

# LECTURE NOTES ON QFT ON CURVED SPACE-TIMES

## 1. INTRODUCTION

The purpose of these notes is to give an introduction to some recent aspects of Quantum Field Theory on curved space-times, emphasizing its relations with partial differential equations and microlocal analysis.

**1.1. Quantum Field Theory.** Quantum Field Theory arose from the need to unify Quantum Mechanics with special relativity. However trying to treat the two basic relativistic field equations, the *Klein-Gordon equation*:

$$\partial_t^2 \phi(t, x) - \Delta_x \phi(t, x) + m^2 \phi(t, x) = 0,$$

and the *Dirac equation*:

$$\gamma^0 \partial_t \psi(t, x) + \gamma^i \partial_{x^i} \psi(t, x) - m \psi(t, x) = 0,$$

(where the  $\gamma^i$  are the Dirac matrices) in a way parallel to the non-relativistic *Schroedinger equation*:

$$\partial_t \psi(t, x) - \frac{i}{2m} \Delta_x \psi(t, x) + iV(x) \psi(t, x) = 0$$

leads to difficulties (see eg [BD]). For the Klein-Gordon equation, there exists a conserved scalar product:

$$\langle \phi_1 | \phi_2 \rangle = i \int_{\mathbb{R}^3} \partial_t \bar{\phi}_1(t, x) \phi_2(t, x) - \bar{\phi}_1(t, x) \phi_2(t, x) dx$$

which is however not positive definite, hence cannot lead to a probabilistic interpretation. However one has

$$\langle \phi | i \partial_t \phi \rangle \geq 0, \text{ (positivity of the energy).}$$

For the Dirac equation the situation is the opposite: the conserved scalar product

$$\langle \psi_1 | \psi_2 \rangle = \int_{\mathbb{R}^3} \bar{\psi}_1(t, x) \cdot \psi_2(t, x) dx$$

is positive, but

$$\langle \psi | i \partial_t \psi \rangle \text{ is indefinite.}$$

The reason behind these difficulties is that, although all these equations are partial differential equations, their nature is very different: the Klein-Gordon and Dirac equations are *classical* equations, while the Schroedinger equation is a *quantum* equation, obtained by quantizing the classical *Newton equation*:

$$\ddot{x}(t) = -\nabla_x V(x(t)), \quad x \in \mathbb{R}^n.$$

or equivalently the Hamilton equations:

$$\begin{cases} \dot{x}(t) = \partial_\xi h(x(t), \xi(t)), \\ \dot{\xi}(t) = -\partial_x h(x(t), \xi(t)) \end{cases}$$

for the classical Hamiltonian:

$$h(x, \xi) = \frac{1}{2}\xi^2 + V(x).$$

We denote by  $X = (x, \xi)$  the points in  $T^*\mathbb{R}^n$  and introduce the coordinate functions

$$q : X \mapsto x, \quad p : X \mapsto \xi.$$

If  $\Phi(t) : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$  is the flow of  $H_h$  and  $q(t) := q \circ \Phi(t)$ ,  $p(t) := p \circ \Phi(t)$  then:

$$\begin{aligned} \partial_t q(t) &= p(t), \\ \partial_t p(t) &= -\nabla V(q(t)). \end{aligned}$$

where  $\{\cdot, \cdot\}$  is the Poisson bracket. Note that

$$\{p_j(t), q_k(t)\} = \delta_{jk}, \quad \{p_j(t), p_k(t)\} = \{q_j(t), q_k(t)\} = 0.$$

To quantize the Liouville equation means to find a Hilbert space  $\mathcal{H}$  and functions  $\mathbb{R} \ni t \mapsto p(t), q(t)$  with values in selfadjoint operators on  $\mathcal{H}$  such that

$$\begin{aligned} [p_j(t), iq_k(t)] &= \delta_{jk} \mathbb{1}, \quad [p_j(t), ip_k(t)] = [q_j(t), iq_k(t)] = 0, \quad \partial_t q(t) = p(t), \\ \partial_t p(t) &= -\nabla V(q(t)). \end{aligned}$$

The last two equations are called *Heisenberg* equations. The solution is as follows:

- (1) Find operators  $p, q$  satisfying

$$[p_j, iq_k] = \delta_{jk} \mathbb{1}, \quad [p_j, ip_k] = [q_j, iq_k] = 0.$$

- (2) Construct the selfadjoint operator on  $\mathcal{H}$

$$H = \frac{1}{2}p^2 + V(q).$$

- (3) Then

$$q(t) := e^{itH} q e^{-itH}, \quad p(t) := e^{itH} p e^{-itH}$$

solve Heisenberg equations.

