Over a field k of char. $\neq 2$ which admits a Galois field extension with group S_3, one knows how to produce a stably rational surface which is not rational. Namely, one takes a conic bundle surface defined by an affine equation $y^2 - az^2 = P(x)$, where $P(x)$ is a separable irreducible polynomial of degree 3 and a is its discriminant (Beauville, CT, Sansuc, Swinnerton-Dyer). B. Hassett has asked whether every stably rational surface over a finite field is actually rational over that field.

Proposition. *Let $\mathbf{F}$ be a finite field of characteristic different from 2 and $X/\mathbf{F}$ be a smooth, projective, geometrically connected surface over a finite field, equipped with a conic bundle structure $X \to \mathbf{P}^1_{\mathbf{F}}$.*

(i) If $\mathrm{Br}(X_{\mathbf{F}'}) = 0$ for any finite field extension $\mathbf{F}'/\mathbf{F}$, then X is $\mathbf{F}$-rational.

(ii) If X is stably $\mathbf{F}$-rational, then X is $\mathbf{F}$-rational.

Proof.

We may assume that the morphism $X \to \mathbf{P}^1_{\mathbf{F}}$ is relatively minimal and that it has no section. This implies that the number r of closed points of $\mathbf{P}^1_{\mathbf{F}}$ with degenerate fibre is at least 1.

We have an exact sequence of Galois modules

$$0 \to P \to \mathbf{Z} \oplus Q \to Pic(\overline{X}) \to \mathbf{Z} \to 0,$$

where P is the permutation module on the $\overline{\mathbf{F}}$-points of $\mathbf{P}^1$ with singular fibre, Q is the permutation module on the components of the singular fibres over $\overline{\mathbf{F}}$, and $\mathbf{Z}$ is generated by a fibre over an $\mathbf{F}$-point of $\mathbf{P}^1$. The map $Pic(\overline{X}) \to \mathbf{Z}$ is the restriction to the generic fibre.

Let M be the kernel of this restriction map. We have short exact sequences

$$0 \to P \to \mathbf{Z} \oplus Q \to M \to 0$$

and

$$0 \to M \to \mathrm{Pic}(\overline{X}) \to \mathbf{Z} \to 0.$$

Galois cohomology then gives exact sequences

$$0 \to \mathbf{Z}/2 \to H^1(\mathbf{F}, M) \to H^1(\mathbf{F}, Pic(\overline{X})) \to 0$$

and

$$0 \to H^1(\mathbf{F}, M) \to H^2(\mathbf{F}, P) \to H^2(\mathbf{F}, \mathbf{Z} \oplus Q).$$

The latter yields an exact sequence

$$0 \to H^1(\mathbf{F}, M) \to \oplus_{i=1}^r \mathbf{Z}/2 \to \mathbf{F}^*/\mathbf{F}^{*2},$$

where i runs through the $r \geq 1$ closed points of $\mathbf{P}^1$ with bad fibre, and the map $\theta_i : \mathbf{Z}/2 \to \mathbf{F}^*/\mathbf{F}^{*2} \simeq \mathbf{Z}/2$ sends 1 to the nontrivial element in $\mathbf{F}^*/\mathbf{F}^{*2}$. We thus have $H^1(\mathbf{F}, M) \simeq (\mathbf{Z}/2)^{r-1}$, and $H^1(\mathbf{F}, \mathrm{Pic}(\overline{X})) \simeq (\mathbf{Z}/2)^{r-2}$ (in particular $r \geq 2$). If $r > 2$, then $\mathrm{Br}(X) = H^1(\mathbf{F}, \mathrm{Pic}(\overline{X})) \neq 0$.

Suppose $r = 2$.

