A survey on unramified cohomology (with special attention to degree 3)

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To a smooth connected variety X over a field F one associates unramified cohomology groups

$$H^i_{nr}(X, \mathbb{Q}/\mathbb{Z}(i-1)) \subset H^i_{gal}(F(X), \mathbb{Q}/\mathbb{Z}(i-1)).$$

For $i = 2$, the group identifies with the Brauer group of X, which one finds in various contexts (rationality questions, Tate's conjecture on cycles of codimension 1, Brauer-Manin obstruction over global fields).

The group $i = 3$ comes up in various contexts

• Rationality questions for algebraic varieties (homogeneous spaces ; Fano hypersurfaces)

• Study of the image of cycle maps on cycles of codimension 2

• Arithmetic geometry over the function field $k(C)$ of a curve C over a global or a local field k

F a field
X/F variety , $n \in \mathbb{N}$ invertible in F, $i \in \mathbb{N}$, $j \in \mathbb{Z}$. Let $\mathcal{H}^i(\mu_n^{\otimes j})$ be the Zariski sheaf associated to the presheaf

$$U \mapsto H^i_{et}(U, \mu_n^{\otimes j}).$$

Definition of unramified cohomology of X

$$H^i_{nr}(X, \mu_n^{\otimes j}) := H^0_{Zar}(X, \mathcal{H}^i(\mu_n^{\otimes j}))$$

Let $\mathbb{Q}/\mathbb{Z}(j) = colim_n \mu_n^{\otimes j}$, where n is prime to the characteristic of the ground field.
Assume X/F smooth.
$H^1_{nr}(X, \mathbb{Q}/\mathbb{Z}) = H^1_{et}(X, \mathbb{Q}/\mathbb{Z})$
$H^2_{nr}(X, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(X)$
We shall be concerned with $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$.

Theorem (Bloch-Ogus, Gersten's conjecture for étale cohomogy). For X/F smooth connected, with field of functions $F(X)$, there is an exact sequence

$$0 \to H^i_{nr}(X, \mu_n^{\otimes j}) \to H^i(F(X), \mu_n^{\otimes j}) \to \oplus_{x \in X^{(1)}} H^{i-1}(F(x), \mu_n^{\otimes j-1}).$$

Here, for any point x of codimension 1 of X, the map

$$H^i(F(X), \mu_n^{\otimes j}) \to H^{i-1}(F(x), \mu_n^{\otimes j-1})$$

is the residue map associated to the DVR $O_{X,x}$.

Corollary : For X/F connected, smooth and proper over a field F, $H^i_{nr}(X, \mu_n^{\otimes j})$ is an F-birational invariant, denoted $H^i_{nr}(F(X)/F, \mu_n^{\otimes j})$. If X is stably F-rational, then $H^i(F, \mu_n^{\otimes j}) \stackrel{\simeq}{\to} H^i_{nr}(F(X)/F, \mu_n^{\otimes j})$.

For algebraically closed fields $k \subset K$, and X/k smooth and projective, on has the "rigidity property" : $H^i_{nr}(k(X)/k, \bullet) = H^i_{nr}(K(X)/K, \bullet)$ (CT, Jannsen ; Suslin's method)

For smooth compactifications of quotients G/H, where G is a connected linear algebraic group over a field F and H a closed subgroup, one would like to have "formulas" for the groups $H^i_{nr}(F(G/H)/F, \mathbb{Q}/\mathbb{Z}(i-1))$, which would in some cases yield the non-F-rationality of G/H.

Much work has been done on the case $i = 2$ (the Brauer group).

Over $F = \mathbb{C}$, for H connected, rationality of G/H is an open question.

In the talk, we ignore p-torsion questions in char. $p > 0$.

Galois descent on the Chow groups

A very long exact sequence

Let F be a field, $\mathrm{char.}(F) = 0$. Let $\overline{F}$ be an algebraic closure of F and $\mathfrak{g} = \mathrm{Gal}(\overline{F}/F)$.
Let X be a smooth, geometrically integral F-variety. We write $\overline{X} = X \times_F \overline{F}$.
Recall that Quillen has associated groups $K_i(A)$ ($i \geq 0$) to any commutative ring A. On a given scheme X, sheafification of the Zariski presheaf $U \mapsto K_i(\Gamma(U, \mathcal{O}_U))$ defines the Zariski sheaf $\mathcal{K}_i$ on X. For X smooth over a field, a famous formula, conjectured by Bloch and proved by Quillen, identifies $H^i_{Zar}(X, \mathcal{K}_i)$ with the Chow group $CH^i(X)$ of codimension i cycles modulo rational equivalence.