The *Stone-von Neumann theorem* says that there is no choice in step 1: modulo some technical conditions and multiplicity one has only one choice, up to unitary equivalence:

$$\mathcal{H} = L^2(\mathbb{R}^n), \quad q = x, \quad p = i^{-1}\nabla_x.$$

Then  $H = -\frac{1}{2}\Delta + V(x)$  is the Schroedinger operator.

The Klein-Gordon equation is also a Hamiltonian equation, however with a *infinite dimensional* phase space, which can be taken for example as  $C_0^\infty(\mathbb{R}^d) \oplus C_0^\infty(\mathbb{R}^d)$ . The classical Hamiltonian is then

$$h(\varphi, \pi) := \frac{1}{2} \int_{\mathbb{R}^d} \pi^2(x) + |\nabla_x \varphi(x)|^2 + m^2 \varphi^2(x) dx,$$

for the linear case, or

$$h(\varphi, \pi) := \frac{1}{2} \int_{\mathbb{R}^d} \pi^2(x) + |\nabla_x \varphi(x)|^2 + m^2 \varphi^2(x) + \varphi^n(x) dx,$$

for some non-linear version. Here the symbols  $\varphi(x)$ ,  $\pi(x)$  are (coordinate) functions, parametrized by a point  $x \in \mathbb{R}^d$ , on the space of smooth solutions of the Klein-Gordon equation, with compactly supported Cauchy data. If  $\phi$  is such a solution then

$$\varphi(x)(\phi) := \phi(0, x), \quad \pi(x)(\phi) := \partial_t \phi(0, x)$$

It is well-known that these are *symplectic coordinates*, i.e.

$$\{\varphi(x), \varphi(x')\} = \{\pi(x), \pi(x')\} = 0, \quad \{\pi(x), \varphi(x')\} = \delta(x, x'), \quad \forall x, x' \in \mathbb{R}^d.$$

One would like to follow the same path and consider families of operators on a Hilbert space  $\mathcal{H}$ ,  $\varphi(x)$ ,  $\pi(x)$ ,  $x \in \mathbb{R}^d$  such that

$$[\pi(x), i\varphi(x')] = \delta(x - x') \mathbb{1}, \quad [\varphi(x), i\varphi(x')] = [\pi(x), i\pi(x')] = 0, \quad \forall x, x' \in \mathbb{R}^d.$$

The fundamental difference with non-relativistic Quantum Mechanics is that, since the phase space is infinite dimensional, the Stone von Neumann theorem cannot be applied anymore: there exists an infinite number of inequivalent representations of commutation relations.

In other words, when one tries to quantize a classical field equation, the Hilbert space has to be constructed *together* with the quantum Hamiltonian: one cannot work on our familiar Hilbert space and then use tools from operator theory to construct the quantum Hamiltonian.

This is the reason why the rigorous construction of Quantum Field Theory models is so difficult, except for *non-interacting* theories. For interacting theories it has been achieved only in 2 and 3 spacetime dimensions, and one has to rely instead on perturbative methods.

Another lesson learned from Quantum Field Theory (and also from Quantum Statistical Mechanics), is that Hilbert spaces do not play such a central role anymore. Instead one focuses on *algebras* and *states*.

Let us finish this discussion by recalling a well-known anecdote: at the Solvay conference in 1927, Dirac told Bohr that he was trying to find a relativistic quantum theory of the electron (i.e. the Dirac equation). Bohr replied that this problem had already been solved by Klein, who had found the Klein-Gordon equation. We know now that these two equations are of a different nature, the first describing fermionic fields, the second bosonic ones, and that they can be interpreted as quantum equations only via Quantum Field Theory.

**1.2. QFT on curved space-times.** Given the difficulties with the construction of interacting field theories on Minkowski space-time, one may wonder why one should consider quantum field theories on *curved space-times*, which have no reason to be simpler.

