INTRODUCTION TO WORK OF HASSETT-PIRUTKA-TSCHINKEL AND SCHREIEDER

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Hassett, Pirutka and Tschinkel [13] gave the first examples of families $X \to B$ of smooth, projective, connected, complex varieties having some rational fibres and some other fibres which are not even stably rational. This used the specialisation method of Voisin, as extended by Pirutka and myself. Under specific circumstances, a simplified version of the specialisation method was produced by Schreieder [18, 19], leading to a simpler proof of the HPT example (no explicit resolution of singularities). In the following note I describe the method in its simplest form. For further developments, the reader is invited to read [2], which offers a different look at [13] as well as some generalizations, [3], and the papers [18, 19, 20] by Schreieder.

I thank Asher Auel for remarks on the typescript.

These notes were written on the occasion of the conference *Quadratic Forms in Chile 2018*, held at IMAFI, Universitad de Talca, 8-12 January 2018. They were further developed on the occasion of the School *Birational geometry of hypersurfaces*, Palazzo Feltrinelli, Gargnano del Garda, 19–23 March 2018.

1. Basics on the Brauer group and on the Chow group of zero-cycles

Grothendieck defined the Brauer group $\mathrm{Br}(X)$ of a scheme X as the second étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of X with values in the sheaf $\mathbb{G}_{m,X}$ on X. This is a contravariant functor with respect to arbitrary morphisms of schemes.

If $X = \mathrm{Spec}(K)$ is the spectrum of a field, then $\mathrm{Br}(X) = \mathrm{Br}(K)$, the more classical cohomological Brauer group $H^2(\mathrm{Gal}(K_s/K), K_s^*)$. Assume $2 \in K^*$. To the quaternion algebra (a, b) one associates a class $(a, b) \in \mathrm{Br}(K)[2]$. The quaternion algebra is isomorphic to a matrix algebra $M_2(K)$ if and only $(a, b) = 0 \in \mathrm{Br}(K)$, if and only if the diagonal quadratic form $< 1, -a, -b >$ has a nontrivial zero over K, if and only if the diagonal quadratic form $< 1, -a, -b, ab >$ has a nontrivial zero over K.

Proposition 1.1. *If X is an integral regular scheme and $K(X)$ is its field of rational functions, then the natural map $\mathrm{Br}(X) \to \mathrm{Br}(K(X))$ is injective.*

Date: June 16th, 2018.

Proposition 1.2. *For R a discrete valuation ring with perfect residue field κ and field of fractions K, there is a natural exact sequence*

$$0 \to \mathrm{Br}(R) \to \mathrm{Br}(K) \to H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to 0.$$

The map $\partial_R : \mathrm{Br}(K) \to H^1(\kappa, \mathbb{Q}/\mathbb{Z})$ is the residue map.

Let R be a discrete valuation ring with perfect residue field κ and field of fractions K. Suppose $2 \in R^*$. Given $a, b \in K^*$ we may consider the element $(a,b) \in \mathrm{Br}(K)[2]$ associated to the quaternion algebra (a,b). Let $v : K^* \to \mathbb{Z}$ be the valuation map. The quotient $a^{v(b)}/b^{v(a)} \in K^*$ belongs to R^*. Let $cl((a^{v(b)}/b^{v(a)})$ denote its class in κ^*/κ^{*2}. One shows :

$$\partial_R((a,b)) = (-1)^{v(a).v(b)} cl((a^{v(b)}/b^{v(a)})) \in \kappa^*/\kappa^{*2} = H^1(\kappa, \mathbb{Z}/2) \subset H^1(\kappa, \mathbb{Q}/\mathbb{Z}).$$

Proposition 1.3. *Let $R \subset S$ be a local inclusion of discrete valuation rings, inducing an inclusion of fields $K \subset L$ and an inclusion of residue fields $\kappa \subset \lambda$. Assume $char(\kappa) = 0$. Let e be the ramification index. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Br}(K) & \to & H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Br}(L) & \to & H^1(\lambda, \mathbb{Q}/\mathbb{Z}) \end{array}$$

where $H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to H^1(\lambda, \mathbb{Q}/\mathbb{Z})$ is $e.Res_{\kappa,\lambda}$.

Let K be a field and X an algebraic variety over K, i.e. a separated K-scheme of finite type. The group of zero-cycles $Z_0(X)$ is the free abelian group on closed points of X. Given any K-morphism $f : Y \to X$ of K-varieties, one defines $f_* : Z_0(Y) \to Z_0(X)$ as the map sending a closed point $M \in Y$ with image the closed point $N = f(M) \in X$ to the zero cycle $[K(M) : K(N)]N \in Z_0(X)$.

Given a normal, connected curve C over K, and a rational function $g \in K(C)^*$, on associates to it its divisor $div_C(g) \in Z_0(C)$. Given a morphism $f : C \to X$, one may then consider the zero-cycle $f_*(div_C(g)) \in Z_0(X)$.

One then defines the Chow group $CH_0(X)$ of zero-cycles on X as the quotient of $Z_0(X)$ by the subgroup spanned by all $f_*(div_C(g))$, for $f : C \to X$ a *proper* K-morphism from a normal, integral K-curve to X and $g \in K(C)^*$.

