GROTHENDIECK–LEFSCHETZ FOR VECTOR BUNDLES

KĘSTUTIS ČESNAVIČIUS

Abstract. According to the Grothendieck–Lefschetz theorem from SGA 2, there are no nontrivial line bundles on the punctured spectrum U_R of a local ring R that is a complete intersection of dimension $\geqslant 4$. Dao conjectured a generalization for vector bundles $\mathscr{V}$ of arbitrary rank on U_R: such a $\mathscr{V}$ is free if and only if $\mathrm{depth}_R(\mathrm{End}_R(\Gamma(U_R, \mathscr{V}))) \geqslant 4$. We use deformation theoretic techniques to settle Dao's conjecture. We also present examples showing that its assumptions are sharp.

1. The conjecture of Dao

1.1. The Grothendieck–Lefschetz theorem. A key result in local commutative algebra, proved by Grothendieck in SGA 2, says that for a Noetherian local ring R that is a complete intersection (in the sense of §1.4) of dimension $\geqslant 4$, every line bundle on the punctured spectrum U_R is trivial, that is, $\mathrm{Pic}(U_R) = 0$ (see [SGA 2_{new}, XI, 3.13 (ii)]). In contrast, nontrivial vector bundles $\mathscr{V}$ may exist on U_R even when R is regular. Nevertheless, a conjecture of Hailong Dao [Dao13, 7.2.2] predicts that

$$\text{if a vector bundle } \mathscr{V} \text{ on } U_R \text{ satisfies} \quad \mathrm{depth}_R(\mathrm{End}_R(\Gamma(U_R, \mathscr{V}))) \geqslant 4, \quad \text{then } \mathscr{V} \text{ is free,} \tag{1.1.1}$$

in which case the depth in question equals $\dim(R)$. When $\mathscr{V}$ is a line bundle, $R \xrightarrow{\sim} \mathrm{End}_R(\Gamma(U_R, \mathscr{V}))$ (see Lemma 2.2), so the prediction (1.1.1) generalizes the Grothendieck–Lefschetz theorem recalled above. The main goal of the present paper is to establish Dao's conjecture in Theorem 2.3.

1.2. The method of proof. Our argument for (1.1.1) is built on the strategy used by Grothendieck for line bundles and rests on the Lefschetz algebraization theorems from SGA 2. More precisely, we begin by using local cohomology to show that the depth assumption implies unobstructed deformations for $\mathscr{V}$ and then, after replacing R by its completion, use this to lift $\mathscr{V}$ to the formal completion along a hypersurface of the punctured spectrum of a complete intersection cut out by fewer hypersurfaces. A Lefschetz theorem from SGA 2 allows us to algebraize the lift and, after taking care to retain the depth assumption, we proceed inductively to eventually reduce to regular R. To conclude, we use a theorem of Huneke–Wiegand: if R is regular and $\mathrm{depth}_R(\mathrm{End}_R(\Gamma(U_R, \mathscr{V}))) \geqslant 3$,

Centre national de la recherche scientifique, Institut de Mathématique d'Orsay, F-91405, France
E-mail address: `kestutis@math.u-psud.fr`.

Date: February 20, 2018.
2010 *Mathematics Subject Classification.* Primary 14B15; Secondary 13D10, 13D45.
Key words and phrases. Depth, Grothendieck–Lefschetz, local cohomology, vector bundle.

then $\mathscr{V}$ is free. Examples coming from Knörrer periodicity for maximal Cohen–Macaulay modules over local hypersurfaces show that the depth assumption in (1.1.1) is optimal, see §3.2.

1.3. A previously known case. When, in addition to $\mathrm{depth}_R(\mathrm{End}_R(\Gamma(U_R, \mathscr{V}))) \geqslant 4$, also $\mathrm{depth}_R(\Gamma(U_R, \mathscr{V})) \geqslant 3$, the conjecture (1.1.1) was established by Dao in [Dao13, 7.2.3]. In this case, the assumption on $\Gamma(U_R, \mathscr{V})$ allows one to transform the depth condition on $\mathrm{End}_R(\Gamma(U_R, \mathscr{V}))$ into

$$\mathrm{Ext}^2_R(\Gamma(U_R, \mathscr{V}), \Gamma(U_R, \mathscr{V})) = 0.$$

Due to the results of Auslander–Ding–Solberg [ADS93], the vanishing of this Ext^2 implies that, after replacing R by its completion, the R-module $\Gamma(U_R, \mathscr{V})$ lifts to a regular ring, and Dao concludes by using the resulting Tor-rigidity of $\Gamma(U_R, \mathscr{V})$. In contrast, we bypass any additional hypotheses on $\Gamma(U_R, \mathscr{V})$ by deforming over U_R instead of over R.