Theorem A (main theorem) *Assume* $H^0(\overline{X}, \mathcal{K}_2)$ *is uniquely divisible. There is an exact sequence*

$$0 \to H^1(X, \mathcal{K}_2) \to H^1(\overline{X}, \mathcal{K}_2)^{\mathfrak{g}} \to$$
$$\to \mathrm{Ker}[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X), \mathbb{Q}/\mathbb{Z}(2))] \to$$
$$\to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \to H^1(\mathfrak{g}, H^1(\overline{X}, \mathcal{K}_2)) \to N(X) \to$$
$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \to H^2(\mathfrak{g}, H^1(\overline{X}, \mathcal{K}_2)).$$

where $N(X)$ *sits in an exact sequence*

$$0 \to \mathrm{Ker}[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))]$$
$$\to N(X) \to \mathrm{Ker}[H^4(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^4(F(X), \mathbb{Q}/\mathbb{Z}(2))].$$

Parts of this sequence : Bloch, 1970's ; CT-Sansuc and CT-Raskind 1980's.
Whole sequence : **B. Kahn 1996** (via Lichtenbaum's $\mathbb{Z}(2)$'s)
CT-Kahn 2013 and CT 2013

The proof relies on the Merkurjev-Suslin theorem and on the Gersten conjecture (Quillen for K-theory, Bloch-Ogus for étale cohomology).
Most of the sequence may be established in a pedestrian manner, combining 1993 results of B. Kahn with the Galois cohomology of the complex

$$K_2(\overline{F}(X)) \to \oplus_{x \in \overline{X}^{(1)}} \overline{F}(x)^{\times} \to \oplus_{x \in \overline{X}^{(2)}} \mathbb{Z},$$

the homology groups of which are precisely the groups $H^i(\overline{X}, \mathcal{K}_2)$.

In each of the following cases, the assumption $H^0(\overline{X}, \mathcal{K}_2)$ uniquely divisible of Theorem A is fulfilled.

• X is smooth and projective over F and $Pic(\overline{X})$ has no torsion, in particular $H^1(X, \mathcal{O}_X) = 0$ (the proof uses Merkurjev-Suslin and Suslin).
If X/F is smooth and projective over F and is geometrically rationally connected, then $K_2F = H^0(X, \mathcal{K}_2)$.

• X is a principal homogeneous space of a simply connected semisimple algebraic group.

• X is a suitable "classifying space", denoted BG, of a semisimple F-group G.

• Combining the rigidity theorem and CT-Raskind results (1985) building on works of Bloch, Merkurjev, Suslin, from Theorem A one deduces :

Theorem B. *Let X/F be a smooth, projective, geometrically connected variety. Assume $X(F) \neq \emptyset$ and :*
(a) $\overline{X}$ rationally connected, hence $\mathrm{Pic}\,(\overline{X})$ is a lattice ;
(b) $\mathrm{Br}(\overline{X}) = 0$;
(c) $H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
Then

$$\mathrm{Pic}(\overline{X}) \otimes \overline{F}^{\times} \stackrel{\simeq}{\to} H^1(\overline{X}, \mathcal{K}_2)$$

and there is an exact sequence

$$0 \to \mathrm{Ker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \xrightarrow{\alpha} H^1(\mathfrak{g}, \mathrm{Pic}\,(\overline{X}) \otimes \overline{F}^{\times}) \to$$
$$\to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to$$
$$\to \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}] \xrightarrow{\beta} H^2(\mathfrak{g}, \mathrm{Pic}\,(\overline{X}) \otimes \overline{F}^{\times}).$$

One way to look at this theorem is as an ordered search to establish the nonrationality of X over F.
Indeed, if any of the hypotheses (a), (b) or (c) is not fulfilled, then already $\overline{X}$ is not rational over $\overline{F}$.
The first three groups in the sequence vanish if X is rational over F.

Linear algebraic groups : T, G, E and BG

Challenge : compute H^3_{nr}, for the sake of it, and also in the hope of detecting nonrationality.

