]). Let us therefore make the statement more precise. Let R be a k-algebra of finite type which is an integral domain. Let $A = R_p$ be the local ring of A at a prime ideal p. We *define* $CH^n(Z \times_k A)$ as the direct limit of the $CH^n(Z \times_k R_f)$ for all $f \notin p$. The transition maps are given by the flat pull-back map ([Fu84], I. 1.7). Similarly, $CH^n(Z \times_k K)$ is defined as the direct limit of the $CH^n(Z \times_k R_f)$ for all non-zero f's.

Let α be in the kernel of $CH^n(Z \times_k A) \longrightarrow CH^n(Z \times_k K)$. Changing R into R_f for suitable $f \notin p$, we may represent α by an element $\beta \in CH^n(Z \times_k R)$. There exists a non-zero $g \in R$ such that β becomes zero when restricted to $CH^n(Z \times_k R_g)$. By Ojanguren's presentation of hypersurfaces in smooth varieties, replacing R by R_f for some suitable $f \notin p$, we may assume that there exists an étale map $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ where $S = k[t_1, \dots, t_d]_r$ and an element $h \in S$ such that the image of h in R differs from g by a unit, and that moreover the induced map $S/h \to R/f$ is an isomorphism. We then have the commutative diagram of localization sequences for the Chow group ([Fu84], I 1.8)

$$\begin{array}{ccccccc}
CH^{n-1}(Z \times_k (S/h)) & \longrightarrow & CH^n(Z \times_k S) & \longrightarrow & CH^n(Z \times_k S_h) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
CH^{n-1}(Z \times_k (R/g)) & \longrightarrow & CH^n(Z \times_k R) & \longrightarrow & CH^n(Z \times_k R_g) & \longrightarrow & 0.
\end{array}$$

In this diagram, the map $CH^{n-1}(Z\times_k(S/h)) \to CH^{n-1}(Z\times_k(R/g))$ is an isomorphism. A straightforward diagram chase then shows that β comes from an element $\gamma \in CH^n(Z\times_k S)$ with image 0 in $CH^n(Z\times_k S_h)$. Now the restrition map $CH^n(Z\times_k k[t_1,\dots,t_n]) \to CH^n(Z\times_k S)$ is (trivially) surjective. The flat pull-back map $CH^n(Z) \to CH^n(Z\times_k k[t_1,\dots,t_d])$ is surjective ([Fu84], I. 1.9). Thus γ comes from an element $\delta \in CH^n(Z)$ with trivial image in $CH^n(Z\times_k S_h)$. Since k is infinite, we may *specialize* at some k-point of the open set $\mathrm{Spec}(S_h)$ of $\mathbb{A}^d_k$ to conclude that $\delta = 0$, hence $\gamma = 0$ hence $\beta = 0$ hence finally $\alpha = 0$. Note that the inclusion of a k-point in $\mathrm{Spec}(S_h)$ is not a flat map and that some care should be exercised here, since Chow groups are not a priori functorial contraviant with respect to arbitrary morphisms. However it is a regular embedding, hence so is the embedding $Z \subset Z\times_k S_h$, and we may use the pull-back maps as defined in [Fu84], chap. VI, and use the functoriality [Fu84] VI. Prop. 6.5 p. 110. ∎

Remark 5.2.6 : A rather convoluted proof of this theorem in a very special case was given in [CT/Pa/Sr89].

THEOREM 5.2.7. — *Let k be an infinite field. Let Z/k be an arbitrary variety. If A is a local ring of a smooth variety over k, and K is its fraction field, then for any integer $n \geq 0$, any positive integer m prime to* char*(k) and any integer j, the maps on étale cohomology groups*

$$H^n(Z\times_k A, \mu_m^{\otimes j}) \longrightarrow H^n(Z\times_k K, \mu_m^{\otimes j})$$

are injective.