1.4. Notation and conventions. A Noetherian local ring $(R, \mathfrak{m})$ is a *complete intersection* if its $\mathfrak{m}$-adic completion is a quotient of a regular local ring by a regular sequence; as is well known, such an R is Cohen–Macaulay. For a local ring $(R, \mathfrak{m})$, we let U_R denote its *punctured spectrum*:

$$U_R := \mathrm{Spec}(R) \backslash \{\mathfrak{m}\}.$$

We use the definition of the (S_n) condition given in [EGA IV$_4$, 5.7.2]: a finite module M over a Noetherian ring R is (S_n) if for every prime ideal $\mathfrak{p} \subset R$ one has

$$\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant \min(n, \dim(M_\mathfrak{p})).$$

We will mostly use this definition when the support of M is $\mathrm{Spec}(R)$, when $\dim(M_\mathfrak{p}) = \dim(R_\mathfrak{p})$.

Acknowledgements. I thank Hailong Dao for very helpful correspondence. I thank the CNRS and the Université Paris-Sud for support.

2. The Grothendieck–Lefschetz theorem for vector bundles of arbitrary rank

In order to implement our deformation theoretic reduction of Dao's conjecture (1.1.1) to the case of a regular R, we need to show that $\mathscr{V}$ deforms and that the deformation inherits the depth assumption. The following lemma uses the Lefschetz theorems from [SGA 2_{new}] to achieve this.

Lemma 2.1. *Let $(\widetilde{R}, \widetilde{\mathfrak{m}})$ be a complete local ring that is a complete intersection, let $f \in \widetilde{\mathfrak{m}}$ be a nonzerodivisor, set $R := \widetilde{R}/(f)$, let $\mathscr{V}$ be a vector bundle on U_R, and consider $j\colon U_R \hookrightarrow \mathrm{Spec}(R)$.*

(a) *If $\dim(R) \geqslant 3$ and $j_*(\mathscr{E}nd(\mathscr{V}))$ is (S_3), then for any lift $\widetilde{\mathscr{V}}$ of $\mathscr{V}$ to a vector bundle on an open neighborhood $\widetilde{U}$ of the closed subscheme $U_R \subset U_{\widetilde{R}}$, we have $\Gamma(\widetilde{U}, \mathscr{E}nd(\widetilde{\mathscr{V}}))/f \cong \Gamma(U_R, \mathscr{E}nd(\mathscr{V}))$.*

(b) *If $\dim(R) \geqslant 3$ and $j_*(\mathscr{E}nd(\mathscr{V}))$ is (S_n) with $n \geqslant 3$, then for any lift $\widetilde{\mathscr{V}}$ as in (a), the pushforward $\widetilde{j}_*(\mathscr{E}nd(\widetilde{\mathscr{V}}))$ along $\widetilde{j}\colon \widetilde{U} \hookrightarrow \mathrm{Spec}(\widetilde{R})$ is also (S_n).*

(c) *If $\dim(R) \geqslant 4$ and $j_*(\mathscr{E}nd(\mathscr{V}))$ is (S_4), then a lift $\widetilde{\mathscr{V}}$ as in (a) exists for some $\widetilde{U}$.*

Proof. By our assumptions, R is a complete intersection of dimension $\geqslant 3$, and so are its thickenings

$$R_n := \widetilde{R}/(f^n) \qquad \text{for} \qquad n \geqslant 1.$$

By the finiteness theorem [SGA 2_{new}, VIII, 2.3], both $j_*(\mathscr{E}nd(\mathscr{V}))$ and $\widetilde{j}_*(\mathscr{E}nd(\widetilde{\mathscr{V}}))$ are coherent.

