# Devoir n° 2
# M2 Introduction à la théorie spectrale
# 2018-2019

A rendre le 5Novembre 2018.

The goal of this homework is to prove Stone's theorem relating selfadjoint operators and unitary groups.

**Exercice 1.** Let $\mathcal{H}$ a Hilbert space and $U(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$.

Show that the restrictions to $U(\mathcal{H})$ of the strong and weak topologies of $B(\mathcal{H})$ are equivalent. We recall that these topologies cannot be caracterized in terms of sequences.

**Solution :** *the strong topology is stronger than the weak topology on $B(\mathcal{H})$ hence on $U(\mathcal{H})$, so it remains to prove that the weak topology is stronger than the strong topology on $U(\mathcal{H})$. Let hence $U_1, U_2 \in U(\mathcal{H})$ and $u \in \mathcal{H}$. We have $\|U_1 u - U_2 u\| = \|Uu - u\|$ for $U = U_2^{-1}U_1 \in U(\mathcal{H})$, so we can assume that $U_1 = U$, $U_2 = \mathbb{1}$. We have then*

$$\|Uu - u\|^2 = 2\|u\|^2 - (Uu|u) - (u|Uu) = (u|(\mathbb{1} - U)u) + ((\mathbb{1} - U)u|u).$$

*Denoting by $\|A\|_{u,v}$ the semi-norm $|(u|Av)|$ and by $\|A\|_u$ the semi-norm $\|Au\|$, we obtain that*

$$\|U - \mathbb{1}\|_u^2 \leq 2\|U - \mathbb{1}\|_{u,Uu},$$

*which shows that the weak topology is stronger than the strong topology on $U(\mathcal{H})$.*

**Exercice 2.** A *strongly continuous unitary group* is a family $\{T_t\}_{t\in\mathbb{R}}$ such that

$i)$ $T_t \in U(\mathcal{H})$,

$ii)$ $T_t T_s = T_{s+s}$, $T_0 = \mathbb{1}$,

$iii)$ $\mathbb{R} \ni t \mapsto T_t \in B(\mathcal{H})$ is continuous for the strong topology.

1) Show that one can replace the strong by the weak topology in *iii)*.

**Solution :** *Follows from Exercice 1.*

2) Let $\{T_t\}_{t\in\mathbb{R}}$ a strongly continuous unitary group (abbreviated 'unitary group 'in the sequel). Let $R_\epsilon u = \epsilon^{-1}\int_0^\epsilon T_s u ds$. Show that s$-\lim_{\epsilon\to 0} R_\epsilon = \mathbb{1}$.

**Solution :** *follows from the strong continuity ot $T_t$ at $t = 0$.*

3) The *generator* $H$ of $\{T_t\}_{t\in\mathbb{R}}$ is defined by :

$$\mathrm{Dom}\, H = \{u \in \mathcal{H} : \lim_{t\to 0} t^{-1}(T_t - \mathbb{1})u =: \mathrm{i}Hu \text{ exists}\}.$$

Show that $R_\epsilon : \mathcal{H} \to \mathrm{Dom}\, H$ and that $\mathrm{Dom}\, H$ is dense.

**Solution :** *We have*

$$\begin{aligned} t^{-1}(T_t - \mathbb{1})R_\epsilon &= t^{-1}\int_t^{t+\epsilon} T_s ds - t^{-1}\int_0^\epsilon T_s ds \\ = \quad & t^{-1}\int_\epsilon^{\epsilon+t} T_s ds - t^{-1}\int_0^t T_s ds \end{aligned}$$

*hence $\lim_{t\to 0} t^{-1}(T_t - \mathbb{1})R_\epsilon u = T_\epsilon u - u$, so $R_\epsilon u \in \mathrm{Dom}\, H$. $\mathrm{Dom}\, H$ is dense by 2).* □

4) Show that $H$ is closed.

**Solution :** *let* $u_n \in \mathrm{Dom}\, H$ *with* $u_n \to u$, $Hu_n \to v$. *Then*

$$t^{-1}(T_t - 1\!\!1)u = \lim_n t^{-1}(T_t - 1\!\!1)u_n = \lim_n t^{-1} \int_0^t T_s \mathrm{i} H u_n ds = t^{-1} \int_0^t T_s \mathrm{i} v ds,$$

*hence* $\lim_{t\to 0} t^{-1}(T_t - 1\!\!1)u = \mathrm{i}v$, *which means that* $u \in \mathrm{Dom}\, H$ *and* $Hu = v$. $\Box$

5) Show that if $u \in \mathrm{Dom}\, H$ then $t \mapsto T_t u$ is strongly differentiable and that

$$\partial_t T_t u = \mathrm{i} H T_t u = \mathrm{i} T_t H u.$$

6) Show that $H$ is symmetric, by differentiating the identity $(u|v) = (T_t u|T_t v)$ for $u, v \in \mathcal{H}$.