Proof : Axiom **A1** is proved in [SGA4] t.2, exp. VII, Cor. 5.9 p. 362. For any scheme X with m invertible on X, the pull-back maps

$$H^n(X, \mu_m^{\otimes j}) \to H^n(X[t], \mu_m^{\otimes j})$$

are isomorphisms ([SGA4], XV, Cor. 2.2; see also [Mi80], VI 4.20 p. 240), hence so are the maps

$$H^n(Z\times_k A, \mu_m^{\otimes j}) \to H^n(Z\times_k A[t_1,\dots,t_r], \mu_m^{\otimes j})$$

for any positive integer r. Axiom **A2** follows just as above from a specialization argument. As for axiom **3** it follows from the excision property in étale cohomology [Mi80], III, 1.27 p.92, completed in [CT/Oj92] p. 114/115. ∎

Remark 5.2.8 : When $\mathrm{char}(k) = 0$ and Z/k is smooth, a similar statement holds with étale cohomology with coefficients in the sheaf $\mathbf{G}_m$ (see [CT/Oj92] loc. cit. in the special case $Z = \mathrm{Spec}(k)$).

Remark 5.2.9 : Just as with G-theory, it is natural to think that the injection in the theorem above is just a special case of a statement that would say : the Bloch-Ogus-Gersten exact sequence for A, with coefficients $\mu_n^{\otimes j}$ remains exact when crossed with an arbitrary k-variety Z. This is proved in [CT/Ho/Ka93].

In all the examples above, the functors F we were considering were from the category of k-algebras to the category of *abelian groups*. The motivation for phrasing the axioms in terms of pointed sets, rather than groups, comes from the study of principal homogeneous spaces under linear algebraic groups. Let k be a field, $\mathrm{char}(k) = 0$, let G be a linear algebraic group over k. If X is a k-scheme, a principal homogeneous space (or torsor) over X under G is a k-scheme Y equipped with a faithfully flat k-morphism $p : Y \to X$ (the structural morphism) and with an action $G \times_k Y \to Y$, $(g, y) \longmapsto g.y$ which respects the projection (namely $p(gy) = p(y)$). Also, the group G must act faithfully and transitively in the fibres of the projection. In other words the map $G \times_k Y \to Y \times_X Y$ given by $(g, y) \longmapsto (gy, y)$ is an isomorphism. Under the assumptions above, one may show that the set of isomorphism classes of torsors over X under G is classified by the Čech cohomology set $\check{\mathrm{H}}^1_{\text{ét}}(X, G)$, henceforth simply denoted $H^1(X, G)$. This is a pointed set, the class $\mathbf{1}$ corresponding to the isomorphism class of the trivial torsor $Y = X \times_k G$ equipped with the projection onto the first factor X. A torsor is in the trivial class if and only if the projection $p : Y \to X$ has a section. (For more details on principal homogeneous spaces, the reader may consult [Mi80], III, § 4 and the literature quoted there.)

In [CT/Oj92], Ojanguren and I study the functor from k-algebras to pointed sets given by $A \longmapsto H^1(A, G)$. For simplicity, let us assume that $\mathrm{char}(k) = 0$ and that G is a reductive k-group. Axioms **A1** and **A3** (patching) may be checked in this context. However, axiom **A2** (the weak homotopy axiom) does not hold in general. It holds when one goes from L to $L[t]$ (one variable) as proved by Raghunathan and Ramanathan ([Ra/Ra84]. However, in the general case it fails as soon as one goes from L to $L[t_1, t_2]$, as demonstrated by various counterexamples due to Parimala, Ojanguren, Sridharan, Raghunathan (see [Ra89]). However, in [Ra89], Raghunathan essentially shows that axiom **A2** holds if all the k-simple components of the derived group of G are k-isotropic (e. g. k is algebraically closed). So the formalism above applies. Using an idea of Raghunathan, Ojanguren and I could produce a variant of the formalism above and prove the general

Theorem 5.2.10 ([CT/Oj92]). — *Let k be a field,* $\mathrm{char}(k) = 0$*, and let G/k be a linear algebraic group over k. Let A be a local ring of a smooth variety over k and let K be its fraction field. The map $H^1(A, G) \to H^1(K, G)$ has trivial kernel. In other words, if a principal homogeneous space $p : Y \to X$ under G over a smooth integral k-variety X is rationally trivial , i.e. has a section over a nonempty open set, then it is everywhere locally trivial, i.e. for any point $x \in X$ there exists a neighbourhood U of x such that $p_U : Y \times_X U \to U$ has a section.*

Variants over a field of arbitrary characteristic have been proved ([CT/Oj92] for k infinite and perfect, [Ra93] for k infinite). The case of a finite ground field has resisted many assaults.