(a) Let $\mathscr{V}_n := \widetilde{\mathscr{V}}/f^n$ be the pullback of $\widetilde{\mathscr{V}}$ to U_{R_n} (so that $\mathscr{V}_1 \cong \mathscr{V}$), and let $\widehat{\mathscr{V}} \cong \varprojlim_n \mathscr{V}_n$ be the formal f-adic completion of $\widetilde{\mathscr{V}}$. The formal f-adic completion $(\mathscr{E}nd(\widetilde{\mathscr{V}}))\hat{}$ is then identified with $\varprojlim_n \mathscr{E}nd(\mathscr{V}_n)$, and each $\mathscr{E}nd(\mathscr{V}_n)$ is a successive extension of copies of $\mathscr{E}nd(\mathscr{V})$. Moreover, since the finite R-module $\Gamma(U_R, \mathscr{E}nd(\mathscr{V}))$ is of depth $\geqslant 3$, we have

$$H^1(U_R, \mathscr{E}nd(\mathscr{V})) \cong H^2_{\mathfrak{m}}(R, \Gamma(U_R, \mathscr{E}nd(\mathscr{V}))) = 0, \quad \text{so also} \quad H^1(U_{R_n}, \mathscr{E}nd(\mathscr{V})) = 0 \quad (2.1.1)$$

for every $n > 0$ (see [SGA 2_{new}, III, 3.3 (iv)]). It follows that

$$\begin{aligned}\Gamma(U_R, (\mathscr{E}nd(\widetilde{\mathscr{V}}))\hat{})/f &\cong (\varprojlim_n \Gamma(U_R, \mathscr{E}nd(\mathscr{V}_n)))/f \\ &\cong \varprojlim_n (\Gamma(U_R, \mathscr{E}nd(\mathscr{V}_n))/f) \cong \Gamma(U_R, \mathscr{E}nd(\mathscr{V})).\end{aligned}$$

To conclude, we use the local Lefschetz theorem [SGA 2_{new}, X, 2.1 (i)] to obtain

$$\Gamma(\widetilde{U}, \mathscr{E}nd(\widetilde{\mathscr{V}})) \xrightarrow{\sim} \Gamma(U_R, (\mathscr{E}nd(\widetilde{\mathscr{V}}))\hat{}).$$

(b) The complement of $\widetilde{U}$ in $U_{\widetilde{R}}$ is a union of finitely many closed points of $U_{\widetilde{R}}$: indeed, the complement of $\widetilde{U}$ in $\operatorname{Spec}(\widetilde{R})$ is of the form $\operatorname{Spec}(\widetilde{R}/I)$ with $(\widetilde{R}/I)/f$ Artinian, so $\dim(\widetilde{R}/I) \leqslant 1$ (see [BouAC, VIII.25, Cor. 2 a)]). Thus, since $\widetilde{R}$ is Cohen–Macaulay and the finite $\widetilde{R}$-module $\widetilde{M} := \Gamma(\widetilde{U}, \mathscr{E}nd(\widetilde{\mathscr{V}}))$ is free on $\widetilde{U}$, we need to show that for every prime $\mathfrak{p} \subset R$ outside $\widetilde{U}$,

$$\operatorname{depth}_{\widetilde{R}_{\mathfrak{p}}}(\widetilde{M}_{\mathfrak{p}}) \geqslant \min(n, \dim(\widetilde{R}_{\mathfrak{p}})). \quad (2.1.2)$$

Scaling by f is injective on $\widetilde{j}_*(\mathscr{E}nd(\widetilde{\mathscr{V}}))$ because it is so locally over $\widetilde{U}$, so f is a nonzerodivisor for $\widetilde{M}$. Moreover, by (a), the R-module $\widetilde{M}/f\widetilde{M}$ is identified with $M := \Gamma(U_R, \mathscr{E}nd(\mathscr{V}))$. Thus, by [EGA IV_1, 0.16.4.10 (i)] and the (S_n) assumption on M,

$$\dim(\widetilde{R}) - \operatorname{depth}_{\widetilde{R}}(\widetilde{M}) = \dim(R) - \operatorname{depth}_R(M) \leqslant \dim(R) - \min(n, \dim(R)). \quad (2.1.3)$$

