THE BOREL FUNCTIONAL CALCULUS FOR SELFADJOINT OPERATORS
AKA THE SPECTRAL THEOREM

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1. Preparations

We first collect various auxilliary results.

Lemma 1.1. *[Borel lemma] Let $(a_n)_{n\in\mathbb{N}}$ a sequence of complex numbers. Then there exist $f \in C_0^\infty(\mathbb{R})$ such that $f^{(n)}(0) = a_n$ for all $n \in \mathbb{N}$.*

Proof. fix a function $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(x) = 1$ near 0. Then on can take:

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \chi(\epsilon_n^{-1} x)$$

One can see that $f \in C_0^\infty(\mathbb{R})$ if the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ converges fast enough to 0. □

Lemma 1.2. *Let $\mathcal{H}$ a Hilbert space and $\{R_i\}_{i\in I}$ a family in $B(\mathcal{H})$ such that*

$$\lim_i (u|R_i v) \text{ exists } \forall u, v \in \mathcal{H}.$$

Then there exists a unique $R \in B(\mathcal{H})$ such that

$$\lim_i (u|R_i v) = (u|Rv) \ \forall u, v \in \mathcal{H}.$$

Proof. apply twice the uniform boundedness principle. □

Theorem 1.3. *[Riesz-Markov theorem] Let X be a locally compact Hausdorff space, and $C_0(X)$ the space of compactly supported continuous functions on X, equipped with the sup norm. Let $\Lambda : C_0(X) \to \mathbb{C}$ a positive linear form, ie such that $\langle\Lambda|f\rangle \geq 0$ if $f \geq 0$. Then there exists a unique regular Borel measure μ on X such that*

$$\langle\Lambda|f\rangle = \int_X f d\mu, \ \forall f \in C_0(X). \tag{1.1}$$

We recall that μ is said regular if

$$\mu(E) = \inf\{\mu(U) : E \subset U \text{ open}\} = \sup\{\mu(K) =: E \supset K \text{ compact}\} \text{ for } E \text{ Borel set.}$$

We recall that if Λ is positive, then Λ is continuous on $C_0(X)$. If we assume only that Λ is a continuous linear form on $C_0(X)$, then there exists a unique complex Borel measure μ such that (1.1) holds. We have then

$$\mu = \sum_{k=0}^{3} i^k \mu_k, \ \mu_k \text{ (positive) Borel measure.}$$

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Theorem 1.4. *[Functional version of the monotone class theorem] Let X a set, C a class of real, bounded functions on X which is an algebra for the pointwise product. Let $\mathcal{C}$ the σ-algebra generated by C, ie the smallest σ-algebra such that all functions of C are $\mathcal{C}$ measurable.*

Let A another class of bounded real functions such that
(1) $C \subset A$,
(2) *A is closed under sequential bounded convergence.*

Then A contains all the $\mathcal{C}$ measurable bounded real functions.

We recall that a sequence (f_n) *converges boundedly* to f, denoted by $f_n \underset{b}{\to} f$ if $f_n \to f$ pointwise and $\sup_n \sup_x |f_n(x)| < \infty$.

2. The smooth functional calculus

Let H a selfadjoint operator on a Hilbert space $\mathcal{H}$. For $z \in \mathbb{C}$ we have

$$\|(H-z)u\|^2 = \|(H - \mathrm{Re}z)u\|^2 + |\mathrm{Im}z|^2\|u\|^2, \ u \in \mathrm{Dom}\, H.$$

This obviously implies that $(H - z) : \mathrm{Dom}\, H \to \mathcal{H}$ is injective if $z \in \mathbb{C} \setminus \mathbb{R}$. Using that H is closed, we also obtain that $\mathrm{Ran}(H - z)$ is closed. Since $\mathrm{Ran}(H - z)^\perp = \mathrm{Ker}(H - \overline{z}) = \{0\}$, we obtain that

$$(H-z) : \mathrm{Dom}\, H \to \mathcal{H}$$

is an isomorphism with

$$\|(H-z)^{-1}\|_{B(\mathcal{H})} \le |\mathrm{Im}z|^{-1}, \ z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.1}$$