In the course of the proof, we shall use the following

Lemma. *Let $f : X \to \mathbf{P}^1_k$ be relatively minimal conic bundle over a perfect field k of characteristic different from 2. Let $M \in \mathbf{P}^1_k$ be a closed point where the fibration has bad reduction. Let $K = k(M)$ be the residue field at M. Let N be a K-point of $\mathbf{P}^1_K$ lying over M. Then the fibration $f_K : X_K \to \mathbf{P}^1_K$ has bad reduction at M.*

Proof. Let $A/k(\mathbf{P}^1)$ be the quaternion algebra associated to the generic fibre of $X \to \mathbf{P}^1_k$. The fibration f has bad reduction at M if and only if the residue $\gamma := \partial_M(A) \in H^1(k(M), \mathbf{Z}/2) = H^1(K, \mathbf{Z}/2)$ is nontrivial. The formula for computing residues under separable extensions gives

$$\partial_N(A_K) = \gamma \in H^1(K(N), \mathbf{Z}/2) = H^1(K, \mathbf{Z}/2).$$

hence the lemma.

If K/k is Galois, as is the case if $k = \mathbf{F}$ is a finite field, then $K \otimes_k K = \prod_{i=1}^{d} K$. Above a closed point of degree $d = [K : k]$, after going over from k to K we thus find d K-points where the fibration has bad reduction.

If we have at least one closed point of $\mathbf{P}^1_{\mathbf{F}}$ of degree ≥ 3 over $\mathbf{F}$ and with bad fibre, then going over to $\mathbf{F}'$, we get a conic bundle over $\mathbf{P}^1_{\mathbf{F}'}$ with at least 3 rational points of $\mathbf{P}^1_{\mathbf{F}'}$ with noncontractible degenerate fibres. We thus have $\mathrm{Br}(X_{\mathbf{F}'}) = H^1(\mathbf{F}', \mathrm{Pic}(\overline{X})) \neq 0$.

We are reduced to the case where all closed points of $\mathbf{P}^1_{\mathbf{F}}$ with bad fibre have degree 1 or 2, and there are exactly two such closed points. If the number of geometric degenerate fibres is at most 3, then one knows that X is $\mathbf{F}$-rational. We are thus reduced to the case where there are exactly two closed points of $\mathbf{P}^1_{\mathbf{F}}$ with bad fibre, and each of them is of degree 2. Their residue fields coincide with the quadratic extension $\mathbf{F}'$ of $\mathbf{F}$. Going over to this extension, we get a conic bundle over $\mathbf{P}^1_{\mathbf{F}'}$ with 4 $\mathbf{F}'$-rational points of $\mathbf{P}^1_{\mathbf{F}'}$ with noncontractible degenerate fibres. We thus have $\mathrm{Br}(X_{\mathbf{F}'}) = H^1(\mathbf{F}', \mathrm{Pic}(\overline{X})) \neq 0$. CQFD

Corollary of Proposition. *Let X be a del Pezzo surface of degree d over a finite field $\mathbf{F}$. For $d \geq 5$, X is $\mathbf{F}$-rational. For $d = 4$, suppose that $\mathbf{F}$ is big enough so that any del Pezzo surface of degree 4 has at least 41 rational points. If such a surface X is stably $\mathbf{F}$-rational, then X is $\mathbf{F}$-rational.*

The case $d \geq 5$ is well known. Suppose $d = 4$. If there is an $\mathbf{F}$-point outside the exceptional lines, then one blows it up and gets a cubic surface with an $\mathbf{F}$-line, hence a surface with a conic bundle structure. If there is an $\mathbf{F}$-point on just one exceptional line, then that line is defined over $\mathbf{F}$ and one blows down that line.

Remark (K. Shramov and A. Trepalin, Oct. 10, 2017). From the tables of Swinnerton-Dyer, Manin, Urabe and recently Bawait, Fité and Loughran it follows that for $\mathbf{F} = \mathbf{F}_q$ a finite field and q sufficiently large, any $\mathbf{F}$-minimal smooth cubic surface X over a finite field $\mathbf{F}$, there exists a finite field extension $\mathbf{F}'/\mathbf{F}$ such that $H^1(\mathbf{F}', \mathrm{Pic}(\overline{X})) \neq 0$. For such surfaces it follows that they are not stably rational over $\mathbf{F}$. It then follows that *if a smooth cubic surface X over $\mathbf{F}$ is not rational, and the cardinality of $\mathbf{F}$ is big enough, then X is not stably rational.*

JLCT, May 4th, 2017. Lemma added, October 8th, 2017. Remark added, October 14th, 2017