The inequality (2.1.2) for $\mathfrak{p} = \widetilde{\mathfrak{m}}$ follows:

$$\operatorname{depth}_{\widetilde{R}}(\widetilde{M}) \geqslant \min(n, \dim(R)) + 1 \geqslant \min(n, \dim(\widetilde{R})).$$

Thus, we may assume that $\mathfrak{p} \in \operatorname{Spec}(\widetilde{R})\backslash(\widetilde{U} \cup \widetilde{\mathfrak{m}})$, so that $\dim(\widetilde{R}_{\mathfrak{p}}) = \dim(\widetilde{R}) - 1 = \dim(R)$. The upper semicontinuity of codepth [EGA IV_2, 6.11.2 (i)] and (2.1.3) then give the desired

$$\operatorname{depth}_{\widetilde{R}_{\mathfrak{p}}}(\widetilde{M}_{\mathfrak{p}}) = \dim(R) - (\dim(\widetilde{R}_{\mathfrak{p}}) - \operatorname{depth}_{\widetilde{R}_{\mathfrak{p}}}(\widetilde{M}_{\mathfrak{p}})) \geqslant \min(n, \dim(R)) = \min(n, \dim(\widetilde{R}_{\mathfrak{p}})).$$

(c) The coherent R-module $\Gamma(U_R, \mathscr{E}nd(\mathscr{V}))$ is of depth $\geqslant 4$, so, analogously to (2.1.1), we have

$$H^2(U_R, \mathscr{E}nd(\mathscr{V})) \cong H^3_{\mathfrak{m}}(R, \Gamma(U_R, \mathscr{E}nd(\mathscr{V}))) = 0. \quad (2.1.4)$$

Consequently, since $f \in \widetilde{R}$ is a nonzerodivisor, there is no obstruction to deforming $\mathscr{V}$ to U_{R_2} (see, for instance, [Ill05, 8.5.3 (b)]), to the effect that $\mathscr{V}$ lifts to a vector bundle $\mathscr{V}_2$ on U_{R_2}. The obstruction to deforming $\mathscr{V}_2$ to U_{R_3} is again controlled by $H^2(U_R, \mathscr{E}nd(\mathscr{V}))$, so $\mathscr{V}_2$ lifts to a vector bundle $\mathscr{V}_3$ on U_{R_3}. Proceeding in this way, we lift $\mathscr{V}$ to a vector bundle $\widehat{\mathscr{V}} := \varprojlim_n \mathscr{V}_n$ on the formal f-adic completion of $U_{\widetilde{R}}$. The local Lefschetz theorem [SGA 2_{new}, X, 2.1 (ii)] then algebraizes $\widehat{\mathscr{V}}$ to a desired $\widetilde{\mathscr{V}}$. □

Geometrically, the depth condition of (1.1.1) amounts to the (S_4) requirement for $j_*(\mathscr{E}nd(\mathscr{V}))$:

Lemma 2.2. *For a Noetherian local ring R that is of dimension $\geqslant 2$ and whose completion $\widehat{R}$ is* (S_2)*, and for vector bundles $\mathscr{V}$ and $\mathscr{V}'$ on U_R,*

$$\text{the } R\text{-modules} \qquad \Gamma(U_R, \mathscr{V}) \quad \text{and} \quad \mathrm{Hom}_R(\Gamma(U_R, \mathscr{V}), \Gamma(U_R, \mathscr{V}')) \qquad \text{are finite and } (\mathrm{S}_2),$$

with the associated coherent sheaves $j_*(\mathscr{V})$ and $j_*(\mathscr{H}om(\mathscr{V}, \mathscr{V}'))$, where $j\colon U_R \hookrightarrow \mathrm{Spec}(R)$.

