UNRAMIFIED WITT GROUPS OF
REAL ANISOTROPIC QUADRICS

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Abstract. The unramified Witt group of the real anisotropic conic and the real
anisotropic quadric surface are known to be finite groups, cyclic of order 4. We
prove that the unramified Witt group of the real anisotropic quadric of dimension
\(d\), \(3 \leq d \leq 6\) is finite, cyclic of order 8, with the form \(<1>\) as the generator.

Let \(k\) be a field of characteristic \(\neq 2\) and \(K/k\) a finitely generated field exten-
sion of \(k\). The unramified Witt group \(W_{nr}(K/k)\) is a subgroup of \(W(K)\), the
Witt group of quadratic forms over \(K\), and has interesting applications (see [CT-
O], [O]). For any positive integer \(i\), one may also define the unramified subgroup
\(H^i_{nr}(\mathbb{R}(Q_3), \mathbb{Z}/2)\) of the étale cohomology group \(H^i(\mathbb{R}(Q_3), \mathbb{Z}/2)\).

If \(X/k\) is a geometrically integral variety over the field \(k\), and \(k(X)\) denotes the
function field of \(X\), by abuse of terminology, we refer to the group \(W_{nr}(k(X)/k)\),
as the unramified Witt group of \(X\). In this note, we prove that the unramified
Witt groups of the real anisotropic quadrics \(Q_d\) of dimension \(d\), \(3 \leq d \leq 6\), are
finite, cyclic of order 8 with the form \(<1>\) as the generator. In lower dimensions,
a similar result was known for the real anisotropic conic (Theorem 6.2 of [P] with
a slight modification for the projective conic) and the anisotropic quadric surface,
the groups then being cyclic of order 4. The finiteness of the unramified Witt group
for anisotropic quadrics of dimension \(\geq 7\) is an open question.

Here is the plan of the paper. Section 1 recalls a few basic facts about quadratic
forms, the unramified Witt group and unramified cohomology. Section 2 starts
with various étale cohomology computations of quadrics (here we use the results of
Artin-Verdier-Cox [C] on the étale cohomological dimension of real varieties without
a real point). These computations together with Karpenko’s computations of the
Chow groups of 3-dimensional quadrics [K] enable us to show that the map

\[
CH^2(Q_3)/2 \rightarrow H^4(Q_3, \mathbb{Z}/2),
\]

which is induced by the cycle map into étale cohomology, is an injection. From the
properties of the Bloch-Ogus spectral sequence it then follows that the group
\(H^3_{nr}(\mathbb{R}(Q_3), \mathbb{Z}/2)\) vanishes. Standard properties of unramified cohomology then
imply \(H^3_{nr}(\mathbb{R}(Q_d), \mathbb{Z}/2) = 0\) for \(3 \leq d \leq 6\), and use of a well-known result of Arason
[A] together with a result of Merkurjev-Suslin [M-S1] and Rost [R] on Milnor’s

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K₃-group lead to $H^d_{nr}(\mathbb{R}(Q_d), \mathbb{Z}/2) = \mathbb{Z}/2$ for $d \geq 7$. Since higher unramified cohomology is rather hard to compute, these results seem to be of independent interest. They also enable us to show that the cycle class map

$$\text{CH}^2(Q_d)/2 \rightarrow H^4(Q_d, \mathbb{Z}/2)$$

is an injection for all $d$.

Section 3 applies the vanishing of $H^d_{nr}(\mathbb{R}(Q_d), \mathbb{Z}/2)$ to the computation of the unramified Witt group $W_{nr}(\mathbb{R}(Q_d))$ (which by a standard argument coincides with $W_{nr}(\mathbb{R}(Q_d))$ for any $d$, $3 \leq d \leq 6$). We use the map $W(\mathbb{R}(Q_3)) \rightarrow W(\mathbb{R}(Q_3 \times Q_2))$, whose kernel is known (Arason, Pfister [A]) and investigate the induced map on unramified Witt groups. The image is easily determined because $W_{nr}(\mathbb{R}(Q_3 \times Q_2))$ is isomorphic to $W_{nr}(\mathbb{R}(Q_2))$. Recall that $I^n(F)$ denotes the $n$-th power of the fundamental ideal of the Witt group of a field $F$. To control the kernel of the above map, we use the injection of $I^3(\mathbb{R}(Q_3))$ into $H^3(\mathbb{R}(Q_3), \mathbb{Z}/2)$. This injection follows from the vanishing of $I^4(\mathbb{R}(Q_3))$ [AEJ] together with the known injectivity of the Milnor map $I^3F/I^4F \rightarrow H^3(F, \mathbb{Z}/2)$ for any field $F$ (char $F \neq 2$) (Merkurjev-Suslin [M-S1], [R]). The final argument is provided by the vanishing of $H^d_{nr}(\mathbb{R}(Q_d), \mathbb{Z}/2)$, established in section 2.

§1. Preliminaries

All fields considered in this note are of characteristic different from 2 unless otherwise mentioned. Given a field $K/k$ and a finitely generated field extension $K/k$, we denote by $V_K$ the set of all rank one discrete valuations on $K$, which are trivial on $k$. Given an element $\nu \in V_K$, there exists a “second residue homomorphism” $\partial_\nu : W(K) \rightarrow W(k(\nu))$ (cf. [Sc, Ch. VI, 2.5]), where $\pi_\nu$ is a parameter for the valuation $\nu$ and $k(\nu)$ is the residue field at $\nu$. The homomorphism $\partial_\nu$ depends on the choice of a parameter but the kernel of $\partial_\nu$ depends only on $\nu$ and hence may be denoted $\ker \partial_\nu$. The unramified Witt group of $K/k$, denoted $W_{nr}(K/k)$, or just $W_{nr}(K)$ when there is no possible confusion of the ground field, is defined as

$$W_{nr}(K/k) = \bigcap_{\nu \in V_K} \ker \partial_\nu.$$

Given a $k$-extension $L$ of $K$, the natural map $W(K) \rightarrow W(L)$ induces a map $W_{nr}(K) \rightarrow W_{nr}(L)$ on the corresponding unramified Witt groups [O, Proposition 7.1].

Let $X/k$ be a smooth geometrically integral variety and $k(X)$ denote its function field. We denote the set of codimension one points of $X$ by $X^{(1)}$. Since $X$ is smooth, the elements of $X^{(1)}$ give elements in $V_{k(X)}$. If $x \in X^{(1)}$ and $\pi_x$ is a fixed uniformizing parameter for the valuation corresponding to $x$, then we denote the corresponding second residue homomorphism by $\partial_x$. Given an element $q \in W(k(X))$, it is easily seen that $\partial_x(q) \neq 0$ for finitely many $x \in X^{(1)}$. Clearly there is a complex

$$0 \rightarrow W_{nr}(k(X)) \rightarrow W(k(X)) \xrightarrow{\bigoplus_{x \in X^{(1)}} \partial_x} \bigoplus_{x \in X^{(1)}} W(k(x))$$
where \( k(x) \) denotes the residue field at a point \( x \in X^{(1)} \).

Recall that a quadratic form \( \phi \) over \( k \) is said to be an \( n \)-fold Pfister form if it is isometric to a form of the type

\[
< 1, a_1 > \otimes < 1, a_2 > \otimes \cdots \otimes < 1, a_n >, \quad a_i \in k^*.
\]

We denote such a form by \( \ll a_1, a_2, \ldots, a_n \gg \). For a Pfister form, the properties of being isotropic and hyperbolic coincide. The following proposition is well-known ([A] Satz 1.3).

**Proposition 1.1.** Let \( k \) be a field and \( \phi \) an \( n \)-fold Pfister form over \( k \), \( n \geq 2 \). Let \( X_\phi \) be the projective quadric defined by \( \phi \) and let \( k(X_\phi) \) denote the function field of \( X_\phi \). Then the function field \( k(X_\phi) \)

is the ideal generated by the class of \( \phi \). \( \square \)

Recall that a quadratic form \( \psi \) defined over \( k \) is said to be a Pfister neighbour of an \( n \)-fold Pfister form \( \phi \) over \( k \) if dimension \( \psi \geq 2^{n-1} + 1 \) and \( \psi \) is isometric to a subform of \( \phi \). A Pfister neighbour \( \psi \) of \( \phi \) has the following property: \( \psi \) is isotropic over a field extension \( K/k \iff \phi \) is isotropic over \( K \). Further, it is easily seen that the function field \( k(X_\phi) \) of the projective quadric defined by a quadratic form \( \rho \) over \( k \) of rank at least three is a purely transcendental extension of \( k \) if and only if the form \( \rho \) is isotropic. These properties are used to prove the following

**Lemma 1.2.** Let \( \phi \) be an \( n \)-fold Pfister form over \( k \) and \( \psi \) a Pfister neighbour of \( \phi \). Let \( X_\phi \) (respectively \( X_\psi \)) denote the corresponding projective quadric and \( k(X_\phi) \) (respectively \( k(X_\psi) \)) be the function field. Then \( W_{nr}(k(X_\phi)) \cong W_{nr}(k(X_\psi)) \).

