Upper large deviations for the maximal flow through a domain of $\mathbb{R}^d$ in first passage percolation

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**Abstract:** We consider the standard first passage percolation model in the rescaled graph $\mathbb{Z}^d/n$ for $d \geq 2$, and a domain $\Omega$ of boundary $\Gamma$ in $\mathbb{R}^d$. Let $\Gamma^1$ and $\Gamma^2$ be two disjoint open subsets of $\Gamma$, representing the parts of $\Gamma$ through which some water can enter and escape from $\Omega$. We investigate the asymptotic behaviour of the flow $\phi_n$ through a discrete version $\Omega_n$ of $\Omega$ between the corresponding discrete sets $\Gamma^1_n$ and $\Gamma^2_n$. We prove that under some conditions on the regularity of the domain and on the law of the capacity of the edges, the upper large deviations of $\phi_n/n^{d-1}$ above a certain constant are of volume order.

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## 1 First definitions and main result

We use many notations introduced in [1] and [2]. Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$ having for vertices $\mathbb{Z}^d_n = \mathbb{Z}^d/n$ and for edges $\mathbb{E}^d_n$, the set of pairs of nearest neighbours for the standard $L^1$ norm. With each edge $e$ in $\mathbb{E}^d_n$ we associate a random variable $t(e)$ with values in $\mathbb{R}^+$. We suppose that the family $(t(e), e \in \mathbb{E}^d_n)$ is independent and identically distributed, with a common law $\Lambda$: this is the standard model of first passage percolation on the graph $(\mathbb{Z}^d_n, \mathbb{E}^d_n)$. We interpret $t(e)$ as the capacity of the edge $e$; it means that $t(e)$ is the maximal amount of fluid that can go through the edge $e$ per unit of time.

We consider an open bounded connected subset $\Omega$ of $\mathbb{R}^d$ such that the boundary $\Gamma = \partial \Omega$ of $\Omega$ is piecewise of class $C^1$ (in particular $\Gamma$ has finite area: $\mathcal{H}^{d-1}(\Gamma) < \infty$). It means that $\Gamma$ is included in the union of a finite number of hypersurfaces of class $C^1$, i.e., in the union of a finite number of $C^1$ submanifolds of $\mathbb{R}^d$ of codimension 1. Let $\Gamma^1$, $\Gamma^2$ be two disjoint subsets of $\Gamma$ that are open in
\section{First Definitions and Main Result}

We want to define the maximal flow from $\Gamma^1$ to $\Gamma^2$ through $\Omega$ for the capacities $(t(e), e \in \mathbb{E}_n^d)$. We consider a discrete version $(\Omega_n, \Gamma_n, \Gamma^1_n, \Gamma^2_n)$ of $(\Omega, \Gamma^1, \Gamma^2)$ defined by:

\[
\begin{align*}
\Omega_n &= \{x \in \mathbb{Z}_n^d \mid d_\infty(x, \Omega) < 1/n\}, \\
\Gamma_n &= \{x \in \Omega_n \mid \exists y \notin \Omega_n, \langle x, y \rangle \in \mathbb{E}_n^d\}, \\
\Gamma^i_n &= \{x \in \Gamma_n \mid d_\infty(x, \Gamma^i) < 1/n, \; d_\infty(x, \Gamma^{3-i}) \geq 1/n\} \text{ for } i = 1, 2,
\end{align*}
\]

where $d_\infty$ is the $L^\infty$-distance, the notation $\langle x, y \rangle$ corresponds to the edge of endpoints $x$ and $y$ (see figure \ref{fig:domain}).

![Figure 1: Domain $\Omega$](image_url)

We shall study the maximal flow from $\Gamma^1_n$ to $\Gamma^2_n$ in $\Omega_n$. Let us define properly the maximal flow $\phi(F_1 \to F_2$ in $C)$ from $F_1$ to $F_2$ in $C$, for $C \subset \mathbb{R}^d$ (or by commodity the corresponding graph $C \cap \mathbb{Z}^d/n$). We will say that an edge $e = \langle x, y \rangle$ belongs to a subset $A$ of $\mathbb{R}^d_2$, which we denote by $e \in A$, if the interior of the segment joining $x$ to $y$ is included in $A$. We define $\mathbb{E}_n^d$ as the set of all the oriented edges, i.e., an element $\tilde{e}$ in $\mathbb{E}_n^d$ is an ordered pair of vertices which are nearest neighbours. We denote an element $\tilde{e} \in \mathbb{E}_n^d$ by $\langle \langle x, y \rangle, y \rangle$, where $x$, $y \in \mathbb{Z}_n^d$ are the endpoints of $\tilde{e}$ and the edge is oriented from $x$ towards $y$. We consider the set $\mathcal{S}$ of all pairs of functions $(g, o)$, with $g : \mathbb{E}_n^d \to \mathbb{R}^+$ and $o : \mathbb{E}_n^d \to \mathbb{E}_n^d$, such that $o(\langle x, y \rangle) = \{\langle \langle x, y \rangle, y \rangle, \lambda(\langle x, y \rangle, y)\}$, satisfying:

- for each edge $e$ in $C$ we have
  \[0 \leq g(e) \leq t(e),\]

- for each vertex $v$ in $C \setminus (F_1 \cup F_2)$ we have
  \[
  \sum_{e \in C : o(e) = \langle \langle v, \rangle \rangle} g(e) = \sum_{e \in C : o(e) = \langle \langle v, \rangle \rangle} g(e),
  \]

where the notation $o(e) = \langle \langle v, \rangle \rangle$ (respectively $o(e) = \langle \langle v, \lambda \rangle \rangle$) means that there exists $y \in \mathbb{Z}_n^d$ such that $e = \langle v, y \rangle$ and $o(e) = \langle \langle v, y \rangle \rangle$ (respectively $o(e) = \langle \langle v, \lambda \rangle \rangle$). A couple $(g, o) \in \mathcal{S}$ is a possible stream in $C$ from $F_1$ to $F_2$: $g(e)$ is the amount of fluid that goes through the edge $e$, and $o(e)$ gives the direction in which the fluid goes through $e$. The two conditions on $(g, o)$ express only the fact
that the amount of fluid that can go through an edge is bounded by its capacity, and that there is no loss of fluid in the graph. With each possible stream we associate the corresponding flow

$$\text{flow}(g, o) = \sum_{u \in F_2, v \in C : (u,v) \in \mathbb{R}^d} g((u,v))\mathbb{I}_{o((u,v))=\langle(u,v)\rangle} - g((u,v))\mathbb{I}_{o((u,v))=\langle(v,u)\rangle}.$$ 

This is the amount of fluid that crosses $C$ from $F_1$ to $F_2$ if the fluid respects the stream $(g, o)$. The maximal flow through $C$ from $F_1$ to $F_2$ is the supremum of this quantity over all possible choices of streams

$$\phi(F_1 \rightarrow F_2 \text{ in } C) = \sup\{\text{flow}(g, o) | (g, o) \in S\}.$$ 

We recall that we consider an open bounded connected subset $\Omega$ of $\mathbb{R}^d$ whose boundary $\Gamma$ is piecewise of class $C^1$, and two disjoint open subsets $\Gamma^1$ and $\Gamma^2$ of $\Gamma$. We denote by

$$\phi_n = \phi(\Gamma^1_n \rightarrow \Gamma^2_n \text{ in } \Omega_n)$$

the maximal flow from $\Gamma^1_n$ to $\Gamma^2_n$ in $\Omega_n$. We will investigate the asymptotic behaviour of $\phi_n/n^{d-1}$ when $n$ goes to infinity. More precisely, we will show that the upper large deviations of $\phi_n$ above a certain constant $\phi_\Omega$ are of volume order. The description of $\phi_\Omega$ will be given in section 2. Here we state the precise theorem:

**Theorem 1.** We suppose that $d(\Gamma^1, \Gamma^2) > 0$. If the law $\Lambda$ of the capacity of an edge admits an exponential moment:

$$\exists \theta > 0 \quad \int_{\mathbb{R}^+} e^{\theta x} d\Lambda(x) < +\infty,$$

then there exists a finite constant $\tilde{\phi}_\Omega$ such that for all $\lambda > \tilde{\phi}_\Omega$,

$$\limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}[\phi_n \geq \lambda n^{d-1}] < 0.$$ 

**Remark 1.** In the theorem 1 we need to impose that $d(\Gamma^1, \Gamma^2) > 0$ because otherwise we cannot be sure that $\tilde{\phi}_\Omega < \infty$, as we will see in section 4. Moreover, if $d(\Gamma^1, \Gamma^2) = 0$, there exists a set of edges of constant cardinality (not depending on $n$) containing paths from $\Gamma^1_n$ to $\Gamma^2_n$ through $\Omega_n$ for all $n$ along the common boundary of $\Gamma^1$ and $\Gamma^2$, and so it may be sufficient for these edges to have a huge capacity to obtain that $\phi_n$ is abnormally big too. Thus, we cannot hope to obtain upper large deviations of volume order (see [2] for a counter-example).