• *X a smooth equivariant F-compactification of an F-torus T*
In this case, a sequence similar to the sequence of Theorem B features in a paper of Blinstein and Merkurjev (2013), but work remains to be done to identify the two sequences.
Let $1 \to R \to P \to T \to 1$ be a flasque resolution, for example with $\hat{R} = \mathrm{Pic}(\overline{X})$. Blinstein-Merkurjev's sequence is

$$0 \to CH^2(\text{``}BR''})_{tors} \to H^1(\mathfrak{g}, \mathrm{Pic}(\overline{X}) \otimes F^{\times}) \to$$
$$\to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to$$
$$\to (S^2(\hat{R}))^{\mathfrak{g}}/Dec) \to H^2(\mathfrak{g}, \mathrm{Pic}(\overline{X}) \otimes F^{\times})$$

There exists a surjective map $S^2(\hat{R}) \to CH^2(\overline{X})$.

• *$X = E$ principal homogeneous space of a semisimple simply connected algebraic group G/F*

Here $Pic(E) = 0$ and $K_2F = H^0(E, \mathcal{K}_2)$. The group $H^1(\overline{E}, K_2)$ is a lattice, it is a $\mathfrak{g}$-permutation module. Thus $H^1(\mathfrak{g}, H^1(\overline{E}, K_2)) = 0$ (Hilbert 90). Theorem A gives an exact sequence

$$0 \to H^1(E, \mathcal{K}_2) \to H^1(\overline{E}, \mathcal{K}_2)^{\mathfrak{g}} \to$$
$$\to Ker[H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(E), \mathbb{Q}/\mathbb{Z}(2))] \to CH^2(E) \to 0.$$

One knows (Panin, Podkopaev, Merkurjev) that $CH^2(E) = 0$. For G absolutely almost simple, $H^1(\overline{E}, K_2)) = \mathbb{Z}$ with trivial action of $\mathfrak{g}$. The image of $1 \in \mathbb{Z}$ in $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ is the celebrated *Rost invariant* of the principal homogeneous space E.

For G and E as above, Theorem A also gives an injection

$$H^3_{nr}(E, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow H^3_{nr}(\overline{E}, \mathbb{Q}/\mathbb{Z}(2)).$$

Going over to a smooth compactification of E, which is $\overline{F}$-rational, one gets the result, already of interest for $E = G$:

Theorem *For E principal homogeneous space of a semisimple simply connected algebraic group G the map $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(F(E)/F, \mathbb{Q}/\mathbb{Z}(2))$ is onto.*

In fact (Merkurjev 1999) one already has $H^3_{nr}(\overline{E}, \mathbb{Q}/\mathbb{Z}(2)) = 0$, hence $H^3_{nr}(E, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = 0$.

There are examples of simply connected groups G over a field F which are not F-rational (e.g. $SL(1, D)$ over special fields, Platonov, Merkurjev).
It is unclear whether they can be detected using

$$H^i_{nr}(E, \mathbb{Q}/\mathbb{Z}(j))/H^i(E, \mathbb{Q}/\mathbb{Z}(j))$$

for some field E/F and some i and j.

Question : For G semisimple but not simply connected, are there "formulas" for $H^3_{nr}(F(G)/F, \mathbb{Q}/\mathbb{Z}(2))$?

- *$X = BG$ with G/F semisimple*

There exists a finite dimensionsional vector space V with a linear action of G and an open set $U \subset V$ of codimension at least 3 stable under G such that $U \to U/G = X$ is a G-torsor. One sets $BG = X$.
We have $K_2 F = H^0(X, K_2)$.

Assume G/F simply connected.
The generic fibre $E/F(X)$ of $U \to X = BG$ is a $G_{F(X)}$-torsor.
Application of the above results shows that over any field F, the group $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ is finite.

For G simply connected, one has $H^1(K, \mathcal{K}_2) = 0$. Theorem A then gives two pieces of information on $X = BG$:

$$CH^2(X) \hookrightarrow CH^2(\overline{X})$$

and

$$Ker[H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))]$$
$$\stackrel{\simeq}{\to} \mathrm{Coker}[CH^2(X) \to CH^2(\overline{X})^{\mathfrak{g}}]$$

Theorem (Merkurjev 2002, Garibaldi)
Let G/F be a simply connected semisimple group and $X = BG$ as above.
(a) If G is split over F, then
$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$.

(b) If G/F is not split, one may have
$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \neq H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$.

Note : For F algebraically closed, G any connected linear algebraic group and $X = BG$, Bogomolov (1987) proved
$H^2_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(1)) = 0$.

More recently (since 2013), Merkurjev studied BG for G semisimple but not necessarily simply connected.
He computed $H^1(X, \mathcal{K}_2)$ for $X = BG$ and obtained a 5 terms exact sequence akin to that of Theorem A. The two sequences are not identical – Merkurjev's sequence gives information when the ground field is algebraically closed, whereas all terms in the above sequence then vanish.

