Zero-cycles and rational points on some surfaces over a global function field

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Abstract

Let $F$ be a finite field of characteristic $p$. We consider smooth surfaces over $F(t)$ defined by an equation $f + tg = 0$, where $f$ and $g$ are forms of degree $d$ in 4 variables with coefficients in $F$, with $d$ prime to $p$. We prove: For such surfaces over $F(t)$, the Brauer-Manin obstruction to the existence of a zero-cycle of degree one is the only obstruction. For $d = 3$ (cubic surfaces), this leads to the same result for rational points.

Soit $F$ un corps fini de caractéristique $p$. Pour une surface lisse sur $F(t)$ définie par une équation $f + tg = 0$, où $f$ et $g$ sont deux formes de degré $d$ sur $F$ en 4 variables, avec $d$ premier à $p$, nous montrons que l’obstruction de Brauer-Manin au principe de Hasse pour les zéro-cycles de degré 1 est la seule obstruction. Pour $d = 3$ (surfaces cubiques), on en déduit le même énoncé pour les points rationnels.


Key words and phrases: local-global principle, Brauer–Manin obstruction, zero-cycles, rational points, global function field, cubic surface.
1 Introduction

Study of the case of curves (Cassels, Tate) and of the case of rational surfaces (Colliot-Thélène et Sansuc [CT/S81]), where a more precise conjecture is made for rational surfaces) has led to the following conjecture for zero-cycles on arbitrary varieties over global fields (Kato and Saito [K/S86], Saito [S89], Colliot-Thélène [CT93], [CT99]).

Conjecture 1.1. Let $X$ be a smooth, projective, geometrically integral variety over a global field $k$. If there exists a family $\{z_v\}_{v \in \Omega}$ of local zero-cycles of degree 1 (here $v$ runs through the set $\Omega$ of places of $k$) such that for all $A \in Br(X)$,

$$\sum_{v \in \Omega} \text{inv}_v(A(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z}$$

holds, then there exists a zero-cycle of degree 1 on $X$. In other words, the Brauer–Manin obstruction to the existence of a zero-cycle of degree 1 on $X$ is the only obstruction.

Over number fields, this conjecture has been established in special cases in work of (alphabetical order, and various combinations) Colliot-Thélène, Frossard, Salberger, Sansuc, Skorobogatov, Swinnerton-Dyer, Wittenberg (see the introduction of [W10]). None of these results applies to smooth surfaces of degree $d$ at least 3 in 3-dimensional projective space – for $d \geq 5$ these surfaces are of general type. In section 2, we establish the conjecture in the special case of a global field $k = \mathbb{F}(t)$ purely transcendental over a finite field $\mathbb{F}$ and of smooth surfaces $X \subset \mathbb{P}^3_k$ defined by an equation $f + tg = 0$, where $f$ and $g$ are two forms of arbitrary degree $d$ over the field $\mathbb{F}$.

According to a conjecture of Colliot-Thélène and Sansuc ([CT/S80]), the Brauer–Manin obstruction to the existence of a rational point on a smooth, geometrically rational surface defined over a global field should be the only obstruction. Such should in particular be the case for smooth cubic surfaces in 3-dimensional projective space $\mathbb{P}^3_k$. In section 3, we establish the conjecture in the special case of a global field $k = \mathbb{F}(t)$ purely transcendental over a finite field $\mathbb{F}$ and of smooth cubic surfaces $X \subset \mathbb{P}^3_k$ defined by an equation $f + tg = 0$, where $f$ and $g$ are two cubic forms over the field $\mathbb{F}$. Simple though they be, such surfaces may fail to obey the Hasse principle.
2 Zero-cycles of degree 1 on surfaces of arbitrary degree

The following theorem is due to S. Saito [S89]. It says that if a strong integral form of the Tate conjecture on 1-dimensional cycles is true, then the above conjecture holds, at least if we stay away from the characteristic of the field. For an alternate proof of Theorem 2.1, see [CT99, Prop. 3.2].