**Remark 2.** The large deviations we obtain are of the relevant order. Indeed, if all the edges in $\Omega_n$ have a capacity which is abnormally big, then the maximal flow $\phi_n$ will be abnormally big too. The probability for these edges to have an abnormally large capacity is of order $\exp -Cn^d$ for a constant $C$, because the number of edges in $\Omega_n$ is $C'n^d$ for a constant $C'$.

**Remark 3.** In the two companion papers [2] and [3], we prove in fact that $\tilde{\phi}_\Omega$ is the almost sure limit of $\phi_n/n^{d-1}$ when $n$ goes to infinity, and that the lower large deviations of $\phi_n/n^{d-1}$ below $\tilde{\phi}_\Omega$ are of surface order.
2 Computation of $\tilde{\phi}_\Omega$

2.1 Geometric notations

We start with some geometric definitions. For a subset $X$ of $\mathbb{R}^d$, we denote by $\mathcal{H}^s(X)$ the $s$-dimensional Hausdorff measure of $X$ (we will use $s = d - 1$ and $s = d - 2$). The $r$-neighbourhood $\mathcal{V}_i(X, r)$ of $X$ for the distance $d_i$, that can be the Euclidean distance if $i = 2$ or the $L^\infty$-distance if $i = \infty$, is defined by

$$\mathcal{V}_i(X, r) = \{ y \in \mathbb{R}^d \mid d_i(y, X) < r \}.$$ 

If $X$ is a subset of $\mathbb{R}^d$ included in an hyperplane of $\mathbb{R}^d$ and of codimension 1 (for example a non degenerate hyperrectangle), we denote by $\text{hyp}(X)$ the hyperplane spanned by $X$, and we denote by $\text{cyl}(X, h)$ the cylinder of basis $X$ and of height $2h$ defined by

$$\text{cyl}(X, h) = \{ x + tv \mid x \in X, \ t \in [-h, h] \},$$

where $v$ is one of the two unit vectors orthogonal to $\text{hyp}(X)$ [see figure 2]. For $x \in \mathbb{R}^d$, $r \geq 0$ and a unit vector $v$, we denote by $B(x, r)$ the closed ball centered at $x$ of radius $r$.

2.2 Flow in a cylinder

Here are some particular definitions of flows through a box. It is important to know them, because all our work consists in comparing the maximal flow $\phi_n$ in $\Omega_n$ with the maximal flows in small cylinders. Let $A$ be a non degenerate hyperrectangle, i.e., a box of dimension $d - 1$ in $\mathbb{R}^d$. All hyperrectangles will be supposed to be closed in $\mathbb{R}^d$. We denote by $v$ one of the two unit vectors orthogonal to $\text{hyp}(A)$. For $h$ a positive real number, we consider the cylinder $\text{cyl}(A, h)$. The set $\text{cyl}(A, h) \setminus \text{hyp}(A)$ has two connected components, which we denote by $C_1(A, h)$ and $C_2(A, h)$. For $i = 1, 2$, let $A^b_i$ be the set of the points in $C_i(A, h) \cap \mathbb{Z}_n^d$ which have a nearest neighbour in $\mathbb{Z}_n^d \setminus \text{cyl}(A, h)$:

$$A^b_i = \{ x \in C_i(A, h) \cap \mathbb{Z}_n^d \mid \exists y \in \mathbb{Z}_n^d \setminus \text{cyl}(A, h), \ (x, y) \in \mathbb{E}_n^d \}.$$ 

Let $T(A, h)$ (respectively $B(A, h)$) be the top (respectively the bottom) of $\text{cyl}(A, h)$, i.e.,

$$T(A, h) = \{ x \in \text{cyl}(A, h) \mid \exists y \notin \text{cyl}(A, h), \ (x, y) \in \mathbb{E}_n^d \text{ and } (x, y) \text{ intersects } A + hv \}.$$
and
\[ B(A, h) = \{ x \in \text{cyl}(A, h) \mid \exists y \notin \text{cyl}(A, h), \langle x, y \rangle \in \mathbb{E}^d_n \text{ and } \langle x, y \rangle \text{ intersects } A - hv \} \).

For a given realisation \((t(e), e \in \mathbb{E}^d_n)\) we define the variable \(\tau(A, h) = \tau(\text{cyl}(A, h), v)\) by
\[ \tau(A, h) = \tau(\text{cyl}(A, h), v) = \phi(A^h_1 \to A^h_2 \text{ in cyl}(A, h)) \],
and the variable \(\phi(A, h) = \phi(\text{cyl}(A, h), v)\) by
\[ \phi(A, h) = \phi(\text{cyl}(A, h), v) = \phi(B(A, h) \to T(A, h) \text{ in cyl}(A, h)) \],
where \(\phi(F_1 \to F_2 \text{ in } C)\) is the maximal flow from \(F_1\) to \(F_2\) in \(C\), for \(C \subset \mathbb{R}^d\) (or by commodity the corresponding graph \(C \cap \mathbb{Z}^d/n)\) defined previously. The dependence in \(n\) is implicit here, in fact we can also write \(\tau_n(A, h)\) and \(\phi_n(A, h)\) if we want to emphasize this dependence on the mesh of the graph.

### 2.3 Max-flow min-cut theorem

The maximal flow \(\phi(F_1 \to F_2 \text{ in } C)\) can be expressed differently thanks to the max-flow min-cut theorem (see [1]). We need some definitions to state this result. A path on the graph \(\mathbb{Z}^d_n\) from \(v_0\) to \(v_m\) is a sequence \((v_0, e_1, v_1, ..., e_m, v_m)\) of vertices \(v_0, ..., v_m\) alternating with edges \(e_1, ..., e_m\) such that \(v_{i-1}\) and \(v_i\) are neighbours in the graph, joined by the edge \(e_i\), for \(i \in \{1, ..., m\}\). A set \(E\) of edges in \(C\) is said to cut \(F_1\) from \(F_2\) in \(C\) if there is no path from \(F_1\) to \(F_2\) in \(C \setminus E\). We call \(E\) an \((F_1, F_2)\)-cut if \(E\) cuts \(F_1\) from \(F_2\) in \(C\) and if no proper subset of \(E\) does. With each set \(E\) of edges we associate its capacity which is the variable
\[ V(E) = \sum_{e \in E} t(e) \].

The max-flow min-cut theorem states that
\[ \phi(F_1 \to F_2 \text{ in } C) = \min \{ V(E) \mid E \text{ is a } (F_1, F_2)\text{-cut} \} \].

### 2.4 Definition of \(\nu\)

The asymptotic behaviour of the rescaled expectation of \(\tau_n(A, h)\) for large \(n\) is well known, thanks to the almost subadditivity of this variable. We recall the following result:

**Theorem 2.** We suppose that
\[ \int_{[0, +\infty)} x d\Lambda(x) < \infty \]
Then for each unit vector \(v\) there exists a constant \(\nu(d, \Lambda, v) = \nu(v)\) (the dependence on \(d\) and \(\Lambda\) is implicit) such that for every non degenerate hyperrectangle \(A\) orthogonal to \(v\) and for every strictly positive constant \(h\), we have
\[ \lim_{n \to \infty} \frac{\mathbb{E}[\tau_n(A, h)]}{n^{d-1}h^{d-1}(A)} = \nu(v) \].
For a proof of this proposition, see [3]. We emphasize the fact that the limit depends on the direction of \( v \), but not on \( h \) nor on the hyperrectangle \( A \) itself.

In fact, Rossignol and Théret proved in [3] that under some moment conditions and/or some condition on \( A \), \( \nu(v) \) is the limit of the rescaled variable \( \tau_n(A, h)/(n^{d-1}H^{d-1}(A)) \) almost surely and in \( L^1 \). We also know, thanks to the works of Kesten [2], Zhang [11] and Rossignol and Théret [3] that the variable \( \phi_n(A, h)/(n^{d-1}H^{d-1}(A)) \) satisfies the same law of large numbers in the particular case where \( A \) is a straight hyperrectangle, i.e., a hyperrectangle of the form \( \prod_{i=1}^{d-1}[0, k_i] \times \{0\} \) for some \( k_i > 0 \).

