Brauer–Manin obstruction and integral points : a survey

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Classical statements.

- p prime, $p = x^2 + y^2$ in $\mathbb{Z}$ iff $p \equiv 1 \bmod 4$.
- $n \in \mathbb{Z}, n = x^2 + y^2 + z^2$ in $\mathbb{Z}$ iff $n > 0$ and $n \neq 4^r(8m + 7)$.
- $n \in \mathbb{Z}$, $n = x^2 + y^2 + z^2 + t^2$ in $\mathbb{Z}$ iff $n > 0$
- $q(x, y, z, t)$ quaternary quadratic form over $\mathbb{Z}$, indefinite over $\mathbb{R}$, then $n \in \mathbb{Z}$ is represented by q over $\mathbb{Z}$ if and only if any congruence $n \equiv q(x, y, z, t) \bmod m$ has a solution.

For each prime p, we have $\mathbb{Z} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$
A polynomial equation $f(x_1, \ldots, x_r)$ with coefficients in $\mathbb{Z}$ has solutions in $\mathbb{Z}_p$ if and only if any congruence $f(x_1, \ldots, x_r) \equiv 0 \mod p^t$ has a solution.
Let us denote by $\mathcal{X}$ the scheme over $\mathbb{Z}$ defined by $f(x_1, \ldots, x_r) = 0$. For any commutative ring A, $\mathcal{X}(A)$ is the set of solutions of $f(x_1, \ldots, x_r) = 0$ with coordinates in A.
By convention, $\mathbb{Z} \subset \mathbb{Z}_\infty = \mathbb{R}$.

We have a diagonal embedding

$$\mathcal{X}(\mathbb{Z}) \hookrightarrow \prod_{p \cup \infty} \mathcal{X}(\mathbb{Z}_p).$$

The classical results mentioned above may each be rewritten as :

LHS not empty iff RHS not empty.
$\mathcal{X}(\mathbb{Z}) \neq \emptyset$ iff for each prime p (also $p = \infty$) $\mathcal{X}(\mathbb{Z}_p) \neq \emptyset$.
In other words, in these cases we have a local-global principle for existence of integral points.

The proof of local-global principles, when they hold, often go hand in hand with density properties.
Let us explain this over an arbitrary number field k. Given a finite set S of places of k, one says that *strong approximation off S holds for a k-variety X* if given any finite set T of places containing S, any integral model $\mathcal{X}$ over O_S of X, for each $v \in T \setminus S$ an open set $U_v \subset X(k_v)$ such that the product

$$\prod_{v \in S} X(k_v) \times \prod_{v \in T \setminus S} U_v \times \prod_{v \notin T} \mathcal{X}(O_v)$$

is not empty, then there exists a point of $X(k)$, hence of $\mathcal{X}(O_T)$, in this product.

Such a statement contains a local-global statement for the existence of O_S-integral points for O_S-models of X.
The archetypical example for strong approximation is the additive group $\mathbb{G}_a$ over a number field.
The first case is $\mathbb{Q}$ with $S = \{\infty\}$. The map $\mathbb{Z} \to \prod_{p \neq \infty} \mathbb{Z}_p$ has dense image.
We may replace $\{\infty\}$ by any finite place ℓ, the map $\mathbb{Z}[1/\ell] \to \mathbb{R} \times \prod_{p \neq \ell} \mathbb{Z}_p$ has dense image.

Classical results

Theorem (Eichler, Kneser, Platonov) *Let G be a semisimple simply connected group over a number field k. Let v be a place. If for each simple factor H of G, the group $H(k_v)$ is noncompact, then strong approximation off v holds for the group G.*

Theorem (Eichler, Kneser) *Let q be a nondegenerate quadratic form in $n \geq 4$ variables over a number field k, and assume q isotropic over k_v for some place v. Then, for any $a \in k^\times$, strong approximation off v holds for*

$$q(x_1, \dots, x_n) = a.$$

Counterexamples to local-global principle for integral points, and an obstruction to strong approximation

Very quickly, one realizes that such local-global principle and strong approximation often fail.
There are solutions in all $\mathbb{Z}_p$ and $\mathbb{R}$ but no solutions in $\mathbb{Z}$ for :
The equation $23 = x(x + 7y)$
The system $\{2x - 5y = 1, xt = 1\}$
The equation $1 = 4x^2 + 25y^2$
The equation $1 = 4x^2 - 475y^2$ (harder)
(Over $\mathbb{Q}$, these are projective conics minus 2 rational points or minus 2 congugate points.)