**Proof.** Consider the product variety \( X_\phi \times_k X_\psi \) and the natural projections given by \( \pi_\phi : X_\phi \times_k X_\psi \to X_\phi \) and \( \pi_\psi : X_\phi \times_k X_\psi \to X_\psi \). Clearly \( k(X_\phi \times_k X_\psi) \), the function field of \( X_\phi \times_k X_\psi \), coincides with \( k(X_\phi)(X_\psi) \) and \( k(X_\phi)(X_\psi) \). Since \( X_\psi \) (respectively \( X_\phi \)) has a \( k(X_\psi) \)-rational point, (respectively \( k(X_\phi) \)-rational point), namely the generic point, the above remarks imply that \( X_\phi \) (respectively \( X_\psi \)) also has a \( k(X_\psi) \)-rational point (respectively \( k(X_\phi) \)-rational point). Thus the field \( k(X_\phi)(X_\psi) \) is a purely transcendental extension of both \( k(X_\phi) \) and \( k(X_\psi) \). Since the unramified Witt groups are invariant under purely transcendental extensions (cf. [O, Proposition 7.2]), we have \( W_{nr}(k(X_\phi)) \cong W_{nr}(k(X_\phi \times_k X_\psi)) \cong W_{nr}(k(X_\psi)) \) and the lemma is proved. \( \square \)

We now recall some results about cohomology groups. Given a field \( k \) and a finitely generated field extension \( K/k \), the “unramified cohomology groups” with coefficients in \( \mathbb{Z}/2 \), denoted \( H^i_{nr}(K/k) \) or just \( H^i_{nr}(K) \), \( i \geq 1 \), if there is no ambiguity of the ground field, are subgroups of the Galois cohomology groups \( H^i(G(K_s/K), \mathbb{Z}/2) \) where \( K_s \) denotes a separable closure of \( K \) (cf. [CT-O]). We denote the Galois cohomology groups \( H^i(G(K_s/K), \mathbb{Z}/2) \) by \( H^i(K) \). The groups \( H^i_{nr}(K) \) are defined as

\[
H^i_{nr}(K) = \bigcap_{\nu \in V_K} \ker \partial_{\nu},
\]

where \( V_K \) is the set of rank one valuations on \( K \) that are trivial on \( k \), \( \nu \in V_K \) and \( \partial_{\nu} \) is the residue homomorphism [CT-O, §1],

\[
\partial_{\nu} : H^i(K) \to H^{i-1}(k(\nu)),
\]
with $k(\nu)$ denoting the residue field of the valuation.

The unramified cohomology groups are invariant under purely transcendental extensions [CT-O, Proposition 1.2]. Thus, in the spirit of Lemma 1.2, we have

**Lemma 1.3.** Let $\phi$ be an $n$-fold Pfister form over $k$ and $\psi$ a Pfister neighbour of $\phi$. Let $X_\phi$ (respectively $X_\phi'$) be the corresponding projective quadric and $k(X_\phi)$ (respectively $k(X_\phi')$) be the function field. Then $H^s_{nr}(k(X_\phi)) \cong H^s_{nr}(k(X_\phi'))$. □

Given an integral variety $X/k$ of dimension $d$, the sheaves $\mathcal{H}^n$, for $n \geq 0$ are the Zariski sheaves associated to the presheaf $U \mapsto H^n(U)$, where $H^n(U) = H^0(U, \mu_2)$, $\mu_2$ denoting the étale sheaf of square roots of unity. By the results of Bloch-Ogus [B-O], if $X$ is smooth and integral, a flasque resolution of the sheaf $\mathcal{H}^n$ is given by

$$0 \rightarrow \mathcal{H}^n \rightarrow \bigoplus_{x \in X(1)} i_x H^{n-1}(k(x)) \rightarrow \cdots \bigoplus_{x \in X(d)} i_x H^{n-d}(k(x)) \rightarrow$$

where $k(X)$ is the function field of $X$, $X^{(i)}$ is the set of points of codimension $i$ in $X$, $\eta$ is the generic point and $k(x)$ is the residue field at the point $x$. Taking the complex of global sections of the above resolution, we see that the homology groups of this complex correspond to the Zariski cohomology groups $H^0(X, \mathcal{H}^n)$ of the sheaf $\mathcal{H}^n$. In fact, if $X$ is also proper, we have $H^n_{nr}(k(X)) = H^0(X, \mathcal{H}^n)$ [CT-O, Remarque 1.1.3]. Thus the groups $H^n(X, \mathcal{H}^n)$ are birational invariants of $X$ (cf. [CT-P, 1.3]).

The “local-to-global” spectral sequence (cf. [B-O, §6 and Remark 6.4])

$$E_2^{p,q} = H^p(X, \mathcal{H}^q) \implies H^n(X) = E^n$$

relates the Zariski cohomology groups of the sheaves $\mathcal{H}^n$ to the étale cohomology groups $H^n_{nr}(X) = H^n_{et}(X, \mu_2)$.

Finally, we observe that the unramified cohomology groups of a smooth, projective, integral variety $X$ can be related to the unramified Witt group of $X$ whenever the homomorphisms $e_n : I^n(k(X)) \rightarrow H^n(k(X))$, (cf. [AEJ]) are well-defined. This is done by considering the induced filtration $(I^m_{nr}(k(X)))_{m \geq 0}$ on $W_{nr}(k(X))$, where, by definition

$$I^m_{nr}(k(X)) = I^m(k(X)) \cap W_{nr}(k(X)),$$

$I(k(X))$ being the fundamental ideal of even dimensional forms and $I^m(k(X))$ its $m$-th power. If the homomorphisms $e_n$ and $e_{n−1}$ are well-defined, we get a commutative diagram [CT-P, Proposition 1.5.1]

$$\begin{array}{ccc}
0 \rightarrow I^m_{nr}(k(X)) & \rightarrow & I^n(k(X)) \\
\downarrow e_n & & \downarrow \text{fection}
\end{array}$$

$$\begin{array}{ccc}
0 \rightarrow H^0(X, \mathcal{H}^n) & \rightarrow & H^n(k(X)) \\
\downarrow e_n & & \downarrow \text{fection}
\end{array}$$

with the top row a complex, the bottom row exact and notation as before. The maps $e_n : I^n F \rightarrow H^n(F)$ are known to be well-defined for all fields $F$, whenever $n \leq 3$, with kernel precisely equal to $I^{n+1}(F)$. Thus there exist isomorphisms $\varpi_n : I^n(F)/I^{n+1}(F) \cong H^n(F)$ for $0 \leq n \leq 3$ (cf. [A], [Me], [M-S1], [R]).
§2. Étale cohomology of quadrics and the cycle map

In this section we compute the étale cohomology groups of certain quadrics. These computations are then used to compute the unramified cohomology group $H^3_{nr}$ of all the real anisotropic quadrics (Proposition 2.7). We also prove the injectivity of the cycle map

$$CH^2(X)/2 \to H^3_{et}(X, \mathbb{Z}/2)$$

for all real anisotropic quadrics (Corollary 2.8). Let $k$ be a field of characteristic zero. We first record a few results from étale cohomology, given a variety $X$ of dimension $d$. Cohomology will always be cohomology with coefficients in $\mathbb{Z}/2$. When using arbitrary finite coefficients, the isomorphisms and exact sequences that follow would require Tate twists for the coefficients. However, since $\mathbb{Z}/2$ is equal to $\mu_2$, all sheaves $\mu_2^i$, $i \in \mathbb{Z}$ coincide with $\mathbb{Z}/2$.

2.a: Let $Y \subseteq X$ be a closed immersion and $U = X \setminus Y$. Then there exists a long exact localisation sequence

$$\ldots \to H^n_Y(X) \to H^n(X) \to H^n(U) \to H^{n+1}_Y(X) \to \ldots$$

with notation as before.