One reason comes from attempts to quantize gravitation, where one starts by linearizing Einstein equations around a curved background metric  $g$ . Another argument is that there are several interesting quantum effects appearing in presence of strong gravitational fields. The most famous one is the *Hawking effect*, which predicts that a black hole can emit quantum particles.

There are several new challenges one has to face when moving from flat Minkowski space-time to an arbitrary curved space-time.

On the computational side, one cannot rely anymore on the *Fourier transform* and related analyticity arguments, which are natural and useful on Minkowski space, since the Klein-Gordon equation has then constant coefficients.

On a more conceptual side, a curved space-time does not have the large group of isometries (the Poincaré group) of the Minkowski space. It follows that on a curved space-time there seems to be no natural notion of a *vacuum state*, which is defined on Minkowski space as the unique state which is invariant under space-time translations, and has an additional *positive energy condition*.

In the eighties, physicists managed to define a class of states, the so-called *Hadamard states*, which were characterized by properties of their two-point functions, which had to have a specific asymptotic expansion near the diagonal, connected with the well-known Hadamard parametrix construction for the Klein-Gordon equation on a curved space-time. Later in 1995, in a seminal paper, Radzikowski reformulated the old Hadamard condition in terms of the *wave front set* of the two-point function. The wave front set of a distribution, introduced in 1970 by Hörmander, is one of the important notions of *microlocal analysis*, a theory which was precisely developed to extend Fourier analysis, in the study of general partial differential equations.

The introduction of tools from microlocal analysis had a great influence on the field, leading for example to the proof of renormalizability of scalar interacting field theories by Brunetti and Fredenhagen.

The goal of these notes is to give an introduction to the modern notion of Hadamard states, for a mathematically oriented audience.

## 2. A QUICK INTRODUCTION TO QUANTUM MECHANICS

This section is supposed to give a very quick introduction to the mathematical formalism of Quantum Mechanics, which is (or is expected to be) still relevant to Quantum Field Theory.

**2.1. Hilbert space approach.** In ordinary Quantum Mechanics, the description of a physical system starts with a Hilbert space  $\mathcal{H}$ , whose scalar product is denoted by  $(u|v)$ . The *states* of the system are described by unit vectors  $\psi \in \mathcal{H}$  with  $\|\psi\| = 1$ .

The various physical quantities which can be measured (like position, momentum, energy, spin) are represented by *selfadjoint operators* on  $\mathcal{H}$ , i.e. (forgetting about important issues with unbounded operators), linear operators  $A$  on  $\mathcal{H}$ , assumed to be bounded for simplicity, such that  $A = A^*$ , (where  $A^*$  is the *adjoint* of  $A$ ), called *observables*.

If  $\psi \in \mathcal{H}$ ,  $\|\psi\| = 1$  is a state vector, then the map:

$$A \mapsto \omega_\psi(A) = (\psi|A\psi)$$

computes the *expectation value* of  $A$  in the state  $\psi$  represent the average value of actual measurements of the physical quantity represented by  $A$ .

Rather quickly people were led to consider also *mixed states*, where the state of the system is only incompletely known. For example if  $\psi_i$ ,  $i \in \mathbb{N}$  is an orthonormal family and  $0 \leq \rho_i \leq 1$  are real numbers with  $\sum_{i=0}^{\infty} \rho_i = 1$ , then we can consider the trace-class

operator:

$$\rho = \sum_{i=0}^{\infty} \rho_i |\psi_i\rangle\langle\psi_i|, \quad \text{Tr}\rho = 1,$$

called a *density matrix* and the map

$$A \mapsto \omega_\rho(A) := \text{Tr}(\rho A)$$

is called a *mixed state*. Vector states are also called *pure states*.

**2.2. Algebraic approach.** The framework above is sufficient to cover all of non-relativistic Quantum Mechanics, i.e. in practice quantum systems consisting of a *finite* number of non-relativistic particles. However when one considers systems with an *infinite* number of particles, like in statistical mechanics, or quantum field theory, where the notion of particles is dubious, an algebraic framework is more relevant. It starts with the following observation about the space  $B(\mathcal{H})$  of bounded operators on  $\mathcal{H}$ :

if we equip it with the operator norm, it is a Banach space, and a Banach algebra, i.e. an algebra with the property that  $\|AB\| \leq \|A\|\|B\|$ . It is also an involutive Banach algebra, i.e. the adjoint operation  $A \mapsto A^*$  has the properties that

$$(AB)^* = B^*A^*, \quad \|A^*\| = \|A\|.$$

Finally one can easily check that:

$$\|A^*A\| = \|A\|^2, \quad A \in B(\mathcal{H}).$$

This last property has very important consequences, for example one can deduce from it the functional calculus and spectral theorem for selfadjoint operators.