If $\phi : X \to Y$ is a proper morphism of K-varieties, there is an induced map $\phi_* : CH_0(X) \to CH_0(Y)$. In particular, if X/K is proper, the structural map $X \to \mathrm{Spec}(K)$ induces a degree map $CH_0(X) \to \mathbb{Z}$.

If $\phi : U \to X$ is an open embedding of K-varieties, the natural restriction map $Z_0(X) \to Z_0(U)$ which forgets closed points outside of U induces a map $CH_0(X) \to CH_0(U)$.

Let X be a K-variety. There is a natural bilinear pairing

$$Z_0(X) \times \mathrm{Br}(X) \to \mathrm{Br}(K)$$

which sends a pair (P,α) with P a closed point of X and $\alpha \in \mathrm{Br}(X)$ to $Cores_{K(P)/K}(\alpha(P))$. If X/K is proper, this pairing induces a bilinear pairing

$$CH_0(X) \times \mathrm{Br}(X) \to \mathrm{Br}(K).$$

See [1, Prop. 3.1].

This pairing satisfies an obvious functoriality property with respect to (proper) K-morphisms of proper K-varieties.

2. Quadric surfaces over a field

The following proposition is classical. See [21], [10], [4, Thm. 3.1].

Proposition 2.1. *Let K be a field, $char(K) \neq 2$, and let $X \subset \mathbb{P}^3_K$ be a smooth quadric surface. It is defined by a quadratic form q, which one may assume to be in diagonal form $q = < 1, -a, -b, abd >$, with $a, b, d \in K^*$. The class of d in K^*/K^{*2} is the discriminant, it does not depend on the choice of the quadratic form q defining the quadric X.*

The natural map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is surjective.

(a) If $d \notin K^{*2}$, the map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is an isomorphism.

(b) If $d \in K^{*2}$, the map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is surjective, and its kernel is of order at most 2, spanned by the class of the quaternion algebra (a, b), which is nontrivial if and only if $X(K) = \emptyset$.

3. A special quadric surface over $\mathbb{P}^2_{\mathbb{C}}$

Reference : [13], [17].

Let $F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$.

Let $X \subset \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ be the family of 2-dimensional quadrics over $\mathbb{P}^2_{\mathbb{C}}$ given by the bihomogeneous equation

$$yzU^2 + zxV^2 + xyW^2 + F(x, y, z)T^2 = 0.$$

This family is smooth over the open set of $\mathbb{P}^2_{\mathbb{C}}$ whose complement is the octic curve defined by the determinant equation

$$\Delta = x^2.y^2.z^2.F(x, y, z) = 0.$$

Note that this is the union of the smooth conic $F = 0$ and (twice) three tangents to this conic. The family is flat over $\mathbb{P}^2_{\mathbb{C}}$ (all fibres are quadrics). The total space is not smooth.

Part (a) of the following proposition is a result of Hassett, Pirutka, Tschinkel [13, Prop. 11].

Part (b) is a special case of the general statement [19, Prop. 7], the proof of which builds upon results of Pirutka ([17, Thm. 3.17], [19, Thm. 4]), for which material is offered in Appendix B below.

As we shall see, the proof given for (a) in [13, Prop. 11] is easily modified to simultaneously give a proof of (b).

The proposition suffices for the special case described in this note; it dispenses us with the recourse to Appendix B.

Proposition 3.1. *Let $\tilde{X}\to X$ be a projective desingularisation of X. Let $\alpha = (x/z, y/z) \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$.*

(a) The image β of α under the inverse map $p_2^* : \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))\to\mathrm{Br}(\mathbb{C}(X))$ is nonzero and lies in the subgroup $\mathrm{Br}(\tilde{X})$.

(b) For each codimension 1 subvariety Y of $\tilde{X}$ which does not lie over the generic point of $\mathbb{P}^2_{\mathbb{C}}$, the element $\beta \in \mathrm{Br}(\tilde{X})$ maps to $0 \in \mathrm{Br}(\mathbb{C}(Y))$.

Proof The equation is symmetrical in (x, y, z). The class $\alpha = (x/z, y/z)$ is given by (x, y) in the open set $z \neq 0$, by $(x/z, 1/z) = (x, z)$ in the open set $y \neq 0$ and by $(1/z, y/z) = (y, z)$ in the open set $x \neq 0$. In view of the symmetry between (x, y, z) in the equation, we may restrict attention to the open set $\mathbb{A}^2_{\mathbb{C}}$ of $\mathbb{P}^2_{\mathbb{C}}$ defined by $z \neq 0$. From now on we use affine coordinates (x, y). In affine coordinates, the quaternion algebra (x, y) has nontrivial residues along $x = 0$ and $y = 0$.

Let $K = \mathbb{C}(\mathbb{P}^2)$. Let X_η/K be the (smooth) generic quadric. The discriminant of the quadratic form $q =< y, x, xy, F(x, y, 1) >$ in K^* is not a square. Thus the map $\mathrm{Br}(K)\to\mathrm{Br}(X_\eta)$ is an isomorphism (see §2). Since the quaternion algebra $\alpha = (x, y) \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$ has some nontrivial residues, it is nonzero in $\mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$. Thus its image $\beta \in \mathrm{Br}(\mathbb{C}(X))$ is nonzero.

