

# THE MANIN CONSTANT IN THE SEMISTABLE CASE

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ABSTRACT. For an optimal modular parametrization  $J_0(n) \rightarrow E$  of an elliptic curve  $E$  over  $\mathbb{Q}$  of conductor  $n$ , Manin conjectured the agreement of two natural  $\mathbb{Z}$ -lattices in the  $\mathbb{Q}$ -vector space  $H^0(E, \Omega^1)$ . Multiple authors generalized his conjecture to higher dimensional newform quotients. We prove the Manin conjecture for semistable  $E$ , give counterexamples to all the proposed generalizations, and prove several semistable special cases of these generalizations. The proofs establish general relations between the integral  $p$ -adic étale and de Rham cohomologies of abelian varieties over  $p$ -adic fields and exhibit a new exactness result for Néron models.

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## 1. INTRODUCTION

By the modularity theorem, every elliptic curve  $E$  over  $\mathbb{Q}$  arises as a quotient

$$\pi: J_0(n) \twoheadrightarrow E$$

of the modular Jacobian  $J_0(n)$  with  $n$  equal to be the conductor of  $E$ . In this situation there are two natural  $\mathbb{Z}$ -lattices in the  $\mathbb{Q}$ -vector space  $\pi^*(H^0(E, \Omega^1))$  generated, respectively, by the pullback of a Néron differential  $\omega_E$  and by the 1-form  $f_E$  associated to the normalized newform determined by the isogeny class of  $E$ . The *Manin constant*  $c_\pi \in \mathbb{Q}^\times$ , defined by

$$\pi^*(\omega_E) = c_\pi \cdot f_E,$$

describes the difference between the two lattices and is the subject of the following conjecture of Manin (see §1.5 for a review of the terminology used in its formulation).

**Conjecture 1.1** ([Man71, 10.3]). *For an elliptic curve quotient*

$$\pi: J_0(n) \twoheadrightarrow E$$

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that is new in the sense that  $n$  equals the conductor of  $E$  and optimal in the sense that  $\text{Ker}(\pi)$  is connected, the two  $\mathbb{Z}$ -lattices in  $\pi^*(H^0(E, \Omega^1))$  described above agree; in other words,

$$c_\pi = \pm 1.$$

Substantial computational evidence supports the conjecture; for instance, Cremona proved that  $c_\pi = \pm 1$  whenever  $n \leq 390000$ , see [Cre16] (and also [ARS06, Thm. 2.6]). The main goal of this paper is to settle Conjecture 1.1 for semistable  $E$ ; more precisely, we prove the following result.

**Theorem 1.2** (Theorem 2.13). *For an  $n \in \mathbb{Z}_{\geq 1}$ , a subgroup  $H \subset \text{GL}_2(\widehat{\mathbb{Z}})$  with  $\Gamma_1(n) \subset H \subset \Gamma_0(n)$ , a new elliptic optimal quotient  $\pi: J_H \rightarrow E$ , and a prime  $p$ ,*

*if  $p^2 \nmid n$ , then  $\text{ord}_p(c_\pi) = 0$  and  $\pi$  induces a smooth morphism on Néron models over  $\mathbb{Z}_p$ .*

*In particular, Conjecture 1.1 holds in the case when  $E$  is semistable (that is, when  $n$  is squarefree).*

The proof of Theorem 1.2 is relatively short, does not rely on any previously known cases, is uniform for all  $p$ , and is given in §2. The key idea is to translate multiplicity one results for differentials in characteristic  $p$  into Hecke-freeness statements about the Lie algebra of the Néron model of  $J_0(n)$ . This provides control on congruences between the “ $f_E$ -isotypic” and the “(non- $f_E$ )-isotypic” parts of this Lie algebra, which combines with comparisons between the modular degree and the congruence number of  $E$  to imply the  $\Gamma_0(n)$  case of Theorem 1.2. The general case reduces to  $\Gamma_0(n)$ .

Previous results [Maz78, Cor. 4.1], [AU96, Thm. A and (ii) on p. 270], [ARS06, Thm. 2.7], and [Čes16, Thm. 1.5] cover many special cases of Theorem 1.2 (in particular, the case of an odd  $p$ ). Nevertheless, some semistable elliptic curves escape the net of these previous results: for instance, this happens for 130.a2, 4930.b1, 182410.a1, and many others in [LMFDB]. Beyond the semistable case, Edixhoven proved in [Edi91, Thm. 3] that a new elliptic optimal quotient  $\pi: J_0(n) \rightarrow E$  and a prime  $p \geq 11$  for which  $E_{\mathbb{Q}_p}$  does not have potentially ordinary reduction of Kodaira type II, III, or IV satisfy  $\text{ord}_p(c_\pi) = 0$ .

**1.3. Counterexamples to proposed generalizations.** Generalizations of Conjecture 1.1 to newform quotients of arbitrary dimension have been put forward by Conrad–Edixhoven–Stein [CES03, Conj. 6.1.7], Joyce [Joy05, Conj. 2], and Agashe–Ribet–Stein [ARS06, Conj. 3.12]. These more general conjectures are supported by a handful of examples [FLSSW01, §4.2], [ARS06, p. 624] and by the semistable case at odd  $p$  that follows from exactness properties of Néron models.

We prove in Theorem 5.10 that all these generalizations of the Manin conjecture fail (at the prime 2) for a 24-dimensional optimal newform quotient of  $J_0(431)$  and also for a 91-dimensional optimal newform quotient of  $J_0(2089)$ .<sup>1</sup> On the positive side, we prove a number of their semistable special cases in Theorem 5.19, which show that in the semistable setting the failure of the generalizations has to involve both the failure of mod  $p$  multiplicity one for  $J_0(n)$  and the failure of the maximality of the order  $\mathcal{O}_f$  determined by the newform  $f$  (in the elliptic curve case  $\mathcal{O}_f$  is always  $\mathbb{Z}$ ).

**1.4. The interplay between integral étale and de Rham cohomologies.** It is a natural idea that relations between the integral étale and de Rham cohomologies of abelian varieties over  $p$ -adic fields are relevant for the Manin conjecture: one guesses that the role of the optimality assumption is to supply exactness on  $H_{\text{ét}}^1(-, \widehat{\mathbb{Z}})$ , whereas similar exactness on  $H_{\text{dR}}^1(-/\mathbb{Z})$  would be related to exactness on Néron models implied by the Manin conjecture (more precisely, the Manin conjecture implies exactness on  $\text{Fil}^1(H_{\text{dR}}^1(-/\mathbb{Z}))$ , that is, on  $\text{Lie}(-/\mathbb{Z})^*$ ). However, the required transfer of

<sup>1</sup>Our counterexamples to the generalizations of the Manin conjecture rely on the correct functioning of the [Sage] commands used in the proof of Theorem 5.10.

an integral étale assumption to an integral de Rham conclusion has been problematic in the past, partly because comparisons of  $p$ -adic Hodge theory tend to fail integrally. The key novelty that underlies our approach is the idea that an arithmetic duality result of Raynaud, when reformulated as Theorem 3.4, supplies an integral link between the two cohomology theories.

Theorem 3.4 is very robust for our purposes and is the backbone of §§3–5. In addition to its role in the proofs of the results mentioned in §1.3, it eventually supplies the exactness on  $H_{\text{dR}}^1(-/\mathbb{Z}_p)$  in Corollary 3.14 and Theorem 5.19 (that is, in the cases of the Manin conjecture and of its generalization proved in this paper) and it leads to cohomology specialization results in the spirit of [BMS16] with no restrictions on the reduction type (see Proposition 3.15), to relations between torsion multiplicities in  $J_0(n)$  and Gorenstein defects of Hecke algebras (see Corollary 3.16), and to an exactness result for Néron models equipped with a “Hecke action” (see Corollary 4.8).

**1.5. Notation and conventions.** For an open subgroup  $H \subset \text{GL}_2(\widehat{\mathbb{Z}})$ , we denote the level  $H$  modular curve over  $\mathbb{Z}$  by  $X_H$  (see [Čes17, 6.1–6.3] for a review of  $X_H$ ), and we denote the Jacobian of  $(X_H)_{\mathbb{Q}}$  by  $J_H := \text{Pic}_{(X_H)_{\mathbb{Q}}/\mathbb{Q}}^0$ . For an  $n \in \mathbb{Z}_{\geq 1}$ , we let  $\Gamma_0(n)$  (resp.,  $\Gamma_1(n)$ ) denote the preimage of  $\left\{ \begin{pmatrix} * & \\ 0 & * \end{pmatrix} \right\} \subset \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$  (resp., of  $\left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \subset \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ ) in  $\text{GL}_2(\widehat{\mathbb{Z}})$ , and we set  $X_0(n) := X_{\Gamma_0(n)}$ , etc. For an  $H$  with  $\Gamma_1(n) \subset H \subset \Gamma_0(n)$ , a quotient  $\pi: J_H \rightarrow A$  is *optimal* (resp., *new* or *newform*) if  $\text{Ker } \pi$  is connected (resp., if up to isogeny  $\pi$  arises from some newform  $f_A$ ). We say that  $f_A$  is *normalized* if its  $q$ -expansion  $a_1q + a_2q^2 + \dots$  at the cusp “ $\infty$ ” has  $a_1 = 1$ . If  $\pi$  is new and optimal,  $f_A$  is normalized, and  $\dim A = 1$ , then we let  $c_\pi$  be the *Manin constant* defined by the equality  $\pi^*(\omega_A) = c_\pi \cdot f_A$ , where  $\omega_A$  is a Néron differential on  $A$  and we have identified  $f_A$  with its associated differential form on  $J_H$ . Up to a sign,  $c_\pi$  does not depend on the choice of  $\omega_A$ .

For a commutative ring  $\mathbb{T}$ , a maximal ideal  $\mathfrak{m} \subset \mathbb{T}$ , and a  $\mathbb{T}$ -module  $M$ , we let  $M_{\mathfrak{m}}$  denote the  $\mathfrak{m}$ -adic completion of  $M$  and we let  $M[\mathfrak{m}^\infty]$  denote the submodule of the elements of  $M$  killed by some power of  $\mathfrak{m}$ . We often consider  $\mathbb{Z}$ -torsion free  $\mathbb{T}$  and  $M$ , for which we repeatedly abuse notation:

$$M[e] := M \cap \text{Ker}(e: M_{\mathbb{Q}} \rightarrow M_{\mathbb{Q}}) \quad \text{for an idempotent } e \in \mathbb{T}_{\mathbb{Q}}. \quad (1.5.1)$$

We let  $\text{ord}_p$  denote the  $p$ -adic valuation normalized by  $\text{ord}_p(p) = 1$ . For a field  $K$ , we let  $\overline{K}$  denote a fixed algebraic closure of  $K$ . For a commutative ring  $R$  and a projective  $R$ -module  $P$ , we set  $P^* := \text{Hom}_R(P, R)$ . For a smooth group scheme  $G \rightarrow S$ , we let  $G^0$  (resp.,  $\text{Lie } G$ ) denote its relative identity component subfunctor (resp., its Lie algebra at the identity section), which in the situations below will always exist as a scheme. For a scheme  $S$  and an  $S$ -scheme  $X$ , we let  $X^{\text{sm}}$  denote the smooth locus of  $X$ . When denoting structure sheaves or sheaves of Kähler differentials, we omit subscripts that may be inferred from the context. We let  $(-)^{\vee}$  denote the dual of an abelian variety, or of a homomorphism of abelian varieties, or of a commutative finite locally free group scheme.

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## 2. THE SEMISTABLE CASE OF THE MANIN CONJECTURE

The main goal of this section is to prove the semistable case of the Manin conjecture in Theorem 2.11. The path to this consists of the notational review in §2.1, of parts (i) and (ii) of Proposition 2.2, and

then of the discussion in between §2.6 and Lemma 2.10. Modulo standard inputs from the literature, the overall argument is brief.

**2.1. The Hecke algebra  $\mathbb{T}$ .** Throughout §2 we fix an  $n \in \mathbb{Z}_{\geq 1}$  and for primes  $\ell \nmid n$  and  $\ell \mid n$ , respectively, we let  $T_\ell$  and  $U_\ell$  be the endomorphisms of  $J_0(n)$  induced via ‘‘Albanese functoriality’’ by their namesake correspondences (see [MW84, Ch. II, §§5.4–5.5]; in the notation there, we choose  $T_{\ell^*}$  and  $U_{\ell^*}$ ). The pullback action of  $T_\ell$  and  $U_\ell$  on  $H^0(J_0(n), \Omega^1)$  agrees with their ‘‘classical’’ action on the space of weight 2 cusp forms, see [MW84, Ch. II, §5.8] (this is our reason for preferring the Albanese functoriality). We let

$$\mathbb{T} \subset \text{End}_{\mathbb{Q}}(J_0(n))$$

be the commutative  $\mathbb{Z}$ -subalgebra generated by all the  $T_\ell$  and  $U_\ell$ , so  $\mathbb{T}$  acts on various objects naturally attached to  $J_0(n)$ , e.g.,

- on the Néron model  $\mathcal{J}$  over  $\mathbb{Z}$  of  $J_0(n)$ ;
- on the tangent space  $\text{Lie } \mathcal{J}$  of  $\mathcal{J}$  at the identity section and on the dual  $H^0(\mathcal{J}, \Omega^1) = (\text{Lie } \mathcal{J})^*$ .

If  $p$  is a prime with  $p^2 \nmid n$ , then, by [DR73, VI.6.7, VI.6.9], the curve  $X_0(n)_{\mathbb{Z}_p}$  is semistable over  $\mathbb{Z}_p$ , so that, by [BLR90, 9.7/2], we have

$$(\text{Pic}_{X_0(n)/\mathbb{Z}}^0)_{\mathbb{Z}_p} \cong \mathcal{J}_{\mathbb{Z}_p}^0.$$

In particular, by [BLR90, 8.4/1 (a)], [Con00, Cor. 5.1.3], and [Con00, Thm. B.4.1] (applied over  $\mathbb{Q}_p$ ), we get the identifications

$$(\text{Lie } \mathcal{J})_{\mathbb{Z}_p} \cong H^1(X_0(n)_{\mathbb{Z}_p}, \mathcal{O}) \quad \text{and} \quad H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1) \cong H^0(X_0(n)_{\mathbb{Z}_p}, \Omega) \quad (2.1.1)$$

that are compatible with duality pairings, where  $\Omega$  denotes the relative dualizing sheaf of  $X_0(n)_{\mathbb{Z}_p}$  over  $\mathbb{Z}_p$ . We transfer the  $\mathbb{T}$ -action across these identifications to endow  $H^1(X_0(n)_{\mathbb{Z}_p}, \mathcal{O})$  and  $H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)$  with a  $\mathbb{T}$ -module structure.

The following result lies at the heart of our approach to the semistable case of the Manin conjecture.

**Proposition 2.2.** *For a maximal ideal  $\mathfrak{m} \subset \mathbb{T}$  of residue characteristic  $p$ ,*

- (i) *if  $\text{ord}_p(n) = 0$ ; or*
- (ii) *if  $\text{ord}_p(n) = 1$  and  $U_p \bmod \mathfrak{m}$  lies in  $\mathbb{F}_p^\times \subset \mathbb{T}/\mathfrak{m}$  (the latter holds if  $\mathfrak{m}$  contains the kernel of the map  $q_f: \mathbb{T} \rightarrow \mathcal{O}_f$  determined by some newform  $f$  of level  $\Gamma_0(n)$  because  $q_f(U_p) = \pm 1$ ); or*
- (iii) *if  $\text{ord}_p(n) = 1$  and  $p$  is odd;*

*then the following equivalent conditions hold:*

- (1) *the  $\mathbb{T}_{\mathfrak{m}}$ -module  $(\text{Lie } \mathcal{J})_{\mathbb{Z}_p} \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}}$  is free of rank 1;*
- (2) *the  $\mathbb{T}_{\mathfrak{m}}$ -module  $H^1(X_0(n)_{\mathbb{Z}_p}, \mathcal{O}) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}}$  is free of rank 1;*
- (3) *multiplicity one for differentials holds at  $\mathfrak{m}$  in the sense that*

$$\dim_{\mathbb{T}/\mathfrak{m}}(H^0(X_0(n)_{\mathbb{F}_p}, \Omega)[\mathfrak{m}]) = 1.$$

*Proof.* The equivalence of (1) and (2) follows from (2.1.1). By the formalism of cohomology and base change (see [Ill05, 8.3.11]) and Grothendieck–Serre duality (see [Con00, Cor. 5.1.3]),

$$H^1(X_0(n)_{\mathbb{Z}_p}, \mathcal{O}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong H^1(X_0(n)_{\mathbb{F}_p}, \mathcal{O}) \quad \text{and} \quad H^0(X_0(n)_{\mathbb{Z}_p}, \Omega) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong H^0(X_0(n)_{\mathbb{F}_p}, \Omega);$$

the latter identification endows  $H^0(X_0(n)_{\mathbb{F}_p}, \Omega)$  with the  $\mathbb{T}$ -action used in (3). Therefore,

$$H^1(X_0(n)_{\mathbb{Z}_p}, \mathcal{O}) \otimes_{\mathbb{T}} \mathbb{T}/\mathfrak{m} \quad \text{is the } \mathbb{F}_p\text{-linear dual of} \quad H^0(X_0(n)_{\mathbb{F}_p}, \Omega)[\mathfrak{m}], \quad (2.2.1)$$

and it follows that (2) implies (3). Conversely, if (3) holds, then, due to (2.2.1), the Nakayama lemma supplies a  $\mathbb{T}_m$ -module surjection

$$s: \mathbb{T}_m \twoheadrightarrow H^1(X_0(n)_{\mathbb{Z}_p}, \mathcal{O}) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_m. \quad (2.2.2)$$

Since  $H^1(X_0(n)_{\mathbb{Z}_p}, \mathcal{O}) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_m$  is a faithful  $\mathbb{T}_m$ -module (see (2.1.1)), the map  $s$  is also injective, and hence is an isomorphism, which proves that (3) implies (2).

