UNIFORM INDEPENDENCE IN LINEAR GROUPS

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Abstract. We show that for any finitely generated group of matrices that is not virtually solvable, there is an integer m such that, given an arbitrary finite generating set for the group, one may find two elements a and b that are both products of at most m generators, such that a and b are free generators of a free subgroup. This uniformity result improves the original statement of the Tits alternative.

Contents

1. Introduction

The main results of this paper were announced in [8]. We will say that two elements x, y in a group Γ are *independent* if they satisfy no relation, i.e. if they generate a non-abelian free subgroup. The classical Tits' alternative [24] says that if Γ is a finitely generated linear group which is not virtually solvable (i.e. does not contain a solvable subgroup of finite index), then Γ contains two independent elements. However, Tits' proof gives no indication of how deep inside the group one has to look in order to find independent elements. The main result of this paper is the following:

Theorem 1.1. *Let Γ be a finitely generated non-virtually solvable linear group. Then there is a constant $m = m(\Gamma) \in \mathbb{N}$ such that for any symmetric generating set Σ ($\Sigma \ni id$) of Γ, there are two words W_1, W_2 of length at most m in the alphabet Σ for which the*

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corresponding elements in Γ are independent. In other words, the set Σ^m contains two independent elements.

By linear group, we mean any subgroup of $\mathrm{GL}_d(K)$ for some integer $d \geq 1$ and some field K. Let now $\mathbb{K}$ be a global field, $\overline{\mathbb{K}}$ its algebraic closure and S a finite set of places of $\mathbb{K}$ containing all infinite ones. We denote by $\mathcal{O}_{\mathbb{K}}(S)$ the ring of S–integers. A subgroup of $GL_d(\overline{\mathbb{K}})$ will be called *irreducible* if it does not leave invariant any non-trivial subspace of $\overline{\mathbb{K}}^d$ (this is sometimes called absolutely irreducible). After passing to a suitable homomorphic image (see Lemma 3.1) of the linear group under consideration, Theorem 1.1 reduces to the following:

Theorem 1.2. *Let $\mathbb{K}$ be a global field, S a finite set of places of $\mathbb{K}$ containing all the infinite ones, and $d \geq 2$ an integer. Then there is a constant $m = m(d, \mathbb{K}, S)$ with the following property. Suppose that $\Sigma \subset SL_d(\mathcal{O}_{\mathbb{K}}(S))$ is a symmetric subset containing the identity and generating an irreducible subgroup whose Zariski closure $\mathbb{G}$ is semisimple and Zariski connected, then Σ^m contains two independent elements.*

Remark 1.3. In characteristic zero we can actually find two independent elements in Σ^m which generate a *Zariski dense* subgroup of $\mathbb{G}$ (see Theorem 7.1 and Remark 7.2).

As in Tits' original proof we use the classical ping-pong lemma (Lemma 2.3) for the action of the subgroup generated by Σ on a projective space over some local field. Since we are in the arithmetic case, there are only finitely many candidates for the local field, namely the completions $\mathbb{K}_v$ with $v \in S$. A substantial part of the proof consists in finding a "good" metric on the projective space. If k is a local field and $H \leq \mathrm{SL}_d(k)$ is a semisimple k–subgroup with corresponding symmetric space (or building) X, any point in X determines a metric on k^d hence on the projective space $\mathbb{P}(k^d)$. For example, the symmetric space $X = SL_d(\mathbb{R})/SO_d(\mathbb{R})$ is the space of scalar products on $\mathbb{R}^d$ with a normalized volume element. Therefore finding a "good" metric on $\mathbb{P}(k^d)$ amounts to finding a "good" point in X.

In Section 4, Lemma 4.2, we establish a useful inequality, a norm-versus-spectrum Comparison Lemma, that relates the displacement of any finite (and more generally compact) set Σ of isometries of the symmetric space (or building) of $\mathrm{SL}_d(k)$ to the displacement of a single element lying in Σ^{d^2}. This Comparison Lemma supplies us with a good metric on $\mathbb{P}(k^d)$ and an element in Σ^{d^2} that has a "large" eigenvalue compared to the Lipschitz constants (for this good metric) of every generator in Σ. With such information, it is not difficult to produce two proximal elements with distinct attracting points that will generate a free semi-group. Hence a consequence of our Comparison Lemma is the Eskin–Mozes–Oh theorem [11] on uniform exponential growth (for details on this implication and improvements in this direction, see our subsequent paper [9]). However, the Comparison Lemma alone is not sufficient to prove Theorem 1.1 and produce the required independent elements. As a matter of fact, it is usually much harder to generate a free subgroup than a free semi–group.