The literature contains such examples as :

For q prime, the equation $q = x^2 + 27y^2$
has solutions in all $\mathbb{Z}_p$ iff $q \equiv 1 \bmod 3$
and if so
it has a solution in $\mathbb{Z}$ iff 2 is a cube in $\mathbb{F}_q$.

(Euler, Gauß, see book by D. Cox)

Let $\mathcal{X}_{n,m}$ with $n, m \in \mathbb{N}$, $(n, m) = 1$ be given by

$$m^2x^2 + n^{2k}y^2 - nz^2 = 1.$$

Then $\mathcal{X}_{n,m}(\mathbb{Z}_p) \neq \emptyset$ for all prime p.
$\mathcal{X}_{n,m}(\mathbb{Z}) = \emptyset$ iff
either 2 divides exactly m and $n \equiv 5 \bmod 8$
or 4 divides m and $n \equiv \pm 3 \bmod 8$.

(Schulze-Pillot and Xu)

(Borovoi–Rudnick)

$$(y-x)(9x+7y)=1-2z^2$$

Solutions (x,y,z) over $\mathbb{Q}$: $(-1/2,1/2,1)$ and $(1/3,0,1)$ hence solution over each $\mathbb{Z}_p$.
For any solution over $\mathbb{Z}_2$, one has $y-x\equiv\pm 3 \bmod 8$.
If solution over $\mathbb{Z}$, if p prime divides $y-x$ then $1-2z^2\equiv 0 \bmod p$ so p odd and 2 square mod p. So (complementary law of quadratic reciprocity) $p\equiv\pm 1 \bmod 8$. So $y-x\equiv\pm 1 \bmod 8$. Contradiction.

All the counterexamples to the local-global principle given above may be explained by means of

the integral Brauer–Manin obstruction

This is a variant, formulated and studied by Fei XU and the speaker (2009), of the Brauer–Manin obstruction to the local-global principle for rational points (1970).

Let k be a number field, $\mathbb{A}_k$ the ring of adèles. Let X be a k-variety. Let $X(\mathbb{A}_k)$ denote the adèles of X and $\mathrm{Br}(X)$ the Brauer group of X. There is a natural pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

$$(\{M_v\}, A) \mapsto \sum_v \mathrm{inv}_v A(M_v),$$

which vanishes on $X(k) \times \mathrm{Br}(X)$ (reciprocity law in class field theory). One lets

$$X(\mathbb{A}_k)^{\mathrm{Br}(X)}$$

denote the kernel on the left. We thus have $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}(X)}$. This inclusion was noticed by Manin (1970) and until ten years ago was mostly used for the study of rational points on smooth, projective varieties.

Let X be a variety over a number field k. Assume $X(\mathbb{A}_k) \neq \emptyset$. Let S be a finite set of places. Let $S \subset T$ with T a finite set of places containing all the archimedean places and let $\mathcal{X}/O_T$ be a model of X/k, then for each $v \in T \setminus S$, let $U_v \subset X(k_v)$ be an open set (for the k_v-topology). Assume that in any such situation, if the set

$$[\prod_{v \in S} X(k_v) \times \prod_{v \in T \setminus S} U_v \times \prod_{v \notin T} \mathcal{X}(O_v)]^{\mathrm{Br}(X)}$$

is not empty, then it contains the diagonal image of a point in $X(k)$ (hence in $\mathcal{X}(O_T)$).

One then says that *strong approximation off S with Brauer–Manin condition holds for X*.

In the counterexamples to the local-global principle given above, one has $k = \mathbb{Q}$, $S = \infty$, $\mathcal{X}$ over $\mathbb{Z}$, and one may show :

$$[X(\mathbb{R}) \times \prod_{p \neq \infty} \mathcal{X}(\mathbb{Z}_p)]^{\mathrm{Br} X} = \emptyset,$$

hence (by reciprocity) $\mathcal{X}(\mathbb{Z}) = \emptyset$.