The arguments above also apply to the minimal regular resolution  $\widetilde{X_0(n)}_{\mathbb{Z}_p}$  in place of  $X_0(n)_{\mathbb{Z}_p}$ , so in the conditions (2)–(3) we could have instead used  $\widetilde{X_0(n)}_{\mathbb{Z}_p}$ . Therefore, the results of [ARS12, §5.2] (which use  $\widetilde{X_0(n)}_{\mathbb{Z}_p}$ ), specifically, [ARS12, Lemma 5.20],<sup>2</sup> show that either (i) or (ii) implies (3). Alternatively, (i) implies (1) by [Par99, Thm. 4.2].

The case (iii) will only be used in Remark 2.3 and Corollary 2.4, so the cases (i) and (ii) suffice for the main results of the paper. To address the case (iii), we now assume that  $p$  is odd with  $\text{ord}_p(n) = 1$ , and we seek to show (1), that is, that  $(\text{Lie } \mathcal{J})_m$  is free of rank 1 as a  $\mathbb{T}_m$ -module. Let

$$\pi_{\text{forg}}, \pi_{\text{quot}}: X_0(n)_{\mathbb{Q}} \rightrightarrows X_0\left(\frac{n}{p}\right)_{\mathbb{Q}}$$

be the degeneracy morphisms characterized as follows in terms of the moduli interpretation on the elliptic curve locus:  $\pi_{\text{forg}}$  forgets the  $p$ -primary factor of the cyclic subgroup of order  $n$ , whereas  $\pi_{\text{quot}}$  quotients the elliptic curve by this  $p$ -primary factor. We will consider the short exact sequence

$$0 \rightarrow K \rightarrow J_0(n) \xrightarrow{((\pi_{\text{forg}})_*, (\pi_{\text{quot}})_*)} J_0\left(\frac{n}{p}\right) \times J_0\left(\frac{n}{p}\right) \rightarrow 0 \quad (2.2.3)$$

in which  $(\pi_{\text{forg}})_*$  and  $(\pi_{\text{quot}})_*$  are induced by the Albanese functoriality, the surjectivity follows from [Rib84, Cor. 4.2], and  $K$  is defined as the kernel. By loc. cit. and [LO91, Thm. 2], the component group scheme  $K/K^0$  is constant, whereas the identity component  $K^0$  is identified with the  $p$ -new subvariety of  $J_0(n)$ . The maps  $(\pi_{\text{forg}})_*$  and  $(\pi_{\text{quot}})_*$  commute with the Hecke operators  $T_\ell$  and  $U_\ell$  provided that  $\ell \neq p$ , so they intertwine the actions of the “ $p$ -anemic” Hecke algebras

$$\mathbb{T}^{(p)} \subset \text{End}(J_0(n)) \quad \text{and} \quad \mathbb{T}_{p\text{-old}}^{(p)} \subset \text{End}\left(J_0\left(\frac{n}{p}\right)\right)$$

that are generated by these operators, and hence they define a surjective ring homomorphism

$$\mathbb{T}^{(p)} \twoheadrightarrow \mathbb{T}_{p\text{-old}}^{(p)}, \quad T_\ell \mapsto T_\ell, \quad U_\ell \mapsto U_\ell. \quad (2.2.4)$$

Moreover, since  $p$  is odd and does not divide  $\frac{n}{p}$ , one knows from [Wil95, Lemma on p. 491] that  $T_p \in \mathbb{T}_{p\text{-old}}^{(p)}$ , that is, that  $\mathbb{T}_{p\text{-old}}^{(p)}$  is in fact the full Hecke algebra  $\mathbb{T}_{p\text{-old}} \subset \text{End}\left(J_0\left(\frac{n}{p}\right)\right)$ .

To determine the endomorphism of  $J_0\left(\frac{n}{p}\right) \times J_0\left(\frac{n}{p}\right)$  that intertwines  $U_p \in \text{End}(J_0(n))$ , we regard  $Y_0(n)_{\mathbb{Q}}$  as the coarse moduli space of pairs

$$(\phi: E_1 \rightarrow E_2, C \subset E_1),$$

where  $\phi$  is a  $p$ -isogeny of elliptic curves and  $C$  is a cyclic subgroup of order  $\frac{n}{p}$ , so that the  $p$ -Atkin–Lehner involution  $w_p$  sends  $(\phi, C)$  to  $(\phi^\vee, \phi(C))$ . In terms of this interpretation,  $U_p$  quotients  $E_1$  by a variable subgroup  $C'_p \subset E_1$  of order  $p$  such that  $C'_p \cap \text{Ker } \phi = 0$ . Therefore,  $U_p + w_p$  sends

$$(\phi: E_1 \rightarrow E_2, C \subset E_1) \quad \text{to} \quad \sum (\psi: E_3 \rightarrow E_1, \psi^{-1}(C) \subset E_3),$$

<sup>2</sup>The key inputs to the proof of loc. cit. are an Eichler–Shimura type congruence relation for  $U_p$  in the style of [Wil80, §5] and arguments from [Wil95, proof of Lemma 2.2] that use the  $q$ -expansion principle as in [Maz77, pp. 94–95].

where the sum runs over the isomorphism classes of all  $p$ -isogenies that cover  $E_1$ . In particular,<sup>3</sup>

$$U_p + w_p = (\pi_{\text{quot}})^* \circ (\pi_{\text{forg}})_* \quad \text{inside} \quad \text{End}(J_0(n)).$$

Thus, since  $T_p = (\pi_{\text{forg}})_* \circ (\pi_{\text{quot}})^*$  in  $\text{End}(J_0(\frac{n}{p}))$  (the switch to ‘‘Picard functoriality’’ here does not matter because  $(p, \frac{n}{p}) = 1$ , see [MW84, Ch. II, §5.4 (2)]) and  $(\pi_{\text{quot}})_* \circ (\pi_{\text{quot}})^* = p + 1$ ,

$$U_p \text{ intertwines the endomorphism } (x, y) \mapsto (T_p x - y, p x) \text{ of } J_0(\frac{n}{p}) \times J_0(\frac{n}{p}). \quad (2.2.5)$$

The  $p$ -new subvariety  $K^0$  is isogenous to a product of newform quotients (with multiplicities) of variable  $J_0(n')$  for divisors  $n' \mid n$  such that  $n' \nmid \frac{n}{p}$ , that is, such that  $p \mid n'$ . Since the  $U_p$  operator commutes with the degeneracy maps towards such  $J_0(n')$ , it acts as  $\pm 1$  on each simple isogeny factor of  $K^0$ . In particular, since  $(\text{Lie } J_0(n))_{\mathbb{Q}_p} \simeq \mathbb{T}_{\mathbb{Q}_p}$  as  $\mathbb{T}_{\mathbb{Q}_p}$ -modules (see [DDT97, 1.34]), the factors  $\mathbb{T}_{\mathfrak{m}'}$  of  $\mathbb{T}_{\mathbb{Z}_p}$  that meet the support of  $(\text{Lie } K)_{\mathbb{Q}_p}$  must satisfy either  $U_p + 1 \in \mathfrak{m}'$  or  $U_p - 1 \in \mathfrak{m}'$ . By (ii), we may assume  $\mathbb{T}_{\mathfrak{m}}$  is not such factor, to the effect that

$$(\text{Lie } K) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} = 0. \quad (2.2.6)$$

Since  $p$  is odd and  $K/K^0$  is constant, the maps  $J_0(n) \rightarrow J_0(n)/K^0$  and  $J_0(n)/K^0 \rightarrow J_0(n)/K$  induce smooth morphisms on Néron models over  $\mathbb{Z}_p$  (see [BLR90, 7.5/4 (ii) and its proof, 7.5/6]). Thus, the sequence (2.2.3) induces a short exact sequence on Lie algebras of Néron models over  $\mathbb{Z}_p$ :

$$0 \rightarrow (\text{Lie } K)_{\mathbb{Z}_p} \rightarrow (\text{Lie } \mathcal{J})_{\mathbb{Z}_p} \rightarrow (\text{Lie } \mathcal{J}' \times \text{Lie } \mathcal{J}')_{\mathbb{Z}_p} \rightarrow 0.$$

The vanishing (2.2.6) then implies that

$$(\text{Lie } \mathcal{J})_{\mathfrak{m}} \xrightarrow{\sim} (\text{Lie } \mathcal{J}' \times \text{Lie } \mathcal{J}')_{\mathfrak{m}}, \quad (2.2.7)$$

so the maximal ideal  $\mathfrak{m}^{(p)} := \mathfrak{m} \cap \mathbb{T}^{(p)}$  of  $\mathbb{T}^{(p)}$  is such that  $(\text{Lie } \mathcal{J}' \times \text{Lie } \mathcal{J}')_{\mathfrak{m}^{(p)}} \neq 0$ , to the effect that  $\mathfrak{m}^{(p)}$  is ‘‘ $p$ -old,’’ that is, is identified with a maximal ideal  $\mathfrak{m}^{(p)} \subset \mathbb{T}_{p\text{-old}}$  via (2.2.4) (and the sentence that follows (2.2.4)). Thus, the case (i) implies that the  $\mathbb{T}^{(p)}/\mathfrak{m}^{(p)}$ -vector space  $(\text{Lie } \mathcal{J}' \times \text{Lie } \mathcal{J}')/\mathfrak{m}^{(p)}$  is of dimension 2. Moreover, due to the formula (2.2.5), the  $U_p$  operator does not act as a  $\mathbb{T}^{(p)}/\mathfrak{m}^{(p)}$ -scalar on this vector space, so the  $\mathbb{T}/\mathfrak{m}$ -vector space  $(\text{Lie } \mathcal{J})/\mathfrak{m}$ , which, due to (2.2.7), is a further quotient of  $(\text{Lie } \mathcal{J}' \times \text{Lie } \mathcal{J}')/\mathfrak{m}^{(p)}$ , is of dimension  $\leq 1$ . It then follows from the Nakayama lemma as in the paragraph after (2.2.2) that  $(\text{Lie } \mathcal{J})_{\mathfrak{m}}$  is free of rank 1 as a  $\mathbb{T}_{\mathfrak{m}}$ -module, as desired.  $\square$

**Remark 2.3.** In [ARS12, §5.2.1], based on computational evidence, Agashe, Ribet, and Stein asked whether (iii) implies (3) and showed that this is not the case if the parity condition is dropped in (iii). There they also implicitly raised the question answered by part (iii) of the following corollary.

**Corollary 2.4.** *For a maximal ideal  $\mathfrak{m} \subset \mathbb{T}$  of residue characteristic  $p$ ,*

- (i) *if  $\text{ord}_p(n) = 0$ ; or*
- (ii) *if  $\text{ord}_p(n) = 1$  and  $U_p \bmod \mathfrak{m}$  lies in  $\mathbb{F}_p^\times \subset \mathbb{T}/\mathfrak{m}$ ; or*
- (iii) *if  $\text{ord}_p(n) = 1$  and  $p$  is odd;*

*then the saturation  $\mathbb{T}' := \mathbb{T}_{\mathbb{Q}} \cap \text{End}_{\mathbb{Q}}(J_0(n))$  of  $\mathbb{T}$  agrees with  $\mathbb{T}$  at  $\mathfrak{m}$ , that is,*

$$\mathbb{T}_{\mathfrak{m}} \xrightarrow{\sim} \mathbb{T}'_{\mathfrak{m}}.$$

*In particular, if  $n$  is odd and squarefree, then the inclusion  $\mathbb{T} \hookrightarrow \mathbb{T}'$  is an isomorphism.*

<sup>3</sup>Compare with the formula in [Rib90, proof of Prop. 3.7] that used the ‘‘Picard functoriality’’ Hecke operators.



*Proof.* Since  $\mathbb{T}'$  acts on  $\text{Lie } \mathcal{J}$  faithfully and  $\mathbb{T}$ -linearly,  $\mathbb{T}'_{\mathfrak{m}}$  acts on

$$(\text{Lie } \mathcal{J})_{\mathbb{Z}_p} \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \stackrel{2.2}{\cong} \mathbb{T}_{\mathfrak{m}}$$

faithfully and  $\mathbb{T}_{\mathfrak{m}}$ -linearly, and the desired conclusion follows.  $\square$

**Remark 2.5.** When  $n$  is a prime, the last claim of Corollary 2.4 also follows from [Maz77, II.9.5].

For our purposes, the significance of Proposition 2.2, especially of its case (ii), is the resulting control of the congruence module  $\text{cong}_{\text{Lie } \mathcal{J}}$  introduced in Definition 2.7 with the following setup.

**2.6. A modular elliptic curve.** For the rest of §2 we

- fix a new elliptic optimal quotient  $\pi: J_0(n) \rightarrow E$ ;
- let  $f \in H^0(X_0(n)_{\mathbb{Q}}, \Omega^1)$  be the normalized newform determined by  $\pi$ ;
- let  $e_f \in \mathbb{T}_{\mathbb{Q}}$  be the idempotent that cuts out the factor of  $\mathbb{T}_{\mathbb{Q}}$  that corresponds to  $f$ ;
- let  $e_{f^\perp} := 1 - e_f$  be the complementary idempotent in  $\mathbb{T}_{\mathbb{Q}}$ .

The idempotents  $e_f$  and  $e_{f^\perp}$  decompose every  $\mathbb{T}$ -module  $M$  rationally:

$$M_{\mathbb{Q}} \cong M_{\mathbb{Q}}[e_f] \oplus M_{\mathbb{Q}}[e_{f^\perp}].$$

The following congruence module measures the failure of an analogous integral decomposition.

**Definition 2.7.** The *congruence module* of a  $\mathbb{Z}$ -torsion free  $\mathbb{T}$ -module  $M$  is the quotient (see (1.5.1))

$$\text{cong}_M := \frac{M}{M[e_f] + M[e_{f^\perp}]}.$$

**Example 2.8.** With the choice  $M = \prod_p H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$ , the congruence module  $\text{cong}_M$  is isomorphic to  $(\mathbb{Z}/\text{deg}_f \mathbb{Z})^2$ , where the *modular degree*  $\text{deg}_f$  is the positive integer that equals  $\pi \circ \pi^\vee$  in  $\text{End}_{\mathbb{Q}}(E)$  (so that  $e_f = \frac{\pi \circ \pi^\vee}{\text{deg}_f}$ ). Indeed, this follows from the optimality of  $\pi$  and from the observation that  $H_{\text{ét}}^1((-)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$  carries short exact sequences of abelian varieties to those of finite free  $\mathbb{Z}_p$ -modules.

This example leads to the following lemma, whose proof is a variant of the proof of [DDT97, Lem. 4.17] (the quotient  $\mathcal{O}/\eta_\Sigma$  used there is a congruence module). The lemma is well known and also follows from [AU96, Lem. 3.2], [CK04, Thm. 1.1], or [ARS12, Thm. 2.1].

**Lemma 2.9.** *In the setup of §2.6, we have  $\text{deg}_f \mid \#(\text{cong}_{\mathbb{T}})$ .*

*Proof.* For every prime  $p$ , the  $\mathbb{T}_{\mathbb{Q}_p}$ -module  $H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$  is free of rank 2 (see [DDT97, Lem. 1.38–1.39]), so the module

$$\frac{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)}{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)[e_f]} \quad \text{is free of rank 2 over} \quad (\mathbb{T}/\mathbb{T}[e_f])_{\mathbb{Z}_p} \cong \mathbb{Z}_p.$$

In particular,  $\text{cong}_{\mathbb{T}_{\mathbb{Z}_p}}^{\oplus 2}$  surjects onto  $\text{cong}_{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)}$  and the claim follows from Example 2.8.  $\square$

We are ready to exploit Proposition 2.2 in the proof of the following key lemma, which will give us the semistable case of the Manin conjecture in Theorem 2.11 and whose proof will simultaneously reprove [ARS12, Thm. 2.1]. An alternative route would be to deduce Theorem 2.11 from the results of §5, e.g., from Theorem 5.19, that are valid for newform quotients of arbitrary dimension.