A particular example : Let k be as above, let D/k be a central simple algebra and let $G = \mathbb{SL}(D)$ be the special linear group on D. We have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{SL}(D) \longrightarrow \mathbb{GL}(D) \longrightarrow \mathbb{G}_{m,k} \longrightarrow 1$$

given by the reduced norm map $\mathbb{GL}(D) \longrightarrow \mathbb{G}_{m,k}$. A variant of Hilbert's theorem 90 says that for any local k-algebra A, the set $H^1(A, \mathbb{GL}(D))$ is reduced to $\mathbf{1}$. One therefore gets a bijection

$$A^*/\mathrm{Nrd}((D \otimes_k A)^*) \simeq H^1(A, \mathbb{SL}(D)).$$

The previous theorem therefore implies that if A is a local ring of a smooth k-variety and K is its field of fractions, then the map

$$A^*/\mathrm{Nrd}((D \otimes_k A)^*) \to K^*/\mathrm{Nrd}((D \otimes_k K)^*)$$

is injective. In words, if an element of A^* is a reduced norm from $D \otimes_k K$, then it is also a reduced norm from $D \otimes_k A$. As a very special case, one may take D to be the usual quaternion algebras, and for A as above, we obtain : if $a \in A^*$ is a sum of 4 squares in K, then it is also a sum of 4 squares in A.

One may wonder whether a similar property holds for the reduced norm of an arbitrary Azumaya algebra over a regular local ring A. This is known when $\dim(A) = 2$ ([Oj82b], [CT/Oj92]) and for A a local ring of a smooth variety of dimension 3 over an infinite field k ([CT/Oj92]).

§ **5.3 Codimension one purity for some functors**

Let k be a field. For any k-variety Z we define the *norm subgroup* $N_Z(k) \subset k^*$ to be the subgroup spanned by the norm subgroups $N_{K/k}(K^*)$, where K/k runs through all finite field extensions of k such that the set $Z(K) = Hom_{\mathrm{Spec}(k)}(\mathrm{Spec}(K), Z)$ of K-rational points is not empty. Equivalently, $N_Z(k) \subset k^*$ is the subgroup spanned by the norm subgroups $N_{k(P)/k}(k(P)^*) \subset k^*$, for P running through the closed points of the k-variety Z – here $k(P)$ denotes the residue field at P. Of course, if $Z(k) \neq \emptyset$, then $N_Z(k) = k^*$. If $L \supset k$ is any field extension, we let $N_Z(L) = N_{Z \times_k L}(L)$.

Here are two concrete examples. If D/k is a central simple algebra and Z/k is the associated Severi-Brauer variety, then $N_Z(k) \subset k^*$ coincides with the subgroup $\mathrm{Nrd}(D^*) \subset k^*$. If q is a nondegenerate quadratic form in $n \geq 3$ variables and q represents 1 over k, and if Z is the projective quadric defined by q, then $N_Z(k)$ is the subgroup of k^* spanned by the non-zero values of q on k^n. If q is a Pfister form, then $N_Z(k)$ is simply the subgroup of (non-zero) values of the Pfister form.

Given any integral variety X/k with function field $k(X)$, we then define two subgroups of $k(X)^*$:

$$D^1_Z(X) = \{f \in k(X)^* \mid \forall M \in X^1 \quad f = u_M g_M\,, \quad u_M \in \mathcal{O}^*_{X,M}\,, \quad g_M \in N_Z(k(X))\}\,,$$

$$D_Z(X) = \{f \in k(X)^* \mid \forall M \in X \quad f = u_M g_M\,, \quad u_M \in \mathcal{O}^*_{X,M}\,, \quad g_M \in N_Z(k(X))\}\,.$$

We trivially have $D_Z(X) \subset D^1_Z(X)$.

The following theorem is due to Markus Rost [Ro90]. Rost's proof relied on Quillen's presentation of hypersurfaces in smooth varieties. The proof given below starts with a reduction also due to Rost, but to complete the proof, I use the formalism described above, applied to the Chow group functor.