Homogeneous spaces of connected linear algebraic groups

For S nonempty, strong approximation off S with Brauer–Manin condition has been established for

X/k homogeneous space of a connected linear algebraic group G/k, with connected geometric stabilizers $\overline{H}$, under a suitable *noncompactness hypothesis for G at the places of S* in the following cases :

• CT/Xu 2005-2009, X homogeneous spaces of G semisimple simply connected, application to the representation of an integer by an indefinite quadratic form of rank at least 3.
Here $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.
In this particular case, the results are a modern version of results of M. Kneser on the genus and the spinor genus of quadratic forms.

The Brauer group point of view leads to a simple numerical test for the representation of an integer by an indefinite ternary integral quadratic form. This accounts in particular for the precise result of Schulze-Pillot and Xu.

Under the further assumption that S contains the archimedean places :

• Harari 2008, X homogeneous space of $G = T$ a torus, $\overline{H} = 1$, e.g. $x^2 - ay^2 = b$, homogeneous space of the torus $x^2 - ay^2 = 1$. Here $\mathrm{Br}(X)/\mathrm{Br}(k)$ is infinite. In some cases, one may nevertheless reduce to finitely many conditions (Wei, Xu; Harpaz) – remember the Euler, Gauss example $p = x^2 + 27y^2$.

• Demarche, Borovoi (2011) G arbitrary, $\overline{H}$ connected arbitrary.

The proofs use :

Hasse principle for semisimple simply connected groups (Kneser, Harder, Chernousov)

Class field theory : Tate-Nakayama duality theorems for tori, non commutative generalization (Kottwiz), extension to duality theorems for complexes of tori (Demarche)

Strong approximation off S for semisimple simply connected groups with *noncompacity condition at some place in* S.

What about integral points when one leaves the world of homogeneous spaces under linear algebraic groups ?

An open question : if X smooth satisfies strong approximation off S and $U \subset X$ is an open set whose complement is of codimension ≥ 2, does strong approximation off S hold for U ?

Known for $X = \mathbb{A}^n_k$ (Yang Cao and Fei Xu, ...).

The same authors give a negative answer to the analogous question for strong approximation with Brauer-Manin condition.

Groupic varieties

Theorem (Yang Cao and Fei Xu 2014/2015) *Let G/k be a conected linear algebraic group. Let S_∞ be the set of archimedean places of k. Assume that for each "simple factor" H of the derived group of G, the group $H(k_\infty)$ is not compact.*
Let X/k be a smooth connected variety which contains G as a dense Zariski open set. Assume the action of G by translation on itself (on the left) extends to an action of G on X. Let S_∞ be the set of archimedean places of k. Then strong approximation off S with Brauer-Manin obstruction holds for X.

Yang Cao and Fei Xu first proved this when G is a torus (toric varieties), then they produced a (partly different) proof for arbitrary G. The proofs are quite elaborate, and involve several new ideas.

Results on strong approximation for groups are of course used.

An important tool on the geometric side is universal torsors (CT-Sansuc, 1976/1987) and the structure of universal torsors on smooth projective compactifications of X (CT, 2008), with some novel insight on the geometric structure of such torsors in the present situation, when restricted to the groupic variety X.

For a finite place v of good reduction for (G, X), there is a delicate description ot the intersection of $G(k_v)$ with the O_v-integral points of X.

Pencils of homogeneous spaces of connected semisimple groups

Theorem (CT-Harari 2013, appears in Crelle 2016).

Let k be a number field. Let X be a smooth geometrically connected k-variety and $f : X \to \mathbb{A}^1_k$ a k-morphism. Assume $X(\mathbb{A}_k) \neq \emptyset$. Let $K = k(\mathbb{A}^1)$ and let G be an absolutely almost simple simply connected semisimple group.

Assume :

(i) The generic fibre X_η/K of f is a homogeneous space of G with connected reductive stabilizers.

(ii) All geometric fibres of f are nonempty and integral.

(iii) There exists a place v_0 such that the fibration acquires a rational section over k_{v_0} and that at almost all points $t \in \mathbb{A}^1(k_{v_0})$, the group G_t is isotropic over k_{v_0}.

(iv) $\mathrm{Br}(k) = \mathrm{Br}(X)$.