**Lemma 2.10.** *For every prime  $p$  such that  $p^2 \nmid n$ ,*

$$\mathrm{ord}_p(\#(\mathrm{cong}_{\mathrm{Lie} \mathcal{J}})) = \mathrm{ord}_p(\mathrm{deg}_f), \quad (2.10.1)$$

*the map  $(\mathrm{Lie} \mathcal{J})_{\mathbb{Z}_p} \rightarrow (\mathrm{Lie} \mathcal{E})_{\mathbb{Z}_p}$  is surjective, and  $((\mathrm{Lie} \pi^\vee)(\mathrm{Lie} \mathcal{E}))_{\mathbb{Z}_p} \xrightarrow{\sim} (\mathrm{Lie} \mathcal{J})_{\mathbb{Z}_p}[e_{f^\perp}]$ .*

*Proof.* The formation of the  $\mathbb{T}$ -module

$$\frac{\mathrm{Lie} \mathcal{J}}{(\mathrm{Lie} \mathcal{J})[e_f] + (\mathrm{Lie} \mathcal{J})[e_{f^\perp}]}$$

commutes with (flat) base change to  $\mathbb{Z}_p$ , so we let  $\mathfrak{n}$  range over the maximal ideals of  $\mathbb{T}$  of residue characteristic  $p$  and decompose  $\mathbb{T}_{\mathbb{Z}_p} \cong \prod \mathbb{T}_{\mathfrak{n}}$ .

Let  $\mathfrak{m} \subset \mathbb{T}$  be the preimage of  $p\mathbb{Z}$  under the surjection  $\mathbb{T} \rightarrow \mathbb{Z}$  determined by  $f$ . The image of  $e_f$  in  $\mathbb{T}_{\mathbb{Z}_p}$  lies in  $(\mathbb{T}_{\mathfrak{m}})_{\mathbb{Q}_p}$ , so for every  $\mathfrak{n} \neq \mathfrak{m}$  the “ $(\mathbb{T}_{\mathfrak{n}})_{\mathbb{Q}_p}$ -coordinate” of  $e_f$  vanishes, to the effect that

$$\mathbb{T}_{\mathfrak{n}} = \mathbb{T}_{\mathfrak{n}}[e_f] \quad \text{and} \quad (\mathrm{Lie} \mathcal{J}) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{n}} = ((\mathrm{Lie} \mathcal{J}) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{n}})[e_f]$$

for every such  $\mathfrak{n}$ . Consequently,

$$\left( \frac{\mathrm{Lie} \mathcal{J}}{(\mathrm{Lie} \mathcal{J})[e_f] + (\mathrm{Lie} \mathcal{J})[e_{f^\perp}]} \right) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \frac{(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}}}{(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}}[e_f] + (\mathrm{Lie} \mathcal{J})_{\mathfrak{m}}[e_{f^\perp}]} \stackrel{2.2}{\simeq} \frac{\mathbb{T}_{\mathfrak{m}}}{\mathbb{T}_{\mathfrak{m}}[e_f] + \mathbb{T}_{\mathfrak{m}}[e_{f^\perp}]} \cong \left( \frac{\mathbb{T}}{\mathbb{T}[e_f] + \mathbb{T}[e_{f^\perp}]} \right) \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

so that

$$\mathrm{ord}_p(\#(\mathrm{cong}_{\mathrm{Lie} \mathcal{J}})) = \mathrm{ord}_p(\#(\mathrm{cong}_{\mathbb{T}})).$$

When combined with Lemma 2.9, this gives the inequality “ $\geq$ ” in (2.10.1).

For the converse inequality, we let  $\mathcal{E}$  denote the Néron model of  $E$  over  $\mathbb{Z}$ , observe that the injection

$$\mathrm{Lie} \pi: \frac{\mathrm{Lie} \mathcal{J}}{(\mathrm{Lie} \mathcal{J})[e_f]} \hookrightarrow \mathrm{Lie} \mathcal{E} \quad \text{gives} \quad \frac{\mathrm{Lie} \mathcal{J}}{(\mathrm{Lie} \mathcal{J})[e_f] + (\mathrm{Lie} \pi^\vee)(\mathrm{Lie} \mathcal{E})} \hookrightarrow \frac{\mathrm{Lie} \mathcal{E}}{\mathrm{deg}_f \cdot \mathrm{Lie} \mathcal{E}} \simeq \frac{\mathbb{Z}}{\mathrm{deg}_f \cdot \mathbb{Z}},$$

and conclude by using the inclusion  $(\mathrm{Lie} \pi^\vee)(\mathrm{Lie} \mathcal{E}) \subset (\mathrm{Lie} \mathcal{J})[e_{f^\perp}]$ .  $\square$

**Theorem 2.11.** *For a new elliptic optimal quotient  $\pi: J_0(n) \rightarrow E$ , the Manin constant  $c_\pi$  satisfies*

$$\mathrm{ord}_p(c_\pi) = 0 \quad \text{for every prime } p \text{ such that } p^2 \nmid n.$$

*Proof.* By Lemma 2.10, the map  $(\mathrm{Lie} \mathcal{J})_{\mathbb{Z}_p} \rightarrow (\mathrm{Lie} \mathcal{E})_{\mathbb{Z}_p}$  is surjective. Thus,  $\frac{H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)}{\pi^*(H^0(\mathcal{E}_{\mathbb{Z}_p}, \Omega^1))}$  is torsion free, that is, that the pullback  $\pi^*(\omega_E)$  of a Néron differential of  $E$  is not divisible by  $p$  in  $H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)$ .

The  $p$ -Atkin–Lehner involution sends  $f$  to  $\pm f$ , so  $f$  lies in

$$H^0(X_0(n)_{\mathbb{Z}_p}, \Omega) \stackrel{(2.1.1)}{\cong} H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)$$

(see [Čes16, 2.7–2.8]). Moreover, by definition,  $\pi^*(\omega_E) = c_\pi \cdot f$  and, by [Edi91, Prop. 2], we have  $c_\pi \in \mathbb{Z}$ . Thus, since  $\pi^*(\omega_E)$  is not divisible by  $p$  in  $H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)$ , the desired  $\mathrm{ord}_p(c_\pi) = 0$  follows.  $\square$

We end §2 with the semistable case of the analogue of the Manin conjecture for parametrizations by modular curves intermediate between  $X_1(n)$  and  $X_0(n)$  (see Theorem 2.13). The following variant of [Čes16, Lem. 4.4] (or of [GV00, Prop. 3.3]) reduces this analogue to the Manin conjecture itself.

**Lemma 2.12.** *Let  $H, H' \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$  be subgroups such that  $\Gamma_1(n) \subset H \subset H' \subset \Gamma_0(n)$  for some  $n \in \mathbb{Z}_{\geq 1}$ , and let*

$$\pi: J_H \rightarrow E \quad \text{and} \quad \pi': J_{H'} \rightarrow E'$$



be new elliptic optimal quotients such that  $E$  and  $E'$  are isogenous over  $\mathbb{Q}$ . There is a unique isogeny  $e$  making the diagram

$$\begin{array}{ccc} J_H & \xrightarrow{\pi} & E \\ \downarrow j^\vee & & \downarrow e \\ J_{H'} & \xrightarrow{\pi'} & E' \end{array} \quad (2.12.1)$$

commute, where  $j^\vee$  is the dual of the pullback map  $j: J_{H'} \rightarrow J_H$ . The kernel  $\text{Ker } e$  is constant and is a subquotient of the Cartier dual of the Shimura subgroup  $\Sigma(n) \subset J_0(n)$ . The Manin constants  $c_\pi$  and  $c_{\pi'}$  are nonzero integers related by the equality

$$c_{\pi'} = c_\pi \cdot \# \text{Coker}(\text{Lie } e: \text{Lie } \mathcal{E} \rightarrow \text{Lie } \mathcal{E}'), \quad (2.12.2)$$

where  $\mathcal{E}$  and  $\mathcal{E}'$  are the Néron models over  $\mathbb{Z}$  of  $E$  and  $E'$ , respectively.

*Proof.* The existence (resp., uniqueness) of  $e$  follows from the multiplicity one theorem (resp., from the surjectivity of  $\pi$ ). For the rest, we lose no generality by assuming that  $H' = \Gamma_0(n)$ . Then  $\text{Ker}(e)^\vee$  is a subgroup of

$$\Sigma(n) := \text{Ker}(J_0(n) \rightarrow J_1(n)).$$

Since  $\Sigma(n)$  is of multiplicative type (see [LO91, Thm. 2]), the claims about  $\text{Ker } e$  follow. The formation of the normalized newform  $f \in H^0(J_{H'}, \Omega^1)$  determined by the isogeny class of  $E$  is compatible with pullback by  $j^\vee$  (see [Čes16, 2.8]), so the comparison of the two ways to pull back a Néron differential of  $E'$  to  $J_H$  in (2.12.1) gives (2.12.2). For the integrality of  $c_\pi$  it then suffices to assume that  $H = \Gamma_1(n)$  and to apply [Ste89, Thm. 1.6].  $\square$

**Theorem 2.13.** *For an  $n \in \mathbb{Z}_{\geq 1}$ , a subgroup  $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$  such that  $\Gamma_1(n) \subset H \subset \Gamma_0(n)$ , and a new elliptic optimal quotient  $\pi: J_H \twoheadrightarrow E$ , if  $p$  is a prime with  $p^2 \nmid n$ , then the Manin constant  $c_\pi$  satisfies  $\text{ord}_p(c_\pi) = 0$  and  $\pi$  induces a smooth morphism  $(\mathcal{J}_H)_{\mathbb{Z}_p} \rightarrow \mathcal{E}_{\mathbb{Z}_p}$  between the Néron models over  $\mathbb{Z}_p$ .*

*Proof.* Let  $\pi': J_0(n) \twoheadrightarrow E'$  be the new elliptic optimal quotient for which  $E'$  is isogenous to  $E$  and let  $e: E \rightarrow E'$  be the isogeny supplied by Lemma 2.12. By Theorem 2.11,  $\text{ord}_p(c_{\pi'}) = 0$ , so, since  $c_\pi \in \mathbb{Z}$ , (2.12.2) implies that  $\text{ord}_p(c_\pi) = 0$ . Thus, since the elements of  $H^0(\mathcal{J}_H, \Omega^1)$  have integral  $q$ -expansions (see [CES03, proof of Lem. 6.1.6]), it follows that the pullback  $\pi^*(\omega_E)$  of a Néron differential of  $E$  is not divisible by  $p$  in  $H^0((\mathcal{J}_H)_{\mathbb{Z}_p}, \Omega^1)$ . Consequently,  $(\text{Lie } \mathcal{J}_H)_{\mathbb{Z}_p} \rightarrow (\text{Lie } \mathcal{E})_{\mathbb{Z}_p}$  is surjective and the smoothness claim follows.  $\square$

One consequence of Theorem 2.13 is the following additivity of Faltings height.

**Corollary 2.14.** *For an  $H$  as in Theorem 2.13 and a new elliptic optimal quotient  $\pi: J_H \twoheadrightarrow E$  such that  $E$  is semistable, the Faltings height  $h(-)$  over  $\mathbb{Q}$  satisfies*

$$h(J_H) = h(\text{Ker } \pi) + h(E).$$

*Proof.* Theorem 2.13 applies at every  $p$ , so it implies that the map  $\mathcal{J}_H \rightarrow \mathcal{E}$  is smooth and hence, by [BLR90, 7.1/6], that the Néron model  $\mathcal{K}$  over  $\mathbb{Z}$  of  $\text{Ker } \pi$  is its kernel. It follows that the sequences

$$0 \rightarrow \text{Lie } \mathcal{K} \rightarrow \text{Lie } \mathcal{J}_H \rightarrow \text{Lie } \mathcal{E} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow H^0(\mathcal{E}, \Omega^1) \rightarrow H^0(\mathcal{J}_H, \Omega^1) \rightarrow H^0(\mathcal{K}, \Omega^1) \rightarrow 0$$

are short exact, so the arguments from [Ull00, proof of Prop. 3.3] give the conclusion.  $\square$

### 3. A LINK BETWEEN THE INTEGRAL $p$ -ADIC ÉTALE AND DE RHAM COHOMOLOGIES

We will analyze the semistable case of a generalization of the Manin conjecture to higher dimensional newform quotients in §5, and this will rest on the present section which establishes relations between the integral  $p$ -adic étale and de Rham cohomologies of abelian varieties over  $p$ -adic fields. Our point of view is that arithmetic duality in the form of a result of Raynaud [Ray85, Thm. 2.1.1], which we recast and mildly generalize in Theorems 3.4 and 3.6, is capable of supplying such relations.

**3.1. The field  $K$ .** Throughout §3 we fix a mixed characteristic  $(0, p)$  complete discretely valued field  $K$  whose residue field  $k$  is perfect. We denote the ring of integers of  $K$  by  $\mathcal{O}$  and we denote Néron models over  $\mathcal{O}$  by calligraphic letters: for instance,  $\mathcal{A}$  and  $\mathcal{B}$  are the Néron models over  $\mathcal{O}$  of abelian  $K$ -varieties  $A$  and  $B$ , whereas  $\mathcal{A}^\vee$  is the Néron model of the dual abelian variety  $A^\vee$ .

The following de Rham lattice  $H_{\text{dR}}^1(-/\mathcal{O})$  constructed by Mazur and Messing will be key for our purposes. In order to emphasize its functoriality, we review its definition.

**3.2. The integral  $H_{\text{dR}}^1$  of an abelian variety.** Let  $A$  be an abelian variety over  $K$ . By [MM74, I.5.1], the Néron model  $\mathcal{A}^\vee$  of the dual abelian variety  $A^\vee$  is identified with the fppf sheaf

$$\mathcal{E}xt^1(\mathcal{A}^0, \mathbb{G}_m) \quad \text{defined by} \quad S \mapsto \text{Ext}_S^1(\mathcal{A}_S^0, (\mathbb{G}_m)_S)$$

on the category of smooth  $\mathcal{O}$ -schemes  $S$ , where  $\text{Ext}_S^1(\mathcal{A}_S^0, (\mathbb{G}_m)_S)$  is the abelian group of extensions  $0 \rightarrow (\mathbb{G}_m)_S \rightarrow \mathcal{E} \rightarrow \mathcal{A}_S^0 \rightarrow 0$  of fppf sheaves of abelian groups.<sup>4</sup> One also considers the sheaf

$$\mathcal{E}xtrig^1(\mathcal{A}^0, \mathbb{G}_m) \quad \text{defined by} \quad S \mapsto \text{Extrig}_S^1(\mathcal{A}_S^0, (\mathbb{G}_m)_S)$$

on the category of smooth  $\mathcal{O}$ -schemes  $S$ , where  $\text{Extrig}_S^1(\mathcal{A}_S^0, (\mathbb{G}_m)_S)$  is the abelian group of extensions as before that are, in addition, equipped with an  $S$ -morphism  $r$ , a rigidification, that fits into a commutative diagram

$$\begin{array}{ccccccc} & & & & \text{Spec}(\mathcal{O}_{\mathcal{A}_S^0}/\mathcal{I}_{e_{\mathcal{A}_S^0}}^2) & & \\ & & & & \downarrow i & & \\ 0 & \longrightarrow & (\mathbb{G}_m)_S & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{A}_S^0 \longrightarrow 0 \\ & & & & \swarrow r & & \uparrow \\ & & & & & & \end{array}$$

in which  $i$  is the first infinitesimal neighborhood of the identity section of  $\mathcal{A}_S^0$ . By [MM74, I.5.2], the functor  $\mathcal{E}xtrig^1(\mathcal{A}^0, \mathbb{G}_m)$  is representable by a smooth  $\mathcal{O}$ -group scheme that fits into a short exact sequence

$$0 \rightarrow \mathbb{V}(\text{Lie } \mathcal{A}) \rightarrow \mathcal{E}xtrig^1(\mathcal{A}^0, \mathbb{G}_m) \rightarrow \mathcal{A}^\vee \rightarrow 0 \tag{3.2.1}$$

in which  $\mathbb{V}(\text{Lie } \mathcal{A})$  is the affine smooth  $\mathcal{O}$ -group scheme that represents the finite free  $\mathcal{O}$ -module  $(\text{Lie } \mathcal{A})^*$  (see [SGA 3I new, I, 4.6.3.1]). By [MM74, I.2.6.7], the  $K$ -base change of the sequence (3.2.1) is identified with the universal vector extension of  $A^\vee$ , so

$$\text{Lie}(\mathcal{E}xtrig^1(\mathcal{A}^0, \mathbb{G}_m)_K) \quad \text{is identified with} \quad H_{\text{dR}}^1(A/K).$$

Therefore, the finite free  $\mathcal{O}$ -module

$$H_{\text{dR}}^1(\mathcal{A}/\mathcal{O}) := \text{Lie}(\mathcal{E}xtrig^1(\mathcal{A}^0, \mathbb{G}_m))$$

is a natural integral structure on  $H_{\text{dR}}^1(A/K)$ . If  $A$  has good reduction, that is, if  $\mathcal{A}$  is an abelian scheme, then, by [MM74, I.§4, esp., I.3.2.3 a), I.4.2.1, and I.4.1.7], the  $\mathcal{O}$ -module  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})$  is identified with the first de Rham cohomology group of  $\mathcal{A}$  over  $\mathcal{O}$ . By construction, the formation

<sup>4</sup>For the sheaf condition of  $\mathcal{E}xt^1(\mathcal{A}^0, \mathbb{G}_m)$ , it is key that every  $0 \rightarrow (\mathbb{G}_m)_S \rightarrow \mathcal{E} \rightarrow \mathcal{A}_S^0 \rightarrow 0$  has no nonidentity automorphisms, as may be checked over  $S_K$  due to the  $\mathcal{O}$ -flatness of  $S$  (each  $\mathcal{E}$  is necessarily a smooth  $S$ -scheme).

of (3.2.1) and of  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})$  is contravariantly functorial in  $A$  and so is the formation of the exact sequence

$$0 \rightarrow (\text{Lie } \mathcal{A})^* \rightarrow H_{\text{dR}}^1(\mathcal{A}/\mathcal{O}) \rightarrow \text{Lie } \mathcal{A}^\vee \rightarrow 0 \quad (3.2.2)$$

of finite free  $\mathcal{O}$ -modules obtained from (3.2.1).

