Identification of the graded pieces

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1. TP for quasiregular semiperfect algebras

We fix a prime number p, recall that an $\mathbb{F}_p$-algebra R is *perfect* if its absolute Frobenius endomorphism $x \mapsto x^p$ is an isomorphism, and consider the following class of $\mathbb{F}_p$-algebras.

Definition 1.1 ([BMS18], 8.8)**.** An $\mathbb{F}_p$-algebra S is *quasiregular semiperfect* if it admits a surjection $R \twoheadrightarrow S$ from a perfect $\mathbb{F}_p$-algebra R such that the cotangent complex $\mathbb{L}_{S/R}$ is quasi-isomorphic to a flat S-module placed in degree -1.

Remark 1.2. The perfectness of R ensures that

$$\mathbb{L}_{S/\mathbb{F}_p} \xrightarrow{\sim} \mathbb{L}_{S/R},$$

so the condition on the cotangent complex does not depend on the choice of R. Moreover, since the absolute Frobenius of S is surjective, a canonical choice for R is

$$S^\flat := \varprojlim_{x \mapsto x^p} S.$$

Example 1.3. Any quotient of a perfect $\mathbb{F}_p$-algebra by a regular sequence is quasiregular semiperfect. Concretely, S could be, for instance,

$$\mathbb{F}_p[T^{1/p^\infty}]/(T-1).$$

Our goal is to review the following identification, established in [BMS18], §8, of the homotopy groups of the topological periodic cyclic homology of a quasiregular semiperfect S:

$$\pi_*(\mathrm{TP}(S)) \cong \widehat{\mathbb{A}}_{\mathrm{cris}}(S)[\sigma, \sigma^{-1}] \qquad \text{with} \qquad \deg(\sigma) = 2,$$

where $\mathbb{A}_{\mathrm{cris}}(S)$ is a certain Fontaine ring that will be reviewed in §2 and $\widehat{\mathbb{A}}_{\mathrm{cris}}(S)$ is its completion for the Nygaard filtration. Thus, concretely,

$$\pi_*(\mathrm{TP}(S)) \cong \begin{cases} \widehat{\mathbb{A}}_{\mathrm{cris}}(S) & \text{for even } *, \\ 0 & \text{for odd } *. \end{cases}$$

Example 1.4. For perfect $\mathbb{F}_p$-algebras, such as $S^\flat$, we have the identification with the p-typical Witt ring:

$$\widehat{\mathbb{A}}_{\mathrm{cris}}(S^\flat) \cong W(S^\flat), \qquad \text{so also} \qquad \pi_*(\mathrm{TP}(S^\flat)) \cong W(S^\flat)[\sigma, \sigma^{-1}].$$

The latter identification is already familiar from the earlier talks of the workshop: to derive it, one analyzes the Tate spectral sequence. This spectral sequence also gives the vanishing of $\pi_{\mathrm{odd}}(\mathrm{TP}(S))$, so we will assume these facts as known.

In the view of Example 1.4, since $\mathrm{TP}(S)$ is always a module over $\mathrm{TP}(S^\flat)$, all we need to discuss is the identification

$$\pi_0(\mathrm{TP}(S)) \cong \widehat{\mathbb{A}}_{\mathrm{cris}}(S). \tag{1.4.1}$$

For this, we will proceed in three steps:

(1) in §2, we will review the construction of the ring $\widehat{\mathbb{A}}_{\mathrm{cris}}(S)$;
(2) in §3, we will review the derived de Rham–Witt complex $LW\Omega_{S/\mathbb{F}_p}$ of S over $\mathbb{F}_p$ and will identify its Nygaard completion as follows:

$$\widehat{\mathbb{A}}_{\mathrm{cris}}(S) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p};$$

(3) in §4, we will conclude by reviewing the identification:

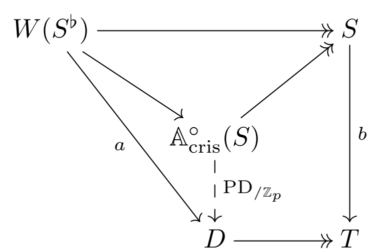
$$\pi_0(\mathrm{TP}(S)) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p}.$$

2. The ring $\mathbb{A}_{\mathrm{cris}}(S)$

For a quasiregular semiperfect $\mathbb{F}_p$-algebra S, we consider the following $\mathbb{Z}_p$-algebras.