Then strong approximation off v_0 holds for X.

(i) and (ii) imply that $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite. If we drop (iv), i.e. do not assume $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$, under a suitable further hypothesis at the place v_0, strong approximation off v_0 with Brauer–Manin condition holds for X.

The proof of the previous results uses many of the techniques used in the study of the Brauer-Manin obstruction for rational points, in particular in work of Harari.

The hypothesis

(ii) All geometric fibres of f are nonempty and integral.

is quite strong.

Special case : **Pencils of quadrics of dimension (at least) 2**

Corollary. *Let $a_i(t), i = 1, 2, 3$ and $p(t)$ be polynomials in $\mathbb{Z}[t]$. Assume that the product $p(t).\prod_i a_i(t)$ is nonconstant and separable in $\mathbb{Q}[t]$. Let $\mathcal{X}/\mathbb{Z}$ be defined by*

$$\sum_{i=1}^{3} a_i(t)x_i^2 = p(t).$$

Assume that for almost all $t \in \mathbb{R}$, the conic $\sum_i a_i(t)x_i^2 = 0$ has a point over $\mathbb{R}$. Then $\mathcal{X}(\mathbb{Z})$ is dense in the product $\prod_p \mathcal{X}(\mathbb{Z}_p)$.

For the special case where each $a_i(t)$ is a constant, a slightly stronger version of that corollary had been obtained by CT-Xu (2011).

The real condition may be replaced by a better noncompacity condition (Fei Xu).

Theorem (Xu 2015) *Let k be a number field. Let $q(x_1, \ldots, x_n)$ a nondegenerate quadratic form with $n \geq 3$ and $p(t)$ a polynomial. Let X be the k-variety defined by*

$$q(x_1, \ldots, x_n) = p(t).$$

Let $\tilde{X} \to X$ be a proper, birational map which is a resolution of the singularities of X.
If $\prod_{v \in \infty} \tilde{X}(k_v)$ is not compact, then strong approximation with Brauer-Manin obstruction off v holds for X holds off ∞.

If $p(t)$ is separable, then X/k is smooth and $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$. In this case the result reduces to strong approximation off ∞.
Special cases of this were obtained by Watson (1960).

Pencils of homogeneous spaces of tori

K/k a number field extension of degree d, $p(t) \in k[t]$ a polynomial.

$\Xi = \sum_{i=1}^{d} x_i\omega_i$ for ω_i a basis of K/k.

Variety X/k defined by :

$$Norm_{K/k}(\Xi) = p(t).$$

Exemple : $x^2 - ay^2 = p(t)$. The study of integral points on such a surface X is quite difficult, already for $deg(p(t)) = 3$ (cubic surface), though, for $p(t)$ separable, $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.

The study of *rational* points on varieties over a number field defined by

$$Norm_{K/k}(\Xi) = p(t)$$

has witnessed progress in recent years, but one is still far from a proof of the hoped for theorem that the Brauer-Manin obstruction for rational points on smooth projective models is the only obstruction for rational points. General results were obtained for $k = \mathbb{Q}$ and $p(t) \in \mathbb{Q}[t]$ a split polynomial (work of Browning, Matthiesen, Skorobogatov, Harpaz, Wittenberg, building upon results of Green, Tao and Ziegler in additive combinatorics).

To study strong approximation off a finite set S of places of k on the affine variety

$$Norm_{K/k}(\Xi) = p(t),$$

one would like to use the fibration $X \to \mathbb{A}^1_k$ given by the coordinate t.

The methods of CT-Xu and CT/Harari do not extend directly, for at least two reasons :

(a) There are several geometrically reducible fibres.

(b) For the smooth fibres X_t, the quotient $\mathrm{Br}(X_t)/\mathrm{Br}(k)$ is infinite.