**3.3. The normalized length  $\text{val}_{\mathcal{O}}(-)$ .** For a finitely generated torsion  $\mathcal{O}$ -module  $M$ , we set

$$\text{val}_{\mathcal{O}}(M) := \frac{1}{e(K/\mathbb{Q}_p)} \cdot \text{length}_{\mathcal{O}}(M), \quad \text{where} \quad e(K/\mathbb{Q}_p) := \text{length}_{\mathcal{O}}(\mathcal{O}/p)$$

is the absolute ramification index of  $K$ . For a bounded complex  $(M_\bullet, d_\bullet)$  of finitely generated  $\mathcal{O}$ -modules such that  $(M_\bullet)_K$  is exact, we set

$$\text{val}_{\mathcal{O}}(M_\bullet) = \sum_i (-1)^i \cdot \text{val}_{\mathcal{O}} \left( \frac{\text{Ker}(d_i: M_i \rightarrow M_{i-1})}{\text{Im}(d_{i+1}: M_{i+1} \rightarrow M_i)} \right). \quad (3.3.1)$$

We prefer the formalism of normalized length  $\text{val}_{\mathcal{O}}(-)$  to that of length  $\text{length}_{\mathcal{O}}(-)$  because the former is insensitive to base change to the ring of integers of a finite extension of  $K$ .

We are ready for the following variant of [Ray85, Thm. 2.1.1] that (through the proof of loc. cit.) uses arithmetic duality results of Bégueri [Bég80]. In its formulation, with an eye towards applications to modular Jacobians, we keep track of an action of a ‘‘Hecke algebra’’  $\mathbb{T}$ ; the basic case is  $\mathbb{T} = \mathbb{Z}$ .

**Theorem 3.4.** *Let  $\mathbb{T}$  be a commutative ring that is finite free as a  $\mathbb{Z}$ -module, let  $A$  and  $B$  be abelian varieties over  $K$  endowed with a  $\mathbb{T}$ -action, and let  $f: A \rightarrow B$  be a  $\mathbb{T}$ -equivariant  $K$ -isogeny. For every maximal ideal  $\mathfrak{m} \subset \mathbb{T}$  of residue characteristic  $p$ , we have*

$$\text{val}_{\mathbb{Z}_p} \left( \frac{H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}}}{f^*(H_{\text{ét}}^1(B_{\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}})} \right) = \text{val}_{\mathcal{O}} \left( \frac{H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})_{\mathfrak{m}}}{f^*(H_{\text{dR}}^1(\mathcal{B}/\mathcal{O})_{\mathfrak{m}})} \right); \quad (3.4.1)$$

in addition, both sides of this equality are equal to  $\text{ord}_p(\#(\text{Ker } f)[\mathfrak{m}^\infty])$ .

*Proof.* Since  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)$  is identified with the  $\mathbb{Z}_p$ -linear dual of the  $p$ -adic Tate module  $T_p(A_{\overline{K}})$  compatibly with the  $\mathbb{T}_{\mathbb{Z}_p}$ -action, and likewise for  $B$ , the left side of (3.4.1) equals  $\text{ord}_p(\#(\text{Ker } f)[\mathfrak{m}^\infty])$ .

If we ignore the  $\mathbb{T}$ -action, more precisely, if we take  $\mathbb{T} = \mathbb{Z}$ , then [Ray85, Thm. 2.1.1] gives

$$\text{val}_{\mathcal{O}} \left( \frac{\text{Lie } \mathcal{B}}{(\text{Lie } f)(\text{Lie } \mathcal{A})} \right) + \text{val}_{\mathcal{O}} \left( \frac{\text{Lie } \mathcal{A}^\vee}{(\text{Lie } f^\vee)(\text{Lie } \mathcal{B}^\vee)} \right) = \text{val}_{\mathcal{O}} \left( \frac{\mathcal{O}}{\#(\text{Ker } f)} \right) = \text{ord}_p(\#(\text{Ker } f)). \quad (3.4.2)$$

Thus, in the  $\mathbb{T} = \mathbb{Z}$  case (3.4.1) follows from (3.4.2) and from the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{Lie } \mathcal{B})^* & \longrightarrow & H_{\text{dR}}^1(\mathcal{B}/\mathcal{O}) & \longrightarrow & \text{Lie } \mathcal{B}^\vee \longrightarrow 0 \\ & & \downarrow (\text{Lie } f)^* & & \downarrow f^* & & \downarrow \text{Lie}(f^\vee) \\ 0 & \longrightarrow & (\text{Lie } \mathcal{A})^* & \longrightarrow & H_{\text{dR}}^1(\mathcal{A}/\mathcal{O}) & \longrightarrow & \text{Lie } \mathcal{A}^\vee \longrightarrow 0. \end{array} \quad (3.4.3)$$

In the general case, both sides of (3.4.1) are additive in composites of isogenies, so we factor  $f$  to assume that  $\text{Ker } f$ , and hence also the left side of (3.4.1), is supported entirely at  $\mathfrak{m}$ . Then we use the decomposition  $\mathbb{T}_{\mathbb{Z}_p} \cong \prod_{\mathfrak{n}} \mathbb{T}_{\mathfrak{n}}$ , where  $\mathfrak{n}$  ranges over the maximal ideals of  $\mathbb{T}$  of residue characteristic  $p$ , to find a  $t \in \mathbb{T}$  that kills  $\text{Ker } f$  but pulls back to a unit in every  $\mathbb{T}_{\mathfrak{n}}$  with  $\mathfrak{n} \neq \mathfrak{m}$  and to a unit in  $\mathbb{T}_{\mathbb{Z}_\ell}$  for some fixed auxiliary prime  $\ell \neq p$ . This  $\ell$ -adic assumption ensures that multiplication by  $t$  is a self-isogeny of  $A$  (of degree prime to  $\ell$ ), whereas the inclusion  $\text{Ker } f \subset \text{Ker } t$  translates into a factorization  $t = g \circ f$  for some isogeny  $g: B \rightarrow A$ . It then follows from the choice of  $t$  that for every  $\mathfrak{n} \neq \mathfrak{m}$  the injection

$$f^*: H_{\text{dR}}^1(\mathcal{B}/\mathcal{O})_{\mathfrak{n}} \hookrightarrow H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})_{\mathfrak{n}}$$

must be surjective. In conclusion, the quotient  $\frac{H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})}{f^*(H_{\text{dR}}^1(\mathcal{B}/\mathcal{O}))}$  is also supported entirely at  $\mathfrak{m}$ , to the effect that (3.4.1) follows from its special  $\mathbb{T} = \mathbb{Z}$  case.  $\square$

**Remark 3.5.** Both sides of (3.4.1) are additive in composites of isogenies and are equal for the multiplication by  $n$ , so “ $\leq$ ” in (3.4.1) implies the equality. In the good reduction case this inequality may be deduced from integral  $p$ -adic Hodge theory (instead of arithmetic duality [Ray85, Thm. 2.1.1]): one may use the results of [BMS16] to build an  $A_{\text{inf}}$ -module  $\frac{H_{A_{\text{inf}}}^1(\mathcal{A})}{f^*(H_{A_{\text{inf}}}^1(\mathcal{B}))}$  that in the key  $\mathbb{T} = \mathbb{Z}$  case realizes the quotient on the right side of (3.4.1) as a specialization of that on the left side.

In order to make Theorem 3.4 applicable more broadly, we turn to its following variant.

**Theorem 3.6.** *Let*

$$A_{\bullet} = \dots \rightarrow A_{i-1} \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \rightarrow \dots,$$

where  $d_i \circ d_{i-1} = 0$  for every  $i$ , be a complex of abelian varieties over  $K$ . Suppose that

- (i) the complex  $A_{\bullet}$  is bounded in the sense that  $A_i = 0$  for all but finitely many  $i$ ; and
- (ii) the complex  $A_{\bullet}$  is exact up to isogeny in the sense that  $\text{Im } d_{i-1} = (\text{Ker } d_i)^0$  for every  $i$ .

Under these assumptions, with the notation of (3.3.1) we have

$$\text{val}_{\mathbb{Z}_p} (H_{\text{ét}}^1((A_{\bullet})_{\overline{K}}, \mathbb{Z}_p)) = \text{val}_{\mathcal{O}} (H_{\text{dR}}^1(\mathcal{A}_{\bullet}/\mathcal{O})), \quad (3.6.1)$$

where the complex  $H_{\text{ét}}^1((A_{\bullet})_{\overline{K}}, \mathbb{Z}_p)$  has the term  $H_{\text{ét}}^1((A_i)_{\overline{K}}, \mathbb{Z}_p)$  in degree  $i$  and similarly for  $H_{\text{dR}}^1(\mathcal{A}_{\bullet}/\mathcal{O})$ ; in addition, both sides of (3.6.1) are equal to

$$\sum_i (-1)^i \text{ord}_p \left( \# \left( \frac{\text{Ker } d_i}{\text{Im } d_{i-1}} \right) \right),$$

so that, in particular, if  $A_{\bullet}$  is exact, then both sides of (3.6.1) vanish.

The reduction of Theorem 3.6 to Theorem 3.4 rests on the following lemma, which, in addition, shows that after inverting  $p$  the étale and the de Rham complexes appearing in (3.6.1) are exact.

**Lemma 3.7.** *For an  $A_{\bullet}$  that satisfies (i) and (ii), there is a  $K$ -morphism*

$$f_{\bullet} : (\tilde{A}_{\bullet}, \tilde{d}_{\bullet}) \rightarrow (A_{\bullet}, d_{\bullet})$$

such that each  $f_i$  is an isogeny and  $\tilde{A}_{\bullet}$  is a split complex of abelian varieties over  $K$  in the sense that there are  $K$ -isomorphisms  $\tilde{A}_i \cong \text{Ker } \tilde{d}_i \times_K \text{Ker } \tilde{d}_{i+1}$  that are compatible with the differentials  $\tilde{d}_i$ .

*Proof.* The maps  $d_i : A_i \rightarrow \text{Im } d_i$  are surjective, so the Poincaré complete reducibility theorem (see [Con06, Cor. 3.20]) supplies an abelian subvariety  $B_i \subset A_i$  for which  $(d_i)|_{B_i} : B_i \rightarrow \text{Im } d_i$  is an isogeny. By letting  $f_i$  be the sum of  $(d_{i-1})|_{B_{i-1}}$  and of  $B_i \hookrightarrow A_i$ , we obtain the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & B_{i-1} \times_K B_i & \xrightarrow{\tilde{d}_i} & B_i \times_K B_{i+1} & \longrightarrow & \dots \\ & & \downarrow f_i & & \downarrow f_{i+1} & & \\ \dots & \longrightarrow & A_i & \xrightarrow{d_i} & A_{i+1} & \longrightarrow & \dots \end{array}$$

whose top row is the candidate  $\tilde{A}_{\bullet}$ . Each  $\text{Ker } f_i$  is finite because so is each  $\text{Im } d_{i-1} \cap B_i$ , and for every  $i$  we have

$$\dim A_i = \dim B_i + \dim B_{i-1}.$$

In conclusion, each  $f_i$  is an isogeny. □

*Proof of Theorem 3.6.* Lemma 3.7 supplies a split bounded complex  $\tilde{A}_\bullet$  of abelian varieties over  $K$  and a  $K$ -isogeny  $f_\bullet: \tilde{A}_\bullet \rightarrow A_\bullet$ . For this split  $\tilde{A}_\bullet$ , both sides of (3.6.1) vanish. Thus, the additivity of  $\text{val}_{\mathbb{Z}_p}(-)$  and  $\text{val}_{\mathcal{O}}(-)$  in short exact sequences of complexes and Theorem 3.4 give (3.6.1). It also follows that the étale side of (3.6.1) is

$$\sum_i (-1)^{i+1} \text{ord}_p(\#(\text{Ker } f_i)),$$

which is the negative of the value of the expression

$$\sum_i (-1)^i \text{ord}_p \left( \# \left( \frac{\text{Ker } \tilde{d}_i}{\text{Im } \tilde{d}_{i-1}} \right) \right) \quad \text{for} \quad (\text{Ker } f_\bullet, \tilde{d}_\bullet).$$

Since this expression is additive in short exact sequences of complexes, the claim about the value of both sides of (3.6.1) follows.  $\square$

### Remarks.

**3.8.** In Theorem 3.6, suppose that the  $A_i$  are endowed with an action of a commutative ring  $\mathbb{T}$  that is finite free as a  $\mathbb{Z}$ -module and that the  $d_i$  are  $\mathbb{T}$ -equivariant. Then, under the assumption that each  $\text{Im } d_{i-1}$  has a  $\mathbb{T}$ -stable isogeny complement  $B_i \subset A_i$ , the proof of Theorem 3.6 gives a  $\mathbb{T}$ -equivariant conclusion: for a maximal ideal  $\mathfrak{m} \subset \mathbb{T}$  of residue characteristic  $p$ , we have

$$\text{val}_{\mathbb{Z}_p} \left( H_{\text{ét}}^1((A_\bullet)_{\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}} \right) = \text{val}_{\mathcal{O}} \left( H_{\text{dR}}^1(\mathcal{A}_\bullet/\mathcal{O})_{\mathfrak{m}} \right).$$

**3.9.** The étale and the de Rham complexes of Theorem 3.6 are exact after inverting  $p$  and perfect, so, as in [KM76, Ch. II, esp. pp. 47–48], they have associated Cartier divisors

$$\text{Div}(H_{\text{ét}}^1(\mathcal{A}_\bullet, \mathbb{Z}_p)) \quad \text{on} \quad \text{Spec } \mathbb{Z}_p \quad \text{and} \quad \text{Div}(H_{\text{dR}}^1(\mathcal{A}_\bullet/\mathcal{O})) \quad \text{on} \quad \text{Spec } \mathcal{O}.$$

Therefore, [KM76, Thm. 3 (vi)] reformulates (3.6.1) into the following relation between the degrees of those divisors:

$$e(K/\mathbb{Q}_p) \cdot \deg_{\mathbb{Z}_p}(\text{Div}(H_{\text{ét}}^1((A_\bullet)_{\overline{K}}, \mathbb{Z}_p))) = \deg_{\mathcal{O}}(\text{Div}(H_{\text{dR}}^1(\mathcal{A}_\bullet/\mathcal{O}))).$$

**3.10.** The étale side of (3.6.1) (or of (3.4.1)) is invariant under passage to a finite extension of  $K$ . Thus, even though the Néron models may change, the de Rham side is invariant as well.

**Example 3.11.** By Theorem 3.6, for a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

of abelian varieties over  $K$ , both sides of (3.6.1) vanish. Thanks to the filtrations (3.2.2), this vanishing of the de Rham side means that

$$\text{length}_{\mathcal{O}} \left( \frac{\text{Lie } \mathcal{C}}{\text{Im}(\text{Lie } \mathcal{B})} \right) - \text{length}_{\mathcal{O}} \left( \frac{\text{Ker}(\text{Lie } \mathcal{B} \rightarrow \text{Lie } \mathcal{C})}{\text{Lie } \mathcal{A}} \right) = \text{length}_{\mathcal{O}} \left( \frac{\text{Lie } \mathcal{A}^\vee}{\text{Im}(\text{Lie } \mathcal{B}^\vee)} \right) - \text{length}_{\mathcal{O}} \left( \frac{\text{Ker}(\text{Lie } \mathcal{B}^\vee \rightarrow \text{Lie } \mathcal{A}^\vee)}{\text{Lie } \mathcal{C}^\vee} \right).$$

It is explained in [LLR04, proof of Thm. 2.1] how one associates a smooth finite type  $\bar{k}$ -group scheme  $D$  (resp.,  $D'$ ) to the morphism  $\mathcal{B} \rightarrow \mathcal{C}$  (resp.,  $\mathcal{B}^\vee \rightarrow \mathcal{A}^\vee$ ) in such a way that

$$D(\bar{k}) \cong \text{Coker}(\mathcal{B}(\widehat{\mathcal{O}^{\text{sh}}}) \rightarrow \mathcal{C}(\widehat{\mathcal{O}^{\text{sh}}})) \quad (\text{resp.}, \quad D'(\bar{k}) \cong \text{Coker}(\mathcal{B}^\vee(\widehat{\mathcal{O}^{\text{sh}}}) \rightarrow \mathcal{A}^\vee(\widehat{\mathcal{O}^{\text{sh}}})),$$

where  $\widehat{\mathcal{O}^{\text{sh}}}$  denotes the completion of the strict Henselization of  $\mathcal{O}$ . From this optic, by [LLR04, Thm. 2.1 (b)] and [BLR90, 7.1/6], Theorem 3.6 proves the equality

$$\dim D = \dim D'.$$

Example 3.11 together with the exactness results [BLR90, 7.5/4 (ii)] and [AU96, Thm. A.1] of Raynaud leads us to the following corollary (compare with [AU96, Cor. A.2]).