We recall some geometric properties of the map \( \nu : v \in S^{d-1} \mapsto \nu(v) \), under the only condition on \( A \) that \( \mathbb{E}(t(e)) < \infty \). They have been stated in section 4.4 of [3]. There exists a unit vector \( v_0 \) such that \( \nu(v_0) = 0 \) if and only if for all unit vector \( v \), \( \nu(v) = 0 \), and it happens if and only if \( \Lambda(0) \geq 1 - p_c(d) \), where \( p_c(d) \) is the critical parameter of the bond percolation on \( \mathbb{Z}^d \). This property has been proved by Zhang in [10]. Moreover, \( \nu \) satisfies the weak triangle inequality, i.e., if \( (ABC) \) is a non degenerate triangle in \( \mathbb{R}^d \) and \( v_A, v_B, v_C \) are the exterior normal unit vectors to the sides \( [BC], [AC], [AB] \) in the plane spanned by \( A, B, C \), then

\[
\mathcal{H}^1([AB])\nu(v_C) \leq \mathcal{H}^1([AC])\nu(v_B) + \mathcal{H}^1([BC])\nu(v_A).
\]

This implies that the homogeneous extension \( \nu_0 \) of \( \nu \) to \( \mathbb{R}^d \), defined by \( \nu_0(0) = 0 \) and for all \( w \) in \( \mathbb{R}^d \),

\[
\nu_0(w) = |w|_2\nu(|w|_2),
\]

is a convex function; in particular, since \( \nu_0 \) is finite, it is continuous on \( \mathbb{R}^d \). We denote by \( \nu_{\min} \) (respectively \( \nu_{\max} \)) the infimum (respectively supremum) of \( \nu \) on \( S^{d-1} \).

The last result we recall is Theorem 4 in [3] concerning the upper large deviations of the variable \( \phi_n(A, h) \) above \( \nu(v) \):

**Theorem 3.** We suppose that

\[
\exists \gamma > 0 \quad \int_{[0, +\infty]} e^{\gamma x} \, d\Lambda(x) < \infty.
\]

Then for every unit vector \( v \) and every non degenerate hyperrectangle \( A \) orthogonal to \( v \), for every strictly positive constant \( h \) and for every \( \lambda > \nu(v) \) we have

\[
\liminf_{n \to \infty} -\frac{1}{n^{d-1}H^{d-1}(A)} \log \mathbb{P} \left[ \frac{\phi_n(A, h)}{n^{d-1}H^{d-1}(A)} \geq \lambda \right] > 0.
\]

We shall rely on this result for proving Theorem 5. Moreover, Theorem 5 is a generalisation of Theorem 3 where we work in the domain \( \Omega \) instead of a parallelepiped.

### 2.5 Continuous min-cut

We give here a definition of \( \tilde{\phi}_0 \) in terms of the map \( \nu \). When a hypersurface \( S \) is piecewise of class \( C^1 \), we say that \( S \) is transverse to \( \Gamma \) if for all \( x \in S \cap \Gamma \), the normal unit vectors to \( S \) and \( \Gamma \) at \( x \) are not collinear; if the normal vector to \( S \) (respectively to \( \Gamma \)) at \( x \) is not well defined, this property must be satisfied by all the vectors which are limits of normal unit vectors to \( S \) (respectively \( \Gamma \)) at \( y \in S \) (respectively \( y \in \Gamma \)) when we send \( y \) to \( x \). There is at most a finite number of such limits. We say that a subset \( P \) of \( \mathbb{R}^d \) is polyhedral if its boundary \( \partial P \) is included in the union of a finite
number of hyperplanes. For each point \( x \) of such a set \( P \) which is on the interior of one face of \( \partial P \), we denote by \( v_P(x) \) the exterior unit vector orthogonal to \( P \) at \( x \). For \( A \subset \mathbb{R}^d \), we denote by \( A \) the interior of \( A \). We define \( \tilde{\phi}_\Omega \) by

\[
\tilde{\phi}_\Omega = \inf \left\{ \mathcal{I}_\Omega(P) \right\}
\]

where

\[
\mathcal{I}_\Omega(P) = \int_{\partial P \cap \Omega} \nu(v_P(x)) \, d\mathcal{H}^{d-1}(x).
\]

See figure 3 to have an example of such a polyhedral set \( P \).

![Figure 3: A polyhedral set \( P \) as in the definition of \( \tilde{\phi}_\Omega \).](image)

The definition of the constant \( \tilde{\phi}_\Omega \) is not very intuitive. We propose to define the notion of a continuous cutset to have a better understanding of this constant. We say that \( S \subset \mathbb{R}^d \) cuts \( \Gamma^1 \) from \( \Gamma^2 \) in \( \overline{\Omega} \) if every continuous path from \( \Gamma^1 \) to \( \Gamma^2 \) in \( \Omega \) intersects \( S \). In fact, if \( P \) is a polyhedral set of \( \mathbb{R}^d \) such that

\[
\overline{\Gamma^1} \subset \overset{\circ}{P} \quad \text{and} \quad \overline{\Gamma^2} \subset \mathbb{R}^d \setminus P,
\]

then \( \partial P \cap \overline{\Omega} \) is a continuous cutset from \( \Gamma^1 \) to \( \Gamma^2 \) in \( \overline{\Omega} \). Since \( \nu(v) \) is the average amount of fluid that can cross a hypersurface of area one in the direction \( v \) per unit of time, it can be interpreted as the capacity of a unitary hypersurface orthogonal to \( v \). Thus \( \mathcal{I}_\Omega(P) \) can be interpreted as the capacity of the continuous cutset \( \partial P \cap \overline{\Omega} \) defined by \( P \). The constant \( \tilde{\phi}_\Omega \) is the solution of a min cut problem, because it is equal to the infimum of the capacity of a continuous cutset that satisfies some specific properties.
3 Sketch of the proof

We first prove that \( \tilde{\phi}_\Omega \) is finite, i.e., that there exists a polyhedral set \( P \subset \mathbb{R}^d \) such that \( \partial P \) is transverse to \( \Gamma \) and

\[
\Gamma^1 \subset \overset{\circ}{P}, \quad \Gamma^2 \subset \mathbb{R}^d \setminus P.
\]

Then, we consider such a polyhedral set \( P \) whose capacity \( \mathcal{I}_\Omega(P) \) is close to \( \tilde{\phi}_\Omega \). We construct a set \( \Omega' \) that contains a small neighbourhood of \( \Omega \), thus \( \Omega' \) contains \( \Omega_n \) for all large \( n \), and such that \( \mathcal{H}^{d-1}(\partial P \cap (\Omega' \setminus \Omega)) \) is very small. We need the property that \( \partial P \) is transverse to \( \Gamma \) to obtain this control on \( \mathcal{H}^{d-1}(\partial P \cap (\Omega' \setminus \Omega)) \). We want to construct a \((\Gamma^1_n, \Gamma^2_n)\)-cut in \( \Omega_n \) that is close to \( \partial P \cap \Omega' \). We cover \( \partial P \cap \Omega' \) with cylinders of arbitrarily small height; this is the reason why we need to consider a polyhedral set \( P \). A part of \( \partial P \cap \Omega' \) of very small area is missing in this covering. We construct then a \((\Gamma^1_n, \Gamma^2_n)\)-cut in \( \Omega_n \) with the help of cutsets in the cylinders constructed on \( \partial P \cap \Omega' \). To achieve this, we have to add edges to cover the part of \( \partial P \cap \Omega' \) missing in the covering by the cylinders, and to glue together the cutsets in the different cylinders. Thanks to the study of the upper large deviations for the maximal flow through cylinders made in \[\Omega\], we obtain that the probability that the flow \( \phi_n \) is greater than \( \mathcal{I}_\Omega(P)n^{d-1} \) goes to zero. We want to prove that this probability decays exponentially fast in \( n^d \). For that purpose, we have to consider a collection of cardinality of order \( n \) of possible sets of edges we can add to construct the cutset in \( \Omega_n \), and to choose the set that has the minimal capacity.

4 The constant \( \tilde{\phi}_\Omega \) is finite

To prove that \( \tilde{\phi}_\Omega < \infty \), it is sufficient to exhibit a set \( P \) satisfying all the conditions given in the definition of \( \tilde{\phi}_\Omega \). Indeed, if such a set \( P \) exists, then

\[
\tilde{\phi}_\Omega \leq \nu_{\max} \mathcal{H}^{d-1}(\partial P \cap \Omega) < \infty
\]

since a polyhedral set has finite perimeter in \( \Omega \). We will construct such a set \( P \). The idea of the proof is the following. We will cover \( \overline{\Gamma^1} \) with small hypercubes which are transverse to \( \Gamma^1 \) and at positive distance of \( \overline{\Gamma^2} \). Then, by compactness, we will extract a finite covering. We will denote by \( P \) the union of the hypercubes of this finite covering. Then \( P \) satisfies the desired properties.

We prove a geometric lemma:

**Lemma 1.** Let \( \Gamma \) be an hypersurface (that is a \( C^1 \) submanifold of \( \mathbb{R}^d \) of codimension 1) and let \( K \) be a compact subset of \( \Gamma \). There exists a positive \( M = M(\Gamma, K) \) such that:

\[
\forall \varepsilon > 0 \quad \exists r > 0 \quad \forall x, y \in K \quad |x - y|_2 \leq r \quad \Rightarrow \quad d_2(y, \tan(\Gamma, x)) \leq M \varepsilon |x - y|_2.
\]

(\( \tan(\Gamma, x) \) is the tangent hyperplane of \( \Gamma \) at \( x \)).