2.b: More generally, let $Z \subseteq Y \subseteq X$ be a sequence of closed immersions. Then there is a long exact sequence

$$\ldots \to H^i_Z(X) \to H^i_Y(X) \to H^i_{Y \setminus Z}(X \setminus Z) \to H^{i+1}_Z(X) \to \ldots$$

with notation as in 2.a.

We refer to [Mi, Ch. VI, §6] for proofs of the purity results cited below, remarking that though it is assumed there that the ground field is algebraically closed, the proofs remain valid over an arbitrary ground field of characteristic zero (see [Ar, SGA 4, XVI 3.8, 3.9, 3.10 and SGA 4, XIX 3.4]).

2.c: (Purity) Let $Y \subseteq X$ be a closed immersion, with both $X$ and $Y$ smooth over $k$, $Y$ connected and everywhere of codimension $c$ in $X$. Then there exists an isomorphism

$$H^n_Y(X) \cong H^{n-2c}(Y)$$

with notation as before. We give a brief description of this isomorphism. Let $\mathcal{H}^c_Y(\mathbb{Z}/2)$ denote the Zariski sheaf on $Y$ defined as in [Mi, p.241]. By [SGA 4, XVI, 3.8], there exists an isomorphism of $\mathcal{H}^c_Y(\mathbb{Z}/2)$ with the locally constant sheaf $(\mathbb{Z}/2)^Y$ on $Y$, i.e. $\mathcal{H}^c_Y(\mathbb{Z}/2) \cong (\mathbb{Z}/2)^Y$. Further, by [SGA 4, XVI, 3.9], we have $\mathcal{H}^c_Y(\mathbb{Z}/2) = 0$ if $i \neq 2c$. Using this information in the spectral sequence [Mi, p.247]

$$H^p_{et}(Y, \mathcal{H}^c_Y(\mathbb{Z}/2)) \Rightarrow H^p_Y(X)$$

we get an isomorphism

$$H^1(Y) \cong H^{3+2c}_Y(X).$$
This isomorphism gives rise to the *Gysin map*

\[ H^i(Y) \cong H^{i+2c}_Y(X) \to H^{i+2c}(X) \]

where the latter map is obtained from the long exact localisation sequence (2.a).

2.d: Let \( X, Y \) as in 2.c. When \( n = 2c \), by the isomorphism described above, we get

\[ \mathbb{Z}/2 \cong H^0(Y) \cong H^{2c}_Y(X). \]

The image of \( 1 \in H^{2c}_Y(X) \) under this isomorphism is called the *fundamental class* \( s_{Y/X} \) of \( Y \) in \( X \).

2.e: Finally, we briefly discuss the cycle map and refer to [Mi, Ch. VI] for details (see also [D, Cycle]). Given a smooth integral variety \( X/k \), let \( Z^n(X) \) denote the free abelian group generated by closed integral subschemes of codimension \( n \) in \( X \).

The cycle map in codimension \( n \) is a homomorphism \( \rho_n : Z^n(X) \to H^{2n}(X) \)

into the étale cohomology group and is briefly described below. Given a smooth integral closed subvariety \( Z \) of codimension \( n \), the element \( \rho_n(Z) \) is defined to be the image in \( H^{2n}(X) \) of the fundamental class \( s_{Z/X} \) in \( H^{2n}(X) \) (cf. 2.d) under the natural map \( H^{2n}_Z(X) \to H^{2n}(X) \) (cf. 2.a). Equivalently, \( \rho_n(Z) \) is the image of the non-trivial element of \( H^0(Z) \) under the Gysin map \( H^0(Z) \to H^{2n}(X) \). Using purity [D], the map so defined can be extended to singular cycles and hence to the whole group \( Z^n(X) \) by linearity.

On the other hand, we have the local-to-global spectral sequence (cf. §1) constructed by Bloch-Ogus [B-O, §3],

\[ H^p_{Zar}(X, \mathcal{H}^q) \implies H^{p+q}_e(X) \]

using the formalism of duality theory and trace map based on [D1, SGA 4, XVIII]. Further, by the results of Bloch-Ogus, we see that \( H^p(X, \mathcal{H}^q) = 0 \) if \( p > q \) [B-O, 6.1 and 6.2] and \( H^p(X, \mathcal{H}^p) = CH^p(X)/2 \) [B-O, 6.3] (see also [B-O, 7.7], whose proof extends to smooth varieties over arbitrary perfect fields), where \( CH^p(X) \) denotes the Chow group of algebraic cycles of codimension \( p \) modulo rational equivalence. Therefore the above spectral sequence gives a natural map

\[ CH^p(X)/2 \to H^{2p}(X). \]

But this map coincides with the “classical” cycle map \( \rho_p \) described above, as is seen using 7.2, 3.7 and 3.9 of [B-O] and working through the definition of \( \rho_p \). In particular, the map \( \rho_p \) factors through rational equivalence and we get a homomorphism

\[ \rho_p : CH^p(X)/2 \to H^{2p}(X). \]

2.f: If \( Y \to X \) is a closed embedding of smooth \( k \)-varieties, \( Y \) connected and everywhere of codimension \( c \), then we have the following commutative diagram:

\[
\begin{array}{ccc}
CH^n(Y) & \xrightarrow{\iota} & CH^{n+c}(X) \\
\downarrow{\rho_n} & & \downarrow{\rho_{n+c}} \\
H^{2n}(Y) & \longrightarrow & H^{2n+2c}(X).
\end{array}
\]
Here the vertical maps are the cycle maps, the top horizontal map is the natural map induced by the inclusion and the lower horizontal map is the Gysin map

\[ H^{2n}(Y) \cong H^{2n+2c}_\text{et}(X) \to H^{2n+2c}(X), \]

That the diagram commutes follows from [Mi, Ch. VI, 9.3] observing that the result remains valid over a ground field of characteristic zero.

For any smooth variety \( X/k \), there is an exact sequence of sheaves in the étale topology as follows,

\[ 0 \to \mu_2 \to \mathbb{G}_m \to \mathbb{G}_m \to 0 \]

where \( \mathbb{G}_m \) is the sheaf associated to \( \mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^* \), the multiplicative group of units of the sections of \( U \), and \( \mu_2 \) is the étale sheaf of square roots of unity, isomorphic to \( \mathbb{Z}/2 \). From Kummer theory and Grothendieck’s Hilbert theorem 90 [Mi, Ch. III, §4], we get exact sequences

\[ 0 \to \Gamma(X, \mathcal{O}_X)^*/\Gamma(X, \mathcal{O}_X)^* \to H^1_\text{ét}(X, \mu_2) \to 2\text{Pic}(X) \to 0 \]

and

\[ 0 \to \text{Pic}(X)/2 \to H^2_\text{ét}(X, \mu_2) \to 2\text{Br}(X) \to 0 \]

where \( \Gamma(X, \mathcal{O}_X)^* \) is the group of units of the global sections over \( X \) and \( 2\text{Pic}(X) \) (respectively \( 2\text{Br}(X) \)) is the 2-torsion subgroup of the Picard group (respectively étale cohomological Brauer group) of \( X \). We abbreviate the étale cohomology groups \( H^n_\text{ét}(U, \mu_2) \) to \( H^n(U) \) for a variety \( U \).

For the rest of the discussion, we fix the ground field \( k \) to be the field \( \mathbb{R} \) of real numbers. The \( d \)-dimensional quadric \( Q_d \) is defined by \( \sum_{i=0}^{d+1} X_i^2 = 0 \) in \( \mathbb{R}^{d+1} \). We denote the corresponding smooth projective complex \( d \)-dimensional quadric \( Q_d \times_{\mathbb{R}} \mathbb{C} \) by \( C_d \). The following well-known lemma computes the étale cohomology groups of the complex quadrics \( C_d \). By the comparison theorem [Mi, Ch. III, Theorem 3.12], these groups coincide with the usual singular cohomology groups of the underlying complex manifold with coefficients in \( \mathbb{Z}/2 \).