Let v be a discrete valuation of rank one on $L := K(X)$, let S be its valuation ring. Let κ_v denote the residue field. If $K \subset S$, then (x, y) is unramified. Suppose $S \cap K = R$ is a discrete valuation ring (of rank one). The image of the closed point of $\mathrm{Spec}(R)$ in $\mathbb{P}^2_{\mathbb{C}}$ is then either a point m of codimension 1 or a (complex) closed point m of $\mathbb{P}^2_{\mathbb{C}}$. By symmetry, for the argument we may assume that these points are in $\mathbb{A}^2_{\mathbb{C}}$.

Consider the first case. If the codimension 1 point m does not belong to $xy = 0$, then $\alpha = (x, y) \in \mathrm{Br}(K)$ is unramified at m on $\mathbb{A}^2_{\mathbb{C}}$ hence also in $\mathrm{Br}(L)$ at v. Moreover, the evaluation of β in $\mathrm{Br}(\kappa_v)$ is just the image under $\mathrm{Br}(\mathbb{C}(m))\to\mathrm{Br}(\kappa_v)$ of the image of α in $\mathrm{Br}(\mathbb{C}(m))$, hence vanishes since $\mathrm{Br}(\mathbb{C}(m)) = 0$ (Tsen).

Suppose m is a generic point of a component of $xy = 0$. By symmetry, it is enough to examine the affine case where the point m of codimension 1 is the generic point of $x = 0$. In the function field L, we have an identity

$$yU^2 + xV^2 + xyW^2 + F(x, y, 1) = 0$$

with $yU^2 + xV^2 \neq 0$. In the completion of K at the generic point of $x = 0$, $F(x, y, 1)$ is a square, because $F(x, y, 1)$ modulo x is equal to $(y - 1)^2$, a nonzero square. Thus in the completion L_v we have an equality (with some other elements $U, V, W \in L_v$).

$$yU^2 + xV^2 + xyW^2 + 1 = 0.$$

This gives $(x, y)_{L_v} = 0 \in \mathrm{Br}(L_v)$. Hence $(x, y)_L$ is unramified at v, thus belongs to $\mathrm{Br}(R)$ and has image 0 in $\mathrm{Br}(\kappa_v)$.

Suppose we are in the second case, i.e. m is a closed point of $\mathbb{A}^2_{\mathbb{C}}$. There is a local map $O_{\mathbb{A}^2_{\mathbb{C}},m}\to S$ which induces a map $\mathbb{C}\to\kappa_v$. If $x \neq 0$, then x

becomes a nonzero square in the residue field $\mathbb{C}$ hence in κ_v, and the residue of $(x,y)_L$ at v is trivial. The analogous argument holds if $y \neq 0$. It remains to discuss the case $x = y = 0$. We have $F(0,0,1) = 1 \in \mathbb{C}^*$. Thus $F(x,y,1)$ reduces to 1 in κ_v, hence is a square in the completion L_v. As above, in the completion L_v we have an equality

$$yU^2 + xV^2 + xyW^2 + 1 = 0,$$

which implies $(x,y)_{L_v} = 0 \in \mathrm{Br}(L_v)$. Hence $(x,y)_L$ is unramified at v, thus belongs to $\mathrm{Br}(S)$ and has image 0 in $\mathrm{Br}(\kappa_v)$. □

As in the reinterpretation [6] of the Artin–Mumford examples, the intuitive idea behind the above result is that the quadric bundle $X \to \mathbb{P}^2_{\mathbb{C}}$ is ramified along $x.y.z.F(x,y,z) = 0$ on $\mathbb{P}^2_{\mathbb{C}}$ and that the ramification of the symbol $(x/z, y/z)$ on $\mathbb{P}^2_{\mathbb{C}}$, which is "included" in the ramification of the quadric bundle $X \to \mathbb{P}^2_{\mathbb{C}}$ disappears over smooth projective models of X : ramification eats up ramification (Abhyankar's lemma). Here one also uses the fact that the smooth conic defined by $F(x,y,z) = 0$ is tangent to each of the lines $x = 0, y = 0, z = 0$, and does not vanish at any of the intersection of these three lines.

4. The specialisation argument

The following theorem is an improvement by S. Schreieder [18, Prop. 26] of the specialisation method, as initiated by C. Voisin [22], in the format later proposed by Colliot-Thélène and Pirutka [7]. The assumptions in [18, Prop. 26] are more general than the ones given here. The generic fibre need not be smooth and one only requires that $f^{-1}(U) \to U$ be universally CH_0-trivial. There is a more general version which involves higher unramified cohomology with torsion coefficients. The proof is identical to the one given here with the Brauer group.

Schreieder's proof is cast in the geometric language of the decomposition of the diagonal. I provide a more "field-theoretic" proof. It is well known that both points of view are equivalent [4, 7]. I add a further, hopefully simplifying, twist by using specialization of R-equivalence on rational points instead of Fulton's specialisation theorem for the Chow group.