**Theorem 2.1.** (Saito) Let $F$ be a finite field and $C/F$ a smooth, projective, geometrically integral curve over $F$. Let $k = F(C)$ be its function field. Let $\mathcal{X}$ be a smooth, projective, geometrically integral $F$-variety of dimension $n$ and $f : \mathcal{X} \to C$ a faithfully flat map whose generic fibre $X/k$ is smooth and geometrically integral.

Assume:

1. For each prime $l \neq \text{char}(F)$, the cycle map
   
   $T_X : \text{CH}^{n-1}(\mathcal{X}) \otimes \mathbb{Z}_l \to H_{\text{et}}^{2n-2}(\mathcal{X}, \mathbb{Z}_l(n-1))$

   from the Chow group of dimension 1 cycles on $\mathcal{X}$ to étale cohomology is onto.

2. There exists a family $\{z_v\}_{v \in \Omega}$ of local zero-cycles of degree 1 (here $v$ runs through the set $\Omega$ of places of $k$) such that for all $A \in \text{Br}(X)$,
   
   $\sum_{v \in \Omega} \text{inv}_v(A(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z}$.

Then there exists a zero-cycle on $X$ of degree a power of $\text{char}(F)$.

In this statement, $A(z_v)$ is the element of the Brauer group of the local field $k_v$ obtained by evaluation of $A$ on the zero-cycle $z_v$. The map $\text{inv}_v : \text{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ is the local invariant of class field theory.

Here is one case where assumption (1) in the previous theorem is fulfilled.

**Theorem 2.2.** Let $F$ be a finite field and $l$ a prime, $l \neq \text{char}(F)$. For a smooth, projective, geometrically integral threefold $\mathcal{X}$ over $F$ which is birational to $\mathbb{P}^3_F$, the cycle map $T_X : \text{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_l \to H_{\text{et}}^4(\mathcal{X}, \mathbb{Z}_l(2))$ is onto.

**Proof.** If $\mathcal{X} = \mathbb{P}^3_F$, then $\text{CH}^2(\mathcal{X}) = \mathbb{Z}$ and one easily checks that the cycle map

$T_X : \text{CH}^2(\mathcal{X}) \otimes \mathbb{Z}_l \to H_{\text{et}}^4(\mathcal{X}, \mathbb{Z}_l(2))$

is simply the identity map $\mathbb{Z}_l = \mathbb{Z}_l$. Using the standard formulas for the computation of Chow groups and of cohomology for a blow-up along a
smooth projective subvariety, as well as the vanishing of Brauer groups of
smooth projective curves over a finite field, one shows: For $X$ a smooth
projective threefold, the cokernel of the above cycle map $T_X$ is invariant
under blow-up of smooth projective subvarieties on $X$.

By a result of Abhyankar ([Abh66, Thm. 9.1.6]), there exists a smooth
projective variety $X'$ which is obtained from $\mathbb{P}^3_F$ by a sequence of blow-
ups along smooth projective $F$-subvarieties, and which is equipped with a
birational $F$-morphism $p : X' \to X$.

There are push-forward maps $\pi_*$ and pull-back maps $\pi^*$ both for Chow
groups and for étale cohomology, and for the birational map $\pi$ we have
$\pi_* \circ \pi^* = \text{id}$. Moreover these maps are compatible with the cycle class
map. Thus the cokernel of $T_X$ is a subgroup of the cokernel of $T_{X'}$, hence is
zero. \hfill \square

Combining Theorems 2.1 and 2.2, we get:

**Theorem 2.3.** Let $F$ be a finite field and $C/F$ a smooth, projective, ge-
ometrically integral curve over $F$. Let $k = F(C)$ be its function field. Let
$X$ be a smooth, projective, geometrically integral $F$-variety of dimension $n$
and $f : X \to C$ a faithfully flat map whose generic fibre $X/k$ is smooth and
geometrically integral.

Assume:

(1) $\dim X = 3$ and $X$ is $F$-rational;
(2) there exists a family $\{z_v\}_{v \in \Omega}$ of local zero-cycles of degree 1 (here $v$
runs through the set $\Omega$ of places of $k$) such that for all $A \in \text{Br}(X),$

$$\sum_{v \in \Omega} \text{inv}_v(A(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z}.$$ 

Then there exists a zero-cycle on $X$ of degree a power of $\text{char}(F)$.