2.1. **Almost analytic extensions.**

Definition 2.1. *Let $f \in C_0^\infty(\mathbb{R})$, $N \in \mathbb{N}$. A function $\tilde{f} \in C_0^\infty(\mathbb{C})$ is an N*-th order almost analytic extension *of f if*

$$\tilde{f}_{|\mathbb{R}} = f, \ |\frac{\partial f}{\partial \overline{z}}(z)| \le C|\mathrm{Im}z|^N. \tag{2.2}$$

If $\tilde{f}$ is an N-th order almost analytic extension of f for any N, it will be called an almost analytic extension *of f*

Lemma 2.2. *Any $f \in C_0^\infty(\mathbb{R})$ admits (N-th order) almost analytic extensions.*

Proof. For $N \in \mathbb{N}$ we take for $z = x + \mathrm{i}y$:

$$\tilde{f}(z) = (\sum_{n=0}^{N} \mathrm{i}^n \partial_x^n f(x) \frac{y^n}{n!})\chi(y), \tag{2.3}$$

for $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ near 0. A routine computation shows that $\tilde{f}$ is an N-th order a.a.e of f. For $N = \infty$ one has to use the Borel lemma in the y variable. The details are left to the reader. □

In practice N-th order a.a.e for $N \ge 2$ suffice.

Lemma 2.3. *Let $f \in C_0^\infty(\mathbb{R})$ with $\tilde{f}$ an a.a.e of f. Then*

$$f(x) = \frac{1}{2\mathrm{i}\pi} \int_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}(z)(z-x)^{-1} d\overline{z} \wedge dz, \ \forall x \in \mathbb{R}.$$

Proof. since $|(z-x)^{-1}| \le |\mathrm{Im}z|^{-1}$ the integral is convergent using (2.2) and equals

$$\lim_{\epsilon \to 0^+} \frac{1}{2\mathrm{i}\pi} \int_{C_\epsilon} \frac{\partial f}{\partial \overline{z}}(z)(z-x)^{-1} d\overline{z} \wedge dz,$$

where $C_\epsilon = [-R, R] + \mathrm{i}[-R, R] \setminus \{|\mathrm{Im}z| \le \epsilon\}$, where R is chosen large enough so that $\mathrm{supp}\, \tilde{f} \subset [-R, R] + \mathrm{i}[-R, R]$.

We set $\omega = \tilde{f}(z)(z-x)^{-1}dz$, so that $d\omega = \frac{\partial f}{\partial \overline{z}}(z)(z-x)^{-1}d\overline{z}\wedge dz$. By Stokes formula

$$\begin{aligned} & \tfrac{1}{2\mathrm{i}\pi}\int_{C_\epsilon} d\omega = \tfrac{1}{2\mathrm{i}\pi}\int_{\partial C_\epsilon}\omega \\ = & \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}}\tilde{f}(x'+\mathrm{i}\epsilon)(x'+\mathrm{i}\epsilon-x)^{-1}dx' - \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}}\tilde{f}(x'-\mathrm{i}\epsilon)(x'-\mathrm{i}\epsilon-x)^{-1}dx'. \end{aligned}$$

We recall the well-known fact:

$$\delta(x) = \frac{1}{2\mathrm{i}\pi}\left(\frac{1}{x+\mathrm{i}0} - \frac{1}{x-\mathrm{i}0}\right) =: \lim_{\epsilon\to 0^+}\frac{1}{2\mathrm{i}\pi}\left(\frac{1}{x+\mathrm{i}\epsilon} - \frac{1}{x-\mathrm{i}\epsilon}\right) \text{ in } \mathcal{D}'(\mathbb{R}). \tag{2.4}$$