Proof. By [EGA IV_2, 5.10.8], the (S_2) assumption implies that $\mathrm{Spec}(\widehat{R})$ has no irreducible component of dimension $\leqslant 1$. Thus, since the formation of $j_*(-)$ commutes with the flat base change to $\widehat{R}$, the finiteness assertion follows from [SGA 2_{new}, VIII, 2.3 (ii)$\Leftrightarrow$(iv)] (see also [SGA 1_{new}, VIII, 1.10]). Since R itself is (S_2) (see [EGA IV_4, 6.4.1 (i)]), the (S_2) assertion and the claim about $j_*(\mathscr{V})$ then follow from [EGA IV_2, 5.10.5]. In general, if a finite R-module M is of depth $\geqslant 2$, then so is any $\mathrm{Hom}_R(M', M)$: if $f \in R$ is a nonzerodivisor for M, then $(\mathrm{Hom}_R(M', M))/f \subset \mathrm{Hom}_R(M', M/f)$, so that any $g \in R$ that is a nonzerodivisor for M/f is also a nonzerodivisor for $(\mathrm{Hom}_R(M', M))/f$. In particular, we conclude that $\mathrm{Hom}_R(\Gamma(U_R, \mathscr{V}), \Gamma(U_R, \mathscr{V}'))$ is of depth $\geqslant 2$. Then, by *loc. cit.*,

$$\mathrm{Hom}_R(\Gamma(U_R, \mathscr{V}), \Gamma(U_R, \mathscr{V}')) \xrightarrow{\sim} \Gamma(R, j_*(\mathscr{H}om(\mathscr{V}, \mathscr{V}'))). \qquad \square$$

We are ready for the promised extension of the Grothendieck–Lefschetz theorem:

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a local ring that is a complete intersection of dimension $\geqslant 4$ and consider the open immersion $j\colon U_R \hookrightarrow \mathrm{Spec}(R)$. A vector bundle $\mathscr{V}$ on U_R is free if and only if $j_*(\mathscr{E}nd(\mathscr{V}))$ is (S_4) (that is, if and only if $\mathrm{depth}_R(\mathrm{End}_R(\Gamma(U_R, \mathscr{V}))) \geqslant 4$, see Lemma 2.2).*

Proof. By Lemma 2.2, both $j_*(\mathscr{V})$ and $j_*(\mathscr{E}nd(\mathscr{V}))$ are coherent. If $\mathscr{V}$ is free, then so is $\mathscr{E}nd(\mathscr{V})$, so that $j_*(\mathscr{E}nd(\mathscr{V}))$ is a direct sum of copies of $\mathscr{O}_{\mathrm{Spec}(R)}$, and hence is (S_n) for any n because R is Cohen–Macaulay. For the converse, we assume that $j_*(\mathscr{E}nd(\mathscr{V}))$ is (S_4).

To establish the freeness of $\mathscr{V}$, we will argue that $j_*(\mathscr{V})$ is free. Flat base change to $\widehat{R}$ commutes with $j_*(-)$, preserves the depth assumption, and descends freeness, so we may assume that R is $\mathfrak{m}$-adically complete. Then $R \cong S/(f_1, \dots, f_n)$ for a complete regular local ring $(S, \mathfrak{n})$ and a regular sequence $f_1, \dots, f_n \in \mathfrak{n}$. We will argue by induction on n, the case $n = 0$ being supplied by [HW97, Cor. 2.9].

Suppose that $n \geqslant 1$ and set $\widetilde{R} := S/(f_1, \dots, f_{n-1})$. By Lemma 2.1, the vector bundle $\mathscr{V}$ lifts to a vector bundle $\widetilde{\mathscr{V}}$ defined on some open neighborhood $\widetilde{U}$ of U_R in $U_{\widetilde{R}}$ and the pushforward $\widetilde{j}_*(\mathscr{E}nd(\widetilde{\mathscr{V}}))$ along $\widetilde{j}\colon \widetilde{U} \hookrightarrow \mathrm{Spec}(\widetilde{R})$ is (S_4). We saw in the proof of Lemma 2.1 (b) that the complement of $\widetilde{U}$ in $U_{\widetilde{R}}$ consists of finitely many prime ideals $\mathfrak{p} \subset \widetilde{R}$ with $\dim(\widetilde{R}_{\mathfrak{p}}) = \dim(R)$. The inductive hypothesis applies to the completion of each such $\widetilde{R}_{\mathfrak{p}}$ equipped with the restriction of $\widetilde{\mathscr{V}}$ to $U_{\widetilde{R}_{\mathfrak{p}}}$, to the effect that the restriction of $\widetilde{j}_*(\widetilde{\mathscr{V}})$ to $U_{\widetilde{R}}$ is a vector bundle. Another application of the inductive assumption, this time to $\widetilde{R}$ equipped with $(\widetilde{j}_*(\widetilde{\mathscr{V}}))|_{U_{\widetilde{R}}}$, then proves that $\widetilde{j}_*(\widetilde{\mathscr{V}})$ is free. It follows that $\widetilde{\mathscr{V}}$ is free as well, and hence that so is its base change $\mathscr{V}$. $\square$

3. The sharpness of the assumptions

The following examples illustrate the optimality of the assumptions of Theorem 2.3.