We may now prove the main result of this section.

**Theorem 2.4.** Let $F$ be a finite field, let $f, g$ be two nonproportional ho-
mogeneous forms in 4 variables, of degree $d$ prime to the characteristic of
$F$. Let $k = F(t)$. Suppose the $k$-surface $X \subset \mathbb{P}^3_k$ defined by $f + tg = 0$
is smooth. If there is no Brauer–Manin obstruction to the Hasse principle for
zero-cycles of degree 1 on $X$, then

(i) there exists a zero-cycle of degree 1 on the $k$-surface $X$;
(ii) there exists a zero-cycle of degree 1 on the $F$-curve $\Gamma$ defined by
$f = g = 0$ in $\mathbb{P}^3_F$. 

Proof. Let $X_1 \subset \mathbb{P}^3_F \times_F \mathbb{P}^1_F$ be the schematic closure of $X \subset \mathbb{P}^3_{F(t)}$. The $F$-variety $X_1$ has an affine birational model with equation

$$\phi(x, y, z) + t\psi(x, y, z) = 0,$$

hence $t$ is determined by $x, y, z$, thus $X$ is $F$-birational to $\mathbb{P}^3_F$. Since $X_1$ admits a smooth projective model over $F$, a result of Cossart ([Co92, Théorème, p. 115]) shows that there exists a smooth projective threefold $\mathcal{X}/F$ and an $F$-birational morphism $\mathcal{X} \to X_1$ which is an isomorphism over the smooth locus of $X_1$, hence in particular which induces an isomorphism over $\text{Spec } F(t) \subset \mathbb{P}^1_F$. That is, the generic fibre of $\mathcal{X} \to \mathbb{P}^1_F$ is $k$-isomorphic to $X/k$.

Statement (i) then follows from Thm. 2.3. Statement (ii) follows from (i) as a special application of a result of Colliot-Thélène and Levine ([CT/L09, Théorème 1, p. 217]).

Remark 2.5. Theorem 2.4 is of interest only in the case where the $F$-curve $\Gamma$ does not contain a geometrically integral component. Otherwise the two statements immediately follow from the Weil estimates for the number of points on geometrically integral curves. These estimates actually provide more: they show that if there exists such a component, then on any field extension $F'$ of $F$ of high enough degree, there exists an $F'$-point on $\Gamma$, hence for any such field there exists an $F'(t)$-point on the $F(t)$-surface $X$.

Remark 2.6. One could try to circumvent the cohomological machinery, i.e. Theorems 2.1 and 2.2. For this, in each of the special cases where there are zero-cycles of degree 1 everywhere locally on $X$ but there is no zero-cycle of degree one on the curve $\Gamma$, one should:

(i) Check that the Brauer group is not trivial, find generators.

(ii) Check that there is a Brauer–Manin obstruction.

Already when the common degree of $f$ and $g$ is 3, which we shall now more particularly examine, this seems no easy enterprise.

3 Rational points on cubic surfaces

The proof of the following theorem is independent of the previous results.

Theorem 3.1. Let $F$ be a finite field, let $f, g$ be two nonproportional cubic forms over $F$ in 4 variables. Assume the characteristic of $F$ is not 3. Let $k = F(t)$. Suppose the $k$-surface $X \subset \mathbb{P}^3_k$ defined by $f + tg = 0$ is smooth.
Let $\Gamma \subset \mathbf{P}_F^3$ be the complete intersection curve defined by $f = g = 0$. The following conditions are equivalent:

(i) There exists a $k$-rational point on the $k$-variety $X$.

(ii) There exists a zero-cycle of degree 1 on the $k$-variety $X$.

(iii) There exists a zero-cycle of degree 1 on the $\mathbf{F}$-curve $\Gamma$.

(iv) There exists a closed point of degree prime to 3 on the $\mathbf{F}$-curve $\Gamma$.

(v) There exists a closed point of degree a power of 2 on the $\mathbf{F}$-curve $\Gamma$.