Theorem (Merkurjev) *Let G be a semisimple group over an an algebraically closed field F of zero characteristic.*
Then $H^3_{nr}(F(BG)/F, \mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$ in each of the following cases :
(a) G simply connected or adjoint
(b) G is simple
(c) $p \neq 2$.

In these cases (except in the simply connected case), one should study whether $H^3_{nr}(L(BG)/L, \mathbb{Q}/\mathbb{Z}(2))/H^3(L, \mathbb{Q}/\mathbb{Z}(2)) = 0$ for *any* field L containing F.

By Theorem C (below) this would follow from Merkurjev's results if one had :
(*) There exists a smooth compactification X of a suitable BG such that the second Chow group of X (is finitely generated and) does not change under extension of algebraically closed ground fields.
[It is known that $CH^2(BG)$ is finitely generated.]

For G a connected subgroup of $GL_{N,\mathbb{C}}$, the rationality of GL_N/G remains a big open problem.

Complex algebraic geometry

Challenge : compute H^3_{nr}, for the sake of it, and also in the hope of detecting nonrationality for some interesting varieties such as Fano hypersurfaces.

Theorem B gives

Theorem C. *Let $X/\mathbb{C}$ be a connected smooth projective variety. Assume*
(a) X is rationally connected, hence $\mathrm{Pic}\,(X)$ is a lattice.
(b) $\mathrm{Br}(X) = 0$
(c) $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
Then for any field $F/\mathbb{C}$, with algebraic closure $\overline{F}$, letting $\mathfrak{g} = \mathrm{Gal}(\overline{F}/F)$, we have an exact sequence

$$0 \to H^3_{nr}(X_F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to$$
$$\to \mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}] \xrightarrow{\beta} H^2(\mathfrak{g}, \mathrm{Pic}\,(X) \otimes \overline{F}^{\times}).$$

One way to look at this theorem is as an ordered search to establish the nonrationality of X.
Indeed, if any of the hypotheses (a), (b) or (c) is not fulfilled, then X is not rational.
For any field $F/\mathbb{C}$, the group $\mathrm{Coker}[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}]$ is a birational invariant of $X/\mathbb{C}$.
The first two groups in the sequence vanish if X is rational.

Théorème D (C. Voisin 2014) . *For any smooth cubic hypersurface* $X \subset \mathbb{P}^5_{\mathbb{C}}$, *for any field* $F/\mathbb{C}$,
$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$.

If X contains a plane Π but is otherwise "very general" this was proved by Auel-CT-Parimala 2013. Here is a proof for any X containing a plane.
1) Using $\mathbb{P}^3$'s containing the plane Π one sees that X is birational to a fibration into 2-dimensional quadrics over $\mathbb{P}^2_{\mathbb{C}}$. Restriction to the generic fibre gives an embedding of
$H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(\mathbb{C}(X)/\mathbb{C}, \mathbb{Q}/\mathbb{Z}(2))$ into
$H^3_{nr}(\mathbb{C}(X)/\mathbb{C}(\mathbb{P}^2), \mathbb{Q}/\mathbb{Z}(2))$. K-theoretical results of Kahn, Rost et Sujatha (for quadrics over any field) show that this last group is in the image of $H^3(\mathbb{C}(P^2), \mathbb{Q}/\mathbb{Z}(2))$, and that group is zero.

2) Theorem C then gives an embedding

$$H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}]$$

For codimension 2 cycles over X as above over an algebraically closed field, rational equivalence, algebraic equivalence and homological equivalence coincide. Thus $CH^2(X) \stackrel{\simeq}{\to} CH^2(X_{\overline{F}})$. Hence $CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}$ is onto. Hence $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$.

By transcendental methods, C. Voisin showed that the integral Hodge conjecture holds for codimension 2 cycles on *any* smooth cubic hypersurface $X \subset \mathbb{P}^5_{\mathbb{C}}$. [She uses this in her proof of Theorem D.] Once this is known, one may use it to get $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$ (CT-Voisin, see below). One can then end the proof of Theorem D by the above method.

In higher dimension, similar arguments give :

Théorème E (2015). *Let* $X \subset \mathbb{P}^n_{\mathbb{C}}$ *be a smooth Fano hypersurface, i.e.of degree* $d \leq n$.
(a) For $n \geq 6$, *for any field* $F/\mathbb{C}$,
$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$.
(b) For $n = 5$,
$H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \oplus H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$.