We have by (2.3)

$$\tilde{f}(x'\pm\mathrm{i}\epsilon) = f(x') \pm \mathrm{i}\epsilon f'(x) + O(\epsilon^2),$$

which using (2.4) implies that $\lim_{\epsilon\to 0}\frac{1}{2\mathrm{i}\pi}\int_{C_\epsilon} d\omega = f(x)$. □

2.2. The smooth functional calculus.

Definition 2.4. *Let H a selfadjoint operator on $\mathcal{H}$. For $f\in C_0^\infty(\mathbb{R})$ we set:*

$$f(H) = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\frac{\partial f}{\partial\overline{z}}(z)(z-H)^{-1}d\overline{z}\wedge dz, \tag{2.5}$$

where $\tilde{f}$ is an almost analytic extension of f.

Using (2.1) and (2.2) we see that the integral converges in norm, as soon as $\tilde{f}$ is an a.a.e of f of order greater than 1. The integrand being norm continuous, the integral is defined as the limit of Riemann sums. We have

$$\|f(H)\| \leq C(f),$$

where the constant $C(f)$ depends only on the sup norm of f and f'.

We still need to check that the rhs in (2.5) does not depend on the choice of $\tilde{f}$, ie that if $\tilde{f}_{|\mathbb{R}} = 0$, then the rhs on (2.5) vanishes. To do this we compute

$$\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\frac{\partial\tilde{f}}{\partial\overline{z}}(z)(u|(z-H)^{-1}v)d\overline{z}\wedge dz,$$

for $u, v\in\mathcal{H}$. The function $g(z) = (u|(z-H)^{-1}v)$ is holomorphic in $\mathbb{C}\setminus\mathbb{R}$, and the argument used in the proof of Lemma 2.3 show that $\int_{\mathbb{C}}\frac{\partial f}{\partial\overline{z}}(z)g(z)d\overline{z}\wedge dz = 0$, since $\tilde{f}_{|\mathbb{R}} = 0$.

We recall that the space $C_0^\infty(\mathbb{R})$ equipped with the usual product and complex conjugation is a $*$-algebra.

Theorem 2.5. *[The smooth functional calculus] The map γ*

$$C_0^\infty(\mathbb{R})\ni f\mapsto\gamma(f) = f(H)\in B(\mathcal{H})$$

is a morphism of $*$-algebras.

Proof. The fact that γ is $\mathbb{C}$-linear is obvious, since $\lambda\tilde{f}+\tilde{g}$ is an a.a.e. of $\lambda f + g$. Next $\tilde{g}(z) = \overline{\tilde{f}(\overline{z})}$ is an a.a.e. of $\overline{f}$ with

$$\frac{\partial\tilde{g}}{\partial\overline{z}}(z) = \overline{\frac{\partial\tilde{f}}{\partial z}(\overline{z})}.$$

Hence using that $H = H^*$ we get:

$$\begin{aligned} f(H)^* = & -\tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\overline{\tfrac{\partial\tilde{f}}{\partial\overline{z}}(z)}(\overline{z}-H)^{-1}dz\wedge d\overline{z} \\ = & \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\overline{\tfrac{\partial\tilde{f}}{\partial z'}(\overline{z}')}(z'-H)^{-1}d\overline{z}'\wedge dz' \\ = & \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\tfrac{\partial\tilde{g}}{\partial\overline{z}'}(z')(z'-H)^{-1}d\overline{z}'\wedge dz' = \overline{f}(H), \end{aligned}$$

where we use the change of variables $z' = \overline{z}$ (which reverses the orientation).