A conditional result

Theorem (Gundlach, 2013) *Let* $P(x) \in \mathbb{Z}[x]$ *be separable as a polynomial in* $\mathbb{Q}[x]$*. Assume either that the degree of* P *is odd or that the leading coefficient of* P *is positive. Let* $\mathcal{X}$ *be the affine* $\mathbb{Z}$*-scheme defined by the equation*

$$y^2 + z^2 = P(x).$$

Let $X = \mathcal{X} \times_{\mathbb{Z}} \mathbb{Q}$*. Assume*

$$[\prod_{p \cup \infty} \mathcal{X}(\mathbb{Z}_p)]^{\mathrm{Br}(X)} \neq \emptyset.$$

If we assume Schinzel's hypothesis, then $\mathcal{X}(\mathbb{Z}) \neq \emptyset$*, and given an element* $\{M_p\} \in [\prod_{p \cup \infty} \mathcal{X}(\mathbb{Z}_p)]^{\mathrm{Br}(X)}$ *one may find an element of* $\mathcal{X}(\mathbb{Z})$ *which is as close to* $\{M_p\}$ *for* p *finite as one wishes, but a priori only in the* x *variable.*

The argument uses the quadratic form $y^2 + z^2$ in a critical way. One produces an integer n such that $P(n)$ is positive and is a sum of two squares in each $\mathbb{Z}_p$. This implies the same statement for $P(n)$ over $\mathbb{Z}$.

For a general $a \in \mathbb{Z}$ there is no local-global statement for the representation of an integer by the form $y^2 + az^2$.

Schinzel's hypothesis is concerned with a system of polynomials in one variable. The Green, Tao, Ziegler results give unconditional versions of Schinzel's hypothesis for systems of polynomials of total degree 1 in at least 2 variables.
This substitute for Schinzel's hypothesis has already been used for the study of rational points (papers mentioned above).

Derenthal and Wei (2014) have used this idea to study strong approximation of varieties over $\mathbb{Q}$ given by an equation

$$Norm_{K/\mathbb{Q}}(\Xi) = P(s,t),$$

where $P(s,t) \in \mathbb{Q}[s,t]$ is a product of polynomials of degree 1 and $K/\mathbb{Q}$ is an arbitrary number field.

Here is one case of their result which is simple to state.

Theorem. *Assume that the irreducible factors of $P(s,t)$ are pairwise nonproportional linear forms. Then X satisfies (smooth) strong approximation with Brauer-Manin obstruction off the real place.*

One "algebraic" step in the proof is a reduction to (smooth) strong approximation on universal torsors, which are given by explicit equations. For varieties given by these equations, the authors use a slight generalization of impressive analytic results of Browning and Matthiesen (2013), in the Green, Tao, Ziegler direction.

Theorem (Harpaz 2015). *Let S be a finite set of places of $\mathbb{Q}$ containing $2, \infty$. Let $f(s,t), g(s,t) \in \mathbb{Z}_S[t]$ be homogeneous forms of degree 2 in (s,t) which split completely over $\mathbb{Z}_S$. Let $\mathcal{Y} \to \mathbb{P}^1_{\mathbb{Z}_S}$ be the pencil of affine conics over $\mathbb{P}^1$ given by the equation*

$$f(s,t)x^2 + g(s,t)y^2 = 1$$

in $\mathbb{A}^2 \times \mathbb{P}^1$ (affine coordinates (x,y), projective coordinates (s,t). Under precise (and generic) conditions on the arithmetic of the coefficients of f and g, if $\mathcal{Y}$ has a $\mathbb{Z}_S$-adelic point, then it has a $\mathbb{Z}_S$-point.

The conditions are of the type : a finite number of classes in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ are independent.

This is an – unconditional - integral point analogue of conditional theorems which were proven for rational points of (some types of) surfaces with a pencil of curves of genus one over $\mathbb{P}^1$. The latter theorems build upon a technique of Swinnerton-Dyer, further developed by him, CT, Skorobogatov, Wittenberg.

These theorems are conditional, they rely on :
1) The Schinzel hypothesis
2) Finiteness of Sha of an elliptic curve E and the perfect, alternate duality on the finite group $Sha(E)$.

In Harpaz' situation, one can do without 1) thanks to Green, Tao, Ziegler.

As for 2), affine conics are simple analogues of curves of genus 1. The aim is to find a point $(s_0, t_0) \in \mathbb{P}^1(\mathbb{Z}_S)$ such that the fibre, which is an affine conic, has an adelic integral point not obstructed by the (infinite !) Brauer group of the fibre, hence has a $\mathbb{Z}_S$-point. For a 1-dimensional torus over $\mathbb{Z}_S$, Harpaz considers multiplication by 2, and develops an analogue of the Selmer group. In the Swinnerton-Dyer case, with multiplication by 2 on elliptic curve, one ultimately has to assume finiteness of Tate-Shafarevich groups. Here there is no such problem, one uses integral version of Tate-Nakayama duality theorems for tori.