(i) The ring $\mathbb{A}^{\circ}_{\mathrm{cris}}(S)$ defined as the divided power envelope over $(\mathbb{Z}_p, p\mathbb{Z}_p)$ of the composite surjection $W(S^\flat) \twoheadrightarrow S^\flat \twoheadrightarrow S$.
(ii) The ring $\mathbb{A}_{\mathrm{cris}}(S)$ defined as the p-adic completion of $\mathbb{A}^{\circ}_{\mathrm{cris}}(S)$.

Thus, the kernel of the surjection $\mathbb{A}^{\circ}_{\mathrm{cris}}(S) \twoheadrightarrow S$ is equipped with a divided power structure that is compatible with the divided power structure on the ideal $p\mathbb{Z}_p \subset \mathbb{Z}_p$, and $\mathbb{A}^{\circ}_{\mathrm{cris}}(S)$ is the initial such $W(S^\flat)$-algebra: for every surjection $D \twoheadrightarrow T$ of $\mathbb{Z}_p$-algebras whose kernel is equipped with a divided power structure over $\mathbb{Z}_p$ and every morphisms a, b that fit into the commutative diagram



there exists a unique divided power $\mathbb{Z}_p$-morphism indicated by the dashed arrow that makes the diagram commute. The ring $\mathbb{A}_{\mathrm{cris}}(S)$ enjoys the analogous universal property among the p-adically complete D. It follows from the definitions that

$$\mathbb{A}_{\mathrm{cris}}(S)/p \cong \mathbb{A}^{\circ}_{\mathrm{cris}}(S)/p \cong \text{PD-envelope}_{/\mathbb{F}_p}(S^\flat \twoheadrightarrow S).$$

By functoriality, $\mathbb{A}_{\mathrm{cris}}(S)$ comes equipped with a Frobenius endomorphism φ. The resulting ideals

$$\mathcal{N}^{\geq n}(\mathbb{A}_{\mathrm{cris}}(S)) := \varphi^{-1}(p^n \mathbb{A}_{\mathrm{cris}}(S)) \subset \mathbb{A}_{\mathrm{cris}}(S) \qquad \text{for} \qquad n \geq 0$$

form an exhaustive, φ-stable filtration of $\mathbb{A}_{\mathrm{cris}}(S)$, the *Nygaard filtration*. The φ-stability implies that the *Nygaard completion*

$$\widehat{\mathbb{A}}_{\mathrm{cris}}(S) := \varprojlim_{n\geq 0} \left(\mathbb{A}_{\mathrm{cris}}(S)/\mathcal{N}^{\geq n}(\mathbb{A}_{\mathrm{cris}}(S))\right)$$

inherits a Frobenius endomorphism from $\mathbb{A}_{\mathrm{cris}}(S)$.

In the case of a perfect ring, such as $S^\flat$, the kernel of the surjection $W(S^\flat) \twoheadrightarrow S^\flat$ carries a unique divided power structure, so

$$\mathbb{A}^\circ_{\mathrm{cris}}(S^\flat) \cong W(S^\flat) \cong \mathbb{A}_{\mathrm{cris}}(S^\flat).$$

Moreover, in this case, the Frobenius φ is an isomorphism, so

$$\mathcal{N}^{\geq n}(\mathbb{A}_{\mathrm{cris}}(S^\flat)) \cong p^n W(S^\flat), \qquad \text{and hence also} \qquad \widehat{\mathbb{A}}_{\mathrm{cris}}(S^\flat) \cong W(S^\flat).$$

3. The derived de Rham–Witt complex

The argument that relates $\pi_0(\mathrm{TP}(S))$ to $\widehat{\mathbb{A}}_{\mathrm{cris}}(S)$ uses the derived de Rham–Witt complex of S over $\mathbb{F}_p$ as an intermediary. To recall the latter, we begin by reviewing the de Rham–Witt complex using the recent approach of Bhatt–Lurie–Mathew [BLM18].

For a fixed prime p, consider the commutative differential graded algebras

$$C^\bullet = (C^0 \xrightarrow{d} C^1 \xrightarrow{d^1} \dots) \qquad \text{with} \qquad C^i[p] = 0 \quad \text{for all} \quad i$$

equipped with an algebra endomorphism $F \colon C^\bullet \to C^\bullet$ such that:

- $F \colon C^0 \to C^0$ lifts the absolute Frobenius endomorphism of C^0/p;
- $dF = pFd$;
- $F \colon C^i \xrightarrow{\sim} d^{-1}(pC^{i+1})$ for all i;
- the unique additive endomorphism $V \colon C^\bullet \to C^\bullet$ such that $FV = p$ (whose existence is ensured by the previous requirement, and which necessarily also satisfies $VF = p$) is such that the following map is an isomorphism:

$$C^\bullet \xrightarrow{\sim} \varprojlim_{n>0} \left(\frac{C^\bullet}{\mathrm{Im}(V^n) + \mathrm{Im}(dV^n)} \right).$$