Let now $f_1, f_2 \in C_0^\infty(\mathbb{R})$ with a.a.e. $\tilde{f}_1, \tilde{f}_2$. We have:

$$\begin{aligned}
f_1(H)f_2(H) = & \ (\tfrac{1}{2\mathrm{i}\pi})^2 \int_{\mathbb{C}^2} \partial_{\overline{z}_1}\tilde{f}_1(z_1)\partial_{\overline{z}_2}\tilde{f}_2(z_2)(z_1-H)^{-1}(z_2-H)^{-1} d\overline{z_1}\wedge dz_1 \wedge d\overline{z}_2 \wedge dz_2 \\
= & \ \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \tfrac{\partial \tilde{f}_1}{\partial \overline{z}_1}(z_1)(z_1-H)^{-1}\left(\tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}_2}\tilde{f}_2(z_2)(z_2-z_1)^{-1}d\overline{z}_2\wedge dz_2\right) d\overline{z}_1\wedge dz_1 \\
& + \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \tfrac{\partial \tilde{f}_2}{\partial \overline{z}_2}(z_2)(z_2-H)^{-1}\left(\tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}_1}\tilde{f}_1(z_1)(z_1-z_2)^{-1}d\overline{z}_1\wedge dz_1\right) d\overline{z}_2\wedge dz_2 \\
= & \ \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \tfrac{\partial \tilde{f}_1}{\partial \overline{z}_1}(z_1)\tilde{f}_2(z_1)(z_1-H)^{-1}d\overline{z}_1\wedge dz_1 \\
& + \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \tfrac{\partial \tilde{f}_2}{\partial \overline{z}_2}(z_2)\tilde{f}_1(z_2)(z_2-H)^{-1}d\overline{z}_2\wedge dz_2 \\
= & \ (f_1f_2)(H),
\end{aligned}$$

where we use successively that

$$(z_1-H)^{-1}(z_2-H)^{-1} = (z_2-z_1)^{-1}(z_1-H)^{-1} + (z_1-z_2)^{-1}(z_2-H)^{-1},$$

ie the first resolvent identity, then the fact that

$$\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{f}(z)(z-z')^{-1}d\overline{z}\wedge dz = \tilde{f}(z'),$$

which is proved as in Lemma 2.3, and finally the fact that $\tilde{f}_1\tilde{f}_2$ is an a.a.e. of f_1f_2.
□

The $*$-algebra $C_0^\infty(\mathbb{R})$ has no unit, but it is easy to add one by considering $C_0^\infty(\mathbb{R})\oplus\mathbb{C}$, identified with the algebra of smooth functions, constant outside a compact set. If $f \in C_0^\infty(\mathbb{R})\oplus\mathbb{C}$, we uniquely write f as $f_0+\lambda$, $f_0\in C_0^\infty(\mathbb{R})$ and $\lambda\in\mathbb{C}$ and set

$$f(H) =: f_0(H) + \lambda 1\!\!1.$$

We easily obtain the following extension of Thm. 2.5.

Corollary 2.6. *The map* γ

$$C_0^\infty(\mathbb{R})\oplus\mathbb{C} \ni f \mapsto \gamma(f) = f(H) \in B(\mathcal{H})$$

is a morphism of $*$*-algebras.*

2.3. **Some further properties.** Three more properties of the smooth functional calculus are important.

Proposition 2.7. *Let* $\chi \in C_0^\infty(\mathbb{R})$ *with* $\chi(0)=1$ *and set* $\chi_R(x)=\chi(R^{-1}x)$*. Then*

$$\mathrm{s-}\lim_{R\to+\infty}\chi_R(H) = 1\!\!1.$$

Proof. Let $\tilde{\chi}$ an a.a.e. of χ. The function $\tilde{\chi}_R(z)=\tilde{\chi}(R^{-1}z)$ is an a.a.e of χ_R. A change of variables shows that

$$\chi_R(H) = \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\frac{\partial\chi}{\partial\overline{z}}(z)(z-H/R)^{-1}d\overline{z}\wedge dz,$$