**Lemma 2.1.** Let \( C_d \) (\( d \geq 2 \)) be the \( d \)-dimensional complex quadric \( Q_d \times_{\mathbb{R}} \mathbb{C} \). Then there exist isomorphisms

\[ H^i(C_d) \cong H^{i-2}(C_{d-2}) \text{ for } i \geq 2, \ d > 2, \ i \neq 2d, \ 2d - 1. \]

Further \( H^0(C_d) \cong \mathbb{Z}/2 \), \( H^1(C_d) = 0 \), \( H^{2d-1}(C_d) = 0 \) for all \( d \geq 1 \) and \( H^2(C_d) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

**Proof.** Clearly \( H^0(C_d) \cong \mathbb{Z}/2 \) for all \( d \) and hence \( H^{2d}(C_d) \cong \mathbb{Z}/2 \), by Poincaré duality. The exact sequence (7) gives \( H^1(C_d) = 0 \), since \( C_d \) is a smooth, projective, rational variety and the torsion in the Picard group is well-known [CT-S, Appendix 2.A] to be a birational invariant of smooth projective varieties. By Poincaré duality, we deduce that \( H^{2d-1}(C_d) = 0 \). The quadric \( C_2 \) is isomorphic to \( \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \) and hence by the Künneth formula, \( H^2(C_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). We now prove the first part of the lemma. By a change of coordinates, we may assume that \( C_d \) is defined by the equation \( X_0 X_1 + \sum_{i=2}^{d+1} X_i^2 = 0 \) in \( \mathbb{P}^{d+1}_{\mathbb{C}} \). Let \( Y = Y_d \) be the hyperplane section of \( C_d \)
defined by the hyperplane $X_1 = 0$. The complement $C_d \setminus Y_d$ is then isomorphic to the $d$-dimensional affine space $k^d$. The point $P = (1 : 0 : 0 \cdots : 0)$ is a singular point on $Y_d$ and $Y_d \setminus P$ is an $A^1$-bundle over the quadric $C_{d-2}$. Consider the inclusions $Y_d \subseteq C_d \hookrightarrow A_{d}^{2}$. We have a long exact sequence (cf. 2.a)

$$
\cdots \to H^0_{Y_d}(C_d) \to H^0(C_d) \to H^0(A_{d}^{2}) \to H^1_C(C_d) \to H^1(C_d) \to \cdots.
$$

Since the étale cohomology groups $H^n(A_{d}^{2})$ are trivial for $n > 0$ and $H^0(C_d)$ surjects onto $H^0(A_{d}^{2})$, we have $H^0_{Y_d}(C_d) = 0$ and isomorphisms

$$
H^0_{Y_d}(C_d) \cong H^0(C_d) \text{ for all } n \geq 1.
$$

We now consider the chain of inclusions $P \subseteq Y_d \subseteq C_d$ and the corresponding long exact sequence in étale cohomology (cf. 2.b). We then have a long exact sequence as follows, (see (5))

$$
\cdots \to H^i_{P}(C_d) \to H^i_{Y_d\setminus P}(C_d \setminus P) \to H^i_{P\setminus Y_d}(C_d \setminus P) \to \cdots.
$$

Using "Purity" (cf. 2.c), we have isomorphisms

$$
H^i_{P}(C_d) \cong H^{n+i}(P),
$$

since $P$ is a smooth closed point on $C_d$. Combining these isomorphisms, we get

$$
H^i_{Y_d\setminus P}(C_d \setminus P) \text{ for all } n \neq 2d, 2d - 1.
$$

Further, since $Y_d \setminus P \subseteq C_d \setminus P$ is smooth of codimension 1, we have

$$
H^i_{Y_d\setminus P}(C_d \setminus P) \cong H^{n-2}(Y_d \setminus P).
$$

But $Y_d \setminus P$ is an $A^1$-bundle over $C_{d-2}$ and since the étale cohomology with finite coefficients of an affine bundle over a smooth variety $X$ is isomorphic to the étale cohomology of $X$, we see that

$$
H^{n-2}(Y_d \setminus P) \cong H^{n-2}(C_{d-2}) \text{ for all } n \geq 2.
$$

The isomorphisms (14), (13) and (10) now complete the proof of the lemma.

As a corollary, we have the following

**Proposition 2.2.** Let $C_d$ be the smooth projective complex quadric of dimension $d$, $d \geq 1$. Then $H^{2i+1}(C_d) = 0$ for $i \geq 0$ and $H^{2i}(C_d) \cong \mathbb{Z}/2$ for $0 \leq i \leq d$, $i \neq d/2$ if $d$ even. If $d$ is even, then we have $H^d(C_d) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Recall that there exists an exact sequence of étale sheaves over a real variety $X$

$$
0 \to \mathbb{Z}/2 \to \pi_*\mathbb{Z}/2 \to \mathbb{Z}/2 \to 0
$$

where $\pi : X_C \to X$ is the étale map of degree 2 obtained by base change to $C$. This induces a long exact sequence

$$
\cdots \to H^n(X) \to H^n(X_C) \to H^n(X) \xrightarrow{(-1)} H^{n+1}(X) \to H^{n+1}(X_C) \to \cdots
$$

which is a generalisation of the corresponding sequence for the Galois extension $C/\mathbb{R}$ [A, Corollary 4.6]. The connecting homomorphism is given by the cup-product with the class of $(-1)$ in $H^1(X)$ under the image of the natural map $H^1(\text{Spec } \mathbb{R}) \to H^1(X)$, noting that $H^1(\text{Spec } \mathbb{R}) \cong H^1(\mathbb{R}) \cong \mathbb{R}^*/\mathbb{R}^2$. We use this sequence for the quadrics $Q_d$ and the preceding lemma to compute the étale cohomology groups of $Q_3$ with $\mu_2$ coefficients. We have
Lemma 2.3. Let $Q_3$ be the 3-dimensional real anisotropic quadric. Then the étale cohomology groups of $Q_3$ with $\mathbb{Z}/2$ coefficients are given by

$$
H^i(Q_3) \cong \mathbb{Z}/2 \text{ for } i = 0, 1, 5, 6,
$$

$$
H^i(Q_3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \text{ for } i = 2, 3, 4, \text{ and}
$$

$$
H^i(Q_3) \cong 0 \text{ for } i \geq 7.
$$

Proof. By a result of Artin-Verdier-Cox [C, Theorem 2.1, the étale 2-cohomological dimension of a real variety of dimension $d$ with no $\mathbb{R}$-rational points is $\leq 2d$. Hence $H^i(Q_3) = 0$ for $i \geq 7$. Clearly $H^0(Q_3) \cong \mathbb{Z}/2$. By the Kummer exact sequence (7), we see that $H^1(Q_3) \cong \mathbb{R}^*/\mathbb{R}^{*2} \cong \mathbb{Z}/2$, since $2\text{Pic}(Q_3) \subseteq 2\text{Pic}(C_3) = 0$, and $C_3$ is rational. Further, by the exact sequence (8), we have $H^2(Q_3) \cong \text{Pic}(Q_3)/2 \oplus 2\text{Br}(Q_3)$. It is well-known that the Picard group $\text{Pic}(C_3)$ is isomorphic to $\mathbb{Z}$, the generator being the class of a hyperplane section. For any proper geometrically integral variety $X/\mathbb{R}$, we have an exact sequence (cf. [CT-S, 1.5.0, p.383]

$$
0 \rightarrow \text{Pic}(X) \rightarrow (\text{Pic}(X_\mathbb{C}))^G \rightarrow \text{Br}(\mathbb{R}) \rightarrow K \rightarrow H^1(G, \text{Pic}(X_\mathbb{C}))
$$

where $G = \text{Gal}(\mathbb{C}/\mathbb{R})$, $X_\mathbb{C} = X \times_\mathbb{R} \mathbb{C}$, $K$ denotes the kernel of the natural map $\text{Br}(X) \rightarrow \text{Br}(X_\mathbb{C})$ and $(\text{Pic}(X_\mathbb{C}))^G$ is the subgroup of elements of $\text{Pic}(X_\mathbb{C})$ fixed under the action of $G$. Since the generator of $\text{Pic}(C_3)$ is defined over $\mathbb{R}$, the inclusion in the above exact sequence gives an isomorphism $\text{Pic}(Q_3) \cong \text{Pic}(C_3) \cong \mathbb{Z}$. Further, we have $\text{Br}(C_3) = 0$, the variety $C_3$ being rational. Clearly, $H^1(G, \text{Pic}(C_3)) = 0$ and hence (17) gives an isomorphism $\text{Br}(\mathbb{R}) \cong \text{Br}(Q_3) \cong \mathbb{Z}/2$. We remark that this computation is valid for $Q_d$, $d \geq 3$ and hence $\text{Br}(Q_d) \cong \mathbb{Z}/2$ for all $d \geq 3$. The exact sequence (16) together with Proposition 2.2 gives an exact sequence

$$
0 \rightarrow H^1(Q_3) \rightarrow H^2(Q_3) \rightarrow H^2(C_3) \rightarrow H^2(Q) \rightarrow H^3(Q_3) \rightarrow 0.
$$