Theorem 4.1. *Let R be a discrete valuation ring, K its field of fractions, κ its residue field. Assume κ is algebraically closed and char$(\kappa) = 0$. Let $\overline{K}$ be an algebraic closure of K. Let $\mathcal{X}/R$ be an integral projective scheme over R, with generic fibre $X = \mathcal{X}_K/K$ smooth, geometrically integral, and with special fibre Z/κ geometrically integral. Assume there exists a nonempty open set $U \subset Z$ and a projective, birational desingularisation $f : \tilde{Z} \to Z$ such that $V := f^{-1}(U) \to U$ is an isomorphism, and such that the complement $\tilde{Z} \setminus V$ is a union $\cup_i Y_i$ of smooth irreducible divisors of $\tilde{Z}$. Assume that the $\overline{K}$-variety $X_{\overline{K}}$ is stably rational. If an element $\alpha \in \mathrm{Br}(\tilde{Z})$ vanishes on each Y_i, then $\alpha = 0 \in \mathrm{Br}(\tilde{Z})$.*

Proof To prove the result, one may assume $R = \kappa[[t]]$ (completion of the original R) and $K = \kappa((t))$. Assume $X_{\overline{K}}$ is stably rational. Then there exists a finite extension $K_1 = \kappa((t^{1/d}))$ of K over which X_{K_1} is K_1-stably rational. We replace $\mathcal{X}/R$ by $\mathcal{X} \times_R \kappa[[t^{1/d}]]$. This does not change the special fibre.

Changing notation once more, we now have $\mathcal{X}/R$ an integral projective scheme whose generic fibre X/K is stably rational over K and whose special fibre Z/κ is just as in the theorem. Fix $m \in V(\kappa)$, mapping to $n \in U(\kappa)$.

Let $L = \kappa(Z)$. We have the commutative diagram of exact sequences

$$\begin{array}{ccccccc} \oplus_i CH_0(Y_{i,L}) & \to & CH_0(\tilde{Z}_L) & \to & CH_0(V_L) & \to & 0 \\ & & \downarrow & & \downarrow \simeq & & \\ & & CH_0(Z_L) & \to & CH_0(U_L) & \to & 0. \end{array}$$

where for each i, the closed embedding $\rho_i = Y_i \to \tilde{Z}$ induces

$$\rho_{i,*} : CH_0(Y_{i,L}) \to CH_0(\tilde{Z}_L),$$

the top exact sequence is the classical localisation sequence for the Chow group, the map $f_* : CH_0(\tilde{Z}_L) \to CH_0(Z_L)$ is induced by the proper map $f : \tilde{Z} \to Z$, the map $CH_0(V_L) \to CH_0(U_L)$ is the isomorphism induced by the isomorphism[1] $f : V \to U$, and the map $CH_0(Z_L) \to CH_0(U_L)$ is the obvious restriction map for the open set $U \subset Z$.

Let ξ be the generic point of $\tilde{Z}$ and η the generic point of Z.

Both η_L and n_L are smooth points of Y_L. There exists an extension $R \to S$ of complete dvr inducing $\kappa \to L$ on residue fields. Let F be the field of fractions of S. By Hensel's lemma, the points η_L and n_L lift to rational points of the generic fibre of $X \times_K F/F$ of $\mathcal{X}_S/S$. Since X/K is stably rational, all points of $X_F(F)$ are R-equivalent ([8, Prop. 10], [14, Cor. 6.6.6]).

It is a well known fact ([15, prop. 3.1], [14, Comments after Thm. 6.6.2]) that for a proper morphism $\mathcal{X}_S \to S$ over a discrete valuation ring S there is an induced map on R-equivalence classes $X(F)/\mathrm{R} \to Z(L)/\mathrm{R}$. This implies $\eta_L - n_L = 0 \in CH_0(Z_L)$. [2]

From the above diagram we conclude that

$$\xi_L = m_L + \sum_i \rho_{i*}(z_i) \in CH_0(\tilde{Z}_L)$$

with $z_i \in CH_0(Y_{i,L})$.

[1] Instead of assuming that $f^{-1}(U) \to U$ is an isomorphism, it would be enough, as in [18], to assume that this morphism is a universal CH_0-isomorphism.

[2] Alternatively, one could argue as follows. Since X is stably rational over K, over any field F containing K, the degree map $CH_0(X_F) \to \mathbb{Z}$ is an isomorphism (for a simple proof, see [7, Lemme 1.5]). One could then invoke Fulton's specialisation theorem for the Chow group of a proper scheme over a dvr [12, §2, Prop. 2.6], to get $\eta_L - n_L = 0 \in CH_0(Z_L)$. Fulton's specialisation theorem is a nontrivial theorem. The argument via R-equivalence (cf. [7, Remarque 1.19]) looks simpler.

For the proper variety $\tilde{Z}_L$, there is a natural bilinear pairing

$$CH_0(\tilde{Z}_L) \times \mathrm{Br}(\tilde{Z}) \to \mathrm{Br}(L).$$

For the smooth, proper, integral variety $\tilde{Z}$, on the generic point $\xi \in \tilde{Z}_L(L)$, this pairing induces the embedding $\mathrm{Br}(\tilde{Z}) \hookrightarrow \mathrm{Br}(\kappa(Z))$. Suppose $\alpha \in \mathrm{Br}(\tilde{Z})$ vanishes in each $\mathrm{Br}(Y_i)$ (which follows from the vanishing in $\mathrm{Br}(\kappa(Y_i))$ because Y_i is smooth). The evaluation of α on m_L is just the image of $\alpha(m) \in \mathrm{Br}(\kappa) = 0$. The above equality implies $\alpha(\xi) = 0 \in \mathrm{Br}(L)$, hence $\alpha = 0 \in \mathrm{Br}(\tilde{Z})$. □

5. Stable rationality is not constant in smooth projective families

We now complete the simplified proof of the theorem of Hassett, Pirutka and Tschinkel [13].