Affine cubic surfaces : computing the Brauer-Manin set of integer points in a classical diophantine problem

An affine cubic surface $X \subset \mathbb{A}^3$ is the complement of a hyperplane section of a projective cubic surface $Y \subset \mathbb{P}^3_k$. Assume Y is smooth. Then the smooth affine cubic surface X is a "log-K3" surface. The study of its integral points is of the same order of difficulty as the study of rational points on projective $K3$-surfaces, such at smooth quartics in $\mathbb{P}^3_k$.

The surfaces studied by Harpaz (above) are also "log-K3 surfaces".

The equation $a = x^3 + y^3 + z^3$*, with* $a \in \mathbb{Z}$ *nonzero.*

There are solutions with $x, y, z \in \mathbb{Q}$.

For $a = 9n \pm 4$ with $n \in \mathbb{Z}$, there are no solutions with $x, y, z \in \mathbb{Z}$.

Famous open question : if a is not of the shape $9n \pm 4$, is there a solution with $x, y, z \in \mathbb{Z}$?

Open already for $a = 33$.

Theorem (CT/Wittenberg 2009) *Let $\mathcal{X}_a$ be the $\mathbb{Z}$-scheme defined by $x^3 + y^3 + z^3 = a$, with $a \neq 0$. Let $X_a = \mathcal{X}_{a,\mathbb{Q}}$. If $a \neq 9n \pm 4$, then*

$$(\prod_p \mathcal{X}_a(\mathbb{Z}_p))^{\mathrm{Br}(X_{a,\mathbb{Q}})} \neq \emptyset.$$

This makes it unlikely that use of a reciprocity law will prevent this equation from having an integral solution.

To prove such a result, one must compute $\mathrm{Br}(X_a)/\mathrm{Br}(\mathbb{Q})$.
Let $X_a^c \subset \mathbb{P}^3_{\mathbb{Q}}$ be the cubic surface with homogeneous equation $x^3 + y^3 + z^3 = at^3$. Let E be the elliptic curve over $\mathbb{Q}$ with equation $x^3 + y^3 + z^3 = 0$. This is the complement of X_a in X_a^c.
There is a localisation exact sequence

$$0 \to \mathrm{Br}(X_a^c) \to \mathrm{Br}(X_a) \to H^1(E, \mathbb{Q}/\mathbb{Z}).$$

The last group classifies abelian unramified covers of E.
We may assume that a is not a cube. An algebraic computation yields $\mathrm{Br}(X_a^c)/\mathrm{Br}(\mathbb{Q}) = \mathbb{Z}/3$, with an explicit generator $\beta \in \mathrm{Br}(X_a^c)$, of order 3.

An algebraic argument shows that the image of $\mathrm{Br}(X_a) \to H^1(E, \mathbb{Q}/\mathbb{Z})$ consist of classes which vanish at each of the points $(1, -1, 0)$, $(0, 1, -1)$, $(1, 0, -1)$.

One then uses arithmetic for the elliptic curve E over $\mathbb{Q}$ (knowledge of all isogeneous curves) to show that such a class in $H^1(E, \mathbb{Q}/\mathbb{Z})$ is zero. Thus $\mathrm{Br}(X_a^c) = \mathrm{Br}(X_a)$.

One then shows that for any $a \in \mathbb{Z}$ not a cube and not of the shape $9n \pm 4$, there exists a prime p such that β takes three distinct values on $\mathcal{X}_a(\mathbb{Z}_p)$.
Thus

$$(\prod_p \mathcal{X}_a(\mathbb{Z}_p))^{\mathrm{Br}(X_a)} = (\prod_p \mathcal{X}_a(\mathbb{Z}_p))^{\beta} \neq \emptyset$$

Warning

For $X \subset X^c$ the complement of a smooth curve C in say a geometrically rational smooth projective surface X^c, the quotient $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ need not be finite.