By the above computations, we have $H^1(Q_3) \cong \mathbb{Z}/2$ and $H^2(Q_3) \cong (\mathbb{Z}/2)^2$. The previous proposition and an easy computation of dimensions using the exact sequence (18) yields isomorphisms

$$
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong H^2(Q_3) \cong H^3(Q_3).
$$

We now use a result of Cox [C, Proposition 2.2], which states that if $X$ is a smooth proper real variety of dimension $d$ such that $X(\mathbb{R}) = \emptyset$, then the natural map $H^{2d}(X) \rightarrow H^{2d}(X_\mathbb{C})$ is zero. The cohomology sequence (16) along with Proposition 2.2 gives exact sequences

$$
0 \rightarrow (\mathbb{Z}/2)^2 \rightarrow H^4(Q_3) \rightarrow \mathbb{Z}/2 \rightarrow H^4(Q_3) \rightarrow H^5(Q_3) \rightarrow 0
$$

and

$$
0 \rightarrow H^5(Q_3) \rightarrow H^6(Q_3) \rightarrow H^6(C_3) \rightarrow H^6(Q_3) \rightarrow H^7(Q_3).
$$

Since $H^7(Q_3) = 0$, the result of Cox quoted above gives isomorphisms $H^6(Q_3) \cong H^6(C_3) \cong \mathbb{Z}/2$ (Lemma 2.1) and $H^5(Q_3) \cong H^6(Q_3) \cong \mathbb{Z}/2$. This is used in (20) to see that $H^3(Q_3) \cong H^4(Q_3) \cong (\mathbb{Z}/2)^2$. Hence the lemma is proved. \qed

We need the following result which computes the cohomology group $H^2(Q_2)$ and $\text{Pic}(Q_2)$. The results about the Picard group are well-known (cf. [K]), but we include them for the sake of completeness.
Proposition 2.4. Let $Q_2$ be the 2-dimensional real anisotropic quadric. Then the Kummer sequence induces an isomorphism

$$Pic(Q_2)/2 \cong H^2(Q_2)$$

and both groups are isomorphic to $(\mathbb{Z}/2)^2$. The generators of $Pic(Q_2)$ are given by the class of the hyperplane section $h$ and the class $\zeta$ of the “double line cycle” defined by $X_0^2 + X_1^2 = 0$, $X_0X_3 + X_1X_2 = 0$, $X_2^2 + X_3^2 = 0$.

Proof. Let $C_2 \subset \mathbb{P}_C^3$ be the complex quadric $Q_2 \times_{\mathbb{R}} \mathbb{C}$. After a $\mathbb{C}$-linear change of coordinates in $\mathbb{P}_C^3$, we see that the image of the Segre embedding $\mathbb{P}_C^1 \times \mathbb{P}_C^1 \rightarrow \mathbb{P}_C^3$ maps isomorphically onto the quadric $C_2$. Consider the two one-parameter families of lines given by $(\mathbb{P}_C^1 \times pt)$ and $(pt \times \mathbb{P}_C^1)$. Clearly two lines intersect if and only if they belong to different families. Further, the group $Pic(C_2)$ is generated by the classes of any two intersecting lines. Let $e_1$ (respectively $e_2$) be the line defined by $X_0 + iX_1 = 0$, $X_2 - iX_3 = 0$ (resp. $X_0 + iX_1 = 0$, $X_2 + iX_3 = 0$). Then $Pic(C_2) \cong \mathbb{Z}[e_1] \oplus \mathbb{Z}[e_2]$ and under the action of the Galois group $G = Gal(\mathbb{C}/\mathbb{R}) = (1, \sigma)$, the line $\sigma(e_1)$ (resp. $\sigma(e_2)$) is defined by $X_0 - iX_1 = 0$, $X_2 + iX_3 = 0$ (resp. $X_0 - iX_1 = 0$, $X_2 - iX_3 = 0$) and hence $\sigma(e_i) = [e_i]$, $i = 1, 2$, where $[e_i]$ denotes the class of $e_i$ in $Pic(C_2)$. Thus the Galois group acts trivially on $Pic(C_2)$ and hence $H^1(G, Pic(C_2)) = 0$. The Brauer group $Br(C_2)$ is trivial, since $C_2$ is birational to $\mathbb{P}_C^2$. Thus the sequence (17) gives an exact sequence

$$(22) \quad 0 \rightarrow Pic(Q_2) \rightarrow (Pic(C_2))^G \rightarrow Br(\mathbb{R}) \rightarrow Br(Q_2) \rightarrow 0.$$ 

The natural map $Br(\mathbb{R}) \rightarrow Br(\mathbb{R}(Q_2))$ factors as

$$Br(\mathbb{R}) \rightarrow Br(Q_2) \rightarrow Br(\mathbb{R}(Q_2))$$

where $\mathbb{R}(Q_2)$ is the function field of the quadric $Q_2$, and the second map is an injection since the Brauer group of a smooth variety injects into the Brauer group of its function field [Mi, Ch. IV, Corollary 2.6]. But the quaternion algebra $(-1, -1)/\mathbb{R}$ which is the non-trivial element of $Br(\mathbb{R})$ splits in $Br(\mathbb{R}(Q_2))$, since -1 is a sum of two squares in $\mathbb{R}(Q_2)$. Hence the group $Br(Q_2)$ is trivial and we get an exact sequence

$$0 \rightarrow Pic(Q_2) \rightarrow (Pic(C_2))^G \xrightarrow{\eta} \mathbb{Z}/2 \rightarrow 0.$$ 

Since $[e_1 + e_2]$ is the class of a hyperplane section, it belongs to $Pic(Q_2)$, hence $[e_1]$ and $[e_2]$ map to the same image under $\eta$. By the exact sequence above, this element must be an element $[ae_1 + be_2]$ in $Pic(C_2)$ maps to the class of $(a + b)$ under the map $\eta$. Thus $Pic(Q_2)$ is identified with the subgroup of $\mathbb{Z}[e_1] \oplus \mathbb{Z}[e_2]$ consisting of elements whose sum of coordinates is even, with the generators being given by the class of the hyperplane $[h] = [e_1] + [e_2]$ and the class $[2e_1]$, which corresponds to the “double line cycle” $\zeta$ defined over $\mathbb{R}$ by

$$X_0^2 + X_1^2 = 0, \quad X_0X_3 + X_1X_2 = 0, \quad X_2^2 + X_3^2 = 0.$$ 

Therefore we have $Pic(Q_2) \cong \mathbb{Z}[h] \oplus \mathbb{Z}[\zeta]$. We now use this and the vanishing of $Br(Q_2)$ proved above in the Kummer sequence (8) to get isomorphisms

$$(\mathbb{Z}/2)[h] \oplus (\mathbb{Z}/2)[\zeta] \cong Pic(Q_2)/2 \cong H^2(Q_2).$$

Hence the proposition is proved. \(\square\)

We now prove the injectivity of the cycle map. This is used subsequently to compute the unramified cohomology group $H^0(Q_3, \mathcal{H}^4)$. 

Proposition 2.5. Let $Q_d$ be the real anisotropic quadric of dimension $d$, $d = 2, 3$. Then the cycle map 
\[ \rho_2 : CH^2(Q_d)/2 \to H^4(Q_d) \]

is injective.

Proof. Recall that for a smooth integral variety $X$, we have the spectral sequence (cf. §2)
\[ E_2^{p,q} = H^p_{zar}(X, H^q) \Rightarrow H^p(X). \]

The above spectral sequence gives an exact sequence
\[ H^3(X) \to H^0(X, H^3) \to CH^2(X)/2 \to H^4(X). \]

By 2.e, the map on the extreme right is precisely the cycle map $\rho_2$. If $X$ is the two dimensional quadric $Q_2$, then we have $H^0(Q_2, H^3) \subset H^3(\mathbb{R}(Q_2)) = 0$ by [CT-P, Proposition 1.2.1]. Using this in the exact sequence (23), we see that the cycle map is injective. We now consider the case of the three dimensional quadric $Q_3$ defined by $\sum_{i=0}^{3} X_i^2 = 0$ in $\mathbb{P}^3_{\mathbb{R}}$. The hyperplane section given by $X_4 = 0$ embeds the two dimensional quadric $Q_2$ as a smooth closed subvariety of $Q_3$. Consider the natural map
\[ CH^1(Q_2) \xrightarrow{\phi} CH^2(Q_3). \]