Theorem 5.1. *There exist a smooth projective family of complex 4-folds $f : X \to T$ parametrized by an open set T of the affine line $\mathbb{A}^1_{\mathbb{C}}$ and points $m, n \in T(\mathbb{C})$ such that the fibre X_n is rational and the fibre X_m is not stably rational.*

Proof One considers the universal family of quadric bundles over $\mathbb{P}^2_{\mathbb{C}}$ given in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ by a bihomogenous form of bidegree $(2, 2)$. This is given by a symmetric $(4, 4)$ square matrix with entries $a_{i,j}(x, y, z)$ homogeneous quadratic forms in three variables (x, y, z). If its determinant is nonzero, it is a homogeneous polynomial of degree 8.

We thus have a parameter space B given by a projective space of dimension 59 (the corresponding vector space being given by the coefficients of 10 quadratic forms in three variables). We have the map $X \to B$ whose fibres X_m are the various quadric bundles $X_m \to \mathbb{P}^2_{\mathbb{C}}$, for $X_m \subset \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ given by the vanishing of a nonzero complex bihomogeneous form of bidegree $(2, 2)$.

Using Bertini's theorem, one shows that there exists a nonempty open set $B_0 \subset B$ such that the fibres of $X \to B$ over points of $m \in B_0$ are flat quadric bundles $X_m \to \mathbb{P}^2_{\mathbb{C}}$ which are smooth as $\mathbb{C}$-varieties.

Using Bertini's theorem, one also shows that there exist points $m \in B_0$ with the property that the corresponding quadric bundle has $a_{1,1} = 0$, which implies that the fibration $X_m \to \mathbb{P}^2_{\mathbb{C}}$ has a rational section (given by the point $(1, 0, 0, 0)$), hence that the generic fibre of $X_m \to \mathbb{P}^2_{\mathbb{C}}$ is rational over $\mathbb{C}(\mathbb{P}^2)$, hence that the $\mathbb{C}$-variety X_m is rational over $\mathbb{C}$. [Warning : this Bertini argument uses the fact that we consider families of quadric surfaces over $\mathbb{P}^2_{\mathbb{C}}$. It does not work for families of conics over $\mathbb{P}^2_{\mathbb{C}}$.]

These Bertini arguments are briefly described in [18, Lemma 20 and Thm. 47] and are tacitly used in [19, Page 3].

By Proposition 3.1, the special example in §3 defines a point $P_0 \in B(\mathbb{C})$ whose fibre is $Z = X_{P_0}$ and which admits a projective birational desingularisation $f : \tilde{Z} \to Z$ satisfying :

(a) there exists a nonempty open set $U \subset Z$, such that the induced map $V := f^{-1}(U) \to U$ is an isomorphism;

(b) the complement $\tilde{Z} \setminus V$ is a union $\cup_i Y_i$ of smooth irreducible divisors of $\tilde{Z}$;

(b) there is a nontrivial element $\alpha \in \mathrm{Br}(\tilde{Z})$ which vanishes on each Y_i.

Theorem 4.1 then implies that the generic fibre of $X \to B$ is not geometrically stably rational. There are various ways to conclude from this that there are many points $m \in B_0(\mathbb{C})$ such that the fibre X_m is not stably rational.

Take one such point $m \in B_0(\mathbb{C})$ and a point $n \in B_0(\mathbb{C})$ such that X_n is rational. Over an open set of the line joining m and n we get a projective family of smooth varieties with one fibre rational and with one fibre not stably rational. □

The proof by Hassett, Pirutka and Tschinkel [13] uses an explicit desingularisation of the variety Z in §3, with a description of the exceptional divisors appearing in the process. Schreieder's improvement of the specialisation method enables one to bypass this explicit desingularisation.

Note that [13] and [19] contain many more results on families of quadrics surfaces over $\mathbb{P}^2$ than Theorem 5.1.

6. Appendix A. Conics over a discrete valuation ring

Let R be a dvr with residue field k of characteristic not 2. Let K be the fraction field. A smooth conic over K admits a regular model $\mathcal{X}$ given in $\mathbb{P}^2_R$ either by an equation

$$x^2 - ay^2 - bz^2 = 0$$

with $a, b \in R^*$ (case (I)) or a regular model $\mathcal{X}$ given by an equation

$$x^2 - ay^2 - \pi z^2 = 0$$

with $a \in R^*$ and π a uniformizing parameter (case (II)). Moreover, in the second case one may assume that a is not a square in the residue field κ.

Proposition 6.1. *Let R be a dvr with residue field k of characteristic not 2. Let K be the fraction field. Let $W \to \mathrm{Spec}(R)$ be a proper flat morphism with W regular and connected. Assume that the generic fibre is a smooth conic over K. Then :*

(a) The natural map $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is onto.

(b) For $Y \subset W$ an integral divisor contained in the special fibre of $W \to \mathrm{Spec}(R)$, and $\beta \in \mathrm{Br}(W)$, the image of β under restriction $\mathrm{Br}(W) \to \mathrm{Br}(Y)$ belongs to the image of $\mathrm{Br}(\kappa) \to \mathrm{Br}(Y)$.