which implies that

$$\|\chi_R(H)\| \leq C, \ \forall R\geq 1. \tag{2.6}$$

On the other hand using $(z-H/R)^{-1} = z^{-1} - z^{-1}(z-H/R)^{-1}H/R$, we obtain

$$\begin{aligned}
\chi_R(H) = & \ \tfrac{1}{2\mathrm{i}\pi}\int_{\mathbb{C}}\tfrac{\partial\chi}{\partial\overline{z}}(z)z^{-1}d\overline{z}\wedge dz + A_RH, \\
A_R = & \ \tfrac{1}{2\mathrm{i}\pi R}\int_{\mathbb{C}}\tfrac{\partial\chi}{\partial\overline{z}}(z)z^{-1}(z-H/R)^{-1}d\overline{z}\wedge dz.
\end{aligned}$$

The first term equals $\chi(0)1\!\!1 = 1\!\!1$, by Lemma 2.3. Using (2.2) we obtain that $\|A_R\|\in O(R^{-1})$. This implies that $\lim_{R\to\infty}\chi_R(H)u = u$ first for $u\in\mathrm{Dom}\,H$ and then for all $u\in\mathcal{H}$ using a density argument and the uniform bound (2.6). □

Proposition 2.8. *Let* $f\in C_0^\infty(\mathbb{R})$ *with* $f\geq 0$*. Then* $f(H)\geq 0$*.*

Proof. For $\epsilon > 0$ $(f+\epsilon)^{\frac{1}{2}} \in C_0^\infty(\mathbb{R}) \oplus \mathbb{C}$ (this is not true for $\epsilon = 0$). Therefore

$$(u|f(H)u) = \lim_{\epsilon\to 0^+}(u|(f(H)+\epsilon)u) = \lim_{\epsilon\to 0^+}\|(f+\epsilon)^{\frac{1}{2}}u^2 \geq 0. \ \Box$$

Proposition 2.9. *One has*

$$\|f(H)\| \leq \|f\|_\infty, \ \forall f \in C_0^\infty(\mathbb{R})$$

Proof. We have for χ as in Prop. 2.7:

$$\begin{aligned} \|f(H)u\|^2 = &\ \ (u||f|^2(H)u) = \|f\|_\infty^2\|u\|^2 - (u|(\|f\|_\infty^2 - |f|^2(H))u) \\ = &\ \ \|f\|_\infty^2\|u\|^2 - \lim_{R\to+\infty}(u|(\|f\|_\infty^2 - |f|^2(H))\chi_R(H)u) \\ \leq &\ \ \|f\|_\infty^2\|u\|^2, \end{aligned}$$

using Prop. 2.8 and the fact that $(\|f\|_\infty^2 - |f|^2)\chi_R$ is a positive C_0^∞ function. $\Box$

3. The Borel functional calculus

Lemma 3.1. *For all $u, v \in \mathcal{H}$ there exists a regular complex Borel measure $\mu_{u,v}$ on $\mathbb{R}$ such that*

$$(u|f(H)v) = \int_{\mathbb{R}} f(\lambda)d\mu_{u,v}(\lambda), \ \forall f \in C_0^\infty(\mathbb{R}). \tag{3.1}$$

Proof. By Prop. 2.8 the linear form

$$C_0^\infty(\mathbb{R}) \ni f \mapsto (u|f(H)v) \in \mathbb{C}$$

is continuous for the sup norm. Then one uses $C_0^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ and the Riesz-Markov theorem Thm. 1.3. $\Box$

Let us denote by $B_{\rm b}(\mathbb{R})$ the $*$-algebra of bounded Borel functions on $\mathbb{R}$.