By Karpenko’s computations of the Chow groups of quadrics, we have an isomorphism $CH^2(Q_3) \cong \mathbb{Z}[h^2] \oplus \mathbb{Z}/2[\zeta']$ [K-Me, 1.8], where $h$ is a hyperplane section, $\zeta' = h^2 - \zeta$, and $\zeta$ is the “double line cycle” discussed in Proposition 2.4, considered now as an element in $CH^2(Q_3)$. By Proposition 2.4, we have $CH^1(Q_2) \cong \mathbb{Z}[h] \oplus \mathbb{Z}[\zeta]$. Under the map $\phi$ above, the class $[h]$ maps to $[h^2]$ and $[h - \zeta]$ maps to $[\zeta']$. Therefore the map $\phi$ is surjective and comparing dimensions, we see that the surjective map gives an isomorphism
\[ Pic(Q_2)/2 \cong CH^1(Q_2)/2 \cong CH^2(Q_3)/2 \cong (\mathbb{Z}/2)^2. \]

We now consider the complement $U = Q_3 \setminus Q_2$ which is the three dimensional affine variety defined by $1 + \sum_{i=0}^{3} X_i^2 = 0$ in $\mathbb{A}_\mathbb{R}^4$. We have the exact sequence (cf. (4))
\[ \cdots \to H^i_{Q_2}(Q_3) \to H^i(Q_3) \to H^i(U) \to H^{i+1}_{Q_2}(Q_3) \to \cdots. \]

Since $U\subset \subset U \times \mathbb{R}$ is affine and three dimensional over an algebraically closed field, $H^i(U\subset) = 0$ for $i > 3$ [Mi, Ch. VI, Theorem 7.2]. We now use these results in the long exact sequence corresponding to the étale map $U \subset \to U$ (cf. (16))
\[ \cdots \to H^i(U) \to H^i(U\subset) \to H^i(U) \overset{\cup(-1)}{\to} H^{i+1}(U) \to \cdots \]

to get isomorphisms $H^i(U) \cong H^{i+r}(U)$ for all $i \geq 4$, $r \geq 0$. But by the result of Cox, [C, Theorem 2.1], since $U(\mathbb{R}) = 0$, we have $H^i(U) = 0$ for $i > 6$. We use these results in the exact sequence (25). Identifying the group $H^1_{Q_2}(Q_3)$ with $H^2(Q_3)$ (cf. 2.c), we obtain a diagram
\[
\begin{array}{ccc}
\text{Pic } (Q_2)/2 & \cong & CH^1(Q_2)/2 \\
\downarrow \rho_1 & & \downarrow \rho_2 \\
H^2(Q_2) & \cong & H^4_{Q_2}(Q_3) \rightarrow H^4(Q_3) \rightarrow H^4(U) = 0.
\end{array}
\]
The horizontal map at the top is an isomorphism by (24). We have an isomorphism \(\text{Pic}(Q_2)/2 \cong H^2(Q_2)\) from the Kummer sequence (8) and Proposition 2.4. Further, by the discussion above, we have \(H^4(U) = 0\) and by Lemma 2.3, \(H^4(Q_3) \cong \mathbb{Z}/2\mathbb{Z}\).

We claim that the above diagram commutes. To prove this, we first remark that for a smooth variety \(X/k\), we have a commutative diagram

\[
\begin{array}{ccc}
\text{Pic}(X)/2 & \xrightarrow{\phi} & H^2(X) \\
\downarrow \rho_1 & & \downarrow \rho_2 \\
CH^1(X)/2 & \xrightarrow{\phi} & H^4(X)
\end{array}
\]

where \(\rho_1\) is the cycle map and \(\phi\) is the boundary map given by the Kummer sequence (cf. (8)). This is obvious by the very definition of the cycle map (cf. [Mi, Ch. VI, Remark 9.6]). Further, by 2.f, the diagram

\[
\begin{array}{ccc}
CH^1(Q_2)/2 & \rightarrow & CH^2(Q_3)/2 \\
\downarrow \rho_1 & & \downarrow \rho_2 \\
H^2(Q_2) & \rightarrow & H^4(Q_3)
\end{array}
\]

is commutative. Hence the claim is proved and the left vertical map is an isomorphism in (26). The bottom map is also an isomorphism, being surjective between two groups of the same order. Thus since all maps except the right vertical map are isomorphisms and the square commutes, we see that

\[\rho_2 : CH^2(Q_3)/2 \rightarrow H^4(Q_3)\]

is also an isomorphism. Therefore the proposition is proved. \(\square\)

We now compute the unramified cohomology group \(H^0(Q_3, \mathcal{H}^3)\). We have

**Theorem 2.6.** Let \(Q_3\) be the three dimensional real anisotropic quadric. Then the unramified cohomology group \(H^0(Q_3, \mathcal{H}^3)\) is zero.

**Proof.** As observed in the proof of Proposition 2.5 (cf. (23)), the local-to-global spectral sequence gives an exact sequence

\[H^3(Q_3) \rightarrow H^0(Q_3, \mathcal{H}^3) \rightarrow CH^2(Q_3)/2 \cong H^4(Q_3).
\]

By Proposition 2.5, the map \(\rho_2\) above is injective. Thus to prove the theorem, it suffices to prove that the homomorphism \(H^3(Q_3) \rightarrow H^0(Q_3, \mathcal{H}^3)\) is zero. We have a commutative diagram as follows:

\[
\begin{array}{ccc}
H^2(Q_3) & \xrightarrow{\cup(-1)} & H^3(Q_3) \\
\downarrow & & \downarrow \\
H^0(Q_3, \mathcal{H}^2) & \xrightarrow{\cup(-1)} & H^0(Q_3, \mathcal{H}^3).
\end{array}
\]

The group \(H^0(Q_3, \mathcal{H}^2)\), which is by definition (cf. §1) the kernel of the homomorphism \(H^2(\mathbb{R}(Q_3)) \xrightarrow{\cup} \oplus_{x \in Q_3} H^1(\mathbb{R}(x))\), is isomorphic to \(2\text{Br}(Q_3)\), the 2-torsion subgroup of the Brauer group, by the Purity theorem for Brauer groups (cf. [Gr, §6...]}
and 7]. The proof of Lemma 2.3 (cf. argument following (17)) gives isomorphisms $\mathbb{Z}/2 \cong \text{Br}(Q_3) \cong H^0(Q_3, \mathcal{H}^2)$, the non-trivial element being the image of the quaternion algebra $(-1, -1)/\mathbb{R}$, under the natural map $\text{Br}(\mathbb{R}) \rightarrow \text{Br}(Q_3)$. Again, by the proof of Lemma 2.3, the top horizontal map is an isomorphism. Now, since the element -1 is a sum of four squares in $\mathbb{R}(Q_3)$, the form $\langle 1, 1, 1 \rangle \in I^3(\mathbb{R}(Q_3))$ is isotropic and hence hyperbolic. The maps $e_n$ are known to be well-defined on $n$-fold Pfister forms (cf. [AEJ]). Since $\langle 1, 1, 1 \rangle$ is isotropic, this implies that $e_3(\langle 1, 1, 1 \rangle) = (-1) \cup (-1) \cup (-1)$ is zero in $H^3(\mathbb{R}(Q_3))$. Thus since $H^0(Q_3, \mathcal{H}^2) \cong \mathbb{Z}/2 < (-1) \cup (-1) >$, the bottom horizontal map in the above commutative square is zero and hence the map $H^3(Q_3) \rightarrow H^0(Q_3, \mathcal{H}^3)$ is zero. Therefore the theorem is proved. □

We now record some corollaries to the theorem.

**Proposition 2.7.** Let $Q_d$ be the real anisotropic quadric of dimension $d$ and $\mathbb{R}(Q_d)$ be its function field. Then we have $H^3_{nr}(\mathbb{R}(Q_d)) = 0$ for $1 \leq d \leq 6$ and $H^3_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/2$ for $d \geq 7$, the non-trivial element being $(-1) \cup (-1) \cup (-1)$.

**Proof.** Clearly $H^0(\mathbb{R}(Q_d), \mathcal{H}^3) = H^3_{nr}(\mathbb{R}(Q_d)) = 0$ for $d \leq 2$, since the 2-cohomological dimension of $\mathbb{R}(Q_d)$ is $d$ [CT-P, Proposition 1.2.1], and $H^0(\mathbb{R}(Q_d), \mathcal{H}^3) \subset H^3(\mathbb{R}(Q_d))$. The quadrics $Q_d$, $3 \leq d \leq 6$ are defined by quadratic forms that are Pfister neighbors of the 3-fold Pfister form $\langle 1, 1, 1 \rangle$ which defines the quadric $Q_6$. By Lemma 1.3 therefore, the unramified cohomology groups of $Q_d$ for $3 \leq d \leq 6$ are equal and by the above theorem, $H^0(Q_d, \mathcal{H}^3) = 0$ for $3 \leq d \leq 6$.