**Corollary 3.12.** *Suppose that  $e(K/\mathbb{Q}_p) \leq p - 1$  and let*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

*be a short exact sequence of abelian varieties over  $K$  such that  $B$  has semiabelian reduction. Then the sequences*

$$0 \rightarrow \mathrm{Lie} \mathcal{A} \rightarrow \mathrm{Lie} \mathcal{B} \rightarrow \mathrm{Lie} \mathcal{C} \quad \text{and} \quad 0 \rightarrow \mathrm{Lie} \mathcal{C}^\vee \rightarrow \mathrm{Lie} \mathcal{B}^\vee \rightarrow \mathrm{Lie} \mathcal{A}^\vee$$

*are left exact and both  $\frac{\mathrm{Lie} \mathcal{C}}{\mathrm{Im}(\mathrm{Lie} \mathcal{B})}$  and  $\frac{\mathrm{Lie} \mathcal{A}^\vee}{\mathrm{Im}(\mathrm{Lie} \mathcal{B}^\vee)}$  are of the form  $(\mathcal{O}/p\mathcal{O})^r$  with the same  $r \in \mathbb{Z}_{\geq 0}$ .  $\square$*

**Remark 3.13.** In Corollary 3.12, suppose that  $A$ ,  $B$ , and  $C$  are endowed with an action of a commutative ring  $\mathbb{T}$  that is finite free as a  $\mathbb{Z}$ -module, that the sequence is  $\mathbb{T}$ -equivariant, and that there is a  $\mathbb{T}$ -stable abelian subvariety  $C' \subset B$  that maps isogenously to  $C$ . Then Remark 3.8 leads to a further  $\mathbb{T}$ -equivariant conclusion: for a maximal ideal  $\mathfrak{m} \subset \mathbb{T}$  of residue characteristic  $p$ ,

$$\mathrm{Coker}((\mathrm{Lie} \mathcal{B})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{C})_{\mathfrak{m}}) \simeq \mathrm{Coker}((\mathrm{Lie} \mathcal{B}^\vee)_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{A}^\vee)_{\mathfrak{m}}).$$

Corollary 3.12 gives the following consequence of the semistable case of the Manin conjecture. For odd  $p$ , this consequence also follows from exactness properties of Néron models [BLR90, 7.5/4].

**Corollary 3.14.** *For a new elliptic optimal quotient  $\pi: J_0(n) \twoheadrightarrow E$  and a prime  $p$ , if  $p^2 \nmid n$ , then  $\pi$  (resp.,  $\pi^\vee$ ) induces a smooth morphism (resp., a closed immersion) on Néron models over  $\mathbb{Z}_p$  and the sequence*

$$0 \rightarrow H_{\mathrm{dR}}^1(\mathcal{E}/\mathbb{Z}_p) \rightarrow H_{\mathrm{dR}}^1(\mathcal{J}/\mathbb{Z}_p) \rightarrow H_{\mathrm{dR}}^1(\mathcal{K}/\mathbb{Z}_p) \rightarrow 0 \quad (3.14.1)$$

*is short exact, where  $\mathcal{E}$ ,  $\mathcal{J}$ , and  $\mathcal{K}$  denote the Néron models over  $\mathbb{Z}_p$  of  $E$ ,  $J_0(n)$ , and  $\mathrm{Ker} \pi$ .*

*Proof.* Theorem 2.13 gives the claim about  $\pi$ . It then follows from Corollary 3.12 that the map  $J_0(n)^\vee \rightarrow (\mathrm{Ker} \pi)^\vee$  also induces a smooth morphism on Néron models over  $\mathbb{Z}_p$ . Thus, by [BLR90, 7.1/6], the map  $\pi^\vee$  induces a closed immersion. Then the Lie algebra complexes that (via (3.2.2)) comprise the graded pieces of the Hodge filtration of (3.14.1) are short exact, so (3.14.1) must be, too.  $\square$

The proof of the following result illustrates Theorem 3.6 beyond isogenies and short exact sequences.

**Proposition 3.15.** *Let  $A$  be an abelian variety over  $K$  endowed with an action of a commutative ring  $\mathbb{T}$  that is finite free as a  $\mathbb{Z}$ -module. For every ideal  $\mathfrak{n} \subset \mathbb{T}$  such that  $n \in \mathfrak{n}$  for some  $n \in \mathbb{Z}_{\geq 1}$ ,*

$$\mathrm{val}_{\mathbb{Z}_p} \left( \frac{H_{\mathrm{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)}{\mathfrak{n} \cdot H_{\mathrm{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)} \right) \leq \mathrm{val}_{\mathcal{O}} \left( \frac{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})}{\mathfrak{n} \cdot H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})} \right). \quad (3.15.1)$$

The following heuristic suggests (3.15.1): one may hope that a suitable formalism of integral  $p$ -adic Hodge theory would realize the quotient on the right side of (3.15.1) as a specialization of the one on the left side, and (normalized) length cannot decrease under specialization. In the good reduction case, one can indeed prove Proposition 3.15 in this way by using the results of [BMS16].

*Proof.* We choose generators  $n_1, \dots, n_m \in \mathbb{T}$  of  $\mathfrak{n}$  and consider the complex of abelian varieties

$$A \xrightarrow{(n_1, \dots, n_m)} \prod_{i=1}^m A \xrightarrow{q} Q, \quad \text{where } Q \text{ is defined to be the cokernel of the first map.}$$

This complex is exact up to isogeny because the kernel of the first map is killed by  $n$ . Therefore, Theorem 3.6 applies and gives the following equality, which implies (3.15.1):

$$\mathrm{val}_{\mathbb{Z}_p} \left( \frac{H_{\mathrm{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)}{\mathfrak{n} \cdot H_{\mathrm{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)} \right) = \mathrm{val}_{\mathcal{O}} \left( \frac{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})}{\mathfrak{n} \cdot H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})} \right) - \mathrm{val}_{\mathcal{O}} \left( \frac{\mathrm{Ker}((n_1, \dots, n_m)^*: \prod_{i=1}^m H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O}) \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O}))}{q^*(H_{\mathrm{dR}}^1(\mathcal{Q}/\mathcal{O}))} \right). \quad \square$$

We end §3 by applying Proposition 3.15 in the context of modular curves to exhibit a relation (3.16.1) between the multiplicity of the mod  $\mathfrak{m}$  torsion of a modular Jacobian and the Gorenstein defect of the  $\mathfrak{m}$ -adic completion of the Hecke algebra. In a variety of settings, relations of this sort that are sharper than (3.16.1) are known—see [KW08, §§1–2] for an overview. Nevertheless, Corollary 3.16, especially its case (ii), seems to cover some situations that are not addressed in the literature.

**Corollary 3.16.** *Fix an  $n \in \mathbb{Z}_{\geq 1}$ , let  $\mathbb{T} \subset \text{End}_{\mathbb{Q}}(J_0(n))$  be the Hecke algebra defined in §2.1, let  $\mathfrak{m} \subset \mathbb{T}$  be a maximal ideal of residue characteristic  $p$ , and let  $\mathcal{J}$  be the Néron model of  $J_0(n)$  over  $\mathbb{Z}$ . Suppose that  $(\text{Lie } \mathcal{J})_{\mathfrak{m}}$  is free of rank 1 as a  $\mathbb{T}_{\mathfrak{m}}$ -module, e.g., that either of the following holds:*

- (i)  $p \nmid n$ ; or
- (ii)  $p^2 \nmid n$  and  $p$  is odd; or
- (iii)  $\text{ord}_p(n) = 1$  and  $U_p \bmod \mathfrak{m}$  lies in  $\mathbb{F}_p^{\times} \subset \mathbb{T}/\mathfrak{m}$ .

Then

$$\dim_{\mathbb{T}/\mathfrak{m}}(J_0(n)[\mathfrak{m}]) \leq \dim_{\mathbb{T}/\mathfrak{m}}((\mathbb{T}/p\mathbb{T})[\mathfrak{m}]) + 1 \quad (3.16.1)$$

and  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein if and only if  $H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}$  is free (of rank 2) as a  $\mathbb{T}_{\mathfrak{m}}$ -module if and only if  $H_{\text{dR}}^1(\mathcal{J}/\mathbb{Z}_p)_{\mathfrak{m}}$  is free (of rank 2) as a  $\mathbb{T}_{\mathfrak{m}}$ -module.

*Proof.* Proposition 2.2 shows that either of (i)–(iii) implies the assumed  $\mathbb{T}_{\mathfrak{m}}$ -freeness of  $(\text{Lie } \mathcal{J})_{\mathfrak{m}}$ .

Let  $w$  denote the Atkin–Lehner involution of  $J_0(n)$ , so that the induced action of  $\mathbb{T}$  on  $J_0(n)^{\vee}$  is identified with the action of  $w\mathbb{T}w$  on  $J_0(n)$  via the isomorphism

$$\mathbb{T} \xrightarrow{t \mapsto wt w} w\mathbb{T}w \quad \text{and the inverse} \quad J_0(n)^{\vee} \xrightarrow{\sim} J_0(n)$$

of the canonical principal polarization (see [MW84, Ch. II, §5.6 (c)]). The automorphism

$$\text{Lie } w: \text{Lie } \mathcal{J} \xrightarrow{\sim} \text{Lie } \mathcal{J}$$

intertwines the actions of  $\mathbb{T}$  and of  $w\mathbb{T}w$ , so the freeness of  $(\text{Lie } \mathcal{J})_{\mathfrak{m}}$  implies that of  $(\text{Lie } \mathcal{J}^{\vee})_{\mathfrak{m}}$ .<sup>5</sup> The filtration (3.2.2) then gives a (necessarily split) extension

$$0 \rightarrow \text{Hom}_{\mathbb{Z}_p}(\mathbb{T}_{\mathfrak{m}}, \mathbb{Z}_p) \rightarrow H_{\text{dR}}^1(\mathcal{J}/\mathbb{Z}_p)_{\mathfrak{m}} \rightarrow \mathbb{T}_{\mathfrak{m}} \rightarrow 0 \quad (3.16.2)$$

of  $\mathbb{T}_{\mathfrak{m}}$ -modules, which proves that

$$\dim_{\mathbb{T}/\mathfrak{m}} \left( \frac{H_{\text{dR}}^1(\mathcal{J}/\mathbb{Z}_p)}{\mathfrak{m} \cdot H_{\text{dR}}^1(\mathcal{J}/\mathbb{Z}_p)} \right) = \dim_{\mathbb{T}/\mathfrak{m}}((\mathbb{T}/p\mathbb{T})[\mathfrak{m}]) + 1. \quad (3.16.3)$$

The modified Weil pairing  $(x, y) \mapsto \langle x, wy \rangle_{\text{Weil}}$  shows that  $H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$  is  $\mathbb{T}_{\mathbb{Z}_p}$ -equivariantly isomorphic to its own  $\mathbb{Z}_p$ -linear dual (see [DDT97, Lem. 1.38]), so

$$\dim_{\mathbb{T}/\mathfrak{m}} \left( \frac{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)}{\mathfrak{m} \cdot H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)} \right) = \dim_{\mathbb{T}/\mathfrak{m}}(J_0(n)[\mathfrak{m}]). \quad (3.16.4)$$

The combination of (3.16.3), (3.16.4), and Proposition 3.15 implies (3.16.1). If  $H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}$  is free as a  $\mathbb{T}_{\mathfrak{m}}$ -module (necessarily of rank 2, see [DDT97, Lem. 1.39]), then

$$\mathbb{T}_{\mathfrak{m}}^{\oplus 2} \simeq (\text{Hom}_{\mathbb{Z}_p}(\mathbb{T}_{\mathfrak{m}}, \mathbb{Z}_p))^{\oplus 2} \quad \text{as } \mathbb{T}_{\mathfrak{m}}\text{-modules,}$$

so  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein. Conversely, if  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein, then, due to (3.16.2), the  $\mathbb{T}_{\mathfrak{m}}$ -module  $H_{\text{dR}}^1(\mathcal{J}/\mathbb{Z}_p)_{\mathfrak{m}}$  is free of rank 2, and hence, by Proposition 3.15 and the Nakayama lemma (compare with the paragraph that follows (2.2.2)), so is  $H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}$ .  $\square$

### Remarks.

<sup>5</sup>To stress the Hecke functoriality we let  $\mathcal{J}^{\vee}$  denote the Néron model of  $J_0(n)^{\vee}$  over  $\mathbb{Z}$ , even though  $\mathcal{J}^{\vee} \cong \mathcal{J}$ .



**3.17.** We are not aware of examples in which the inequality (3.16.1) is sharp.

**3.18.** Corollary 3.16 leads to examples of non-Gorenstein  $\mathbb{T}_{\mathfrak{m}}$ . For instance, if an odd  $n$  is squarefree and  $\dim_{\mathbb{T}/\mathfrak{m}}(J_0(n)[\mathfrak{m}]) > 2$  for some  $\mathfrak{m}$  (as happens for  $n = 19 \cdot 41$  and an Eisenstein  $\mathfrak{m}$  of residue characteristic 5, see [Yoo16, Ex. 4.7]), then, by Corollary 3.16, the ring  $\mathbb{T}_{\mathfrak{m}}$  is not Gorenstein.

#### 4. THE ÉTALE AND THE DE RHAM CONGRUENCES

The goal of this section is to use the results of §3, especially Theorem 3.4, to derive an exactness result for Néron models in Corollary 4.8 in the setting of an abelian variety equipped with a “Hecke action.” This exactness result and its applicability criterion given by Proposition 4.10 will be useful in §5 for proving new cases of the generalization of the Manin conjecture to higher dimensional newform quotients. To capture the relevant axiomatics, we begin with an abstract local setup.

**4.1. An abelian variety equipped with rational idempotents.** Throughout §4, we fix

- a mixed characteristic  $(0, p)$  complete discretely valued field  $K$  whose residue field  $k$  is perfect;
- an abelian variety  $A$  over  $K$  and its Néron model  $\mathcal{A}$  over the ring of integers  $\mathcal{O}$  of  $K$ ;
- idempotents  $e_1, e_2 \in \text{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$  that satisfy  $e_1 + e_2 = 1$ ;
- a commutative subring  $\mathbb{T} \subset \text{End}_K(A)$  whose elements commute with  $e_1$  and  $e_2$ .

As in §3, we let calligraphic letters indicate Néron models over  $\mathcal{O}$  (see §3.1).

We are primarily interested in the case when  $e_1, e_2 \notin \text{End}_K(A)$ —then various “rational objects,” e.g.,  $p$ -adic étale and de Rham cohomologies, attached to  $A$  decompose into summands cut out by the  $e_i$ , but their integral counterparts typically do not decompose, which produces interesting congruences. As we will see in §5, Jacobians of modular curves provide a rich supply of examples of the situation above. Similarly to Theorem 3.4, remembering  $\mathbb{T}$  leads to finer statements than with  $\mathbb{T} = \mathbb{Z}$ , and this will be important in applications to newform quotients of modular Jacobians.

**4.2. The étale congruences.** The *étale congruence module* is the finite quotient

$$\frac{H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)}{H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_1] + H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_2]}$$

(see (1.5.1) for the  $(-)[e_i]$  notation), and the *étale congruence number* is its order. The ring  $\mathbb{T}_{\mathbb{Z}_p}$  acts, and the étale congruence module decomposes compatibly with the decomposition  $\mathbb{T}_{\mathbb{Z}_p} \cong \prod \mathbb{T}_{\mathfrak{m}}$ , where  $\mathfrak{m} \subset \mathbb{T}$  ranges over the maximal ideals of residue characteristic  $p$ .