THEOREM 5.3.1. — *Assume that Z/k is a proper equidimensional variety. If X/k is a smooth integral k-variety, then $D_Z(X) = D^1_Z(X)$.*

Proof : Let $p : Z \to \mathrm{Spec}(k)$ be the structural morphism. Let d be the dimension of Z. Let R be the local ring of X at a point $M \in X$. Consider the commutative diagram

$$\begin{array}{ccc} Z_R & \supset & Z_K \\ {\scriptstyle p_R}\downarrow & & \downarrow{\scriptstyle p_K} \\ \mathrm{Spec}\, R & \supset & \mathrm{Spec}\, K \end{array}$$

where the vertical maps are proper. From [Fu84], I, Prop. 1. 4, we then get the commutative diagram

$$\begin{array}{ccccccc} & & \bigoplus_{P\in Z_R^{(d)}} k(P)^* & \longrightarrow & \bigoplus_{P\in Z_R^{(d+1)}} \mathbb{Z} & \longrightarrow & CH^{d+1}(Z_R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & & & \\ R^* & \longrightarrow & K^* & \longrightarrow & \bigoplus_{p\in R^{(1)}} \mathbb{Z} & & & & \end{array}$$

where the middle horizontal maps are the divisor maps. The top row is exact by the definition of Chow groups. The bottom row is exact since an element of K^* with trivial divisor is a unit in the regular ring R (a very special case of Gersten's conjecture!). We have

$$p_*(\bigoplus_{P\in Z_R^{(d)}} k(P)^*) = p_*(\bigoplus_{P\in Z_K^{(d)}} k(P)^*) = N_Z(K).$$

On the other hand, if $f \in K^*$ belongs to $D^1_Z(X)$, by the analogous and compatible diagram over each local ring R_p for p prime of height one, we conclude that $\mathrm{div}(f) = p_*(z)$ for some $z \in \bigoplus_{P\in Z_R^{(d+1)}} \mathbb{Z}$.

By Theorem 5.2.5 (whose proof uses Ojanguren's presentation and the formalism described above) we know that $CH^{d+1}(Z_R) = 0$. From the commutative diagram we conclude that there exists $g \in p_*(\bigoplus_{P\in Z_R^{(d)}} k(P)^*) = N_Z(K)$ such that $\mathrm{div}(f/g) = 0$, hence by the exactness of the sequence below, $f = ug$ with $u \in R^*$. $\square$

Corollary 5.3.2. — *Let Z/k be a proper k-variety. Let X and Y be smooth proper integral k-varieties. If X is k-birational to Y, then the group $D^1_Z(X)$ is isomorphic to $D^1_Z(Y)$, and the quotient $D^1_Z(X)/N_Z(k(X))$ is isomorphic to $D^1_Z(Y)/N_Z(k(Y))$. If $X = \mathbb{P}^d_k$, then the natural map $k^*/N_Z(k) \to D_Z(X)/N_Z(k(X))$ is an isomorphism.*

Proof : Only the last statement has not been proved. But if in the diagram above one lets $R = k[t_1, \dots, t_d]$ the proof follows, since $k^* = R^*$. □

Remark 5.3.3 : Let $k = \mathbf{R}$ and let Z be the smooth projective quadric over k defined by the Pfister form $< 1, 1 >^{\otimes d}$. If $X/\mathbf{R}$ is a smooth, projective, geometrically integral variety of dimension d, then the group $D_Z(X)/N_Z(k(X))$ may be identified with the group $(\mathbb{Z}/2)^s$ where s is the number of connected components of the topological space $X(\mathbf{R})$ of real points of X [CT78]. This number is a birational invariant. Rost's theorem is a far reaching generalization of this fact.

Let now D be a central simple algebra over a field k. Let Z/k be the associated Severi-Brauer variety. Let $G = \mathbb{SL}(D)$ be the special linear group on D. Let X be a smooth integral k-variety, and let $E/k(X)$ be a principal homogeneous space under G over the generic point of X. Assume that E may be extended to a principal homogeneous space in some neighbourhood of each codimension 1 point on X. Then Theorem 5.3.1 implies that everywhere locally on X the principal homogeneous space $E/k(X)$ may be extended (up to isomorphism) to a principal homogeneous space. This result, first proved in a special case in [CT/Pa/Sr89] has also been proved in a more K-theoretical way in [CT/Oj92],Theorem 5.3 (the proof there uses Quillen's approach).

Whether the same codimension 1 purity statement holds true for principal homogeneous spaces under an arbitrary linear algebraic group G/k over a smooth k-variety X is an open question. Only the case $\dim(X) = 2$ is known [CT/Sa79] ; the basic ingredient of the proof being that reflexive modules over two-dimensional regular local rings are free.

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