The last requirement implies that each C^i is an inverse limit of p^n-torsion abelian groups, and hence is p-adically complete (the unique limit of a p-adic Cauchy sequence exists already in each term of the inverse limit). The map F does not respect the differentials, but the Frobenius endomorphism

$$\varphi := (p^i F \text{ in degree } i) \colon C^\bullet \to C^\bullet$$

does. The resulting ideals

$$\mathcal{N}^{\geq n}(C^\bullet) := \varphi^{-1}(p^n C^\bullet) \subset C^\bullet \qquad \text{for} \qquad n \geq 0$$

form a separated, exhaustive, φ-stable filtration of $C^\bullet$, the *Nygaard filtration.*

Theorem 3.1 (Bhatt–Lurie–Mathew). *The functor*

$$\{C^\bullet \text{ as above}\} \xrightarrow{C^\bullet \mapsto C^0/VC^0} \mathbb{F}_p\text{-algebras}$$

admits a left adjoint

$$R \mapsto W\Omega^\bullet_{R/\mathbb{F}_p},$$

so that

$$\mathrm{Hom}_{F\text{-}cdga}(W\Omega^\bullet_{R/\mathbb{F}_p}, C^\bullet) \cong \mathrm{Hom}_{\mathbb{F}_p\text{-}alg.}(R, C^0/VC^0).$$

Moreover, for a regular $\mathbb{F}_p$-algebra R, the complex $W\Omega^\bullet_{R/\mathbb{F}_p}$ agrees with the de Rham–Witt complex of Deligne–Illusie that was defined and studied in [Ill79].

Remark 3.2. The last aspect implies that for regular R one has a quasi-isomorphism

$$\Omega^\bullet_{R/\mathbb{F}_p} \xrightarrow{\sim} W\Omega^\bullet_{R/\mathbb{F}_p}/p.$$

Definition 3.3. The *derived de Rham–Witt complex* $LW\Omega_{R/\mathbb{F}_p}$ of a simplicial $\mathbb{F}_p$-algebra R is the value at R of the left Kan extension along the vertical inclusion

$$\begin{array}{ccc} \{\text{simplicial } \mathbb{F}_p\text{-algebras}\} & \xrightarrow{R\mapsto (LW\Omega_{R/\mathbb{F}_p},\,\mathcal{N}^{\geq\bullet})} & \left\{\begin{array}{l}p\text{-complete } E_\infty\text{-algebras in}\\ \text{the filtered derived } \infty\text{-}\\ \text{category of } \mathbb{Z}_p\text{-modules}\end{array}\right\} \\ \big\uparrow & \nearrow_{R\mapsto (W\Omega^\bullet_{R/\mathbb{F}_p},\,\mathcal{N}^{\geq\bullet})} & \\ \left\{\begin{array}{l}\text{polynomial } \mathbb{F}_p\text{-algebras}\\ \text{of finite type}\end{array}\right\} & & \end{array}$$

of the indicated diagonal functor, and its *Nygaard completion* $\widehat{LW\Omega}_{R/\mathbb{F}_p}$ is the completion of $LW\Omega_{R/\mathbb{F}_p}$ with respect to the filtration $\mathcal{N}^{\geq\bullet}$.

Remark 3.4. Using the left Kan extension, one may analogously define the derived de Rham complex $L\Omega_{R/\mathbb{F}_p}$ and its Hodge completion $\widehat{L\Omega}_{R/\mathbb{F}_p}$. Remark 3.2 implies the canonical identification

$$LW\Omega_{R/\mathbb{F}_p}/p \cong L\Omega_{R/\mathbb{F}_p}$$

and further arguments imply that also

$$\widehat{LW\Omega}_{R/\mathbb{F}_p}/p \cong \widehat{L\Omega}_{R/\mathbb{F}_p}.$$

For us, the key significance of the derived de Rham–Witt complex comes from the following relation to the construction $\mathbb{A}_{\mathrm{cris}}$ discussed in §2.