**4.3. The de Rham congruences.** The *de Rham congruence module* is the finite quotient

$$\frac{H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})}{H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})[e_1] + H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})[e_2]}$$

that we temporarily denote by  $M$  (see §3.2 for a review of the lattice  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{O}) \subset H_{\text{dR}}^1(A/K)$ ), and the *de Rham congruence number* is  $p^{\text{val}_{\mathcal{O}}(M)}$  with  $\text{val}_{\mathcal{O}}(\cdot)$  of §3.3. Like its étale counterpart,  $M$  decomposes into  $\mathfrak{m}$ -primary pieces compatibly with  $\mathbb{T}_{\mathbb{Z}_p} \cong \prod \mathbb{T}_{\mathfrak{m}}$ .

We will see in Corollary 4.8 that relations between the étale and the de Rham congruence numbers are intricately linked to exactness properties of Néron models and in Remark 4.9 that the two numbers differ in general. We begin with a key comparison result.

**Theorem 4.4.** *In the setting of §4.1, for every maximal ideal  $\mathfrak{m} \subset \mathbb{T}$  of residue characteristic  $p$ ,*

$$\mathrm{val}_{\mathbb{Z}_p} \left( \left( \frac{H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)}{H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_1] + H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_2]} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right) \geq \mathrm{val}_{\mathcal{O}} \left( \left( \frac{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})}{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})[e_1] + H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})[e_2]} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right).$$

The following notation will be useful for the proof of Theorem 4.4 and for the subsequent discussion.

**4.5. The abelian varieties  $A_i$  and  $Q_i$ .** For an  $i \in \{1, 2\}$ ,

- we let  $A_i \subset A$  be the image of  $A$  under any  $\mathbb{Z}_{>0}$ -multiple of  $e_i$  that lies in  $\mathrm{End}_K(A)$ ;
- we set  $Q_i := A/A_i$ .

Effectively,  $A_i$  is the abelian subvariety of  $A$  “cut out by  $e_i$ ” and is  $\mathbb{T}$ -stable, so  $A_i$  and  $Q_i$  inherit a  $\mathbb{T}$ -action. The inclusion and quotient homomorphisms  $j_i: A_i \hookrightarrow A$  and  $q_i: A \rightarrow Q_i$  are  $\mathbb{T}$ -equivariant.

**Remark 4.6.** Any abelian subvariety  $B \subset A$  is cut out (in the sense of §4.5) by some idempotent  $e \in \mathrm{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Indeed, this property is isogeny invariant—if  $f: A \rightarrow A'$  is an isogeny and  $e$  cuts out  $B$ , then  $\frac{1}{\deg f} \cdot f \circ e \circ f'$  cuts out  $f(B)$ , where  $f': A' \rightarrow A$  is the isogeny such that  $f' \circ f = \deg f$ —and it clearly holds for the abelian variety  $B \times_K A/B$  that is isogenous to  $A$ .

*Proof of Theorem 4.4.* By construction,

$$q_i^*(H_{\acute{e}t}^1((Q_i)_{\overline{K}}, \mathbb{Z}_p)) = H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_i],$$

so Theorem 3.4 applied to the isogeny  $A \xrightarrow{q=(q_1, q_2)} Q_1 \times_K Q_2$  proves that

$$\mathrm{val}_{\mathbb{Z}_p} \left( \left( \frac{H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)}{H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_1] + H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_2]} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right) = \mathrm{val}_{\mathcal{O}} \left( \left( \frac{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})}{q_1^*(H_{\mathrm{dR}}^1(\mathcal{Q}_1/\mathcal{O})) + q_2^*(H_{\mathrm{dR}}^1(\mathcal{Q}_2/\mathcal{O}))} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right).$$

It remains to note that  $\frac{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})}{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})[e_1] + H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})[e_2]}$  is a quotient of  $\frac{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})}{q_1^*(H_{\mathrm{dR}}^1(\mathcal{Q}_1/\mathcal{O})) + q_2^*(H_{\mathrm{dR}}^1(\mathcal{Q}_2/\mathcal{O}))}$ .  $\square$

**Remark 4.7.** It follows from the proof and from the last claim of Theorem 3.4 that the  $\mathfrak{m}$ -primary factor of the étale congruence number equals  $\#((A_1 \cap A_2)[\mathfrak{m}^\infty])$ .

**Corollary 4.8.** *In Theorem 4.4, if the equality holds:*

$$\mathrm{val}_{\mathbb{Z}_p} \left( \left( \frac{H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)}{H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_1] + H_{\acute{e}t}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_2]} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right) = \mathrm{val}_{\mathcal{O}} \left( \left( \frac{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})}{H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})[e_1] + H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})[e_2]} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right),$$

*then the maps  $(\mathrm{Lie} \mathcal{A})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{Q}_i)_{\mathfrak{m}}$  are surjective. In particular, if the displayed equality holds for every  $\mathfrak{m}$ , then the  $\mathcal{O}$ -morphisms  $\mathcal{A} \rightarrow \mathcal{Q}_i$  (resp.,  $\mathcal{A}_i \rightarrow \mathcal{A}$ ) are smooth (resp., closed immersions).*

*Proof.* By the proof of Theorem 4.4, the displayed equality holds if and only if the sequences

$$0 \rightarrow H_{\mathrm{dR}}^1(\mathcal{Q}_i/\mathcal{O})_{\mathfrak{m}} \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}/\mathcal{O})_{\mathfrak{m}} \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}_i/\mathcal{O})_{\mathfrak{m}} \quad \text{for } i \in \{1, 2\}$$

are left exact. Due to the filtrations (3.2.2), this exactness implies the left exactness of the sequences

$$0 \rightarrow (\mathrm{Lie} \mathcal{Q}_i)_{\mathfrak{m}}^* \rightarrow (\mathrm{Lie} \mathcal{A})_{\mathfrak{m}}^* \rightarrow (\mathrm{Lie} \mathcal{A}_i)_{\mathfrak{m}}^*,$$

and hence also the surjectivity of  $(\mathrm{Lie} \mathcal{A})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{Q}_i)_{\mathfrak{m}}$ . For the last claim, it remains to recall that  $\mathcal{A} \rightarrow \mathcal{Q}_i$  is smooth if and only if  $\mathrm{Lie} \mathcal{A} \rightarrow \mathrm{Lie} \mathcal{Q}_i$  is surjective and that the smoothness of  $\mathcal{A} \rightarrow \mathcal{Q}_i$  implies that  $\mathcal{A}_i \rightarrow \mathcal{A}$  is a closed immersion (see [BLR90, 7.1/6]).  $\square$

**Remark 4.9.** Corollary 4.8 implies that the étale and the de Rham congruence numbers differ in general. Indeed, in the case when  $e(K/\mathbb{Q}_p) \geq p - 1$ , there are examples of inclusions  $A_1 \subset A$  of abelian varieties (with good reduction) that fail to induce closed immersions on Néron models over  $\mathcal{O}$  (see [BLR90, 7.5/8]) and, by Remark 4.6, any such  $A_1$  arises from some  $e_1$  (with  $\mathbb{T} = \mathbb{Z}$ ).

We end §4 with a criterion for the étale and the de Rham congruence numbers to be equal.

**Proposition 4.10.** *In the setup of §4.1, suppose that  $\mathfrak{m} \subset \mathbb{T}$  be a maximal ideal of residue characteristic  $p$  for which there exists an  $r \in \mathbb{Z}_{\geq 0}$  such that*

- (i) *the  $(\mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathcal{O})$ -module  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})_{\mathfrak{m}}$  is free of rank  $r$ ; and*
- (ii) *the  $(\mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ -module  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}}$  is free of rank  $r$ .*

*Then the  $\mathbb{T}_{\mathfrak{m}}$ -module  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}}$  is free of rank  $r$  and the equality holds in Theorem 4.4:*

$$\text{val}_{\mathbb{Z}_p} \left( \left( \frac{H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)}{H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_1] + H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)[e_2]} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right) = \text{val}_{\mathcal{O}} \left( \left( \frac{H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})}{H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})[e_1] + H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})[e_2]} \right) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \right).$$

*Proof.* The assumption on  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})_{\mathfrak{m}}$  gives the equality

$$\text{val}_{\mathcal{O}} \left( \frac{H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})}{\mathfrak{m} \cdot H_{\text{dR}}^1(\mathcal{A}/\mathcal{O})} \right) = r \cdot \dim_{\mathbb{F}_p}(\mathbb{T}/\mathfrak{m}).$$

Therefore, by Proposition 3.15,

$$\dim_{\mathbb{T}/\mathfrak{m}} \left( \frac{H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)}{\mathfrak{m} \cdot H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)} \right) \leq r,$$

to the effect that, by the Nakayama lemma,  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}}$  is generated by  $r$  elements as a  $\mathbb{T}_{\mathfrak{m}}$ -module. Due to (ii), these  $r$  elements are  $\mathbb{T}_{\mathfrak{m}}$ -independent, so the desired  $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}} \simeq (\mathbb{T}_{\mathfrak{m}})^{\oplus r}$  follows. Consequently, both sides of the claimed equality are equal to

$$r \cdot \text{val}_{\mathbb{Z}_p} \left( \frac{\mathbb{T}_{\mathfrak{m}}}{\mathbb{T}_{\mathfrak{m}}[e_1] + \mathbb{T}_{\mathfrak{m}}[e_2]} \right). \quad \square$$

**Remark 4.11.** Modular Jacobians endowed with their Hecke action tend to satisfy (ii) (see [DDT97, Lem. 1.38–1.39]). Thus, loosely speaking, for them Proposition 4.10 proves that the Hecke-freeness of the integral de Rham cohomology implies the Hecke-freeness of the integral  $p$ -adic étale cohomology.

## 5. THE SEMISTABLE CASE OF THE HIGHER DIMENSIONAL MANIN CONJECTURE

The Manin conjecture has been generalized to newform quotients of arbitrary dimension (see Conjecture 5.2), and our goal is to address this generalization. More precisely, we prove in Theorem 5.10 that in the higher dimensional case the conjecture fails even at a prime of good reduction and we prove many of its semistable cases in Theorem 5.19. Our techniques also supply general relations between the modular degree and the congruence number (see Corollary 5.9 and Theorems 5.15 and 5.17) and determine the endomorphism rings of suitable newform quotients (see Corollary 5.18).

### 5.1. A newform of level $\Gamma_0(n)$ . Throughout §5,

- we fix an  $n \in \mathbb{Z}_{\geq 1}$  and let  $\mathcal{J}$  be the Néron model over  $\mathbb{Z}$  of  $J_0(n)$ ;
- as in §2.1, we let  $\mathbb{T} \subset \text{End}_{\mathbb{Q}}(J_0(n))$  be the  $\mathbb{Z}$ -subalgebra generated by all the  $T_{\ell}$  and  $U_{\ell}$ ;
- we fix a normalized weight 2 newform  $f$  of level  $\Gamma_0(n)$ , let  $e_f \in \mathbb{T}_{\mathbb{Q}}$  be the idempotent that cuts out the factor of  $\mathbb{T}_{\mathbb{Q}}$  determined by  $f$ , and set  $e_{f^{\perp}} := 1 - e_f$ ;
- we set  $\mathcal{O}_f := \mathbb{T}/\mathbb{T}[e_f]$  (see (1.5.1)), so that  $\mathcal{O}_f$  is an order in a totally real number field;
- we let  $\pi_f: J_0(n) \rightarrow A_f$  be the optimal newform quotient determined by  $f$ , so that

$$\mathcal{O}_f \hookrightarrow \text{End}_{\mathbb{Q}}(A_f);$$

- we let  $\mathcal{A}_f, \mathcal{A}_f^{\vee}, \mathcal{K}$ , and  $\mathcal{K}^{\vee}$  be the Néron models over  $\mathbb{Z}$  of  $A_f, A_f^{\vee}, \text{Ker } \pi_f$ , and  $(\text{Ker } \pi_f)^{\vee}$ ;

- we let  $S_2(\Gamma_0(n), \mathbb{Z})$  be the module of those weight 2 cusp forms of level  $\Gamma_0(n)$  whose  $q$ -expansion at the cusp “ $\infty$ ” lies in  $\mathbb{Z}[[q]]$ , and for a commutative ring  $R$  we set

$$S_2(\Gamma_0(n), R) := S_2(\Gamma_0(n), \mathbb{Z}) \otimes_{\mathbb{Z}} R.$$

The Manin conjecture for  $A_f$  is the following generalization of Conjecture 1.1.

**Conjecture 5.2** ([Joy05, Conj. 2] or [ARS06, Conj. 3.12]). *In the setting of §5.1 (see also (1.5.1)),*

$$\pi_f^*(H^0(\mathcal{A}_f, \Omega^1)) = S_2(\Gamma_0(n), \mathbb{Z})[e_{f\perp}] \quad \text{inside} \quad H^0(J_0(n), \Omega^1) \cong S_2(\Gamma_0(n), \mathbb{Q}).$$

**Remark 5.3.** As in [Edi91, proof of Prop. 2], since “ $\infty$ ” extends to a  $\mathbb{Z}$ -point of  $X_0(n)^{\text{sm}}$ , the Néron property gives the inclusion

$$\pi_f^*(H^0(\mathcal{A}_f, \Omega^1)) \subset S_2(\Gamma_0(n), \mathbb{Z})[e_{f\perp}],$$

so Conjecture 5.2 amounts to the vanishing of the finite quotient

$$\frac{S_2(\Gamma_0(n), \mathbb{Z})[e_{f\perp}]}{\pi_f^*(H^0(\mathcal{A}_f, \Omega^1))}.$$

Due to the following standard lemma and the exactness property [BLR90, 7.5/4 (ii) and its proof] of Néron models, the  $p$ -primary part of this quotient vanishes for every odd prime  $p$  with  $p^2 \nmid n$  (compare with [ARS06, Cor. 3.7]). We will see in Theorem 5.10 that, in contrast, the 2-primary part need not vanish even when  $2 \nmid n$ .

**Lemma 5.4.** *In the setting of §5.1, if  $p$  is a prime with  $p \nmid n$ , then*

$$S_2(\Gamma_0(n), \mathbb{Z}_p) = H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1) \quad \text{inside} \quad H^0(J_0(n)_{\mathbb{Q}_p}, \Omega^1) \cong S_2(\Gamma_0(n), \mathbb{Q}_p);$$

*if  $p$  is a prime with  $p^2 \nmid n$ , then*

$$S_2(\Gamma_0(n), \mathbb{Z}_p)[e_{f\perp}] = H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)[e_{f\perp}] \quad \text{inside} \quad H^0(J_0(n)_{\mathbb{Q}_p}, \Omega^1) \cong S_2(\Gamma_0(n), \mathbb{Q}_p).$$

*Proof.* If  $p \nmid n$ , then, due to [Edi06, 2.5] and (2.1.1),

$$S_2(\Gamma_0(n), \mathbb{Z}_p) = H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1).$$

By loc. cit., if  $p \mid n$  but  $p^2 \nmid n$ , then

$$S_2(\Gamma_0(n), \mathbb{Z}_p) = H^0(U^\infty, \Omega^1),$$

where  $U^\infty \subset X_0(n)_{\mathbb{Z}_p}$  is the open complement of the irreducible component of  $X_0(n)_{\mathbb{F}_p}$  that does not contain the reduction of the cusp “ $\infty$ .” Thus, since the  $p$ -Atkin–Lehner involution  $w_p$  interchanges the two irreducible components of  $X_0(n)_{\mathbb{F}_p}$  and acts as  $\pm 1$  on  $S_2(\Gamma_0(n), \mathbb{Z}_p)[e_{f\perp}]$ , we get

$$S_2(\Gamma_0(n), \mathbb{Z}_p)[e_{f\perp}] = H^0(X_0(n)_{\mathbb{Z}_p}, \Omega)[e_{f\perp}]$$

(see [Čes16, proof of Lem. 2.7]). The claim then follows from another application of (2.1.1).  $\square$

Similarly to the elliptic curve case discussed in §2, the strategy of our analysis of Conjecture 5.2 is to relate it to a comparison of the congruence number and the modular degree of  $f$ . We use the following standard lemma to introduce these numbers in Definition 5.6.

**Lemma 5.5.**

- The composition  $A_f^\vee \xrightarrow{\pi_f^\vee} J_0(n)^\vee \cong J_0(n)$  is  $\mathbb{T}$ -equivariant.
- The finite  $\mathbb{Q}$ -group scheme  $\pi_f^\vee(A_f^\vee) \cap \text{Ker } \pi_f$  carries a perfect alternating bilinear pairing for which the action of  $\mathbb{T}$  is self adjoint.