Proof By purity for the Brauer group of a 2-dimensional regular scheme, to prove (a), one may assume that $W = \mathcal{X}$ as above. Let $X = \mathcal{X} \times_R K$. It is well known that the map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is onto, with kernel spanned by the quaternion symbol $(a, b)_K$ in case (I) and by $(a, \pi)_K$ in case (II).

Let $\beta \in \mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(X)$. Let $\alpha \in \mathrm{Br}(K)$ be some element with image β_K. We have the exact sequence

$$0 \to \mathrm{Br}(R)\{2\} \to \mathrm{Br}(K)\{2\} \to H^1(\kappa, \mathbb{Q}_2/\mathbb{Z}_2)$$

Comparison of residues on $\mathrm{Spec}(R)$ and on $\mathcal{X}$ shows that the residue $\delta_R(\alpha)$ is either 0 or is equal to the nontrivial class in $H^1(k(\sqrt{a})/k, \mathbb{Z}/2)$, and this last case may happen only in case (II). In the first case, we have $\alpha \in \mathrm{Br}(R)$, hence $\beta - \alpha_{\mathcal{X}} = 0$ in $\mathrm{Br}(X_K)$ hence also in $\mathrm{Br}(X)$ since X is regular. In the second case, we have

$$\delta_R(\alpha) = \delta_R((a, \pi))$$

hence $\alpha = (a, \pi) + \gamma$ with $\gamma \in \mathrm{Br}(R)$. We then get

$$\beta = (a, \pi)_{K(X)} + \gamma_{K(X)} \in \mathrm{Br}(K(X)).$$

But $(a, \pi)_{K(X)} = 0$. Thus $\beta - \gamma_{\mathcal{X}} \in \mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(K(X))$ vanishes, hence $\beta = \gamma_{\mathcal{X}} \in \mathrm{Br}(\mathcal{X})$. The map $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$ is thus surjective. This gives (a) for $\mathcal{X}$ hence for W, and (b) immediately follows. □

Exercise Artin-Mumford type examples are specific singular conic bundles X in the total space of a rank 3 projective bundle over $\mathbb{P}^2_{\mathbb{C}}$ whose unramified Brauer group is non trivial. Using Proposition 6.1 and Theorem 4.1, deform such examples into conic bundles of the same type with smooth ramification locus and whose total space is not stably rational. As in Section 3, there is no need to compute an explicit resolution of singularities of X.

7. Appendix B. Quadric surfaces over a discrete valuation ring

The following section was written up to give details on some tools and results used in [19, Thm. 4]. As demonstrated above, this section turns out not to be necessary to vindicate the HPT example. But it is useful for more general examples.

References : [21], [9, §3], [10, Thm. 2.3.1], [17, Thm. 3.17].

Let R be a discrete valuation ring, K its fraction field, π a uniformizer, $\kappa = R/(\pi)$ the residue field. Assume $char(\kappa) \neq 2$.

Let $X \subset \mathbb{P}^3_K$ be a smooth quadric, defined by a nondegenerate 4-dimensional quadratic form q. Up to scaling and changing of variables, there are four possibilities.

(I) $q = < 1, -a, -b, abd >$ with $a, b, d \in R^*$.

(II) $q = < 1, -a, -b, \pi >$ with $a, b \in R^*$ and π a uniformizing parameter of R.

(III) $q = < 1, -a, \pi, -\pi.b >$ with $a, b \in R^*$ and π a uniformizing parameter of R. The class of $a.b \in R^*$ represents the discriminant of the quadratic form. Its image $\overline{a}.\overline{b} \in \kappa^*$ is a square if and only if the discriminant of q is a square in the completion of K for the valuation defined by R.

Let $\mathcal{X} \subset \mathbb{P}^3_R$ be the subscheme cut out by q. Let Y/κ be the special fibre.

In case (I), $\mathcal{X}/R$ is smooth.

In case (II), X is regular, the special fibre Y is a cone over a smooth conic.

In case (III), the special fibre is given by the equation $x^2 - \overline{a}y^2 = 0$ in $\mathbb{P}^3_\kappa$. If $\overline{a}$ is a square, this is the union of two planes intersection along the line $x = y = 0$. If If $\overline{a}$ is not a square, this is an integral scheme which over $\kappa(\sqrt{a})$ breaks up as the union of two planes. In both cases, the scheme $\mathcal{X}$ is singular at the points $x = y = 0, z^2 - \overline{d}t^2 = 0$. See [21, §2].

Proposition 7.1. *Let us assume char$(\kappa) = 0$.*

In case (III), let $W \to \mathcal{X}$ be a projective, birational desingularisation of $\mathcal{X}$.

In case (I), the map $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$ is onto. If $d \in R$ is not a square, it is an isomorphism. If d is a square, the kernel is spanned by the class $(a, b) \in \mathrm{Br}(R)$.

In case (II), the map $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$ is an isomorphism.

In case (III), assume $\overline{a}.\overline{b}$ is not a square in κ. Then $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is onto.

In case (III), if either $\overline{a}$ or $\overline{b}$ is a square, or if $\overline{a}.\overline{b}$ is not a square, then $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is onto. An element of $\mathrm{Br}(K)$ whose image in $\mathrm{Br}(X)$ lies in $\mathrm{Br}(W)$ belongs to $\mathrm{Br}(R)$.