Theorem 3.5 ([BMS18], 8.14)**.** *For a quasiregular semiperfect $\mathbb{F}_p$-algebra S, there is a canonical identification*

$$\mathbb{A}_{\mathrm{cris}}(S) \cong LW\Omega_{S/\mathbb{F}_p}$$

that is compatible with the Nygaard filtrations; *in particular, one also has*

$$\widehat{\mathbb{A}}_{\mathrm{cris}}(S) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p}.$$

Proof sketch. One eventually bootstraps the conclusion from the identifications

$$\mathbb{A}_{\mathrm{cris}}(S)/p \cong L\Omega_{S/\mathbb{F}_p} \overset{3.4}{\cong} LW\Omega_{S/\mathbb{F}_p}/p,$$

the first of which follows from [Bha12], 3.27. A key reduction is to the case of the $\mathbb{F}_p$-algebra $S^\flat[X_i^{1/p^\infty} \mid i \in I]/(X_i \mid i \in I)$, where $I := \mathrm{Ker}(S^\flat \twoheadrightarrow S)$. □

4. The relation to $\pi_0(\mathrm{TP}(S))$

We fix a quasiregular semiperfect $\mathbb{F}_p$-algebra S and seek to review in Theorem 4.3 the identification (1.4.1). For this, we rely on the following lemmas.

Lemma 4.1 ([BMS18], 5.13)**.** *Letting* HP *indicate periodic cyclic homology, we have a natural identification*

$$\pi_0(\mathrm{HP}(S/\mathbb{F}_p)) \cong \widehat{L\Omega}_{S/\mathbb{F}_p}.$$

Proof sketch. One combines:

(1) the Tate spectral sequence that relates HP to the Hochschild homology HH;
(2) the Hochschild–Kostant–Rosenberg theorem that gives the identification

$$\pi_{2i}(\mathrm{HH}(S/\mathbb{F}_p)) \simeq \left(\bigwedge\nolimits^i \mathbb{L}_{S/\mathbb{F}_P}\right)[-i]. \qquad \square$$

Lemma 4.2 ([BMS18], 6.7)**.** *We have a natural identification*

$$\pi_0(\mathrm{TP}(S))/p \cong \pi_0(\mathrm{HP}(S/\mathbb{F}_p)).$$

Proof sketch. By Bökstedt's computation, one has the fiber sequence

$$\mathrm{THH}(\mathbb{F}_p)[2] \to \mathrm{THH}(\mathbb{F}_p) \to \mathrm{HH}(\mathbb{F}_p/\mathbb{F}_p).$$

By base changing to $\mathrm{THH}(S)$ over $\mathrm{THH}(\mathbb{F}_p)$, one obtains the fiber sequence

$$\mathrm{THH}(S)[2] \to \mathrm{THH}(S) \to \mathrm{HH}(S/\mathbb{F}_p).$$

Upon applying the Tate construction, the latter becomes the fiber sequence

$$\mathrm{TP}(S)[2] \xrightarrow{p\cdot\sigma} \mathrm{TP}(S) \to \mathrm{HP}(S/\mathbb{F}_p).$$

Since the odd homotopy groups vanish, one concludes by applying π_0. $\square$

Theorem 4.3 ([BMS18], 8.15)**.** *We have a natural identification*

$$\pi_0(\mathrm{TP}(S)) \cong \widehat{LW\Omega}_{S/\mathbb{F}_p} \overset{3.5}{\cong} \widehat{\mathbb{A}}_{\mathrm{cris}}(S).$$

Proof sketch. The lemmas imply the desired identification modulo p:

$$\pi_0(\mathrm{TP}(S))/p \cong \widehat{L\Omega}_{S/\mathbb{F}_p} \overset{3.4}{\cong} \widehat{LW\Omega}_{S/\mathbb{F}_p}/p \cong \widehat{\mathbb{A}}_{\mathrm{cris}}(S)/p.$$

To bootstrap from this, one relies on the universal property of $\mathbb{A}_{\mathrm{cris}}(S)$ via the identification $LW\Omega_{S/\mathbb{F}_p} \cong \mathbb{A}_{\mathrm{cris}}(S)$ of Theorem 3.5. The key intermediate case is that of

$$\mathbb{F}_p[T^{\pm 1/p^\infty}]/(T-1) \cong \mathbb{F}_p[\mathbb{Q}_p/\mathbb{Z}_p],$$

in which one uses the descent of the group algebra $\mathbb{F}_p[\mathbb{Q}_p/\mathbb{Z}_p]$ to its counterpart over the sphere spectrum in order to argue the identification

$$\mathrm{TP}(\mathbb{F}_p[\mathbb{Q}_p/\mathbb{Z}_p]) \cong \mathrm{HP}(\mathbb{Z}[\mathbb{Q}_p/\mathbb{Z}_p]). \qquad \square$$

References

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