**Proof:**

By a standard compactness argument, it is enough to prove the following local property:

\[
\forall x \in \Gamma \quad \exists M(x) > 0 \quad \forall \varepsilon > 0 \quad \exists r(x, \varepsilon) > 0 \quad \forall y, z \in \Gamma \cap B(x, r(x, \varepsilon)) \quad d_2(y, \tan(\Gamma, z)) \leq M(x) \varepsilon |y - z|_2.
\]
Indeed, if this property holds, we cover \( K \) by the open balls \( \tilde{B}(x, r(x, \varepsilon)/2), x \in K \), we extract a finite subcover \( B(x_i, r(x_i, \varepsilon)/2), 1 \leq i \leq k \), and we set
\[
M = \max \{ M(x_i) : 1 \leq i \leq k \}, \quad r = \min \{ r(x_i, \varepsilon)/2 : 1 \leq i \leq k \}.
\]
Let now \( y, z \) belong to \( K \) with \( |y - z|_2 \leq r \). Let \( i \) be such that \( y \) belongs to \( B(x_i, r(x_i, \varepsilon)/2) \). Since \( r \leq r(x_i, \varepsilon)/2 \), then both \( y, z \) belong to the ball \( B(x_i, r(x_i, \varepsilon)) \) and it follows that
\[
d_2(y, \tan(\Gamma, z)) \leq M(x_i) |y - z|_2 \leq M \varepsilon |y - z|_2.
\]

We turn now to the proof of the above local property. Since \( \Gamma \) is an hypersurface, for any \( x \) in \( \Gamma \) there exists a neighbourhood \( V \) of \( x \) in \( \mathbb{R}^d \), a diffeomorphism \( f : V \to \mathbb{R}^d \) of class \( C^1 \) and a \( (d - 1) \) dimensional vector space \( Z \) of \( \mathbb{R}^d \) such that \( Z \cap f(V) = f(\Gamma \cap V) \) (see for instance [1], 3.1.19). Let \( A \) be a compact neighbourhood of \( x \) included in \( V \). Since \( f \) is a diffeomorphism, the maps \( y \in A \mapsto df(y) \in \mathrm{End}(\mathbb{R}^d), u \in f(A) \mapsto df^{-1}(u) \in \mathrm{End}(\mathbb{R}^d) \) are continuous. Therefore they are bounded:
\[
\exists M > 0 \quad \forall y \in A \quad ||df(y)|| \leq M, \quad \forall u \in f(A) \quad ||df^{-1}(u)|| \leq M
\]
(here \( ||df(x)|| = \sup \{ |df(x)(y)|_2 : |y|_2 \leq 1 \} \) is the standard operator norm in \( \mathrm{End}(\mathbb{R}^d) \)). Since \( f(A) \) is compact, the differential map \( df^{-1} \) is uniformly continuous on \( f(A) \):
\[
\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall u, v \in f(A) \quad |u - v|_2 \leq \delta \quad \Rightarrow \quad ||df^{-1}(u) - df^{-1}(v)|| \leq \varepsilon.
\]
Let \( \varepsilon \) be positive and let \( \delta \) be associated to \( \varepsilon \) as above. Let \( \rho \) be positive and small enough so that \( \rho < \delta/2 \) and \( B(f(x), \rho) \subset f(A) \) (since \( f \) is a \( C^1 \) diffeomorphism, \( f(A) \) is a neighbourhood of \( f(x) \)). Let \( r \) be such that \( 0 < r < \rho/M \) and \( B(x, r) \subset A \). We claim that \( M \) associated to \( x \) and \( r \) associated to \( \varepsilon, x \) answer the problem. Let \( y, z \) belong to \( \Gamma \cap B(x, r) \). Since \( |y, z| \subset B(x, r) \subset A \), and \( ||df(\zeta)|| \leq M \) on \( A \), then
\[
|f(y) - f(x)|_2 \leq M|y - x|_2 \leq Mr < \rho, \quad |f(z) - f(x)|_2 < \rho, \quad |f(y) - f(z)|_2 < \delta, \quad |f(y) - f(z)|_2 < M|z - y|_2.
\]
We apply next a classical lemma of differential calculus (see [1], I, 4, Corollary 2) to the map \( f^{-1} \) and the interval \( [f(z), f(y)] \) (which is included in \( B(f(x), \rho) \subset f(A) \)) and the point \( f(z) \):
\[
|y - z - df^{-1}(f(z))(f(y) - f(z))|_2 \leq |f(y) - f(z)|_2 \sup \{ ||df^{-1}(\zeta) - df^{-1}(f(z))|| : \zeta \in [f(z), f(y)] \}.
\]
The right–hand member is less than \( M|y - z|_2 \varepsilon \). Since \( z + df^{-1}(f(z))(f(y) - f(z)) \) belongs to \( \tan(\Gamma, z) \), we are done.

We come back to our case. The boundary \( \Gamma \) of \( \Omega \) is piecewise of class \( C^1 \), i.e., it is included in a finite union of \( C^1 \) hypersurfaces, which we denote by \( (S_1, \ldots, S_p) \). The hypersurfaces \( S_1, \ldots, S_p \) being \( C^1 \) and the set \( \Gamma \) compact, the maps \( x \in \Gamma \mapsto v_{S_k}(x), 1 \leq k \leq p \) (where \( v_{S_k}(x) \) is the unit normal vector to \( S_k \) at \( x \)) are uniformly continuous:
\[
\forall \delta > 0 \quad \exists \eta > 0 \quad \forall k \in \{ 1, \ldots, p \} \quad \forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \leq \eta \Rightarrow |v_{S_k}(x) - v_{S_k}(y)|_2 < \delta.
\]
Let $\eta^*$ be associated to $\delta = 1$ by this property. Let $k \in \{1, \ldots, p\}$. The set $S_k \cap \Gamma$ is a compact subset of the hypersurface $S_k$. Applying the previous lemma, we get:

$$\exists M_k \forall \delta_0 > 0 \exists \eta_k > 0 \forall x, y \in S_k \cap \Gamma \quad |x - y|_2 \leq \eta_k \Rightarrow d_2(y, \tan(S_k, x)) \leq M_k \delta_0 |x - y|_2.$$  

Let $M_0 = \max_{1 \leq k \leq p} M_k$ and let $\delta_0$ in $[0, 1/2]$ be such that $M_0 \delta_0 < 1/2$. For each $k$ in $\{1, \ldots, p\}$, let $\eta_k$ be associated to $\delta_0$ as in the above property and let

$$\eta_0 = \min \left( \min_{1 \leq k \leq p} \eta_k, \eta^*, \frac{1}{8d} \text{dist}(\Gamma^1, \Gamma^2) \right).$$

We build a family of cubes $Q(x, r)$, indexed by $x \in \Gamma$ and $r \in [0, r_T]$ such that $Q(x, r)$ is a cube centered at $x$ of side length $r$ which is transverse to $\Gamma$. For $x \in \mathbb{R}^d$ and $k \in \{1, \ldots, p\}$, let $p_k(x)$ be a point of $S_k \cap \Gamma$ such that

$$|x - p_k(x)|_2 = \inf \{ |x - y|_2 : y \in S_k \cap \Gamma \}.$$  

Such a point exists since $S_k \cap \Gamma$ is compact. We define then for $k \in \{1, \ldots, p\}$

$$\forall x \in \mathbb{R}^d \quad v_k(x) = v_{S_k}(p_k(x)).$$

We define also

$$d_r = \inf_{v_1, \ldots, v_p \in S^{d-1}} \max_{b \in B_d} \min_{1 \leq k \leq r} \left( \frac{|e - v_k|_2}{|e - b|} \right)$$

where $B_d$ is the collection of the orthonormal basis of $\mathbb{R}^d$ and $S^{d-1}$ is the unit sphere of $\mathbb{R}^d$. Let $\eta$ be associated to $d_r/4$ as in the above continuity property. We set

$$r_\Gamma = \frac{\eta}{2d}.$$

Let $x \in \Gamma$. By the definition of $d_r$, there exists an orthonormal basis $b_x$ of $\mathbb{R}^d$ such that

$$\forall e \in b_x \quad \forall k \in \{1, \ldots, p\} \quad \min_{1 \leq k \leq r} \left( \frac{|e - v_k(x)|_2}{|e - v_k(x)|_2} \right) > \frac{d_r}{2}.$$  

Let $Q(x, r)$ be the cube centered at $x$ of sidelength $r$ whose sides are parallel to the vectors of $b_x$. We claim that $Q(x, r)$ is transverse to $\Gamma$ for $r < r_\Gamma$. Indeed, let $y \in Q(x, r) \cap \Gamma$. Suppose that $y \in S_k$ for some $k \in \{1, \ldots, p\}$, so that $v_k(y) = v_{S_k}(y)$ and $|x - p_k(x)|_2 < 2d_\Gamma < \eta$ and $|v_{S_k}(y) - v_k(x)|_2 < d_r/4$. For $e \in b_x$,

$$\frac{d_r}{2} \leq |e - v_k(x)|_2 \leq |e - v_{S_k}(y)|_2 + |v_{S_k}(y) - v_k(x)|_2$$

whence

$$|e - v_{S_k}(y)|_2 \geq \frac{d_r}{2} - \frac{d_r}{4} = \frac{d_r}{4}.$$  

This is also true for $-e$, therefore the faces of the cube $Q(x, r)$ are transverse to $S_k$.