3.1. The dimension requirement is sharp. For a field k, consider the local ring

$$R := (k[x, y, z, t]/(xy - zt))_{(x,y,z,t)}$$

that is a complete intersection of dimension 3. We claim that $\mathrm{Pic}(U_R) \cong \mathbb{Z}$, to the effect that the dimension $\geqslant 4$ condition of Theorem 2.3 cannot be weakened to $\geqslant 3$: indeed, for any line bundle $\mathscr{L}$ on U_R, we have $\mathscr{O}_{U_R} \xrightarrow{\sim} \mathscr{E}nd(\mathscr{L})$, so $j_*(\mathscr{E}nd(\mathscr{L}))$ is (S_n) for every n, but $\mathscr{L}$ will need not be $\mathscr{O}_{U_R}$.

The equation $xy - zt$ cuts out $X := \mathbb{P}^1_k \times \mathbb{P}^1_k$ sitting in $\mathbb{P}^3_k$ via its Segre embedding. Since $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, with the hyperplane class spanning the diagonal copy of $\mathbb{Z}$ (see [Har77, Eg. II.6.6.2]), we conclude that the Picard group of the punctured spectrum of the local ring of the vertex of the affine cone over $X \subset \mathbb{P}^3_k$ is $\mathbb{Z}$ (see [Har77, Ex. II.6.3]). Since this local ring is R, we obtain the claimed $\mathrm{Pic}(U_R) \cong \mathbb{Z}$.

3.2. The (S_4) requirement is sharp. For every $n \geqslant 1$ and every algebraically closed field k of characteristic different from 2, following a suggestion of Hailong Dao, we will construct a *nonfree* module M_n over the local, $(2n-1)$-dimensional, complete intersection ring

$$R_n := k[\![x, y, u_1, v_1, \ldots, u_{n-1}, v_{n-1}]\!]/(xy + u_1 v_1 + \cdots + u_{n-1} v_{n-1})$$

such that M_n is Cohen–Macaulay of depth $2n - 1$ (that is, M_n is "maximal Cohen–Macaulay") and the R_n-module $\mathrm{End}_{R_n}(M_n)$ is (S_3). Since U_{R_n} is regular, the Auslander–Buchsbaum formula will ensure that M_n defines a vector bundle $\mathscr{V}_n$ on U_{R_n}. For $n \geqslant 2$, the pushforward $(j_n)_*(\mathscr{V}_n)$ along $j_n \colon U_{R_n} \hookrightarrow \mathrm{Spec}(R_n)$ will be given by M_n (see [EGA IV$_2$, 5.10.5]), so $\mathscr{V}_n$ will be nonfree but $(j_n)_*(\mathscr{E}nd(\mathscr{V}_n))$ will be (S_3) (see Lemma 2.2). Thus, for $n \geqslant 3$ (when $\dim(R_n) \geqslant 4$), this will show that the (S_4) requirement in Theorem 2.3 cannot be weakened to (S_3) (even when $j_*(\mathscr{V})$ itself is (S_n) for every n).

For $n = 1$, we set $M_1 := k[\![y]\!]$ with $R_1 = k[\![x, y]\!]/(xy)$, so that M_1 is a nonfree maximal Cohen–Macaulay R_1-module, $\mathrm{End}_{R_1}(M_1)$ is (S_3) (equivalently, (S_1)), and M_1 admits the free resolution

$$\ldots \xrightarrow{y} k[\![x, y]\!]/(xy) \xrightarrow{x} k[\![x, y]\!]/(xy) \xrightarrow{y} k[\![x, y]\!]/(xy) \xrightarrow{x} k[\![x, y]\!]/(xy).$$