In case (III), assume $\overline{a}.\overline{b}$ is a square in κ. Then the image of $(a, \pi) \in \mathrm{Br}(K)$ in $\mathrm{Br}(X)$ belongs to $\mathrm{Br}(W)$. It spans the quotient of $\mathrm{Br}(W)$ by the image of $\mathrm{Br}(R)$. If moreover $\overline{a}$ is not a square in κ, then it does not belong to the image of $\mathrm{Br}(R)$.

Proof Let x be a codimension 1 regular point on $\mathcal{X}$ or on W, lying above the closed point of $\mathrm{Spec}(R)$. Let e_x denote its multiplicity in the fibre. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(K) & \to & H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Br}(X) & \to & H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \end{array}$$

The kernel of $Br(K) \to H^1(\kappa, \mathbb{Q}/\mathbb{Z})$ is $\mathrm{Br}(R)$.

In case (I) and (III), the special fibre Y is geometrically integral over κ, the multiplicity is 1, the map $H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$ is thus injective. This is enough to prove the claim.

Let us consider case (III). The map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is onto. Let $\alpha \in \mathrm{Br}(K)$. Let $\rho \in H^1(\kappa, \mathbb{Q}/\mathbb{Z})$ be its residue. On the (singular) normal model given by $q =< 1, -a, \pi, -\pi.b >$ over R, if $\overline{a} \in \kappa$ is a square, the fibre Y contains geometrically integral components of multiplicity 1 given by the components of $x^2 - \overline{a}y^2 = 0$. By the above diagram, $\rho = 0 \in H^1(\kappa, \mathbb{Q}/\mathbb{Z})$. We can also use the model given by $q =< 1, -b, \pi, -\pi.a >$. If $\overline{b} \in \kappa$ is a square, we conclude that $\rho = 0 \in H^1(\kappa, \mathbb{Q}/\mathbb{Z})$. Let us assume that $\rho \neq 0 \in H^1(\kappa, \mathbb{Q}/\mathbb{Z})$. Thus $\overline{a}$ and $\overline{b}$ are nonsquares. On the first model, the kernel of $H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to H^1(\kappa(Y), \mathbb{Q}/\mathbb{Z})$ coincides with the kernel of $H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to H^1(\kappa(\sqrt{\overline{a}}, \mathbb{Q}/\mathbb{Z})$, which is the $\mathbb{Z}/2$-module spanned by the class of $\overline{a}$ in $\kappa^*/\kappa^{*2} = H^1(\kappa, \mathbb{Z}/2)$. On the second model, the kernel of $H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to H^1(\kappa(Y), \mathbb{Q}/\mathbb{Z})$ is the $\mathbb{Z}/2$-module spanned by the class of $\overline{b}$ in $\kappa^*/\kappa^{*2} = H^1(\kappa, \mathbb{Z}/2)$. We thus conclude that $\overline{a}.\overline{b}$ is a square in κ, and that the residue of α coincides with $\overline{a}$, i.e. is equal to the residue of $(a, \pi) \in \mathrm{Br}(K)$ (or to the residue of (b, π)).

It remains to show that if $\overline{a}.\overline{b}$ is a square in κ, then (a, π) has trivial residues on W and more generally with respect to any rank one discrete valuation v on the function field $K(X)$ of X. One may restrict attention to those v which induce the R-valuation on K. Let $S \subset K(X)$ be the valuation ring of v and let λ be its residue field. There is an inclusion $\kappa \subset \lambda$. In $K(X)$ we have an equality

$$(x^2 - ay^2) = \pi.(z^2 - b),$$

where both sides are nonzero. Thus in $\mathrm{Br}(K(X))$, we have the equality

$$(a, \pi) = (a, x^2 - ay^2) + (a, z^2 - b) = (a, z^2 - b),$$

where the last equality comes from the classical $(a, x^2 - ay^2) = 0$. To compute residues, we may go over to completions. In the completion of R, $a.b$ is a square. It is thus a square in the completion of $K(X)$ at v. But then in this completion $(a, z^2 - b) = (b, z^2 - b) = 0$ Hence the residue of (a, π) at v is zero. □

Proposition 7.2. *Assume char$(\kappa) = 0$. Let $\mathcal{X} \subset \mathbb{P}^3_R$ be as above, and let $W \to \mathcal{X}$ be a proper birational map with W regular. Let $\beta \in \mathrm{Br}(W)$ and let $Y \subset W$ be an integral divisor contained in the special fibre of $W \to \mathrm{Spec}(R)$. Then the image of β in $\mathrm{Br}(\kappa(Y))$ belongs to the image of $\mathrm{Br}(\kappa) \to \mathrm{Br}(\kappa(Y))$.*

Proof In case (I) and (II), and in case (III) when $\overline{a}.\overline{b}$ is a square in κ, this is clear since then the map $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is onto.

Suppose we are in case (III). To prove the result, we may make a base change from R to its henselisation. Then ab is square in R. The group $\mathrm{Br}(W)$ is spanned by the image of $\mathrm{Br}(R)$ and the image of the class (a, π). The equation of the quadric may now be written

$$X^2 - aY^2 + \pi Z^2 - a\pi T^2 = 0.$$

This implies that $(a, -\pi)$ vanishes in the Brauer group of the function field $\kappa(W)$ of W. Since W is regular, the map $\mathrm{Br}(W) \to \mathrm{Br}(\kappa(W))$ is injective. Since $(a, -\pi)_{\kappa(W)}$ belongs to $\mathrm{Br}(W)$ and spans $\mathrm{Br}(W)$ modulo the image of $\mathrm{Br}(R)$, this completes the proof. □

One may rephrase the above results in a simpler fashion.