*Proof.*

- (a) The Atkin–Lehner involution  $w$  of  $J_0(n)$  acts as  $\pm 1$  on  $A_f$ . Moreover, under the canonical principal polarization  $\theta: J_0(n) \xrightarrow{\sim} J_0(n)^\vee$ , the action of a  $t \in \mathbb{T}$  corresponds to that of  $wt^\vee w$  (see [MW84, Ch. II, §5.6 (c)]), so the claimed  $\mathbb{T}$ -equivariance follows.
- (b) Let  $\lambda := \pi_f \circ \theta^{-1} \circ \pi_f^\vee$  be the pullback of  $\theta^{-1}$  to a polarization of  $A_f^\vee$ , so that  $A_f^\vee[\lambda]$  carries a perfect alternating bilinear Weil pairing (see [Pol03, §10.4, esp. Prop. 10.3]). Since  $\lambda$  is  $\mathbb{T}$ -equivariant, [Oda69, Cor. 1.3 (ii)] ensures that the action of any  $t \in \mathbb{T}$  is self adjoint with respect to this pairing. It remains to observe that

$$A_f^\vee[\lambda] \cong \pi_f^\vee(A_f^\vee) \cap \text{Ker } \pi_f. \quad \square$$

**Definition 5.6** (Compare with [ARS12, §3]). The *congruence number*  $\text{cong}_f$  of  $f$  and the *modular degree*  $\text{deg}_f$  of  $f$  are the positive integers (see (1.5.1) and Lemma 5.5 (b))

$$\text{cong}_f := \# \left( \frac{\mathbb{T}}{\mathbb{T}[e_f] + \mathbb{T}[e_{f^\perp}]} \right) \quad \text{and} \quad \text{deg}_f := \left( \# \left( \pi_f^\vee(A_f^\vee) \cap \text{Ker } \pi_f \right) \right)^{\frac{1}{2}};$$

for a maximal ideal  $\mathfrak{m} \subset \mathbb{T}$ , the  *$\mathfrak{m}$ -primary factors*  $\text{cong}_{f, \mathfrak{m}}$  and  $\text{deg}_{f, \mathfrak{m}}$  are the positive integers

$$\text{cong}_{f, \mathfrak{m}} := \# \left( \frac{\mathbb{T}_{\mathfrak{m}}}{\mathbb{T}_{\mathfrak{m}}[e_f] + \mathbb{T}_{\mathfrak{m}}[e_{f^\perp}]} \right) \quad \text{and} \quad \text{deg}_{f, \mathfrak{m}} := \left( \# \left( \left( \pi_f^\vee(A_f^\vee) \cap \text{Ker } \pi_f \right) [\mathfrak{m}^\infty] \right) \right)^{\frac{1}{2}}.$$

**Remark 5.7.** One may view  $\text{deg}_{f, \mathfrak{m}}^2$  as (a factor of) an étale congruence number: the abelian subvariety of  $J_0(n)$  cut out by  $e_f$  (resp., by  $e_{f^\perp}$ ) as in §4.5 is identified with  $\pi_f^\vee(A_f^\vee)$  (resp.,  $\text{Ker } \pi_f$ ), so

$$\text{deg}_{f, \mathfrak{m}}^2 \stackrel{4.7}{=} \# \left( \frac{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}}{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}[e_f] + H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}[e_{f^\perp}]} \right) \quad \text{with } p := \text{char}(\mathbb{T}/\mathfrak{m}). \quad (5.7.1)$$

In particular,

$$\text{deg}_f^2 = \prod_{\text{primes } p} \# \left( \frac{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)}{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)[e_f] + H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)[e_{f^\perp}]} \right).$$

Agashe, Ribet, and Stein proved in [ARS12, Thm. 3.6] that the exponent of  $\pi_f^\vee(A_f^\vee) \cap \text{Ker } \pi_f$  divides the exponent of  $\frac{\mathbb{T}}{\mathbb{T}[e_f] + \mathbb{T}[e_{f^\perp}]}$  and that both exponents have the same  $p$ -adic valuation for every prime  $p$  with  $p^2 \nmid n$ . As a key step towards the semistable case of Conjecture 5.2, we wish to complement these results with relations between  $\text{cong}_f$  and  $\text{deg}_f$  themselves.

**Proposition 5.8.** *If  $\mathfrak{m} \subset \mathcal{O}_f$  is a maximal ideal of residue characteristic  $p$  with  $p^2 \nmid n$ , then*

$$\text{deg}_{f, \mathfrak{m}} = \text{cong}_{f, \mathfrak{m}} \cdot \# \text{Coker} \left( (\text{Lie } \mathcal{J})_{\mathfrak{m}} \rightarrow (\text{Lie } \mathcal{A}_f)_{\mathfrak{m}} \right). \quad (5.8.1)$$

*Proof.* Corollary 3.12 gives the left exact sequences

$$0 \rightarrow (\text{Lie } \mathcal{K})_{\mathbb{Z}_p} \rightarrow (\text{Lie } \mathcal{J})_{\mathbb{Z}_p} \rightarrow (\text{Lie } \mathcal{A}_f)_{\mathbb{Z}_p} \quad \text{and} \quad 0 \rightarrow (\text{Lie } \mathcal{A}_f^\vee)_{\mathbb{Z}_p} \rightarrow (\text{Lie } \mathcal{J})_{\mathbb{Z}_p} \rightarrow (\text{Lie } \mathcal{K}^\vee)_{\mathbb{Z}_p},$$

which give the following identifications:

$$(\text{Lie } \mathcal{K})_{\mathbb{Z}_p} \cong (\text{Lie } \mathcal{J})_{\mathbb{Z}_p}[e_f] \quad \text{and} \quad (\text{Lie } \mathcal{A}_f^\vee)_{\mathbb{Z}_p} \cong (\text{Lie } \mathcal{J})_{\mathbb{Z}_p}[e_{f^\perp}].$$

Therefore, Theorem 3.4 and (3.4.3) applied to the  $\mathbb{T}$ -equivariant (see Lemma 5.5 (a)) isogeny  $\text{Ker } \pi_f \times A_f^\vee \rightarrow J_0(n)$  give the equality

$$\text{deg}_{f, \mathfrak{m}}^2 = \# \left( \left( \frac{\text{Lie } \mathcal{J}}{(\text{Lie } \mathcal{J})[e_f] + (\text{Lie } \mathcal{J})[e_{f^\perp}]} \right) \otimes_{\mathbb{T}} \mathbb{T}_{\mathfrak{m}} \right) \cdot \# \left( \left( \frac{\text{Lie}(\mathcal{K}^\vee) \oplus \text{Lie}(\mathcal{A}_f)}{\text{Lie } \mathcal{J}} \right) \otimes_{\mathbb{T}} \mathbb{T}_{\mathfrak{m}} \right). \quad (5.8.2)$$

By Proposition 2.2 (i)–(ii), we have  $(\text{Lie } \mathcal{J})_{\mathfrak{m}} \simeq \mathbb{T}_{\mathfrak{m}}$ , so the first factor on the right side of (5.8.2) equals  $\text{cong}_{f, \mathfrak{m}}$ . In addition, by Remark 3.13,

$$\# \text{Coker}((\text{Lie } \mathcal{J})_{\mathfrak{m}} \rightarrow (\text{Lie } \mathcal{A}_f)_{\mathfrak{m}}) = \# \text{Coker}((\text{Lie } \mathcal{J})_{\mathfrak{m}} \rightarrow (\text{Lie } \mathcal{K}^{\vee})_{\mathfrak{m}}),$$

so, since the kernels of these maps are  $(\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_f]$  and  $(\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_{f\perp}]$ , respectively, the second factor on the right side of (5.8.2) equals

$$(\# \text{Coker}((\text{Lie } \mathcal{J})_{\mathfrak{m}} \rightarrow (\text{Lie } \mathcal{A}_f)_{\mathfrak{m}}))^2 \cdot \# \text{Coker}\left((\text{Lie } \mathcal{J})_{\mathfrak{m}} \hookrightarrow \frac{(\text{Lie } \mathcal{J})_{\mathfrak{m}}}{(\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_f]} \oplus \frac{(\text{Lie } \mathcal{J})_{\mathfrak{m}}}{(\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_{f\perp}]}\right).$$

It remains to observe the short exact sequence

$$0 \rightarrow (\text{Lie } \mathcal{J})_{\mathfrak{m}} \rightarrow \frac{(\text{Lie } \mathcal{J})_{\mathfrak{m}}}{(\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_f]} \oplus \frac{(\text{Lie } \mathcal{J})_{\mathfrak{m}}}{(\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_{f\perp}]} \xrightarrow{(x,y) \mapsto x-y} \frac{(\text{Lie } \mathcal{J})_{\mathfrak{m}}}{(\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_f] + (\text{Lie } \mathcal{J})_{\mathfrak{m}}[e_{f\perp}]} \rightarrow 0$$

and recall that  $(\text{Lie } \mathcal{J})_{\mathfrak{m}} \simeq \mathbb{T}_{\mathfrak{m}}$ . □

**Corollary 5.9.** *If  $p$  is a prime with  $p^2 \nmid n$ , then*

$$\text{ord}_p(\deg_f) = \text{ord}_p(\text{cong}_f) + \text{ord}_p(\# \text{Coker}(\text{Lie } \mathcal{J} \rightarrow \text{Lie } \mathcal{A}_f)); \quad (5.9.1)$$

*in particular, if, in addition,  $p$  is odd, then  $\text{ord}_p(\deg_f) = \text{ord}_p(\text{cong}_f)$ .*

*Proof.* The product of the equalities (5.8.1) over all  $\mathfrak{m}$  of residue characteristic  $p$  gives (5.9.1). In the case when  $p$  is odd, [BLR90, 7.5/4 (ii) and its proof] ensure the smoothness of  $\mathcal{J}_{\mathbb{Z}_p} \rightarrow (\mathcal{A}_f)_{\mathbb{Z}_p}$ , so the second summand of the right side of (5.9.1) vanishes. □

With Corollary 5.9 in hand, we are ready to present a counterexample to Conjecture 5.2.

**Theorem 5.10.** *Conjecture 5.2 fails for a 24-dimensional optimal newform quotient of  $J_0(431)$  and also for a 91-dimensional optimal newform quotient of  $J_0(2089)$  (both 431 and 2089 are primes).*

*Proof.* By [ARS06, Rem. 3.7], there is a weight 2 newform  $f$  of level  $\Gamma_0(431)$  with  $\dim A_f = 24$  and

$$\deg_f = 2^{11} \cdot 6947 \quad \text{and} \quad \text{cong}_f = 2^{10} \cdot 6947.$$

The following Sage code [Sage] confirms these values  $\deg_f$  and  $\text{cong}_f$ .

```
J = J0(431).decomposition();
degf = J[5].modular_degree();
C = ModularSymbols(431).cuspidal_subspace().decomposition();
congf = C[5].congruence_number(C[5].complement().cuspidal_subspace());
```

Alternatively, the following Magma code [Magma] computes  $\deg_f^2$  with a faster runtime.

```
C := NewformDecomposition(CuspidalSubspace(ModularSymbols(431)));
degf2 := #ModularKernel(C[6]);
```

These means also give us a weight 2 newform  $f'$  of level  $\Gamma_0(2089)$  with  $\dim A_{f'} = 91$  and

$$\deg_{f'} = 2^{80} \cdot 3 \cdot 5 \cdot 11 \cdot 19 \cdot 73 \cdot 139 \quad \text{and} \quad \text{cong}_{f'} = 2^{79} \cdot 3 \cdot 5 \cdot 11 \cdot 19 \cdot 73 \cdot 139.$$

By Corollary 5.9, the map

$$(\text{Lie } \mathcal{J})_{\mathbb{Z}_2} \rightarrow (\text{Lie } \mathcal{A}_f)_{\mathbb{Z}_2}$$

is not surjective, to the effect that  $\frac{H^0(\mathcal{J}_{\mathbb{Z}_2}, \Omega^1)}{\pi_f^*(H^0((\mathcal{A}_f)_{\mathbb{Z}_2}, \Omega^1))}$  has nontrivial 2-torsion. Since, by Lemma 5.4,

$$H^0(\mathcal{J}_{\mathbb{Z}_2}, \Omega^1) = S_2(\Gamma_0(431), \mathbb{Z}_2) \quad \text{and} \quad \frac{S_2(\Gamma_0(431), \mathbb{Z}_2)}{S_2(\Gamma_0(431), \mathbb{Z}_2)[e_{f\perp}]} \text{ is torsion free,}$$

we get that

$$\pi_f^*(H^0(\mathcal{A}_f, \Omega^1)) \neq S_2(\Gamma_0(431), \mathbb{Z})[e_{f\perp}]$$

(and likewise for  $f'$ ), contrary to Conjecture 5.2.  $\square$

**Remarks.**

5.11. Theorem 5.17 below and [Kil02] suggest considering the levels  $\Gamma_0(431)$ ,  $\Gamma_0(503)$ , and  $\Gamma_0(2089)$ .

5.12. For the  $f$  considered in the proof of Theorem 5.10, let  $\tilde{\pi}_f: J_1(431) \rightarrow \tilde{A}_f$  be the resulting optimal newform quotient of  $J_1(431)$ . As in the proof of Lemma 2.12, there is a commutative diagram

$$\begin{array}{ccc} J_1(431) & \xrightarrow{\tilde{\pi}_f} & \tilde{A}_f \\ \downarrow & & \downarrow a \\ J_0(431) & \xrightarrow{\pi_f} & A_f \end{array}$$

in which  $a$  is an isogeny and  $\text{Ker } a$  is a quotient of the Cartier dual of the Shimura subgroup  $\Sigma(431) \subset J_0(431)$ . Moreover, by [LO91, Cor. 1 to Thm. 1], we have  $2 \nmid \#\Sigma(431)$ , so  $\text{Lie } a$  is an isomorphism on Lie algebras of Néron models over  $\mathbb{Z}_2$ , and hence  $\text{Lie } \tilde{\pi}_f$  is not surjective on such Lie algebras. Then, as in the proof of Theorem 5.10, the quotient  $\frac{S_2(\Gamma_1(431), \mathbb{Z})}{(\tilde{\pi}_f)^*(H^0(\tilde{\mathcal{A}}_f, \Omega^1))}$  has nontrivial 2-torsion, where  $\tilde{\mathcal{A}}_f$  denotes the Néron model over  $\mathbb{Z}$  of  $\tilde{A}_f$  and [CES03, Lem. 6.1.6] supplies the inclusion

$$(\tilde{\pi}_f)^*(H^0(\tilde{\mathcal{A}}_f, \Omega^1)) \subset S_2(\Gamma_1(431), \mathbb{Z}).$$

This is a counterexample to the analogue [CES03, Conj. 6.1.7] of Conjecture 5.2 for newform quotients of  $J_1(n)$ .

In the case when  $A_f$  is an elliptic curve, one knows that  $\deg_f \mid \text{cong}_f$  (see Lemma 2.9). Even though this divisibility fails in the higher dimensional case (see the proof of Theorem 5.10), we wish to generalize it as follows.

**Proposition 5.13.** *For a maximal ideal  $\mathfrak{m} \subset \mathcal{O}_f$  of residue characteristic  $p$  such that  $(\mathcal{O}_f)_{\mathfrak{m}}$  is a discrete valuation ring,  $\deg_{f, \mathfrak{m}} \mid \text{cong}_{f, \mathfrak{m}}$ . In particular, if the order  $\mathcal{O}_f$  is maximal, then  $\deg_f \mid \text{cong}_f$ .*

*Proof.* It will suffice to mildly generalize the proof of Lemma 2.9. Namely,

$$H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \otimes_{\mathbb{T}_{\mathbb{Z}_p}} \mathbb{T}_{\mathfrak{m}} \quad \text{is a free } \mathbb{T}_{\mathfrak{m}}[\frac{1}{p}]\text{-module of rank 2}$$

(see [DDT97, Lem. 1.38–1.39]) and  $\mathcal{O}_f = \mathbb{T}/\mathbb{T}[e_f]$ , so, since  $(\mathcal{O}_f)_{\mathfrak{m}}$  is a discrete valuation ring, we check over  $(\mathcal{O}_f)_{\mathfrak{m}}[\frac{1}{p}]$  that

$$\frac{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}}{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}[e_f]} \quad \text{is a free } (\mathcal{O}_f)_{\mathfrak{m}}\text{-module of rank 2.}$$

Then the further quotient

$$\frac{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}}{H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}[e_f] + H_{\text{ét}}^1(J_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}}[e_{f\perp}]} \quad \text{admits a surjection from} \quad \left( \frac{\mathbb{T}_{\mathfrak{m}}}{\mathbb{T}_{\mathfrak{m}}[e_f] + \mathbb{T}_{\mathfrak{m}}[e_{f\perp}]} \right)^2,$$

and the desired conclusion follows from (5.7.1).  $\square$

**Remark 5.14.** Proposition 5.13 implies that for the newform  $f$  used in the proof of Theorem 5.10, the order  $\mathcal{O}_f$  is nonmaximal. Indeed, according to [Ste99, Table 2],  $\mathcal{O}_f$  has index 4 in its normalization.

As we will see in Theorem 5.19, the following consequence of the work above proves the semistable case of Conjecture 5.2 under the assumption that  $\mathcal{O}_f \otimes_{\mathbb{Z}} \mathbb{Z}_2$  is regular. Furthermore, its freeness aspect supplies additional information that seems new in the case of an odd  $p$ .