An abstract algebra  $\mathfrak{A}$  equipped with a norm and an involution with these properties, which is moreover complete is called a *C\* algebra*. The typical example of a *C\** algebra is of course the algebra  $B(\mathcal{H})$  of bounded operators on a Hilbert space.

If  $\mathcal{H}$  is a Hilbert space, a  $*$ -homomorphism

$$\mathfrak{A} \ni A \mapsto \pi(A) \in B(\mathcal{H})$$

is called a *representation* of  $\mathfrak{A}$  in  $\mathcal{H}$ . An injective representation is called *faithful*.

The need for such change of point of view comes from the fact that a physical system, like a gas of electrons, can exist in many different physical realizations, for example at different temperatures. In other words it does not come equipped with a canonical Hilbert space.

Observables, like for example the electron density, have a meaning irrelevant of the realizations, and are described by selfadjoint elements in some *C\** algebra  $\mathfrak{A}$ . However the various Hilbert spaces and the representations of the observables on them are very different from one temperature to another.

One can also describe the possible physical realizations of a system with the language of *states*. A *state*  $\omega$  on  $\mathfrak{A}$  is a linear map:

$$\omega : \mathfrak{A} \mapsto \mathbb{C}$$

such that

$$\omega(A^*A) \geq 0, \quad A \in \mathfrak{A}.$$

Assuming that  $\mathfrak{A}$  has a unit (which can always be assumed by adjoining one), one also requires that

$$\omega(\mathbb{1}) = 1.$$

The set of states on a  $C^*$  algebra is a convex set, its extremal points are called *pure states*. If  $\mathfrak{A} \subset B(\mathcal{H})$  and  $\psi$  is a unit vector, or if  $\rho$  is a density matrix, then

$$\omega_\psi(A) := (\psi|A\psi), \quad \omega_\rho(A) := \text{Tr}(\rho A)$$

are states on  $\mathfrak{A}$ . If  $\mathfrak{A} = B(\mathcal{H})$ , then  $\omega_\psi$  is a pure state. It is important to be aware of the fact that if  $\mathfrak{A}$  is only a  $C^*$  subalgebra of  $B(\mathcal{H})$ , then  $\omega_\psi$  may *not* be a pure state on  $\mathfrak{A}$ .

**2.3. The GNS construction.** After being told that one should use  $C^*$  algebras and states, one can wonder where the Hilbert spaces have gone. Actually given a  $C^*$  algebra  $\mathfrak{A}$  and a state  $\omega$  on it, it is quite easy to construct a canonical Hilbert space and a representation of  $\mathfrak{A}$  on it, as proved by Gelfand, Naimark and Segal. There exist a triple  $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$  where  $\mathcal{H}_\omega$  is a Hilbert space,  $\pi_\omega : \mathfrak{A} \mapsto B(\mathcal{H}_\omega)$  is a faithful representation,  $\Omega_\omega \in \mathcal{H}_\omega$  is a unit vector such that:

$$\omega(A) = (\Omega_\omega | \pi_\omega(A) \Omega_\omega), \quad A \in \mathfrak{A}.$$

### 3. NOTATION

In this section we collect some notation that will be used in these notes. If  $\mathcal{X}$  is a real or complex vector space we denote by  $\mathcal{X}^\#$  its dual. Bilinear forms on  $\mathcal{X}$  are identified with elements of  $L(\mathcal{X}, \mathcal{X}^\#)$ , which leads to the notation  $x_1 \cdot b x_2$  for  $b \in L(\mathcal{X}, \mathcal{X}^\#)$ ,  $x_1, x_2 \in \mathcal{X}$ . The space of symmetric (resp. anti-symmetric) bilinear forms on  $\mathcal{X}$  is denoted by  $L_s(\mathcal{X}, \mathcal{X}^\#)$  (resp.  $L_a(\mathcal{X}, \mathcal{X}^\#)$ ).