Proposition 7.3. *Assume char$(\kappa) = 0$. Let $\mathcal{X} \subset \mathbb{P}^3_R$ be as above, and let $W \to \mathcal{X}$ be a proper birational map with W regular.*

(i) If R is henselian, then the map $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is onto.

(ii) For any element $\beta \in \mathrm{Br}(W)$ and $Y \subset W$ an integral divisor contained in the special fibre of $W \to \mathrm{Spec}(R)$, the image of β under restriction $\mathrm{Br}(W) \to \mathrm{Br}(Y)$ belongs to the image of $\mathrm{Br}(\kappa) \to \mathrm{Br}(Y)$.

Upon use of Merkurjev's geometric lemmas [16, §1], and use of Tsen's theorem, one then gets [19, Prop. 7] of Schreieder.

8. Appendix C. A remark on the vanishing of unramified elements on components of the special fibre

The following proposition, found in June 2017, gives some partial explanation for the vanishing on components of the special fibre which occurs in [18, Prop. 6, Prop. 7] [19, Prop. 7] or in Proposition 3.1 above. Unfortunately the proof requires that the component be of multiplicity one in the fibre. Since this was written, in the case of quadric bundles, S. Schreieder [20, §9.2] has managed to use arguments as in [9, §3] to get information on what happens with the other components.

Proposition 8.1. *Let $A \hookrightarrow B$ be a local homomorphism of discrete valuation rings and let $K \subset L$ be the inclusion of their fraction fields. Let $\kappa \subset \lambda$ be the induced inclusion on their residue fields.*

Let ℓ be a prime invertible in A.

Let $i \geq 2$ be an integer and let $\alpha \in H^i(K, \mu_\ell^{\otimes i})$.

Assume:

(i) B is unramified over A.

(ii) The image of α in $H^i(L, \mu_\ell^{\otimes i})$ is unramified, and in particular is the image of a (well defined) element $\beta \in H^i(B, \mu_\ell^{\otimes i})$.

Then $\beta(\lambda) \in H^i(\lambda, \mu_\ell^{\otimes i})$ is in the image of $H^i(\kappa, \mu_\ell^{\otimes i}) \to H^i(\lambda, \mu_\ell^{\otimes i})$.

Proof We may assume that A and B are henselian. Then the residue map $\partial_A : H^i(K, \mu_\ell^{\otimes i}) \to H^{i-1}(\kappa, \mu_\ell^{\otimes(i-1)})$ is part of a split exact sequence ([5, Appendix B], [11, Cor. 6.8.8])

$$0 \to H^i(A, \mu_\ell^{\otimes i}) \to H^i(K, \mu_\ell^{\otimes i}) \to H^{i-1}(\kappa, \mu_\ell^{\otimes(i-1)}) \to 0,$$

and all reduction maps $H^j(A, \mu_\ell^{\otimes i}) \to H^j(\kappa, \mu_\ell^{\otimes i})$, denoted $\rho \mapsto \rho(\kappa)$, are isomorphisms. We have the analogous split exact sequence

$$0 \to H^i(B, \mu_\ell^{\otimes i}) \to H^i(L, \mu_\ell^{\otimes i}) \to H^{i-1}(\lambda, \mu_\ell^{\otimes(i-1)}) \to 0,$$

Let $\pi \in A$ be a uniformizer. Given $\alpha \in H^i(K, \mu_\ell^{\otimes i})$, the residue $\partial_A(\alpha) \in H^{i-1}(\kappa, \mu_\ell^{\otimes(i-1)})$ is the image of some unique $\gamma \in H^{i-1}(A, \mu_\ell^{\otimes(i-1)})$. Let us denote by (π) the class of π in $K^*/K^{*\ell} = H^1(K, \mu_\ell)$. Then $\alpha - (\pi) \cup \gamma \in H^i(K, \mu_\ell^{\otimes i})$ has trivial residue. Thus there exists $\zeta \in H^i(A, \mu_\ell^{\otimes i})$ such that

$$\alpha = \zeta + (\pi) \cup \gamma \in H^i(K, \mu_\ell^{\otimes i}).$$

By hypothesis, the restriction β of α to L is unramified. Thus $(\pi) \cup \gamma \in H^i(L, \mu_\ell^{\otimes i})$ is unramified. Since B is unramified over A, the uniformizer π is also a uniformizer of B. Thus

$$0 = \partial_B((\pi) \cup \gamma) = \gamma_\lambda \in H^{i-1}(\lambda, \mu_\ell^{\otimes(i-1)})$$

hence $\gamma = 0 \in H^{i-1}(B, \mu_\ell^{\otimes(i-1)})$, from which follows $\beta = \zeta \in H^i(B, \mu_\ell^{\otimes i})$ and $\beta(\lambda) \in H^i(\lambda, \mu_\ell^{\otimes i})$ is the image of $\zeta(\kappa) \in H^i(\kappa, \mu_\ell^{\otimes i})$. $\square$

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