TD 6 : Rectifiability

Exercise 1.— *Cantor "4–coins".*

1. Let $E \subset \mathbb{R}^2$ be a Borel set and assume that there exists $\theta_1 \neq \theta_2 \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\mathcal{H}^1(\pi_{\theta_1}(E)) = \mathcal{H}^1(\pi_{\theta_2}(E)) = 0$, where π_{θ_i} is the orthogonal projection onto the vector line whose angle with the horizontal axis is θ_i. Show that E is purely 1–unrectifiable.

2. Let $C_4 = K \times K$ where K is the self-similar Cantor set with $r_n = \lambda^n$ for $\lambda = \frac{1}{4}$. Show (again?) that C_4 is purely 1–unrectifiable and that $0 < \mathcal{H}^1(C_4) < \infty$.

3. Let E be a compact purely 1–unrectifiable set. We define $L(E) \subset \mathbb{R}^2$ as the union of lines $y = ax + b$ with $(a, b) \in E$, i.e.
$$L(E) = \{(x, y) \in \mathbb{R}^2 \ : \ \exists (a, b) \in E, \ y = ax + b\}.$$
Show that $\mathcal{L}^2(L(E)) = 0$.
Hints:
   - (i) Check that $L(E)$ is a Borel set.
   - (ii) Let $c \in \mathbb{R} \setminus \{0\}$, $\theta = \arctan\left(\frac{1}{c}\right)$ and $F_c = L(E) \cap \{(x, y) \ : \ x = c\}$. Show that
$$\mathcal{H}^1(F_c) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1\left(\pi_\theta(E)\right) = 0\,.$$

4. Construct a *Besicovitch set*, i.e. a Borel set B such that $\mathcal{L}^2(B) = 0$ and containing a unit segment (and even a line here) in every direction.

Exercise 2.— *tangent plane to a d–regular set.*
Let $E \subset \mathbb{R}^n$ be a *d–regular* set, that is, E is closed and there exists a constant $C_0 \geq 1$ (*regularity constant* of E) such that for all $x \in E$, $0 < r < \text{diam}(E)$,
$$\frac{1}{C_0} r^d \leq \mathcal{H}^d(E \cap B(x, r)) \leq C_0 r^d.$$
Show that if P is an approximate tangent d–plane at a point $x \in E$, then P is a tangent d–plane to E at x.