**Theorem 5.15.** *If  $\mathfrak{m} \subset \mathcal{O}_f$  is a maximal ideal of residue characteristic  $p$  with  $p^2 \nmid n$  such that either  $p$  is odd or  $(\mathcal{O}_f)_{\mathfrak{m}}$  is a discrete valuation ring, then*

$$(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}}$$

*is surjective,  $\mathrm{deg}_{f, \mathfrak{m}} = \mathrm{cong}_{f, \mathfrak{m}}$ , and  $(\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}}$  is free of rank 1 as an  $(\mathcal{O}_f)_{\mathfrak{m}}$ -module.*

*Proof.* In both cases, by Proposition 2.2 (i)–(ii),

$$(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}} \simeq \mathbb{T}_{\mathfrak{m}} \quad \text{as } \mathbb{T}_{\mathfrak{m}}\text{-modules.}$$

If  $p$  is odd, then the surjectivity of  $(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}}$  follows from the smoothness of the map  $\mathcal{J}_{\mathbb{Z}_p} \rightarrow (\mathcal{A}_f)_{\mathbb{Z}_p}$  supplied by [BLR90, 7.5/4 (ii) and its proof]. If  $(\mathcal{O}_f)_{\mathfrak{m}}$  is a discrete valuation ring, then the surjectivity follows by combining Propositions 5.8 and 5.13. Therefore, in both cases  $\mathrm{deg}_{f, \mathfrak{m}} \stackrel{5.8}{=} \mathrm{cong}_{f, \mathfrak{m}}$  and  $(\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}}$  is isomorphic to

$$\mathbb{T}_{\mathfrak{m}}/\mathbb{T}_{\mathfrak{m}}[e_f] \cong (\mathcal{O}_f)_{\mathfrak{m}}$$

as a  $\mathbb{T}_{\mathfrak{m}}$ -module (so also as an  $(\mathcal{O}_f)_{\mathfrak{m}}$ -module).  $\square$

Our next aim is to show in Theorem 5.17 that conclusions like those of Theorem 5.15 may also be drawn if the assumption on  $(\mathcal{O}_f)_{\mathfrak{m}}$  is replaced by the Gorensteinness of  $\mathbb{T}_{\mathfrak{m}}$ . To put the latter condition into context, we now review some cases in which it is known to hold.

**5.16. Multiplicity one.** In the setting of §5.1, suppose that  $p^2 \nmid n$ , let  $\mathfrak{m} \subset \mathcal{O}_f$  be a maximal ideal of residue characteristic  $p$ , and let

$$\rho_{\mathfrak{m}}: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\mathcal{O}_f/\mathfrak{m})$$

be the associated semisimple modular mod  $\mathfrak{m}$  Galois representation. By Corollary 3.16 and (3.16.4),

$$\mathbb{T}_{\mathfrak{m}} \quad \text{is Gorenstein if and only if} \quad \dim_{\mathbb{T}/\mathfrak{m}}(J_0(n)[\mathfrak{m}]) = 2.$$

Therefore,  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein in any of the following cases:

- (1) if  $\mathrm{ord}_p(n) = 0$  and  $\rho_{\mathfrak{m}}$  is absolutely irreducible, except possibly when, in addition,  $p = 2$ , the ideal  $\mathfrak{m}$  contains  $T_2 - 1$ , the restriction of  $\rho_{\mathfrak{m}}$  to  $\mathrm{Gal}(\overline{\mathbb{Q}}_2/\mathbb{Q}_2)$  is unramified, and  $\rho_{\mathfrak{m}}(\mathrm{Frob}_2)$  lies in the center of  $\mathrm{GL}_2(\mathbb{T}/\mathfrak{m})$ , see [Rib90, Thm. 5.2 (b)], [Edi92, Thm. 9.2] (and possibly also [Gro90, Thm. 12.10 (1)]), and [RS01, Thm. 6.1] in the appendix by Buzzard;
- (2) if  $\mathrm{ord}_p(n) = 1$  and  $\rho_{\mathfrak{m}}$  is absolutely irreducible and not of level  $n/p$ , see [MR91, Main Thm.];
- (3) if  $\mathrm{ord}_p(n) = 1$  with  $p$  odd,  $\rho_{\mathfrak{m}}$  is irreducible,  $U_p \notin \mathfrak{m}$ , and the semisimplification

$$(\rho_{\mathfrak{m}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})^{\mathrm{ss}}$$

is not of the form  $\chi \oplus \chi$  for some character  $\chi: \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow (\mathbb{T}/\mathfrak{m})^{\times}$ , see [Wil95, Thm. 2.1 (ii)].

In addition, in the case when  $n$  is a prime,  $\mathbb{T}_{\mathfrak{m}}$  is also Gorenstein when  $\mathfrak{m}$  is Eisenstein, see [Maz77, 16.3]. Also, in contrast to (1), if  $p = 2$ , then even when  $n$  is a prime  $\mathbb{T}_{\mathfrak{m}}$  need not be Gorenstein—see [Kil02]; alternatively, the possible failure of Gorensteinness in such situations may be deduced by combining the examples in the proof of Theorem 5.10 with the following result.

**Theorem 5.17.** *If  $\mathfrak{m} \subset \mathcal{O}_f$  is a maximal ideal of residue characteristic  $p$  with  $p^2 \nmid n$  such that  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein (see §5.16), then*

$$(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}}$$

*is surjective,  $\mathrm{deg}_{f, \mathfrak{m}} = \mathrm{cong}_{f, \mathfrak{m}}$ , the  $(\mathcal{O}_f)_{\mathfrak{m}}$ -modules  $(\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}}$  and  $(\mathrm{Lie} \mathcal{A}_f^{\vee})_{\mathfrak{m}}^*$  are free of rank 1, where  $(-)^*$  denotes the  $\mathbb{Z}_p$ -linear dual, and the  $(\mathcal{O}_f)_{\mathfrak{m}}$ -module  $H_{\mathrm{dR}}^1(\mathcal{A}_f^{\vee}/\mathbb{Z}_p)_{\mathfrak{m}}$  is free of rank 2.*

*Proof.* By Proposition 2.2 (i)–(ii),

$$(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}} \simeq \mathbb{T}_{\mathfrak{m}} \quad \text{as } \mathbb{T}_{\mathfrak{m}}\text{-modules,}$$

so, since  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein, also

$$(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}}^* \simeq \mathbb{T}_{\mathfrak{m}} \quad \text{as } \mathbb{T}_{\mathfrak{m}}\text{-modules}$$

and, by the proof of Corollary 3.16,

$$(\mathrm{Lie} \mathcal{J}^{\vee})_{\mathfrak{m}} \simeq \mathbb{T}_{\mathfrak{m}} \quad \text{as } \mathbb{T}_{\mathfrak{m}}\text{-modules.}^6$$

The filtration (3.2.2) then proves that  $H_{\mathrm{dR}}^1(\mathcal{J}/\mathbb{Z}_p)_{\mathfrak{m}}$  is free of rank 2 as a  $\mathbb{T}_{\mathfrak{m}}$ -module. Therefore, Proposition 4.10 and Remark 4.11 imply that the  $\mathfrak{m}$ -primary parts of the étale and the de Rham congruence numbers of  $J_0(n)_{\mathbb{Q}_p}$  formed with respect to the idempotents  $e_f$  and  $e_{f\perp}$  are equal. Thus, since the abelian subvariety of  $J_0(n)$  cut out by  $e_f$  (resp., by  $e_{f\perp}$ ) as in §4.5 is identified with  $\pi_f^{\vee}(A_f^{\vee})$  (resp., with  $\mathrm{Ker} \pi_f$ ), Corollary 4.8 proves the surjectivity of the maps

$$(\mathrm{Lie} \mathcal{J})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}} \quad \text{and} \quad (\mathrm{Lie} \mathcal{J}^{\vee})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{K}^{\vee})_{\mathfrak{m}}.$$

Proposition 5.8 then gives  $\mathrm{deg}_{f, \mathfrak{m}} = \mathrm{cong}_{f, \mathfrak{m}}$  and Corollary 3.12 gives the short exact sequences

$$0 \rightarrow (\mathrm{Lie} \mathcal{K})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{J})_{\mathfrak{m}} \rightarrow (\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow (\mathrm{Lie} \mathcal{K}^{\vee})_{\mathfrak{m}}^* \rightarrow (\mathrm{Lie} \mathcal{J}^{\vee})_{\mathfrak{m}}^* \rightarrow (\mathrm{Lie} \mathcal{A}_f^{\vee})_{\mathfrak{m}}^* \rightarrow 0,$$

which show that

$$(\mathrm{Lie} \mathcal{K})_{\mathfrak{m}} \cong (\mathrm{Lie} \mathcal{J})_{\mathfrak{m}}[e_f] \quad \text{and} \quad (\mathrm{Lie} \mathcal{K}^{\vee})_{\mathfrak{m}}^* \cong (\mathrm{Lie} \mathcal{J}^{\vee})_{\mathfrak{m}}^*[e_f].$$

Since  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein,  $(\mathrm{Lie} \mathcal{J}^{\vee})_{\mathfrak{m}}^*$  inherits  $\mathbb{T}_{\mathfrak{m}}$ -freeness from  $(\mathrm{Lie} \mathcal{J}^{\vee})_{\mathfrak{m}}$ , and it follows that  $(\mathrm{Lie} \mathcal{A}_f)_{\mathfrak{m}}$  and  $(\mathrm{Lie} \mathcal{A}_f^{\vee})_{\mathfrak{m}}^*$  are free of rank 1 as  $(\mathcal{O}_f)_{\mathfrak{m}}$ -modules. The filtration (3.2.2) then gives the claim about  $H_{\mathrm{dR}}^1(\mathcal{A}_f^{\vee}/\mathbb{Z}_p)_{\mathfrak{m}}$ .  $\square$

The following corollary supplies information about endomorphism rings of newform quotients of  $J_0(n)$  (we leave the explication of its “one  $\mathfrak{m}$  at a time” generalization to an interested reader).

**Corollary 5.18.** *If  $n$  is squarefree and for every maximal ideal  $\mathfrak{m} \subset \mathcal{O}_f$  of residue characteristic 2 either  $(\mathcal{O}_f)_{\mathfrak{m}}$  is a discrete valuation ring or  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein, then the inclusion  $\mathcal{O}_f \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A_f)$  is an isomorphism.*

*Proof.* Theorems 5.15 and 5.17 imply that the  $\mathcal{O}_f$ -module  $\mathrm{Lie} \mathcal{A}_f$  is locally free of rank 1, to the effect that the inclusion  $\mathcal{O}_f \hookrightarrow \mathrm{End}_{\mathcal{O}_f}(\mathrm{Lie} \mathcal{A}_f)$  is an isomorphism. Since the action of  $\mathrm{End}_{\mathbb{Q}}(A_f)$  on  $\mathrm{Lie} \mathcal{A}_f$  is faithful and  $\mathcal{O}_f$ -linear, the desired conclusion follows.  $\square$

We are ready to put the preceding results together to prove suitable semistable cases of Conjecture 5.2.

**Theorem 5.19.** *In the setting of §5.1, suppose that  $p$  is a prime with  $p^2 \nmid n$  such that either*

- (i)  $p$  is odd, or
- (ii) for every maximal ideal  $\mathfrak{m} \subset \mathcal{O}_f$  of residue characteristic  $p$ , either  $(\mathcal{O}_f)_{\mathfrak{m}}$  is a discrete valuation ring or  $\mathbb{T}_{\mathfrak{m}}$  is Gorenstein.

Then the  $p$ -primary aspect of Conjecture 5.2 holds for  $A_f$  in the sense that

$$p \nmid \# \left( \frac{S_2(\Gamma_0(n), \mathbb{Z})[e_{f\perp}]}{\pi_f^*(H^0(\mathcal{A}_f, \Omega^1))} \right)$$

(see Remark 5.3); in addition, the map  $\pi_f: \mathcal{J}_{\mathbb{Z}_p} \rightarrow (\mathcal{A}_f)_{\mathbb{Z}_p}$  is smooth, the map  $\pi_f^{\vee}: (\mathcal{A}_f^{\vee})_{\mathbb{Z}_p} \rightarrow (\mathcal{J})_{\mathbb{Z}_p}$  is a closed immersion, and the sequence

$$0 \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}_f/\mathbb{Z}_p) \rightarrow H_{\mathrm{dR}}^1(\mathcal{J}/\mathbb{Z}_p) \rightarrow H_{\mathrm{dR}}^1(\mathcal{K}/\mathbb{Z}_p) \rightarrow 0 \tag{5.19.1}$$

is short exact.

*Proof.* By Theorems 5.15 and 5.17, the map

$$(\mathrm{Lie} \mathcal{J})_{\mathbb{Z}_p} \rightarrow (\mathrm{Lie} \mathcal{A}_f)_{\mathbb{Z}_p}$$

is surjective, so  $\mathcal{J}_{\mathbb{Z}_p} \rightarrow (\mathcal{A}_f)_{\mathbb{Z}_p}$  is smooth and  $\frac{H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)}{\pi_f^*(H^0((\mathcal{A}_f)_{\mathbb{Z}_p}, \Omega^1))}$  is  $p$ -torsion free, to the effect that

$$\pi_f^*(H^0((\mathcal{A}_f)_{\mathbb{Z}_p}, \Omega^1)) = H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)[e_{f^\perp}].$$

Since

$$H^0(\mathcal{J}_{\mathbb{Z}_p}, \Omega^1)[e_{f^\perp}] \stackrel{5.4}{=} S_2(\Gamma_0(n), \mathbb{Z}_p)[e_{f^\perp}],$$

the claim about Conjecture 5.2 follows. The additional claims then follow from Corollary 3.12, similarly to the proof of Corollary 3.14.  $\square$

**Remark 5.20.** The short exactness of the sequence (5.19.1) fails in the counterexamples of Theorem 5.10 when  $p = 2$ . Indeed, due to (3.2.2), such exactness would imply the surjectivity of the map

$$(\mathrm{Lie} \mathcal{J})_{\mathbb{Z}_2} \rightarrow (\mathrm{Lie} \mathcal{K}^\vee)_{\mathbb{Z}_2},$$

that is, by Corollary 3.12, the surjectivity of the map

$$(\mathrm{Lie} \mathcal{J})_{\mathbb{Z}_2} \rightarrow (\mathrm{Lie} \mathcal{A}_f)_{\mathbb{Z}_2}.$$

In [CES03, Conj. 6.1.7], Conrad, Edixhoven, and Stein proposed an analogue of Conjecture 5.2 for newform quotients of  $J_1(n)$ . We now deduce several special cases of their conjecture from Theorem 5.19 (Remark 5.12 exhibited a counterexample to the general case).

**Corollary 5.21.** *Let  $p$  be a prime as in Theorem 5.19, fix a subgroup  $H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$  such that  $\Gamma_1(n) \subset H \subset \Gamma_0(n)$ , and let  $\pi: J_H \twoheadrightarrow A$  be the optimal newform quotient that is isogenous to  $A_f$ . Then*

$$p \nmid \# \left( \frac{S_2(\Gamma_0(n), \mathbb{Z})[e_{f^\perp}]}{\pi^*(H^0(\mathcal{A}, \Omega^1))} \right)$$

and  $\pi: (\mathcal{J}_H)_{\mathbb{Z}_p} \rightarrow \mathcal{A}_{\mathbb{Z}_p}$  is smooth, where  $\mathcal{J}_H$  and  $\mathcal{A}$  denote the Néron models over  $\mathbb{Z}$  of  $J_H$  and  $A$ .

*Proof.* As in Lemma 2.12, the multiplicity one theorem supplies a unique isogeny  $a$  such that

$$\begin{array}{ccc} J_H & \xrightarrow{\pi} & A \\ \downarrow & & \downarrow a \\ J_0(n) & \xrightarrow{\pi_f} & A_f \end{array} \quad (5.21.1)$$

commutes. The analogous statement also holds with  $J_0(n)$  replaced by  $J_{H'}$  for some  $H \subset H' \subset \Gamma_0(n)$ , so the inclusion

$$\pi^*(H^0(\mathcal{A}, \Omega^1)) \subset S_2(\Gamma_0(n), \mathbb{Z})[e_{f^\perp}]$$

supplied by [CES03, Lem. 6.1.6] in the case  $H = \Gamma_1(n)$  implies this inclusion for general  $H$ . Moreover, (5.21.1) gives the equality

$$\# \left( \frac{S_2(\Gamma_0(n), \mathbb{Z})[e_{f^\perp}]}{\pi_f^*(H^0(\mathcal{A}_f, \Omega^1))} \right) = \# \left( \frac{S_2(\Gamma_0(n), \mathbb{Z})[e_{f^\perp}]}{\pi^*(H^0(\mathcal{A}, \Omega^1))} \right) \cdot \# \left( \frac{\mathrm{Lie} \mathcal{A}_f}{(\mathrm{Lie} a)(\mathrm{Lie} \mathcal{A})} \right),$$

so the claim follows from Theorem 5.19.  $\square$

**Remark 5.22.** The proof of Corollary 5.21 also shows that the induced map  $\mathcal{A}_{\mathbb{Z}_p} \xrightarrow{a} (\mathcal{A}_f)_{\mathbb{Z}_p}$  is étale.

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