Now we consider the collection

$$(Q(x, r), x \in \Gamma, r < r_\Gamma).$$
5 Definition of the set \( \Omega' \)

Let \( \lambda \) be in \( [\widetilde{\phi}_\Omega, +\infty[ \). We are studying

\[ \mathbb{P}[\phi_n \geq \lambda n^{d-1}] . \]

Suppose first that \( \widetilde{\phi}_\Omega > 0 \). There exists a positive \( s \) such that \( \lambda > \widetilde{\phi}_\Omega(1 + s)^2 \). By definition of \( \widetilde{\phi}_\Omega \), for every positive \( s \), there exists a polyhedral subset \( P \) of \( \mathbb{R}^d \), such that \( \partial P \) is transverse to \( \Gamma \),

\[ \overline{\Gamma} \subset \overset{\circ}{P} , \]

and since \( d(P, \overline{\Gamma}) > 0 \) we also obtain that

\[ \overline{\Gamma} \subset \mathbb{R}^d \setminus P . \]

Thus, for any set \( \Omega' \) satisfying \( \Omega \subset \Omega' \subset \nu_2(\Omega, \delta_0) \), we have

\[ \int_{\partial P \cap \Omega'} \nu(v_P(x)) dH^{d-1}(x) \leq I_\Omega(P)(1 + s/2) , \]

then \( \lambda > (1 + s/2)(\int_{\partial P \cap \Omega'} \nu(v_P(x)) dH^{d-1}(x)) \) and

\[ \mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P} \left[ \phi_n \geq \left( \int_{\partial P \cap \Omega'} \nu(v_P(x)) dH^{d-1}(x) \right)(1 + s/2)n^{d-1} \right] . \]

Suppose now that \( \widetilde{\phi}_\Omega = 0 \). Then for an arbitrarily fixed \( s \in ]0, 1[ \), there exists a polyhedral subset \( P \) of \( \mathbb{R}^d \), such that \( \partial P \) is transverse to \( \Gamma \),

\[ \overline{\Gamma} \subset \overset{\circ}{P} , \overline{\Gamma} \subset \mathbb{R}^d \setminus P . \]
and

\[ I_\Omega(P) \leq \frac{\lambda}{1 + s}, \]

and thus \( \lambda > I_\Omega(P)(1 + s) \). If \( I_\Omega(P) > 0 \), we can use exactly the same argument as previously. We suppose that \( I_\Omega(P) = 0 \). We know as previously that there exists \( \delta_0 > 0 \) (depending on \( \lambda, P \) and \( \Gamma \)) such that for all \( \delta \leq \delta_0 \),

\[ \mathcal{H}^{d-1}(\partial P \cap (\mathcal{V}_2(\Omega, \delta) \setminus \Omega)) < \frac{\lambda}{\nu_{\max}(1 + s/2)}. \]

Thus, in any case, we obtain that there exists \( \delta_0 > 0 \) such that, for any set \( \Omega' \) satisfying \( \Omega \subset \Omega' \subset \mathcal{V}_2(\Omega, \delta_0) \), we have

\[ \mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P} \left[ \phi_n \geq \left( \int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) \right) (1 + s/2)n^{d-1} \right]. \]

We will construct a particular set \( \Omega' \) satisfying \( \Omega \subset \Omega' \subset \mathcal{V}_2(\Omega, \delta_0) \). In the previous section, we have associated to each couple \( (x, r) \) in \( \Gamma \times [0, r] \) a hypercube \( Q(x, r) \) centered at \( x \), of sidelength \( r \), and which is transverse to \( \Gamma \). Using exactly the same method, we can build a family of hypercubes

\[ (Q'(x, r), x \in \Gamma, r < r_{(\Gamma, P)}) \]

such that \( Q'(x, r) \) is centered at \( x \), of sidelength \( r \), and it is transverse to \( \Gamma \) and \( \partial P \). The family

\[ (Q'(x, r), x \in \Gamma, r < \min(r_{(\Gamma, P)}, \delta_0/(2d))) \]

is a covering of the compact set \( \Gamma \), thus we can extract a finite covering from this collection, we denote it by \( (Q'(x_i, r_i), i \in J) \). We define

\[ \Omega' = \Omega \cup \bigcup_{i \in J} Q'(x_i, r_i). \]

Since \( r_i \leq \delta_0/(2d) \) for all \( i \in J \), we have \( \Omega' \subset \mathcal{V}_2(\Omega, \delta_0) \). Moreover, \( \partial P \) is transverse to the boundary \( \Gamma' \) of \( \Omega' \). Finally, if we define

\[ \delta_1 = \min_{i \in J} r_i / 2, \]

we know that \( \mathcal{V}_2(\Omega, \delta_1) \subset \Omega' \), and thus for all \( n \geq 2d/\delta_1 \), we have \( \Omega_n \subset \Omega' \).

6 Existence of a family of \((\Gamma_1^1, \Gamma_2^1)-cuts\)

In this section we prove that we can construct a family of disjoint \((\Gamma_1^1, \Gamma_2^1)-cuts\) in \( \Omega_n \). Let \( \zeta \) be a fixed constant larger than 2d. We consider a parameter \( h < h_0 = d(\partial P, \Gamma^1 \cup \Gamma^2) \). For \( k \in \{0, ..., \lfloor h n / \zeta \rfloor \} \) we define

\[ P(k) = \{ x \in \mathbb{R}^d | d(x, P) \leq k \zeta / n \}, \]

and for \( k \in \{0, ..., \lfloor h n / \zeta \rfloor - 1 \} \) we define

\[ U(k) = (\mathbb{R}^d \setminus P_{k+1}) \setminus P_k = \{ x \in \mathbb{R}^d | k \zeta / n \leq d(x, P) < (k+1) \zeta / n \}, \]

and \( M'(k) = U(k) \cap \Omega' \) (see figure 3). We will prove the following lemma:
Lemma 2. There exists $N$ large enough such that for all $n \geq N$, every path on the graph $(\mathbb{Z}_n^d, \mathbb{R}^d)$ from $\Gamma_1^n$ to $\Gamma_2^n$ in $\Omega_n$ contains at least one edge which is included in the set $\mathcal{M}'(k)$ for $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$.

This lemma states precisely that for all $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$, $\mathcal{M}'(k)$ contains a $(\Gamma_1^n, \Gamma_2^n)$-cut in $\Omega_n$.

Proof:
Let $k \in \{0, ..., \lfloor hn/\zeta \rfloor - 1\}$. Let $\gamma$ be a discrete path from $\Gamma_1^n$ to $\Gamma_2^n$ in $\Omega_n$. In particular, $\gamma$ is continuous, so we can parametrise it: $\gamma = (\gamma_t)_{0 \leq t \leq 1}$. There exists $N$ large enough such that for all $n \geq N$, we have

$$\Omega_n \subset \Omega', \quad \Gamma_1^n \subset V_2(\Gamma_1, 2d/n) \subset \overset{\circ}{P}_k, \quad \text{and} \quad \Gamma_2^n \subset V_2(\Gamma_2, 2d/n) \subset \mathbb{R}^d \setminus \overset{\circ}{P}_{k+1}.$$

Since $\gamma$ is continuous, we know that there exists $t_1, t_2 \in [0, 1]$ such that

$$t_1 = \sup \{t \in [0, 1] \mid \gamma_t \in \overset{\circ}{P}_k\},$$

$$t_2 = \inf \{t \geq t_1 \mid \gamma_t \in \mathbb{R}^d \setminus \overset{\circ}{P}_{k+1}\}.$$