This resolution shows that

$$\mathrm{Ext}^{2i-1}_{R_1}(M_1, M_1) = 0 \quad \text{and} \quad \mathrm{Ext}^{2i}_{R_1}(M_1, M_1) \cong k \qquad \text{for} \qquad i \geqslant 1. \tag{3.2.1}$$

To construct the remaining M_n from M_1, we will use the Knörrer periodicity theorem [Knö87, Thm. 3.1]: for every $n \geqslant 1$, the stable category $\underline{\mathrm{MCM}}(R_n)$ of maximal Cohen–Macaulay R_n-modules[1] is equivalent to its counterpart $\underline{\mathrm{MCM}}(R_{n+1})$. Explicitly, in terms of matrix factorizations

$$\widetilde{R}^a_n \xrightarrow{\varphi} \widetilde{R}^a_n \xrightarrow{\psi} \widetilde{R}^a_n \qquad \text{with} \qquad \psi \circ \varphi = \varphi \circ \psi = xy + u_1 v_1 + \cdots + u_{n-1} v_{n-1}$$

with $\widetilde{R}_n := k[\![x, y, u_1, v_1, \ldots, u_{n-1}, v_{n-1}]\!]$, Knörrer's functor maps the maximal Cohen–Macaulay module $\mathrm{Coker}(\varphi)$ to $\mathrm{Coker}\left(\begin{pmatrix} u_n & \psi \\ \varphi & -v_n \end{pmatrix}\right)$, where $\begin{pmatrix} u_n & \psi \\ \varphi & -v_n \end{pmatrix}$ is a map in the matrix factorization

$$\widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \xrightarrow{\begin{pmatrix} u_n & \psi \\ \varphi & -v_n \end{pmatrix}} \widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \xrightarrow{\begin{pmatrix} v_n & \psi \\ \varphi & -u_n \end{pmatrix}} \widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \qquad \text{of} \qquad xy + u_1 v_1 + \cdots + u_n v_n.$$

By [Buc87, 4.4.1 (3)], the category $\underline{\mathrm{MCM}}(R_n)$ is naturally triangulated, with the translation being given by the syzygy functor $\mathrm{Coker}(\varphi) \mapsto \mathrm{Coker}(\psi)$ (that is, by $(\varphi, \psi) \mapsto (\psi, \varphi)$ on matrix

[1] The objects of $\underline{\mathrm{MCM}}(R_n)$ are the maximal Cohen–Macaulay R_n-modules and the morphisms are given by $\mathrm{Hom}_{\underline{\mathrm{MCM}}(R_n)}(M, M') := \mathrm{Hom}_{R_n}(M, M')/\{f \colon M \to M' \text{ such that } f \text{ factors through a finite free } R_n\text{-module}\}$, see [Buc87, 2.1.1 and 4.2.1] for more details.

factorizations), which is its own inverse. Thus, the commutativity of the diagram

$$\begin{tikzcd}
\widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \arrow[r, "\left(\begin{smallmatrix} u_n & \varphi \\ \psi & -v_n \end{smallmatrix}\right)"] \arrow[d, "\sim"', "\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)"] & \widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \arrow[r, "\left(\begin{smallmatrix} v_n & \varphi \\ \psi & -u_n \end{smallmatrix}\right)"] \arrow[d, "\sim"', "\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)"] & \widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \arrow[d, "\sim"', "\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)"] \\
\widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \arrow[r, "\left(\begin{smallmatrix} v_n & \psi \\ \varphi & -u_n \end{smallmatrix}\right)"] & \widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1} \arrow[r, "\left(\begin{smallmatrix} u_n & \psi \\ \varphi & -v_n \end{smallmatrix}\right)"] & \widetilde{R}^a_{n+1} \oplus \widetilde{R}^a_{n+1}
\end{tikzcd}$$

shows that the Knörrer equivalence commutes with translations.