Proof. That (i) implies (ii) is trivial. That (ii) implies (iii) is a special case of [CT/L09]. Statements (iii) and (iv) are equivalent, since $\Gamma$ is a curve of degree 9. If (v) holds, then $\Gamma$ has a point in a tower of quadratic extensions of $\mathbf{F}$, hence the cubic surface $X$ has a point in a tower of quadratic extensions of $k$. An extremely well known argument shows that if a cubic surface over a field has a point in a separable quadratic extension of that field, then it has a rational point: the line joining two conjugate points is defined over the ground field, either it is entirely contained in the cubic surface or it meets it in a third, rational point. Iterating this remark, we see that $X$ has a rational point, i.e. (i) holds.

Let us prove that (iii) implies (v). To prove this, one may replace $\mathbf{F}$ by its maximal pro-2-extension extension $F$, which we now do. For an odd integer $n$, we let $F_n/F$ be the unique, cyclic, field extension of $F$ of degree $n$.

For $Z/L$ a variety over a field $L$, the index $\text{ind}(Z) = \text{ind}(Z/L)$ is the gcd of the $L$-degrees of closed points on $Z$. The index of an $L$-variety is equal to the index of its reduced $L$-subvariety. The index of an $L$-variety which is a finite union of $L$-varieties is the gcd of the indices of each of them. The assumption made in (iii) is precisely that the index of the curve $\Gamma$ is 1.

Since $F$ has no quadratic or quartic extension, an effective zero-cycle of degree 1, 2, 4 contains an $F$-rational point, and an effective zero-cycle of degree 3, 6 or 9 either contains an $F$-point or has index a multiple of 3.

If $\Gamma$ contains a geometrically integral component, then $\Gamma(F) \neq \emptyset$ (Weil estimates, see the remark after Theorem 2.4).

Suppose $\Gamma$ does not contain a geometrically integral component. One then easily checks that the degree 9 curve $\Gamma$ can break up only in one of the following ways:

\[ 9 = 3(1 + 1 + 1) \]
\[ 9 = 2(1 + 1 + 1) + (1 + 1 + 1) \]
\[ 9 = (2 + 2 + 2) + (1 + 1 + 1) \]
\[ 9 = (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1) \]
\[ 9 = (1 + \cdots + 1) \text{ (9 times)} \]
9 = (3 + 3 + 3)

Here \(m(a + a + a)\) means the sum of three conjugate integral curves of degree \(a\) over \(\overline{F}\) with multiplicity \(m\).

An integral curve of degree 2 over \(\overline{F}\) is a smooth plane conic, contained in a well-defined plane. An integral curve of degree 3 over \(\overline{F}\) is either a plane cubic or a smooth twisted cubic.

Let the integral curve \(C \subset \mathbb{P}_F^3\) break up as \((1+1+1)\). The singular set consists of at most 3 points. Then either \(C(F) \neq \emptyset\) or 3 divides \(\text{ind}(C)\).

Let the integral curve \(C \subset \mathbb{P}_F^3\) break up as \((2+2+2)\). Each conic is defined over \(F_3\). Two distinct smooth conics on \(f = 0\) define two distinct planes, hence they intersect in at most 2 geometric points. Such points must already be in \(F_3\). Thus any closed point in the singular locus of \(C\) has degree 1 or 3. One concludes that either \(C(F) \neq \emptyset\) or 3 divides \(\text{ind}(C)\).

Let the integral curve \(\Gamma \subset \mathbb{P}_F^3\) break up as \((3 + 3 + 3)\), and assume that this corresponds to a decomposition as three conjugate plane cubics. Each of these is defined over \(F_3\). The intersection number of two of these cubics is 3. The points of intersection of two such curves are thus defined over \(F_3\). We conclude that the singular locus of \(\Gamma\) splits over \(F_3\). This implies that any singular closed point on \(\Gamma\) has degree a power of 3. Thus \(\Gamma(F) \neq \emptyset\) or 3 divides \(\text{ind}(\Gamma)\).