We now assume that $d \geq 7$. For any field $F$, char $F \neq 2$, there exists an exact sequence [A]

$$\begin{align*}
0 \rightarrow \mathbb{Z}/2 < (-1) \cup (-1) \cup (-1) > & \rightarrow H^3(F) \rightarrow H^3(F(Q_6))
\end{align*}$$

where $F(Q_6)$ is the function field of the quadric $\sum_{i=1}^{7} X_i^2 = 0$ in $\mathbb{P}_F^7$. In particular, if $F = \mathbb{R}(Q_d)$, $d \geq 7$, restriction of the above exact sequence to the unramified cohomology groups gives an exact sequence

$$\begin{align*}
0 \rightarrow \mathbb{Z}/2 < (-1) \cup (-1) \cup (-1) > & \rightarrow H^3_{nr}(\mathbb{R}(Q_d)) \rightarrow H^3_{nr}(\mathbb{R}(Q_d \times Q_6)).
\end{align*}$$

But $\mathbb{R}(Q_d \times Q_6)$ is a purely transcendental extension of $\mathbb{R}(Q_6)$. (In general, $\mathbb{R}(Q_m \times Q_n)$ is a purely transcendental extension of $\mathbb{R}(Q_m)$ when $n \geq m$, since the quadric $Q_n$ has an $\mathbb{R}(Q_m)$-rational point if $n \geq m$.) Since the unramified cohomology groups remain invariant under purely transcendental extensions, we see that $H^3_{nr}(\mathbb{R}(Q_6)) \cong H^3_{nr}(\mathbb{R}(Q_d \times Q_6)) = 0$, and the above sequence yields an isomorphism $H^3_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/2 < (-1) \cup (-1) \cup (-1) >$. We claim that the element $(-1) \cup (-1) \cup (-1)$ is non-zero in $H^3(\mathbb{R}(Q_d))$. To prove this, we use the Merkurjev-Suslin-Rost isomorphism [M-S1] $K_3(F)/2 \cong H^3(F)$, where for a field $F$, char $F \neq 2$, $K_n(F)$ is the $n$-th Milnor $K$-group of $F$ and the isomorphism is obtained by mapping the class of the symbol \{a, b, c\} in $K_3(F)/2$ to $(a) \cup (b) \cup (c)$ in $H^3(F)$. Now $(-1) \cup (-1) \cup (-1) = 0$ in $H^3(\mathbb{R}(Q_d))$ implies that $\{-1, -1, -1\}$ belongs to $2K_3(\mathbb{R}(Q_d))$ and this in turn implies that -1 is a reduced norm of the quaternion algebra $(-1, -1)/\mathbb{R}(Q_d)$ [M-S, Theorem 12.1]. But this is impossible as the element -1 is not a sum of four squares in $\mathbb{R}(Q_d)$ if $d \geq 7$ (Pfister, cf. [Sc, Ch. IV, Theorem 4.3]). Hence $H^0(Q_d, \mathcal{H}^3) \cong \mathbb{Z}/2$ and the proposition is proved. □

We also have the following
Corollary 2.8. Let $Q_d$ be the real anisotropic quadric of dimension $d$, $d \geq 2$. Then the cycle map 
\[ \rho_2 : CH^2(Q_d)/2 \to H^4(Q_d, \mathbb{Z}/2) \]
is injective.

Proof. We use the exact sequence (23). For $2 \leq d \leq 6$, by Proposition 2.7, 
$H^0(Q_d, \mathcal{H}^3) = 0$ and hence the result holds. For $d \geq 7$, we have 
$H^0(Q_d, \mathcal{H}^3) \cong \mathbb{Z}/2$, the non-trivial element being $(-1) \cup (-1) \cup (-1)$ which is the image of the non-zero element in $H^3(\mathbb{R})$ under the natural map $H^3(\mathbb{R}) \to H^3(Q_d)$. This implies that the map $H^3(Q_d) \to H^0(Q_d, \mathcal{H}^3)$ in (23) is surjective and hence the result follows. □

§3. Unramified Witt groups of $Q_d$, $3 \leq d \leq 6$.

In this section, we prove that $W_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/8$ for $3 \leq d \leq 6$. We continue with the notation used in the previous sections. Associated to the $d$-dimensional quadric $Q_d$ is its defining quadratic form of dimension $d+2$, given by $(d+2) < 1$, which we denote by $\phi_d$. Recall that the level of a field $F$ is defined to be the least integer $n$ such that $-1$ is a sum of $n$ squares in $F$. A field $F$ is said to be non-formally real if its level is finite. The field $\mathbb{R}(Q_n)$ is non-formally real and has level $2^l n$, where $2^l n < n + 1 < 2^l n + 1$ (Pfister, cf. [Sc, Ch. IV, Theorem 4.3]). Thus the quadrics $Q_n$ can be grouped as follows:

| Level of the function field $2^l n = 1$ | Quadrics $Q_0$ |
| 2 \|$Q_1, Q_2$ |
| 2 \$Q_3, \cdots, Q_6$ |
| $\vdots$ | $\vdots$ |
| $2^k$ | $Q_{2^k-1}, \cdots, Q_{2^k+1-2}$ |
| $\vdots$ | $\vdots$ |

Let $B_k$ denote the bunch of quadrics whose function fields are of level $2^k$. The quadratic form $\phi_{2^k+1-2}$, attached to the “topmost” quadric $Q_{2^k+1-2}$ of $B_k$ is the $(k+1)$-fold Pfister form $<1, 1, \cdots, 1>$ and those defining all the other quadrics in $B_k$ are Pfister neighbours of $\phi_{2^k+1-2}$. By Lemma 1.2, we see that the unramified Witt group is an invariant of the bunch $B_k$. We prove the following well-known lemma, first proved by Arason [A1], and which computes the unramified Witt group of the anisotropic conic. The proof below was outlined to us by Parimala and is valid over any ground field $k$, char $k \neq 2$.

Lemma 3.1. Let $C/k$ be a smooth projective anisotropic conic and $k(C)$ be its function field. Then any unramified quadratic form $\phi$ over $k(C)$ comes from a quadratic form over $k$. In particular, if a quadratic form $\phi$ over $k$ becomes isotropic over $k(C)$, then the anisotropic part of $\phi \otimes k(C)$ is defined over $k$.

Proof. We shall argue by induction on the rank of the unramified quadratic form $\phi$, the property being clearly true if the rank is zero. If $\phi = \phi' \perp \mathbb{H}$ over $k(C)$, where $\phi'$ is the anisotropic part, $\mathbb{H}$ denotes the hyperbolic form $<1, -1>$ (defined
over $k$) and $n$ is positive, then $\phi'$ is unramified of rank smaller than $\phi$, hence the result follows from the induction hypothesis. We may therefore assume that $\phi$ is anisotropic.

For every closed point $x \in C$, we have $\partial_x(\phi) = 0$, where $\partial_x$ is the second residue homomorphism (cf. §1) corresponding to the point $x$. This implies (cf. [Se, Ch. VI, §2]) that given any closed point $x \in C$, there exists a regular quadratic space $E_x$ over the discrete valuation ring $\mathcal{O}_{C,x}$, whose image under the natural map $\mathcal{O}_{C,x} \hookrightarrow k(C)$ coincides with $\phi$. By an easy patching theorem (valid over any integral Dedekind scheme) we get an anisotropic quadratic bundle $E$ on $C$ such that the restriction of $E$ at a closed point $x \in C$ is $E_x$ and $E \otimes k(C) \cong \phi$.