Since

$$\overset{\circ}{P}_k \cup U(k) \cup \mathbb{R}^d \setminus \overset{\circ}{P}_{k+1}$$

is a partition of $\mathbb{R}^d$, we know that $(\gamma_t)_{t_1 \leq t \leq t_2}$, which is a continuous path, is included in $U(k)$. The length of $(\gamma_t)_{t_1 \leq t \leq t_2}$ is larger than $d(\gamma_{t_1}, \gamma_{t_2})$. The segment $[\gamma_{t_1}, \gamma_{t_2}]$ intersects

$$\{x \in \mathbb{R}^d \mid d(x, P) = (k + 1/2)\zeta/n\}$$

at a point $z$, and we know that

$$V_2(z, \zeta/(2n)) \subset V(k).$$
Thus \( d(\gamma_{t_1}, \gamma_{t_2}) \geq \zeta/n \), and then the length of \((\gamma_t)_{t_1 \leq t \leq t_2}\) is larger than \(\zeta/n\). Finally, \(\gamma\) is composed of edges of length \(1/n\), and \(\zeta \geq 2d\), so \((\gamma_t)_{t_1 \leq t \leq t_2}\), and thus \(\gamma\), contains at least one edge which is included in \(\mathcal{U}(k)\). Noticing that for all \(n \geq N\),

\[
\gamma \subset \Omega_n \subset \Omega',
\]

we obtain that this edge belongs to \(\mathcal{U}(k) \cap \Omega' = \mathcal{M}'(k)\).

\[ \square \]

7 Covering of \(\partial P \cap \Omega'\) by cylinders

From now on we only consider \(n \geq N\). According to lemma \(\square\) we know that each set \(\mathcal{M}'(k)\) for \(k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}\) contains a \((\Gamma_1, \Gamma_2)\)-cut in \(\Omega_n\), thus if we denote by \(M'(k)\) the set of the edges included in \(\mathcal{M}'(k)\), we obtain

\[
\phi_n \leq \min \{V(M'(k)) : k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}\}.
\]

However, we do not have estimates on \(V(M'(k))\) that allow us to control \(\phi_n\) using only the previous inequality. The estimates we can use are the one of the upper large deviations for the maximal flow from the top to the bottom of a cylinder (Theorem \(\square\)). In this section, we will transform our family of cuts \((M'(k))\) by replacing a huge part of the edges in each \(M'(k)\) by the edges of minimal cut sets in cylinders.

We denote by \(H_i, i = 1, \ldots, N\) the intersection of the faces of \(\partial P\) with \(\Omega'\). For each \(i = 1, \ldots, N\), we denote by \(v_i\) the exterior normal unit vector to \(P\) along \(H_i\). We will cover \(\partial P \cap \Omega'\) by cylinders, except a surface of \(H^{d-1}\) measure controlled by a parameter \(\varepsilon\). To explain the construction of a cut set we will do with a huge number of cylinders, we present first the simpler construction of a cut set using one cylinder. Let \(R\) be a hyperrectangle that is included in \(H_j\) for a \(j \in \{1, \ldots, N\}\), and let \(B\) be the cylinder defined by

\[
B = \{x + tv_j \mid x \in R, \ t \in [0, h]\},
\]

where \(h \leq h_0\) is the same parameter as previously. The cylinder \(B\) is built on \(\partial P \cap \Omega\), in \(\mathbb{R}^d \setminus P\).

We recall that \(h_0 = d(\partial P, \Gamma_1 \cup \Gamma_2) > 0\), so we know that \(d(B, \Gamma_1 \cup \Gamma_2) > 0\). We denote by \(E_a\) the set of the edges included in

\[
\mathcal{E}_a = \{x + tv_j \mid x \in R, \ d(x, \partial R) < \zeta/n, \ t \in [0, h]\}.
\]

The set \(\mathcal{E}_a\) is a neighbourhood in \(B\) of the "vertical" faces of \(B\), i.e., the faces of \(B\) that are collinear to \(v_j\). We denote by \(E_b\) a set of edges in \(B\) that cuts the top \(R + hv_j\) from the bottom \(R\) of \(B\). Let \(M'(k)\) be the set of the edges included in \(\mathcal{M}'(k)\), for a \(k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}\). Let \(B'\) be the thinner cylinder

\[
B' = \{x + tv_j \mid x \in R, \ d(x, \partial R) \geq \zeta/n, \ t \in [0, h]\}.
\]

Thus for all \(k \in \{0, \ldots, \lfloor hn/\zeta \rfloor - 1\}\), the set of edges

\[
(M'(k) \cap (\mathbb{R}^d \setminus B')) \cup E_a \cup E_b
\]
covers $\Gamma_2^n$ from $\Gamma_1^n$ in $\Omega_n$. Indeed, the set of edges $M'(k)$ is already a cut between $\Gamma_1^n$ and $\Gamma_2^n$ in $\Omega_n$. We remove from it the edges that are inside $B'$ which is in the interior of $B$, and we add to it a cutset $E_b$ from the top to the bottom of $B$, and the set of edges $E_a$ that glue together $E_b$ and $M'(k) \cap (\mathbb{R}^d \setminus B')$. This property is illustrated in the figure 5.

Remark 4. In this figure, we have represented $E_b$ as a surface (so a path in dimension 2) that separates the top from the bottom of the cylinder to illustrate the fact that $E_b$ cuts all discrete paths from the bottom to the top of $B$. Actually, we can mention that it is possible to define an object which could be the dual of an edge in dimension $d \geq 2$ (as a generalization of the dual of a planar graph). This object is a plaquette, i.e., a hypersquare of sidelength 1/n that is orthogonal to the edge and cuts it in its middle, and whose sides are parallel to the hyperplanes of the axis. Then the dual of a cutset is a hypersurface of plaquettes, thus the figure 5 is somehow intuitive.

We do exactly the same construction, but with a large number of cylinders, that will almost cover $\partial P \cap \Omega'$. We consider a fixed $\varepsilon > 0$. There exists a $l$ sufficiently small (depending on $F$, $P$ and $\varepsilon$) such that there exists a finite collection $(R_{i,j}, i = 1, \ldots, N, j = 1, \ldots, N_i)$ of hypersquares of side $l$ of disjoint interiors satisfying $R_{i,j} \subset H_i$ for all $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, N_i\}$, and for all $i \in \{1, \ldots, N\}$,

$$\{x \in H_i \mid d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-1} \} \subset \bigcup_{j=1}^{N_i} R_{i,j} \subset \{x \in H_i \mid d(x, \partial H_i) \geq \varepsilon \mathcal{H}^{d-2}(\partial H_i)^{-1} \mathcal{N}^{-2} \}.$$
We immediately obtain that

\[ \mathcal{H}^{d-1} \left( (\partial P \cap \Omega') \setminus \bigcup_{i=1}^{N} \bigcup_{j=1}^{N_i} R_{i,j} \right) \leq \varepsilon. \]

We remark that

\[ \int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) \geq \sum_{i=1}^{N} N_i \mathcal{H}^{d-1}(\nu(v_i)), \]

so that

\[ \mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P} \left[ \phi_n \geq (1 + s/2) n^{d-1} \sum_{i=1}^{N} N_i \mathcal{H}^{d-1}(\nu(v_i)) \right]. \]

Let \( h < h_0 \). For all \( i \in \{1, \ldots, N\} \) and \( j \in \{1, \ldots, N_i\} \), we define

\[ B_{i,j} = \{ x + tv_i | x \in R_{i,j}, t \in [0, h] \}. \]

Since all the \( B_{i,j} \) are at strictly positive distance of \( \partial H_i \), there exists a positive \( h_1 \) such that for all \( h < h_1 \), the cylinders \( B_{i,j} \) have pairwise disjoint interiors. We thus consider \( h < \min(h_0, h_1) \) (see figure E for example). At this point, we could define a neighbourhood of the vertical faces of each

![Figure 6: Covering of \( \partial P \cap \Omega' \) by cylinders.](image)

cylinder \( B_{i,j} \), and do the same construction as in the previous example with one cylinder. Actually, we need to choose a little bit more carefully the sets of edges we define along the vertical faces of the
cylinders. We will not consider only each cylinder $B_{i,j}$, but also thinner versions of these cylinders of the type

$$B_{i,j}(k) = \{ x + tv_i \mid x \in R_{i,j}, d(x, \partial R_{i,j}) > k\zeta/n, \ t \in [0, h] \}$$

for different values of $k$. We will then consider the edges included in a neighbourhood of the vertical faces of each $B_{i,j}(k)$ (see the set $W_{i,j}(k)$ above), and choose $k$ to minimize the capacity of the union over $i$ and $j$ of these edges. The reason why we need this optimization is also the reason why we built a family $(M'(k))$ of cutsets and not only one cutset from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$, we will try to explain it in remark \[3\].