Example : complement X of a smooth conic $q(x, y, t) = 0$ in $\mathbb{P}^2_{\mathbb{Q}}$. In this case $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

Given by the quaternion classes $(q(x, y, t)/t^2, a)$, with $a \in \mathbb{Q}^\times$.

An example for which failure of the local-global principle for integral points is not accounted for by the Brauer-Manin obstruction on the variety.

Let $q(x,y,z) = 16x^2 + 9y^2 - 3z^2$. Consider the $\mathbb{Z}$-scheme $\mathcal{X} \subset \mathbb{P}^2_{\mathbb{Z}}$ defined by $q(x,y,z) \neq 0$. Let $\mathcal{Q}$ be the affine quadric over $\mathbb{Z}$ defined by $q(x,y,z) = 1$.

Using the obvious μ_2-covering $\mathcal{Q}_{\mathbb{Q}} \to \mathcal{X}_{\mathbb{Q}}$, one shows :

$$[\prod_p \mathcal{X}(\mathbb{Z}_p)]^{\mathrm{Br}\mathcal{X}_{\mathbb{Q}}} \neq \emptyset \text{ but } \mathcal{X}(\mathbb{Z}) = \emptyset.$$

This provides a rather simple "Skorobogatov" type of example in the affine context : after a descent under a finite group, on reduces to the Brauer-Manin condition on covers.
Other examples were found by Kresch and Tschinkel.

Hyperbolic curves

$\mathbb{P}^1$ *minus three points*

Let $X \subset \mathbb{P}^1_k$ be an open set whose geometric complement consist of at least 3 points. One may view X as a closed curve in a k-torus T. The whole situation may be realized over the ring O_S of S-integers, for some finite set S of places. We thus have $\mathcal{X} \subset \mathcal{T}$.

Conjecture (Harari and Voloch 2009, extending a conjecture by Skolem (1937))

$$\mathcal{X}(O_S) = [\prod_{v \notin S} \mathcal{X}(O_v)] \cap \mathcal{T}(O_S)^{closure} \subset \prod_{v \notin S} \mathcal{T}(O_v).$$

They show that $[\prod_{v \notin S} \mathcal{X}(O_v)] \cap \mathcal{T}(O_S)^{closure}$ may be interpreted as a Brauer-Manin set of $\mathcal{X}$ (analogue of a statement by Scharashkin and Skorobogatov for rational points on projective curves of genus at least 2 with an embedding into its jacobian).

The analogue of this conjecture over global fields of positive characteristic has been proved by Q. Liu and F. Xu (2014). They also have a partial result over $k = \mathbb{Q}$ – and here the fact that $\mathbb{Z}$ has only finitely many units plays a key rôle.

One might be tempted to produce a similar conjecture for integral points of arbitrary hyperbolic curves.
For a finite closed subscheme $Z \subset W$, where W/k is either an abelian variety or a torus, letting $\overline{W(k)} \subset W(\mathbb{A}_k)$ be the topological closure, one shows (Stoll; Liu-Xu) :

$$Z(k) = Z(\mathbb{A}_k) \cap \overline{W(k)}.$$

$\overline{W(k)}$ is closely related to $W(\mathbb{A})^{\mathrm{Br}(W)}$.
Using this property, given a quasi-finite k-morphism $X \to Y$ of hyperbolic curves, one might show that the conjecture for Y implies the conjecture for X.

Harari and Voloch however have the following example.
Let X be the affine curve $\mathcal{X}/\mathbb{Z}$ given by $y^2 = x^3 + 3$. Over $\mathbb{Q}$, this is the complement of one rational point in an elliptic curve E. Let P be the point $(x, y) = (1, 2)$. One has $E(\mathbb{Q}) = \mathbb{Z}.P$ and $\mathcal{X}(\mathbb{Z}) = \{\pm P\}$. Let p run through primes $p \equiv 3 \bmod 8$. Take a subsequence of the pP in $E(\mathbb{Q})$ so that its image under the embedding $E(\mathbb{Q}) \hookrightarrow E(A_{\mathbb{Q}})$ converges.
Claim :
(i) The limit is in $\prod_p \mathcal{X}(\mathbb{Z}_p)^{\mathrm{Br}\mathcal{X}_{\mathbb{Q}}}$.
(ii) This limit is neither P nor $-P$.