Proposition 3.2. *There exists a unique map γ:*

$$B_{\rm b}(\mathbb{R}) \ni f \mapsto f(H) \in B(\mathcal{H})$$

such that

$$(u|f(H)v) = \int_{\mathbb{R}} f(\lambda)d\mu_{u,v}(\lambda), \ \forall f \in B_{\rm b}(\mathbb{R}), \ u, v \in \mathcal{H}. \tag{3.2}$$

Proof. For $f \in B_{\rm b}(\mathbb{R})$ the rhs of (3.2) defines a sesquilinear form $f(H)$ on $\mathcal{H}$ We will use the monotone class theorem Thm. 1.4 with

$$A = \{f \in B_{\rm b}(\mathbb{R}) : f(H) \text{ is bounded on } \mathcal{H}\}.$$

Let $C = C_0^\infty(\mathbb{R})$. It is easy to see that the σ-algebra generated by $C_0^\infty(\mathbb{R})$ and by $C_0(\mathbb{R})$ are the same, ie the Borel σ-algebra. By Prop. 2.9 we know that $C \subset A$. Let $(f_n) \in A$ such that $f_n \underset{b}{\to} f$. Clearly $(u|f_n(H)v) \to (u|f(H)v)$, using the rhs of (3.2). Applying Lemma 1.2 we obtain that $f(H)$ is bounded, hence $f \in A$, which shows that A is closed under bounded sequential convergence. Therefore A contains all the bounded Borel functions, ie $A = B_{\rm b}(\mathbb{R})$. $\Box$

Theorem 3.3. *[The Borel functional calculus] The map γ:*

$$B_{\rm b}(\mathbb{R}) \ni f \mapsto f(H) \in B_{\rm b}(\mathbb{R})$$

is a morphism of $*$-algebras with

$$f \geq 0 \Rightarrow f(H) \geq 0, \ \|f(H)\| \leq \|f\|_\infty, \ f \in B_{\rm b}(\mathbb{R}).$$

Proof. From (3.2) we see immediately that γ is $\mathbb{C}$-linear and $\overline{f}(H) = f^*(H)$. Let us prove that $(fg)(H) = f(H)g(H)$ for $f, g \in B_{\rm b}(\mathbb{R})$. Let

$$A_1 = \{f \in B_{\rm b}(\mathbb{R}) : (fg)(H) = f(H)g(H) \forall g \in C_0^\infty(\mathbb{R})\}.$$

Clearly $C_0^\infty(\mathbb{R}) \subset A_1$. Let $(f_n) \in A_1$ such that $f_n \underset{b}{\to} f$. Using the rhs of (3.2) we have for $g \in C_0^\infty(\mathbb{R})$:

$$(u|(f_n g)(H)v) = \int_{\mathbb{R}} f_n(\lambda)g(\lambda)d\mu_{u,v}(\lambda) \to \int_{\mathbb{R}} f(\lambda)g(\lambda)d\mu_{u,v}(\lambda) = (u|(fg)(H)v)$$

$$(u|f_n(H)g(H)v) = \int_{\mathbb{R}} f_n(\lambda)d\mu_{u,g(H)v}(\lambda) \to \int_{\mathbb{R}} f(\lambda)d\mu_{u,g(H)v}(\lambda) = (u|f(H)g(H)v).$$

Since $(f_n g)(H) = f_n(H)g(H)$ we obtain that $f \in A_1$. By the monotone class theorem we obtain that $A_1 = B_{\rm b}(\mathbb{R})$.

Let now

$$A_2 = \{f \in B_{\rm b}(\mathbb{R}) : (fg)(H) = f(H)g(H) \forall g \in B_{\rm b}(\mathbb{R})\}.$$

By the previous step $C_0^\infty(\mathbb{R}) \subset A_2$, and by the same argument as above A_2 is closed under bounded sequential convergence, hence $A_2 = B_{\rm b}(\mathbb{R})$.