Here are the precise definitions of the sets of edges. We still consider the same constants $\zeta$ bigger than $2d$ and $h < \min(h_0, h_1)$. We define another positive constant $\eta$ that we will choose later (depending on $P$, $s$ and $\Omega$). For $i$ in $\{1, \ldots, N\}$ and $j$ in $\{1, \ldots, N_i\}$ we recall the definition of $B_{i,j}$:

$$B_{i,j} = \{ x + tv_i \mid x \in R_{i,j}, \ t \in [0, h] \},$$

and we define the following subsets of $\mathbb{R}^d$:

$$B'_{i,j} = \{ x + tv_i \mid x \in R_{i,j}, \ d(x, \partial R_{i,j}) > \eta, \ t \in [0, h] \},$$

$$\forall k \in \{0, \ldots, \lfloor \eta n/\zeta - 1 \rfloor \}, \ W_{i,j}(k) = \{ x \in B_{i,j} \mid k\zeta/n \leq d_2(x, \partial R_{i,j} + \mathbb{R}v_i) < (k+1)\zeta/n \},$$

$$\forall k \in \{0, \ldots, \lfloor h n k/\zeta - 1 \rfloor \}, \ M(k) = \mathcal{M}(k) \setminus \left( \bigcup_{i,j} B'_{i,j} \right),$$

(see figures 4 and 5). We denote by $W_{i,j}(k)$ the set of the edges included in $W_{i,j}(k)$ and we define $W(k) = \cup_{i,j} W_{i,j}(k)$. We also denote by $M(k)$ the edges included in $M(k)$. Exactly as in

\[7 \text{ COVERING OF } \partial P \cap \Omega' \text{ BY CYLINDERS}\]
the construction of a cutset with one cylinder, we obtain a cutset that is built with cutsets in each cylinders $B_{i,j}$. Indeed, if we denote by $E_{i,j}$ a set of edges that is a cutset from the top to the bottom of $B_{i,j}$ (oriented towards the direction given by $v_i$), then for each $k_1 \in \{0, \ldots, \lfloor \eta n / \zeta - 1 \rfloor \}$ and $k_2 \in \{0, \ldots, \lfloor hn / \zeta - 1 \rfloor \}$, the set of edges:

$$
\bigcup_{i=1, \ldots, \mathcal{N}} \bigcup_{j=1, \ldots, \mathcal{N}_i} E_{i,j} \cup W(k_1) \cup M(k_2)
$$

contains a cutset from $\Gamma_n^1$ to $\Gamma_n^2$ in $\Omega_n$. We deduce that

$$
\phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + \min_{k_1} V(W(k_1)) + \min_{k_2} V(M(k_2)).
$$

(1)
8 Control of the cardinality of the sets of edges \( W \) and \( M \)

For the sake of clarity, we do not recall the sets in which the parameters take its values, we always assume that they are the following: \( i \in \{1, ..., \mathcal{N}\} \), \( j \in \{1, ..., \mathcal{N}_i\} \), \( k_1 \in \{0, ..., \lfloor \eta n / \zeta - 1 \rfloor \} \) and \( k_2 \in \{0, ..., \lfloor \eta n / \zeta - 1 \rfloor \} \). We have to evaluate the number of edges in the sets \( W(k_1) \) and \( M(k_2) \) to control the terms \( \min_{k_1} V(W(k_1)) \) and \( \min_{k_2} V(M(k_2)) \) in (\( \Phi \)). There exist constants \( c_1(d, \Omega) \), \( c_2(P, d, \Omega) \) such that

\[
\text{card } W(k_1) \leq c_1 \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega')}{ld-1} \zeta^{d-2} \eta n^{d-1} \leq c_2 l^{-1} \eta n^{d-1}.
\]

The cardinality of \( M(k_2) \) is a little bit more complicated to control. We will divide \( M(k) \) (respectively \( \mathcal{M}(k) \)) into three parts: \( M(k) \subset M_1(k) \cup M_2(k) \cup M_3(k) \) (respectively \( \mathcal{M}(k) \subset \mathcal{M}_1(k) \cup \mathcal{M}_2(k) \subset \mathcal{M}_3(k) \)), that are represented in figure [\( \Phi \)]

We define \( R'_{i,j} = \{ x \in R_{i,j} | d(x, \partial R_{i,j}) > \eta \} \) which is the basis of \( B'_{i,j} \). The set \( \mathcal{M}_1(k) \) is a translation of the sets \( H_i \setminus (\cup_{j=1}^{\mathcal{N}_i} R'_{i,j}) \) along the direction given by \( v_i \) enlarged with a thickness \( \zeta / (n\kappa) \):

\[
\mathcal{M}_1(k) \subset \bigcup_{i=1}^{\mathcal{N}} \{ x + tv_i | x \in H_i \setminus (\cup_{j=1}^{\mathcal{N}_i} R'_{i,j}), \, t \in [k\zeta / n, (k+1)\zeta / n] \}.
\]

Here we have an inclusion and not an equality because \( \mathcal{M}_1(k) \) can be a truncated version of this set (truncated at the junction between the translates of two different faces). Since we know that

\[
\mathcal{H}^{d-1} \left( (\partial P \cap \Omega') \setminus \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{\mathcal{N}_i} R_{i,j} \right) \leq \varepsilon,
\]

and

\[
\mathcal{H}^{d-1} \left( \bigcup_{i=1}^{\mathcal{N}} \bigcup_{j=1}^{\mathcal{N}_i} (R_{i,j} \setminus R'_{i,j}) \right) \leq \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega')}{ld-1} l^{d-2} \eta = \mathcal{H}^{d-1}(\partial P \cap \Omega') l^{-1} \eta,
\]

we have the following bound on the cardinality of \( \mathcal{M}_1(k) \):

\[
\text{card } (\mathcal{M}_1(k)) \leq c_3(\varepsilon + l^{-1} \eta)n^{d-1},
\]

for a constant \( c_3(d, P, \Omega, \Omega') \).

The part \( M_2(k) \) corresponds to the edges included in the "bends" of the neighbourhood of \( \partial P \) located around the boundary of the faces of \( \partial P \) in \( \Omega' \), denoted by \( \mathcal{M}_2(k) \), i.e.:

\[
\mathcal{M}_2(k) \subset \bigcup_{i,j} (\mathcal{V}_2(H_i \cap H_j, (k+1)\zeta / n) \setminus \mathcal{V}_2(H_i \cap H_j, k\zeta / n))
\]

and there exists a constant \( c_4(d, P, \Omega') \) such that

\[
\text{card } M_2(k) \leq c_4|k\zeta / n|^{d-2} n^{d-1} \leq c_4 h^{d-2} n^{d-1}.
\]

The last part \( M_3(k) \) corresponds to the part of \( \mathcal{M}(k) \) that is near the boundary \( \Gamma' \) of \( \Omega' \). Indeed, \( \Gamma' \) is not orthogonal to \( \partial P \), thus for some \( k \), the set \( \mathcal{M}(k) \) may contain edges that are not included in

\[
\bigcup_{i=1}^{\mathcal{N}} \{ x + tv_i | x \in H_i \setminus (\cup_{j=1}^{\mathcal{N}_i} R'_{i,j}), \, t \in [k\zeta / n, (k+1)\zeta / n] \},
\]

for a constant \( c_5(d, P, \Omega, \Omega') \).
neither in
\[ \bigcup_{i,j} (V_2(H_i \cap H_j, (k + 1)\zeta/n) \setminus V_2(H_i \cap H_j, k\zeta/n) ) , \]
(see figure [5]). However, \( M(k) \subset \mathcal{U}(k) \), the problem is to evaluate the difference of cardinality between the different \( M(k) \) due to the intersection of \( \mathcal{U}(k) \) with \( \Omega' \). We have constructed \( \Omega' \) such that \( \Gamma' \) is transverse to \( \partial P \) precisely to obtain this control. The sets \( \Gamma' \) and \( \partial P \) are polyhedral surfaces which are transverse. We denote by \( (\mathcal{H}_i, i \in I) \) (resp. \( (\mathcal{H}'_j, j \in J) \)) the hyperplanes that contain \( \partial P \) (resp. \( \Gamma' \)), and by \( v_i \) (resp. \( v'_j \)) the exterior normal unit vector to \( P \) along \( \mathcal{H}_i \) (resp. \( \Omega' \) along \( \mathcal{H}'_j \)). The set \( \Gamma' \cap \partial P \) is included in the union of a finite number of intersections \( \mathcal{H}_i \cap \mathcal{H}'_j \) of transverse hyperplanes. To each such intersection \( \mathcal{H}_i \cap \mathcal{H}'_j \), we can associate the angles between \( v_i \) and \( v'_j \), and between \( v_i \) and \( -v'_j \), in the plane of dimension 2 spanned by \( v_i \) and \( v'_j \). Each such angle is strictly positive because \( \mathcal{H}_i \) is transverse to \( \mathcal{H}'_j \), and so the minimum \( \theta_0 \) over the finite number of defined angles is strictly positive. This \( \theta_0 \) and the measure \( \mathcal{H}^{d-2}(\partial P \cap \Gamma') \) give to us a control on the volume of \( M_3(k) \), and thus on \( \text{card}(M_3(k)) \), as soon as these sets belong to a neighbourhood of \( \partial P \cap \Gamma' \) (see figure [9]). Thus, there exist \( h_2(\Omega', P) > 0 \) and a constant \( c_5(d, P, \Omega, \Omega') \) such that for all \( h \leq h_2 \),
\[ \text{card}(M_3(k)) = c_5 h n^{d-1}. \]
We conclude that there exists a positive constant \( c_6(d, P, \Omega, \Omega') \) such that
\[ \text{card } M(k) \leq c_6(\varepsilon + l^{-1} \eta + h^{d-2} + h) n^{d-1}. \]
9 Calibration of the constants