What about *smooth cubic hypersurfaces* $X \subset \mathbb{P}^4_{\mathbb{C}}$?
None is rational (Clemens-Griffiths) but could some, all be stably rational ? Could none be ?

For any such X and any field $F/\mathbb{C}$, using Theorem C one may show :

$$H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))/H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \stackrel{\simeq}{\to} Coker[CH^2(X_F) \to CH^2(X_{\overline{F}})^{\mathfrak{g}}]$$

Could this group be non zero (hence X not stably rational) ?
The relevant field F to look at is the field of rational functions of the intermediate jacobian $J^3(X)$. This is an abelian variety which parametrizes cycles of codimension 2 on X which are homologically equivalent to zero. There is an obvious class in $CH^2(X_{\overline{F}})^{\mathfrak{g}}$ and one wonders whether it comes from $CH^2(X_F)$, thus defining on $X \times J^3(X)$ what C. Voisin calls a “universal codimension 2 cycle”.

C. Voisin (2014) gives further equivalent conditions for this property of X.
She shows that the existence of a universal codimension 2 cycle on X is actually equivalent to
$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_{nr}(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$ for any field $F/\mathbb{C}$.

She also shows that there exist cubic hypersurfaces $X \subset \mathbb{P}^4_{\mathbb{C}}$ for which this holds, and indeed for which the stronger property $CH_0(X_F) \simeq \mathbb{Z}$ holds for every $F/\mathbb{C}$.

H^3_{nr} **and the image of codimension 2 cycles in various cohomology groups : over $\mathbb{C}$ and over a finite field**

For X/F smooth, there are cycle maps
$CH^i(X) = \mathbb{H}^{2i}_{Zar}(X, \mathbb{Z}(i)) \to \mathbb{H}^{2i}_{et}(X, \mathbb{Z}(i))$
with values in the hypercohomology of the motivic complexes $\mathbb{Z}(i)$

Pour $i = 1$, isomorphism, the group is $Pic(X)$
Pour $i = 2$, fundamental exact sequence (Lichtenbaum, Kahn, uses Merkurjev-Suslin)

$$0 \to CH^2(X) \to \mathbb{H}^4_{et}(X, \mathbb{Z}(2)) \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \to 0.$$

One may use this sequence to prove the two results below on the quotient of $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$ modulo its maximal divisible subgroup the first one over $\mathbb{C}$, the second one over a finite field $\mathbb{F}$, by a unified method (Kahn, CT-Kahn).

$F = \mathbb{C}$

$X/\mathbb{C}$ projective and smooth.
$H^{2i}_{Hodge}(X(\mathbb{C}), \mathbb{Z}(i)) \subset H^{2i}_{Betti}(X(\mathbb{C}), \mathbb{Z}(i))$
$cl_i : CH^i(X) \to H^{2i}_{Hodge}(X(\mathbb{C}), \mathbb{Z}(i))$
If cl_i is onto, one says that the integral Hodge conjecture holds.
This is so for $i = 1$ (Lefschetz).
Conjecturally, the cokernel is finite : this is the standard Hodge conjecture.

Theorem (CT-Voisin 2012) *The quotient of* $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))$ *by its maximal divisible subgroup is finite and is equal to the torsion subgroup of* $H^4_{Betti}(X(\mathbb{C}), \mathbb{Z}(2))/Im[CH^2(X)]$.

CT-Voisin use the Bloch-Kato conjecture in degree 3. Bruno Kahn gives a proof which uses "only" the degree 2 conjecture (Merkurjev-Suslin).

A starting point for this result was the birational invariance of both groups (first one via Gersten conjecture ; second one via resolution of singularities and analysis of blow-up along a smooth subvariety).

(CT-Ojanguren) There exists a smooth, projective, unirational variety $X/\mathbb{C}$ of dimension 6 such that $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \neq 0$, hence (by CT-Voisin) the integral Hodge conjecture does not hold for codimension 2 cycles on rationally connected X of dimension at least 6.

(Kollár) There exist smooth hypersurfaces $X \subset \mathbb{P}^4_{\mathbb{C}}$ (of high degree) for which $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) \neq 0$ and the integral Hodge conjecture fails for codimension 2 cycles.

Theorem (Voisin, via Hodge theory). *If $X/\mathbb{C}$ is of dimension 3 and uniruled, the integral Hodge conjecture holds for codimension 2 cycles, hence $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.*

Conjecture (Voisin). On (smooth, projective) rationally connected of arbitrary dimension, the integral Hodge conjecture holds for dimension 1 cycles.