By (3.1) we obtain that $\mu_{u,u}$ is a positive Borel measure, which shows that $f(H) \geq 0$ if $f \geq 0$. The second statement is proved as in Prop. 2.3. □

Remark 3.4. *One can show that $\gamma : f \mapsto f(H)$ is the unique extension of the map γ in Thm. 2.5 to $B_{\rm b}(\mathbb{R})$ such that if $f_n \underset{b}{\to} f$ then $f(H) = {\rm w-}\lim f_n(H)$.*

3.1. **The traditional formulation.** Usually the spectral theorem for selfadjoint operators is formulated with projection valued measures. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra on $\mathbb{R}$.

Definition 3.5. *A map $E : \mathcal{B}(\mathbb{R}) \to B(\mathcal{H})$ is called a* projection valued measure *if:*

(1) *$E(I)$ is a selfadjoint projection for all $I \in \mathcal{B}(\mathbb{R})$;*
(2) *$E(\emptyset) = 0$, $E(\mathbb{R}) = \mathbb{1}$;*
(3) *if $I_i \cap I_j = \emptyset$ for $i \neq j$ then*

$$E(\cup_{i=0}^\infty I_i) = {\rm s} - \sum_{i=0}^\infty E(I_i),$$

(where the symbol ${\rm s} - \sum$ means that the series is strongly convergent).

(4) *$E(I \cap J) = E(I)E(J)$ for $I, J \in \mathcal{B}(\mathbb{R})$.*

One sets then $E_\lambda = E(]-\infty, \lambda])$ and defines

$$f(H) := \int_{\mathbb{R}} f(\lambda)dE_\lambda, f \in B_{\rm b}(\mathbb{R}).$$

If the rhs is interpreted as in Subsect. 3.2 below. The relationship between the two formulations is of course that

$$E(I) = \mathbb{1}_I(H),\ I \in \mathcal{B}(\mathbb{R}).$$

3.2. **Extending the Borel functional calculus.** Let $B(\mathbb{R})$ be the $*$-algebra of (arbitrary) Borel functions. It is easy to extend the functional calculus to $B(\mathbb{R})$.

Definition 3.6. *Let $f \in B(\mathbb{R})$. We define the (possibly unbounded) operator $f(H)$ by:*

$$\begin{aligned}{\rm Dom}\, f(H) &= \{u \in \mathcal{H} : \textstyle\int_{\mathbb{R}} |f|^2(\lambda)d\mu_{u,u}(\lambda) < \infty\},\\ (u|f(H)v) &:= \textstyle\int_{\mathbb{R}} f(\lambda)d\mu_{u,v}(\lambda),\ u \in \mathcal{H}, v \in {\rm Dom}\, H.\end{aligned}$$

The following extension of Thm. 3.3 is left for the reader as an exercise.

Theorem 3.7. (1) *the operator $f(H)$ is closed, densely defined;*

(2) *One has*

$$f(H)^* = \overline{f}(H),\ (\lambda f+g)(H) = \lambda f(H) + g(H) \text{ on } \operatorname{Dom} f(H)\cap \operatorname{Dom} g(H);$$

(3) *One has*

$$(fg)(H) = f(H)g(H),$$

with the convention that $\operatorname{Dom} AB = \{u \in \operatorname{Dom} B : Bu \in \operatorname{Dom} A\}$.

3.3. **Some more properties.** It is necessary for consistency to check various natural properties of the functional calculus. Here are two of them.

Proposition 3.8. *Let* $z_0 \in \mathbb{C}\backslash\mathbb{R}$ *and* $f_{z_0}(\lambda) = (z_0-\lambda)^{-1}$. *Then*

$$f_{z_0}(H) = (z_0 - H)^{-1}.$$

Proof. Note that above the right hand side is the resolvent, the left hand side is defined by the Borel functional calculus. It suffices to show that for any $\chi \in C_0^\infty(\mathbb{R})$ one has $\chi(H)f_{z_0}(H) = (z_0-H)^{-1}\chi(H)$ (use that both sides of the equality are bounded operators and that $\chi_R(H) \to \mathbb{1}$ strongly when $R\to+\infty$ if $\chi\in C_0^\infty(\mathbb{R})$ with $\chi(0)=1$). We have