We remark that the sets $W(k)$ (resp., the sets $M(k)$) are pairwise disjoint for different $k$. Then we obtain that

$$
P[\phi_n \geq \lambda n^{d-1}] \leq \mathbb{P} \left[ \phi_n \geq (1 + s/2) n^{d-1} \sum_{i=1}^N N_i \ell_i^{-1} \nu(v_i) \right]
$$

$$
\leq \mathbb{P} \left[ \sum_{i=1}^N \sum_{j=1}^{N_i} \phi_{B_{i,j}} \geq (1 + s/4) n^{d-1} \sum_{i=1}^N N_i \ell_i^{-1} \nu(v_i) \right]
$$

$$
+ \mathbb{P} \left[ \min_{k_1} V(W(k_1)) \geq (s/8) n^{d-1} \sum_{i=1}^N N_i \ell_i^{-1} \nu(v_i) \right]
$$

$$
+ \mathbb{P} \left[ \min_{k_2} V(M(k_2)) \geq (s/8) n^{d-1} \sum_{i=1}^N N_i \ell_i^{-1} \nu(v_i) \right]
$$

$$
\leq \sum_{i=1}^N \sum_{j=1}^{N_i} \left( \max_{i,j} \mathbb{P}[\phi_{B_{i,j}} \geq \ell_i^{-1} \nu(v_i)(1 + s/4) n^{d-1}] \right)
$$

$$
+ \mathbb{P} \left[ \sum_{i=1}^N t(e_i) \geq (s/8) n^{d-1} \sum_{i=1}^N N_i \ell_i^{-1} \nu(v_i) \right]^{\eta n/\zeta}
$$

$$
+ \mathbb{P} \left[ \sum_{i=1}^N t(e_i) \geq (s/8) n^{d-1} \sum_{i=1}^N N_i \ell_i^{-1} \nu(v_i) \right]^{2(h n/\zeta)}.
$$

The terms

$$
P[\phi_{B_{i,j}} \geq \ell_i^{-1} \nu(v_i)(1 + s/4) n^{d-1}]
$$

have already been studied in [9] (we recalled it as Theorem 3 in this paper).

It remains to study two terms of the type

$$
P(n) = \mathbb{P} \left( \sum_{i=1}^{\alpha n^{d-1}} t(e_i) \geq \beta n^{d-1} \right).
$$

As soon as $\beta > \alpha \mathbb{E}(t)$ and the law of the capacity of the edges admits an exponential moment, the Cramér theorem in $\mathbb{R}$ allows us to affirm that

$$
\limsup_{n \to \infty} \frac{1}{n^{d-1}} \log P(n) < 0.
$$

Moreover, for all

$$
\varepsilon \leq \varepsilon_0 = \frac{1}{2\nu_{\max}} \int_{\mathbb{R} \cap \Omega} \nu(v_P(x)) d\mathcal{H}^{d-1}(x),
$$
we have
\[
\sum_{i=1}^{N} N_i t^{d-1} \nu(v_i) \geq \int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x) - \varepsilon \nu_{\text{max}}
\]
\[
\geq \frac{1}{2} \int_{\partial P \cap \Omega'} \nu(v_P(x)) d\mathcal{H}^{d-1}(x)
\]
\[
\geq \frac{\nu_{\text{min}}}{2} \mathcal{H}^{d-1}(\partial P \cap \Omega').
\]
Thus, for all \(\varepsilon < \varepsilon_0\) and \(h < \min(h_0, h_1, h_2)\), if the constants satisfy the two following conditions:
\[
c_2 l^{-1} h < \mathcal{H}^{d-1}(\partial P \cap \Omega') \nu_{\text{min}} \mathbb{E}(t(e)) s/16,
\]
and
\[
c_6 (\varepsilon + l^{-1} \eta + h^{d-2} + h) < \mathcal{H}^{d-1}(\partial P \cap \Omega') \nu_{\text{min}} \mathbb{E}(t(e)) s/16,
\]
thanks Theorem 3 and the Cramèr theorem in \(\mathbb{R}\), we obtain that
\[
\limsup_{n \to \infty} \frac{1}{n^d} \log \mathbb{P}[\phi_n \geq \lambda n^{d-1}] < 0,
\]
and theorem 4 is proved. We claim that it is possible to choose the constants such that conditions 2 and 3 are satisfied. Indeed, we first choose \(\varepsilon < \varepsilon_0\) such that
\[
\varepsilon < \frac{1}{4} \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega') \nu_{\text{min}} \mathbb{E}(t(e)) s}{16 c_6}.
\]
To this fixed \(\varepsilon\) corresponds a \(l\). Knowing \(\varepsilon\) and \(l\), we choose \(h < \min(h_0, h_1, h_2)\) and \(\eta\) such that
\[
\max(h, h^{d-2}, l^{-1} h, l^{-1} \eta) < \frac{1}{4} \frac{\mathcal{H}^{d-1}(\partial P \cap \Omega') \nu_{\text{min}} \mathbb{E}(t(e)) s}{16 \max(c_2, c_6)}.
\]
This ends the proof of theorem 1.

**Remark 5.** We try here to explain why we built several sets \(W(k_1)\) and \(M(k_2)\), and not only one couple of such sets, that would have been sufficient to construct a cutset from \(\Gamma^1_n\) to \(\Gamma^2_n\) in \(\Omega_n\). To use estimates of upper large deviations of maximal flows in cylinder we already know, we want to compare \(\phi_n\) with \(\sum_{i,j} \phi_{B_{i,j}}\). Heuristically, to construct a \((\Gamma^1_n, \Gamma^2_n)\)-cut in \(\Omega_n\) from the union of cutsets in each cylinder \(B_{i,j}\), we have to add edges to glue together the different cutsets at the common boundary of the small cylinders, and to extend these cutsets to \((\partial P \cap \Omega_n) \setminus \bigcup_{i=1}^{N} \bigcup_{j=1}^{N} R_{i,j}\). Yet we want to prove that the upper large deviations of \(\phi_n\) are of volume order. If we only consider one possible set \(E\) of edges such that
\[
\phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + V(E),
\]
we will obtain that
\[
\mathbb{P}[\phi_n \geq \lambda n^{d-1}] \leq \sum_{i,j} \mathbb{P}[\phi_{B_{i,j}} \geq t^{d-1} \nu(v_i)(1 + s/4)n^{d-1}]
\]
\[
+ \mathbb{P}\left[V(E) \geq n^{d-1} \sum_{i=1}^{N} N_i t^{d-1} \nu(v_i)s/4\right].
\]
We can choose such a set $E$ so that it contains less than $\delta n^{d-1}$ edges for a small $\delta$ (E is equal to $W(k_1) \cup M(k_2)$ for a fixed couple $(k_1, k_2)$ for example), but the probability

$$P \left[ \sum_{i=1}^{\delta n^{d-1}} t(e_i) \geq Cn^{d-1} \right]$$

does not decay exponentially fast with $n^d$ in general. To obtain this speed of decay, we have to make an optimization over the possible choices of the set $E$, i.e., we choose $E$ among a set of $C'n$ possible disjoint sets of edges $E_1, \ldots, E_{C'n}$; in this case, we obtain that

$$\phi_n \leq \sum_{i,j} \phi_{B_{i,j}} + \min_{k=1, \ldots, C'n} V(E_k),$$

and so

$$P[\phi_n \geq \lambda n^{d-1}] \leq \sum_{i,j} P[\phi_{B_{i,j}} \geq l^{d-1} \nu(v_i)(1 + s/4)n^{d-1}] + \prod_{k=1}^{C'n} P \left[ V(E_k) \geq n^{d-1} \sum_{i=1}^{N} N_i l^{d-1} \nu(v_i)s/4 \right]. \tag{4}$$

It is then sufficient to prove that for all $k$, $P[V(E_k) \geq C''n^{d-1}]$ decays exponentially fast with $n^{d-1}$ to conclude that the last term in (4) decays exponentially fast with $n^d$. Theorem 3 gives a control on the terms

$$P[\phi_{B_{i,j}} \geq l^{d-1} \nu(v_i)(1 + s/4)n^{d-1}].$$

The conclusion is that to obtain the volume order of the upper large deviations, the optimization over the different possible values of $k_1$ and $k_2$ is really important, even if it is not needed if we only want to prove that $P(\phi_n \geq \lambda n^{d-1})$ goes to zero when $n$ goes to infinity.

References