If  $\sigma \in L_s(\mathcal{X}, \mathcal{X}^\#)$  is non-degenerate, we denote by  $O(\mathcal{X}, \sigma)$  the linear (pseudo-)orthogonal group on  $\mathcal{X}$ . Similarly if  $\sigma \in L_a(\mathcal{X}, \mathcal{X}^\#)$  is non-degenerate, i.e.  $(\mathcal{X}, \sigma)$  is a symplectic space, we denote by  $Sp(\mathcal{X}, \sigma)$  the linear symplectic group on  $\mathcal{X}$ .

If  $\mathcal{Y}$  is a complex vector space, we denote by  $\mathcal{Y}_\mathbb{R}$  its *realification*, i.e.  $\mathcal{Y}$  considered as a real vector space. We denote by  $\overline{\mathcal{Y}}$  a *conjugate vector space* to  $\mathcal{Y}$ , i.e. a complex vector space  $\overline{\mathcal{Y}}$  with an anti-linear isomorphism  $\mathcal{Y} \ni y \mapsto \overline{y} \in \overline{\mathcal{Y}}$ . The *canonical conjugate vector space* to  $\mathcal{Y}$  is simply the real vector space  $\mathcal{Y}_\mathbb{R}$  equipped with the complex structure  $-i$ , if  $i$  is the complex structure of  $\mathcal{Y}$ . In this case the map  $y \rightarrow \overline{y}$  is chosen as the identity. If  $a \in L(\mathcal{Y}_1, \mathcal{Y}_2)$ , we denote by  $\overline{a} \in L(\overline{\mathcal{Y}}_1, \overline{\mathcal{Y}}_2)$  the linear map defined by:

$$(3.1) \quad \overline{a y_1} := \overline{a} \overline{y_1}, \quad \overline{y_1} \in \overline{\mathcal{Y}}_1.$$

We denote by  $\mathcal{Y}^*$  the *anti-dual* of  $\mathcal{Y}$ , i.e. the space of anti-linear forms on  $\mathcal{Y}$ . Clearly  $\mathcal{Y}^*$  can be identified with  $\overline{\mathcal{Y}^\#} \sim \overline{\mathcal{Y}^\#}$ .

Sesquilinear forms on  $\mathcal{Y}$  are identified with elements of  $L(\mathcal{Y}, \mathcal{Y}^*)$ , and we use the notation  $(y_1 | b y_2)$  or  $\overline{y_1} \cdot b y_2$  for  $b \in L(\mathcal{Y}, \mathcal{Y}^*)$ ,  $y_1, y_2 \in \mathcal{Y}$ .

The space of hermitian (resp. anti-hermitian) sesquilinear forms on  $\mathcal{Y}$  is denoted by  $L_s(\mathcal{Y}, \mathcal{Y}^*)$  (resp.  $L_a(\mathcal{Y}, \mathcal{Y}^*)$ ).

If  $q \in L_h(\mathcal{Y}, \mathcal{Y}^*)$  is non-degenerate, i.e.  $(\mathcal{Y}, q)$  is a pseudo-unitary space, we denote by  $U(\mathcal{Y}, q)$  the linear pseudo-unitary group on  $\mathcal{Y}$ .

If  $b$  is a bilinear form on the real vector space  $\mathcal{X}$ , its canonical sesquilinear extension to  $\mathbb{C}\mathcal{X}$  is by definition the sesquilinear form  $b_{\mathbb{C}}$  on  $\mathbb{C}\mathcal{X}$  given by

$$(w_1|b_{\mathbb{C}}w_2) := x_1 \cdot bx_2 + y_1 \cdot by_2 + ix_1 \cdot by_2 - iy_1 \cdot bx_2, \quad w_i = x_i + iy_i$$

for  $x_i, y_i \in \mathcal{X}$ ,  $i = 1, 2$ . This extension maps (anti-)symmetric forms on  $\mathcal{X}$  onto (anti-)hermitian forms on  $\mathbb{C}\mathcal{X}$ .

Conversely if  $\mathcal{Y}$  is a complex vector space and  $\mathcal{Y}_{\mathbb{R}}$  is its realification, i.e.  $\mathcal{Y}$  considered as a real vector space, then for  $b \in L_{\text{h/a}}(\mathcal{Y}, \mathcal{Y}^*)$  the form  $\text{Re}b$  belongs to  $L_{\text{s/a}}(\mathcal{Y}_{\mathbb{R}}, \mathcal{Y}_{\mathbb{R}}^{\#})$ .