In summary, the image of M_1 under the $(n-1)$-fold Knörrer equivalence is a maximal Cohen–Macaulay R_n-module M_n such that the "stabilized Ext's" defined as in [Buc87, 6.1.1] by

$$\underline{\mathrm{Ext}}^i_{R_n}(M, M') := \mathrm{Hom}_{\underline{\mathrm{MCM}}(R_n)}(M, M'[i])$$

satisfy

$$\underline{\mathrm{Ext}}^i_{R_n}(M_n, M_n) \cong \underline{\mathrm{Ext}}^i_{R_1}(M_1, M_1) \qquad \text{for every } i \text{ and } n.$$

Since each M_n is maximal Cohen–Macaulay, [Buc87, 6.4.1 (i)] ensures that for $i > 0$ the stabilized Ext's in question agree with their usual nonstable counterparts, so that (3.2.1) gives

$$\mathrm{Ext}^{2i-1}_{R_n}(M_n, M_n) = 0 \quad \text{and} \quad \mathrm{Ext}^{2i}_{R_n}(M_n, M_n) \neq 0 \qquad \text{for every} \qquad n, i \geqslant 1.$$

In particular, $\mathrm{Ext}^2_{R_n}(M_n, M_n) \neq 0$, so each M_n is nonfree. On the other hand, the vanishing of $\mathrm{Ext}^1_{R_n}(M_n, M_n)$ implies that $\mathrm{End}_{R_n}(M_n)$ fits into an R_n-module exact sequence

$$0 \to \mathrm{End}_{R_n}(M_n) \to M_n^{\oplus r_1} \to M_n^{\oplus r_2} \to M_n^{\oplus r_3} \to Q \to 0.$$

Since M_n is Cohen–Macaulay, it follows that $H^j_{\mathfrak{m}}(R_n, \mathrm{End}_{R_n}(M_n)) = 0$ for $n \geqslant 2$ and $j \leqslant 2$ (see [SGA 2_{new}, III, 3.3]), so that $\mathrm{End}_{R_n}(M_n)$, which is free over U_{R_n}, is (S_3), as desired.

Remark 3.3. The dimensions of the rings R_n are odd. Thus, the failure of the freeness of the modules M_n should be contrasted with the following result [Dao13, 7.2.5]: for a local ring R of *even* dimension $\geqslant 4$ whose completion is a quotient of an either equicharacteristic or unramified regular local ring by a nonzerodivisor, a vector bundle $\mathscr{V}$ on U_R is free if and only if $\mathrm{depth}_R(\mathrm{End}_R(\Gamma(U_R, \mathscr{V}))) \geqslant 3$.

References

[ADS93] Maurice Auslander, Songqing Ding, and Øyvind Solberg, *Liftings and weak liftings of modules*, J. Algebra **156** (1993), no. 2, 273–317. MR1216471

[BouAC] Nicolas Bourbaki, *Éléments de mathématique. Algèbre commutative*, chap. I-VII, Hermann (1961, 1964, 1965); chap. VIII-X, Springer (2006, 2007) (French).

[Buc87] Ragnar-Olaf Buchweitz, *Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings*, unpublished manuscript (1987). Available at https://www.researchgate.net/publication/268205854.

[Dao13] Hailong Dao, *Some homological properties of modules over a complete intersection, with applications*, Commutative algebra, Springer, New York, 2013, pp. 335–371. MR3051378

[EGA IV_1] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Inst. Hautes Études Sci. Publ. Math. **20** (1964), 259 (French). MR0173675 (30 #3885)

[EGA IV_2] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965), 231 (French). MR0199181 (33 #7330)

[EGA IV_4] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 361 (French). MR0238860 (39 #220)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[HW97] Craig Huneke and Roger Wiegand, *Tensor products of modules, rigidity and local cohomology*, Math. Scand. **81** (1997), no. 2, 161–183. MR1612887

[Ill05] Luc Illusie, *Grothendieck's existence theorem in formal geometry*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 179–233. With a letter (in French) of Jean-Pierre Serre. MR2223409

[Knö87] Horst Knörrer, *Cohen-Macaulay modules on hypersurface singularities. I*, Invent. Math. **88** (1987), no. 1, 153–164. MR877010

[SGA 1_{new}] *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, Société Mathématique de France, Paris, 2003 (French). Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61]; Directed by A. Grothendieck; With two papers by M. Raynaud; Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. MR2017446 (2004g:14017)

[SGA 2_{new}] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], vol. 4, Société Mathématique de France, Paris, 2005 (French). Séminaire de Géométrie Algébrique du Bois Marie, 1962; Augmenté d'un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud]; With a preface and edited by Yves Laszlo; Revised reprint of the 1968 French original. MR2171939