$$\begin{aligned}
&(z_0-H)^{-1}\chi(H) = \tfrac{1}{2\mathrm{i}\pi}\textstyle\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z-H_0)^{-1}(z-H)^{-1}dz\wedge d\overline{z}\\
= &\tfrac{1}{2\mathrm{i}\pi}\textstyle\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z_0-z)^{-1}((z-H)^{-1}-(z_0-H)^{-1})dz\wedge d\overline{z}\\
= &\tfrac{1}{2\mathrm{i}\pi}\textstyle\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z_0-z)^{-1}(z-H)^{-1}dz\wedge d\overline{z}\\
&-(z_0-H)^{-1}\tfrac{1}{2\mathrm{i}\pi}\textstyle\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z_0-z)^{-1}dz\wedge d\overline{z}.
\end{aligned}$$

Noting that $(z_0-z)^{-1}\tilde{\chi}(z)$ is an almost analytic extension of $f_{z_0}\chi$, with $\partial_{\overline{z}}(z_0-z)^{-1}\tilde{\chi}(z) = \tilde{\chi}(z)(z_0-z)^{-1}$, the first integral equals $(f_{z_0}\chi)(H) = f_{z_0}(H)\chi(H)$. The second integral is zero by integration by parts. □

Proposition 3.9. *Let* $f(\lambda)=\lambda$. *Then* $f(H)=H$.

One can show similarly that if $f(\lambda)=\lambda^n$, then $f(H)=H^n$.

Proof. Note that an argument similar to the one in Prop. 3.8 gives:

$$(H-z_0)\chi(H) = g_{z_0}(H),\ g_{z_0}(\lambda) = (\lambda - z_0)\chi(\lambda) \tag{3.3}$$

We first show that $H\subset f(H)$. Let $\chi\in C_0^\infty(\mathbb{R})$ with $\chi(0)=1$ and $\lambda\chi'(\lambda)\geq 0$. If $u\in\operatorname{Dom} H$ we have

$$\sup_{n\in\mathbb{N}} \|H\chi(n^{-1}H)u\|^2 < \infty.$$

Since $\|H\chi(n^{-1}H)u\|^2 = \int \lambda^2\chi^2(n^{-1}\lambda)d\mu_{u,u}(\lambda)$, this shows that $u\in\operatorname{Dom} f(H)$ by the monotone convergence theorem. The same argument shows that $(v|Hu) = \lim_{n\to\infty}(v|H\chi(n^{-1}H)u) = (v|f(H)u)$, hence $H\subset f(H)$.

We show that $f(H)\subset H$. Let $u\in\operatorname{Dom} f(H)$ and consider the anti-linear form

$$\varphi : \mathcal{H}\ni v\mapsto \int_{\mathbb{R}}(\lambda+\mathrm{i})d\mu_{v,u}(\lambda).$$

Since $u\in\operatorname{Dom} f(H)$ we have $|\varphi(v)|\leq C\|v\|$ hence $\chi(v) = (v|w)$ for some $w\in\mathcal{H}$. We have $(\lambda+\mathrm{i})d\mu_{v,u} = d\mu_{v,w}$ hence $d\mu_{v,u} = (\lambda+\mathrm{i})^{-1}d\mu_{v,w}$ for all $v\in\mathcal{H}$, hence $u = g(H)w$ for $g(\lambda) = (\lambda+\mathrm{i})^{-1}$. By Prop. 3.8 we have $u = (H+\mathrm{i})^{-1}w$ hence $u\in\operatorname{Dom} H$.

Next we write for $\chi\in C_0^\infty(\mathbb{R})$:

$$(v|(f(H)-H)\chi(H)u) = \int(\lambda-\lambda)\chi(\lambda)d\mu_{v,u}(\lambda) = 0,$$

using (3.3), hence $f(H)u = Hu$ so $f(H)\subset H$. □