## 4. CCR AND CAR ALGEBRAS

**4.1. Introduction.** It is useful to discuss the CCR and CAR without making reference to a Fock space. There are some mathematical subtleties with CCR algebras, coming from the fact that the field operators are 'unbounded'. These subtleties can mostly be ignored for our purposes.

**4.2. Algebras generated by symbols and relations.** In physics many algebras are defined by specifying a set of generators and the relations they satisfy. This is completely sufficient to do computations, but mathematicians may feel uncomfortable with such an approach. However it is easy (and actually rather useless) to give a rigorous definition.

Assume that  $\mathcal{A}$  is a set. We denote by  $c_c(\mathcal{A}, \mathbb{K})$  the vector space of functions  $\mathcal{A} \rightarrow \mathbb{K}$  with finite support (usually  $\mathbb{K} = \mathbb{C}$ ). If for  $A \in \mathcal{A}$ , we denote the indicator function  $\mathbb{1}_{\{A\}}$  simply by  $A$ , we see that any element of  $c_c(\mathcal{A}, \mathbb{K})$  can be written as  $\sum_{A \in \mathcal{B}} \lambda_A A$ ,  $\mathcal{B} \subset \mathcal{A}$  finite,  $\lambda_A \in \mathbb{K}$ .

Then  $c_c(\mathcal{A}, \mathbb{K})$  can be seen as the vector space of finite linear combinations of elements of  $\mathcal{A}$ . We set

$$\mathfrak{A}(\mathcal{A}, \mathbb{1}) := \overset{\text{al}}{\otimes} c_c(\mathcal{A}, \mathbb{K}),$$

called the *universal unital algebra over  $\mathbb{K}$  with generators  $\mathcal{A}$* . Usually one write  $A_1 \cdots A_n$  instead of  $A_1 \otimes \cdots \otimes A_n$  for  $A_i \in \mathcal{A}$ .

Let us denote by  $\overline{\mathcal{A}}$  another copy of  $\mathcal{A}$ . We denote by  $\bar{a}$  the element  $a \in \overline{\mathcal{A}}$ . We set then  $*a := \bar{a}$ ,  $*\bar{a} := a$  and extend  $*$  to  $\mathfrak{A}(\mathcal{A} \sqcup \overline{\mathcal{A}}, \mathbb{1})$  by setting

$$(b_1 b_2 \cdots b_n)^* = b_n^* \cdots b_2^* b_1^*, \quad b_i \in \mathcal{A} \sqcup \overline{\mathcal{A}}, \quad \mathbb{1} = \mathbb{1}^*.$$

The algebra  $\mathfrak{A}(\mathcal{A} \sqcup \overline{\mathcal{A}}, \mathbb{1})$  equipped with the involution  $*$  is called the *universal unital  $*$ -algebra over  $\mathbb{K}$  with generators  $\mathcal{A}$* .

Let now  $\mathfrak{R} \subset \mathfrak{A}(\mathcal{A}, \mathbb{1})$  (the set of 'relations'). We denote by  $\mathfrak{I}(\mathfrak{R})$  the ideal of  $\mathfrak{A}(\mathcal{A}, \mathbb{K})$  generated by  $\mathfrak{R}$ . Then the quotient

$$\mathfrak{A}(\mathcal{A}, \mathbb{1}) / \mathfrak{I}(\mathfrak{R})$$

is called the *unital algebra with generators  $\mathcal{A}$  and relations  $R = 0$ ,  $R \in \mathfrak{R}$* .

Similarly if  $\mathfrak{R} \subset \mathfrak{A}(\mathcal{A} \sqcup \overline{\mathcal{A}}, \mathbb{1})$  is  $*$ -invariant, then  $\mathfrak{A}(\mathcal{A} \sqcup \overline{\mathcal{A}}, \mathbb{1}) / \mathfrak{I}(\mathfrak{R})$  is called the *unital  $*$ -algebra with generators  $\mathcal{A} \sqcup \overline{\mathcal{A}}$  and relations  $R = 0$ ,  $R \in \mathfrak{R}$* .