Let the curve \(\Gamma \subset \mathbb{P}_F^3\) break up as \((3 + 3 + 3)\), and assume that \(\Gamma\) breaks up as the sum of three conjugate twisted cubics. The curve \(\Gamma\) lies on the smooth cubic surface \(X \cap F(t)\) defined by \(f + tg = 0\). Each twisted curve is defined over \(F_3\). Let \(\sigma\) be a generator of \(\text{Gal}(F_3(t)/F(t))\). Write \(\Gamma = C + \sigma(C) + \sigma^2C\) on \(X_{F_3(t)}\). Using intersection theory on the smooth surface \(X_{F_3(t)}\), which is invariant under the action of \(\text{Gal}(F_3(t)/F(t))\), and letting \(H\) be the class of a plane section, we find \(2\sigma = (3H.3H) = (\Gamma.\Gamma) = 3(C.C) + 6(\sigma(C).C)\). The curve \(C\) is a twisted cubic, hence a smooth curve of genus zero on the smooth cubic surface \(X\), whose canonical bundle \(K\) is given by \(-H\). The formula for the arithmetic genus of a curve on a surface, namely \(2(p_a(C) - 1) = (C.C) + (C.K)\) gives \((C.C) = 1\). This implies \((C.\sigma(C)) = 4\), hence \((\sigma(C).\sigma^2(C)) = 4\) and \((\sigma^2(C).C) = 4\). Since each of these twisted cubics is defined over \(F_3\) and since \(F_3\) has no field extension of degree 2 or 4, this implies that the points of intersection of any two of these
twisted cubics are defined over $F_9$. We conclude that the singular locus of $\Gamma$ splits over $F_9$. This implies that the degree of any closed point in that locus is a power of 3. Thus either $\Gamma(F) \neq \emptyset$ or 3 divides $\text{ind}(\Gamma)$.

In all cases we have proved: Either $\Gamma(F) \neq \emptyset$ or 3 divides $\text{ind}(\Gamma)$. The assumption $\text{ind}(\Gamma) = 1$ now implies $\Gamma(F) \neq \emptyset$.

Remark 3.2. If the order $q$ of the finite field $F$ is large enough and $f + tg = 0$ is soluble in $F(t)$, a variant of the proof for the equivalence of (iv) and (v) shows that $f + tg = 0$ has a solution in polynomials of degree at most 5. This raises the interesting general question whether there are integers $N(d)$ with the following property: Suppose that $G(X_0, \ldots, X_4, t)$ is a polynomial defined over $F$, homogeneous of degree 3 in the $X_i$ and of degree $d$ in $t$; if $G = 0$ is soluble in $F(t)$, then it has a solution in polynomials of degree at most $N(d)$.

We may now prove:

Theorem 3.3. Let $F$ be a finite field, let $f, g$ be two nonproportional cubic forms in 4 variables. Assume the characteristic of $F$ is not 3. Let $k = F(t)$. Suppose the cubic surface $X \subset P^3_k$ over $k$ defined by $f + tg = 0$ is smooth. If there is no Brauer–Manin obstruction to the Hasse principle for rational points on $X$, then there exists a $k$-rational point on $X$.

Proof. Combine Theorem 2.4 and Theorem 3.1.

Remark 3.4. Again, it would be nice to avoid the cohomological machinery, i.e. Theorems 2.1 and 2.2. When $X$ has no rational points over $F(t)$ but points in all the completions of $F(t)$ one should exhibit an explicit Brauer–Manin obstruction for $X$. For this purpose, it would probably be helpful to use [SD93]. Down to earth computations, which we shall not insert here, have led to the following result. If a smooth cubic surface $X$ given by $f + tg = 0$ is a counterexample to the Hasse principle over $F(t)$, then, after replacing $F$ by its maximal pro-2-extension $F$, the following holds: When going over to the algebraic closure of $F$, the curve $\Gamma$ in the proof of Theorem 3.1 breaks up as a sum of 9 conjugate lines, or a sum of three twisted cubics, or a sum of three conjugate conics plus a sum of three coplanar conjugate lines; when using the word “conjugate” we mean that the Galois action is transitive. Only in these three cases may we expect a Brauer–Manin obstruction.

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References