Given such a bundle $E$, since $E \cong E^*$, the degree of the bundle $E$ denoted $\text{deg } E$, which is the degree of the determinant bundle of $E$, is zero. Hence by the Riemann-Roch theorem, we have

$$h^0(E) \geq \text{deg } E + (\text{rk } E)(1 - g(C)) \geq 1,$$

where $h^0(E)$ is the dimension of the $k$-vector space of global sections of $E$, $\text{rk}$ is the rank and $g(C)$ is the genus of $C$ which is zero. This implies that the bundle $E$ has a non-trivial global section, say $s$. The bilinear structure gives a homomorphism

$$b_0 : \Gamma(C, E) \times \Gamma(C, E) \to \Gamma(C, \mathcal{O}_C)$$

where for a sheaf $\mathcal{G}$ on $C$, $\Gamma(C, \mathcal{G})$ is the $k$-vector space of global sections of $\mathcal{G}$ and $\mathcal{O}_C$ is the structure sheaf. Since $C$ is smooth, projective, and geometrically integral, the space $\Gamma(C, \mathcal{O}_C)$ is isomorphic to $k$. Hence the evaluation of the section gives a splitting $E \cong < a > \oplus E'$ as bilinear spaces, with $0 \neq a = b_0(s,s) \in k$ since $E \otimes k(C)$ is anisotropic. By the induction hypothesis, $E' \otimes k(C)$ comes from $k$, and the lemma is proved. □

**Remark.** Let $K/k$ be a field extension. If given any anisotropic quadratic form $q$ over $k$, the anisotropic part of $q_K = q \otimes K$ comes from a quadratic form defined over $k$, the field $K$ is called an excellent extension of $k$. The excellence of the function field of an anisotropic conic over $k$ has been proved by various people. We refer to [VG] and the references cited there for a more extensive treatment of this property.

**Corollary 3.2.** (Parimala [P]) Let $Q_d$ be the real anisotropic quadric over $\mathbb{R}$, $d = 1, 2$. Then we have $W_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/4$, with the form $< 1 >$ as the generator.

**Proof.** By the remark at the beginning of this section, it suffices to prove the result for the conic $Q_1$. By the above lemma, the elements of $W_{nr}(\mathbb{R}(Q_1))$ are defined over $\mathbb{R}$, i.e. the natural map $W(\mathbb{R}) \to W_{nr}(\mathbb{R}(Q_1))$ is surjective. But $W(\mathbb{R}) \cong \mathbb{Z}$ with the form $< 1 >$ as the generator. The form $4 < 1 > = < 1, 1 >$ is isotropic and hence hyperbolic, since $-1$ is a sum of two squares in $\mathbb{R}(Q_1)$. But $2 < 1 > = < 1, 1 >$ is not zero in $W(\mathbb{R}(Q_1))$ since $-1$ is not a square in $\mathbb{R}(Q_1)$. Hence $W_{nr}(Q_1) \cong \mathbb{Z}/4$. The result for $Q_2$ follows from Lemma 1.2. □

We now prove the main theorem.

**Theorem 3.3.** Let $Q_d$ be the $d$-dimensional real anisotropic quadric, $3 \leq d \leq 6$. Then $W_{nr}(\mathbb{R}(Q_d)) \cong \mathbb{Z}/8$, the form $< 1 >$ being the generator.

**Proof.** By the remarks at the beginning of this section, it suffices to prove the theorem for the three dimensional anisotropic quadric $Q_3$, since the quadrics $Q_d$
with $3 \leq d \leq 6$ are all in $B_2$. By Proposition 1.1, there is an exact sequence of groups

$$0 \to W(\mathbb{R}(Q_1)) \ll 1, 1 \gg \to W(\mathbb{R}(Q_3)) \to W(\mathbb{R}(Q_3 \times Q_2))$$

(30)

We remark that the kernel $W(\mathbb{R}(Q_3)) \ll 1, 1 \gg$ is a 2-torsion group, since the level of $\mathbb{R}(Q_3)$ is 4 and hence $2. \ll 1, 1 \gg = \ll 1, 1 \gg$ is hyperbolic. Restricting the above exact sequence to the unramified subgroups, we get an exact sequence

(31)

$$0 \to |W(\mathbb{R}(Q_3)) \ll 1, 1 \gg| \cap W_{nr}(\mathbb{R}(Q_3)) \to W_{nr}(\mathbb{R}(Q_3)) \to W_{nr}(\mathbb{R}(Q_3 \times Q_2)).$$

But $\mathbb{R}(Q_3 \times Q_2) = \mathbb{R}(Q_3)(Q_2) = \mathbb{R}(Q_2)(Q_3)$ is a purely transcendental extension of $\mathbb{R}(Q_2)$ and hence $W_{nr}(\mathbb{R}(Q_3 \times Q_2)) \cong W_{nr}(\mathbb{R}(Q_2)) \cong \mathbb{Z}/4 < 1$, by Corollary 3.2. We claim that $|W(\mathbb{R}(Q_3)) \ll 1, 1 \gg| \cap W_{nr}(\mathbb{R}(Q_3)) \cong \mathbb{Z}/2 \ll 1, 1 \gg$. But $I(\mathbb{R}(Q_3)) \subseteq W(\mathbb{R}(Q_3))$ and the quotient $W(\mathbb{R}(Q_3))/I(\mathbb{R}(Q_3)) \cong \mathbb{Z}/2$, with the non-trivial element being the class of the form $< 1 >$. Therefore to prove the claim, it suffices to prove that the group

$$|I(\mathbb{R}(Q_3)) \ll 1, 1 \gg| \cap W_{nr}(\mathbb{R}(Q_3)) = 0.$$

But this group is a subgroup of the group $I_{nr}(\mathbb{R}(Q_3))$. Further, $I^4(\mathbb{R}(Q_3)) = 0$, this observation being a consequence of the general result (cf. [AEJ]) that if $F$ is a field of transcendence degree $m$ over $\mathbb{R}$, then $I^{m+1}(F)$ is torsion-free. In our situation, since $\mathbb{R}(Q_3)$ is non-formally real, the group $W(\mathbb{R}(Q_3))$ is torsion [Sc, Ch. II, Theorem 7.1]. Also, as mentioned in §1, the maps $\tau_n : I^n(F)/I^{n+1}(F) \to H^n(F)$ are well-defined isomorphisms for $n = 2, 3$. Thus from these remarks and (3) of §1, we see that there is an injection $I_{nr}^3(\mathbb{R}(Q_3)) \hookrightarrow H_{nr}^3(\mathbb{R}(Q_3))$. But the unramified cohomology group $H_{nr}^3(\mathbb{R}(Q_3)) \cong H^3(Q_3, \mathbf{G}_m)$ is torsion [Proposition 2.5]. The element $\ll 1, 1 \gg$ is not zero in $W(\mathbb{R}(Q_3))$, since $-1$ is a sum of no less than four squares in $\mathbb{R}(Q_3)$. Thus, the exact sequence (31) reads as

$$0 \to \mathbb{Z}/2 \ll 1, 1 \gg \cap W_{nr}(\mathbb{R}(Q_3)) \to \mathbb{Z}/4 < 1 > \to < 1 >$$

But the form $< 1 > \in W_{nr}(\mathbb{R}(Q_3))$ is an element of order 8 since the level of $\mathbb{R}(Q_3)$ is 4, and hence the above sequence is short exact, non-split and $W_{nr}(\mathbb{R}(Q_3)) \cong \mathbb{Z}/8$ as was to be proved. \hfill \Box

Finally, we note the following Corollary to Theorem 3.3. Given a finitely generated field extension $K/k$, an element $f \in K^*$ is said to be unramified if the form $< f >$ belongs to $W_{nr}(K/k)$. Let $Q_d$ be the real anisotropic quadric of dimension $d$, $3 \leq d \leq 6$ and $\mathbb{R}(Q_d)$ be its function field. Since the level of $\mathbb{R}(Q_d)$ is 4, the form $< 1, 1, 1, 1 >$ is isotropic and hence universal, i.e. it represents every element in $\mathbb{R}(Q_d)^*$.

This implies that any element $f \in \mathbb{R}(Q_d)^*$ is a sum of five squares. If further $f$ is unramified, then we have the following

**Corollary 3.4.** Let $f \in \mathbb{R}(Q_d)$, $3 \leq d \leq 6$ be unramified. Then $f$ can be represented as a sum of four squares in $\mathbb{R}(Q_d)$. Also, $-1$ is unramified and cannot be represented by less than four squares.
Proof. If $f$ is unramified then the form $\langle 1, 1, -f \rangle \in I^3(\mathbb{R}(Q_d))$ is actually an element of $I^3_{nr}(\mathbb{R}(Q_d))$. The homomorphism $e_3$ maps $I^3_{nr}(\mathbb{R}(Q_d))$ into $H^3(Q_d, \mathcal{H}^d)$ and the latter group is zero by Proposition 2.7. Thus $e_3(\langle 1, 1, -f \rangle)$ is zero. But the homomorphism $e_3$ is injective on Pfister forms [A2, Proposition 2]. Hence the form $\langle 1, 1, -f \rangle$ is hyperbolic and $f$ can be represented as a sum of four squares. Further, the element $-1$, which is clearly unramified, is a sum of no less than four squares in $\mathbb{R}(Q_d)$. Thus the corollary is proved. □

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References


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