**Solution :** *a simple computation.*

7) Show that $R(\mathrm{i}) = \mathrm{i}^{-1} \int_0^{+\infty} \mathrm{e}^{-t} T_t dt \in B(\mathcal{H})$ and that $R(\mathrm{i}) = (H + \mathrm{i})^{-1}$. Show similarly that $R(-\mathrm{i}) = -\mathrm{i} \int_{-\infty}^0 \mathrm{e}^{t} T_t dt \in B(\mathcal{H})$ and that $R(-\mathrm{i}) = (H - \mathrm{i})^{-1}$. Deduce from this that $H$ is selfadjoint.

**Solution :** *apply* $t^{-1}(T_t - 1\!\!1)$ *to* $R(\mathrm{i})$ *or* $R(-\mathrm{i})$, *use the group property and decompose the integral one obtains.*

8) Show that $u \in \mathrm{Dom}\, H$ iff $t^{-1}(T_t - 1\!\!1)u$ is bounded for $|t| \leq 1$ (to show $\Leftarrow$ implication show first that $u \in \mathrm{Dom}\, H^*$).

**Solution :** *the* $\Rightarrow$ *implication is clear. If* $t^{-1}(T_t - 1\!\!1)u$ *is bounded for* $|t| \leq 1$ *and* $v \in \mathrm{Dom}\, H$ *we have*

$$||(u|Hv) = \lim_{t\to 0}(u|t^{-1}(T_t - 1\!\!1)v)| = \lim_{t\to 0}|(t^{-1}(T_{-t} - 1\!\!1)u|v)| \leq C\|v\|,$$

*hence* $u \in \mathrm{Dom}\, H^* = \mathrm{Dom}\, H$. $\Box$

**Exercice 3.** Let now $H$ a selfadjoint operator on $\mathcal{H}$. Let us set $T_t = \mathrm{e}^{\mathrm{i}tH}$, defined by the functional calculus.

1) Show that $\{T_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group.

**Solution :** *the only thing that deserves to be explained is the strong (or weak) continuity. It follows from writing* $(u|f(H)v) = \int_{\mathbb{R}} f(\lambda) d\mu_{u,v}(\lambda)$, *and Lebesgue dominated convergence theorem.*

2) Let $\tilde{H}$ the generator of $\{T_t\}_{t\in\mathbb{R}}$, constructed in the previous exercice. Show that $H \subset \tilde{H}$ and then that $H = \tilde{H}$.

**Solution :** *We have for* $u \in \mathrm{Dom}\, H$ : $t^{-1}(T_t - 1\!\!1)u - \mathrm{i}Hu = f_t(H)u$, *for* $f_t(\lambda) = t^{-1}(\mathrm{e}^{\mathrm{i}t\lambda} - 1) - \mathrm{i}\lambda$. *We have* $|f_t(\lambda)| \leq 2|\lambda|$ *and* $f_t \to 0$ *pointwise when* $t \to 0$. *Next*

$$\|f_t(H)u\|^2 = \int_{\mathbb{R}} |f_t(\lambda)|^2 d\mu_u(\lambda),$$

*and* $\int \lambda^2 d\mu_u(\lambda) < \infty$ *since* $u \in \mathrm{Dom}\, H$, *so by dominated convergence we have* $\lim_{t\to 0} t^{-1}(T_t - 1\!\!1)u = \mathrm{i}Hu$, *hence* $H \subset \tilde{H}$. *Taking adjoints we have* $\tilde{H}^* = \tilde{H} \subset H^* = H$.$\Box$

**Exercice 4.** Let $\mathcal{H} = l^2(\mathbb{R})$, ie the space of functions $u : \mathbb{R} \to \mathbb{C}$ such that $\sum_{x\in\mathbb{R}} |f(x)|^2 < \infty$ (of course if $u \in l^2(\mathbb{R})$ then $\{x : u(x) \neq 0\}$ is at most countable.

1) Let us set $T_t u(x) := u(x - t)$. Show that $T_t$ satisfies conditions *i)* and *ii)* in Exercise 2. Show that if $u_0(x) = 1\!\!1_{\{0\}}(x)$ then $(u_0|T_t u_0) = 0$ for $t \neq 0$. Is $T_t$ strongly continuous ?

**Solution :** $T_t u_0 = u_t$, $u_t$ *and* $u_0$ *are orthogonal. Of course* $T_t$ *cannot be strongly continuous at* $t = 0$.