Conditional proof by Voisin by reduction to a result of C. Schoen which depends on the Tate conjecture on divisors on surfaces over a finite field.

$F = \mathbb{F}$ finite, l prime, $l \neq \mathrm{char}(\mathbb{F})$

$CH^i(X) \otimes \mathbb{Z}_l \to H^{2i}_{et}(X, \mathbb{Z}_l(i))$

Theorem (Kahn, CT-Kahn). *The quotient of $H^3_{nr}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$ by its maximal divisible subgroup is isomorphic to the finite, torsion subgroup of $H^4_{et}(X, \mathbb{Z}_l(2))/Im(CH^2(X) \otimes \mathbb{Z}_l)$.*

Note : For $X/\mathbb{F}$ projective, $H^4_{et}(X, \mathbb{Z}_l(2))/Im(CH^2(X) \otimes \mathbb{Z}_l)$ is conjecturally finite (standard Tate conjecture).
Question : is $H^3_{nr}(X, \mathbb{Q}_l/\mathbb{Z}_l(2))$ finite ?

Results and questions from CT-Kahn 2012

Theorem (uses Schoen). If Tate's conjecture holds for divisors on surfaces over a finite field, then for $X/\overline{\mathbb{F}}$ uniruled of dimension 3, $H^3_{nr}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) = 0$.

Basic question : If $X/\mathbb{F}$ smooth and projective of dimension 3, is $H^3_{nr}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) = 0$?
[It is known that $H^4_{nr}(X, \mathbb{Q}_l/\mathbb{Z}_l(3)) = 0$.]
Is this at least true if X is geometrically uniruled ?

In any dimension – but the critical case is for $dim(X) = 3$ – does the integral Tate conjecture hold for cycles of dimension 1 ? [A conditional, weak version is proved by Schoen.]

One nontrivial case is known.

Theorem (Parimala-Suresh, 2013). *For X a threefold which admits a conic bundle structure over a surface over a finite field $\mathbb{F}$, $l \neq \mathrm{char}(\mathbb{F}) \neq 2$, $H^3_{nr}(X, \mathbb{Q}_l/\mathbb{Z}_l(2)) = 0$.*

This result is used in CT-Kahn to prove a by now "standard" local-global conjecture for zero-cycles on varieties over a global field k (CT, Sansuc, Kato, Saito) in the special case of a surface with a conic bundle structure over $\mathbb{P}^1_k$ and $k = \mathbb{F}(C)$ is the function field of a curve.
The analogous result over k a number field is a theorem of Salberger (1988) proved by a quite different method.

Other occurences of the groups $H^i_{nr}(\bullet, \mathbb{Q}/\mathbb{Z}(2))$

For a smooth projective variety of dimension d over the reals, let s denote the number of connected components of $X(\mathbb{R})$. Then

$$H^n_{nr}(X, \mathbb{Z}/2) = (\mathbb{Z}/2)^s$$

for any $n > d$. (CT-Parimala 1990)

For a smooth projective curve C over k a p-adic field, the group $H^3_{nr}(C, \mathbb{Q}/\mathbb{Z}(2))$ can be computed in terms of the components of the special fibre of a regular model of C over the ring of integers (Kato 1986 ; Ducros 2002, 2008 via Berkovich spaces).

Let $K = k(C)$ as above, for each closed point P of the curve C, let K_P denote the completion of K at P, equipped with the topology defined by the valuation associated to P.
For a smooth variety X over K, one may consider the diagonal embedding

$$X(K) \to \prod_{P \in C^{(1)}} X(K_P).$$

Putting on each $X(K_P)$ the topology inherited by that of K_P and on the product the product topology, one may wonder whether the LHS is dense in the RHS. This combines a question on a local-global principle for rational points and a question on weak approximation.

For projective homogeneous spaces of connected linear algebraic groups, work has been done on this problem by Harbater-Hartmann-Krashen and CT-Parimala-Suresh.

Harari, Scheiderer and Szamuely study the case of principal homogeneous spaces of tori over K. In their work, the group $H^3_{nr}(K(X)/K, \mathbb{Q}/\mathbb{Z}(2))$ plays a crucial rôle.

Roughly speaking, over $K = k(C)$ which is of cohomological dimension 3, $H^3_{nr}(\bullet, \mathbb{Q}/\mathbb{Z}(2))$ plays the rôle which the Brauer group $H^2_{nr}(\bullet, \mathbb{Q}/\mathbb{Z}(1))$ plays for similar problems over global fields, which are essentially of cohomological dimension 2.