In Section 5 we prove the following theorem. Let $\mathbb{G}$ be a semisimple algebraic $\mathbb{K}$–subgroup of SL_d. Let $G = \prod_{v \in S} \mathbb{G}(\mathbb{K}_v)$ and $\Gamma = \mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ be a corresponding S–arithmetic group, which we view as a discrete subgroup of G via the diagonal embedding. By the Borel Harish-Chandra theorem Γ is a lattice in G, i.e. the quotient space G/Γ carries a finite G–invariant measure. Let X be the product of symmetric spaces and affine buildings associated to G with a base point x_0.

Theorem 1.4. *There are positive constants c_1 and c_2 such that for any finite subset Σ in Γ generating a subgroup whose Zariski closure is connected semisimple and not contained in a proper parabolic subgroup of $\mathbb{G}$, we have for all $x \in X$:*

$$d_\Sigma(x) \geq c_1 d_{X/\Gamma}(\pi(x), \pi(x_0)) - c_2,$$

where $d_\Sigma(x) = \max\{d(g \cdot x, x), g \in \Sigma\}$, $\pi(x)$ is the projection of x to the locally symmetric space X/Γ and $d_{X/\Gamma}$ is the induced metric on X/Γ.

In other words, the displacement in X of a finite set of lattice points must grow at a fixed linear rate independently of the finite set, as one tends to infinity in the locally symmetric space X/Γ, provided that it generates a "large enough" subgroup. Note that this theorem is trivial when Γ is uniform. Moreover, the analogous result holds also for non-arithmetic lattices (see Remark 5.5) and the constant c_1 can actually be taken to be independent of the choice of the lattice inside a given group G.

At the beginning of the argument proving Theorem 1.4, we establish Lemma 5.6, a quantitative version of the Kazhdan–Margulis theorem (namely, if $g \in G$ is "far" from Γ then $g\Gamma g^{-1}$ contains a non-trivial unipotent "close" to the identity), which is itself of independent interest.

As a corollary of Theorem 1.4 we obtain Proposition 5.9, an arithmetic variant of the Comparison Lemma. Hence, the outcome of Section 5 is that we can choose the "good" metric on $\mathbb{P}(k^d)$ to be arithmetically defined. This will turn out to be crucial when constructing the ping–pong players.

Section 6 is devoted to the construction of the desired independent elements as ping–pong players on $\mathbb{P}(k^d)$. This is done in four steps. First, we construct a proximal element, second, a very contracting one, third, a very proximal one, and fourth, a conjugate of the very proximal element which will form the second ping–pong partner (see Section 2 for this terminology). This construction relies on the study of the dynamics of projective transformations carried out in [6], and in particular the relation (first used by Tits in his original proof) between the contraction properties of a transformation and the Lipschitz constant of its restriction to an open subset (see Proposition 2.2). The arithmetically defined metric that we get from Section 5 supplies us with the two main ingredients needed to construct the desired ping–pong pair, namely control on proximality and control on the ability to separate projective points from projective hyperplanes. The guiding idea is that the distance between two arithmetically defined objects is either zero or can be bounded from below by arithmetic data.

In Section 7 we restrict to the characteristic 0 case and show that the bounded independent elements can be chosen to generate a Zariski dense subgroup.

In Section 8 we describe some consequences of Theorem 1.1. One of the main application is that a finitely generated non–amenable linear group is uniformly non–amenable and has uniform Cheeger constant, i.e. the family of all Cayley graphs associated with finite generating sets forms a uniform family of expanders, see Section 8.1. One important consequence is the following:

Theorem 1.5. *Let Γ be a non–virtually solvable linear group. Then there is a positive constant ϵ such that for any finite (not necessarily symmetric) generating set Σ of Γ, and any finite set $A \subset \Gamma$, there is some $\sigma \in \Sigma$ for which*

$$\frac{|\sigma A \triangle A|}{|A|} > \epsilon.$$

Theorem 1.5 has several consequences, for instance, for the growth function of Γ with respect to a varying generating set. Clearly it implies that Γ has uniform exponential growth, but in addition it shows that the growth function gets larger when the generating set get larger. Moreover since in Theorem 1.5 we do not assume, in contrast to the situation in [11], that the generating set Σ is symmetric, we obtain a uniform exponential growth result for semi–groups rather than for groups. As another example, note that it implies uniform exponential growth for spheres rather than for balls. For more results in this vein see Section 8.2.

We will also show that Theorem 1.1 implies the connected case of the Topological Tits Alternative from [6] and [7]. Recall that the connected case of the Topological Tits Alternative had several interesting consequences such as the Connes–Sullivan conjecture about amenable actions of subgroups of real Lie groups, and the Carrière conjecture about the polynomial versus exponential dichotomy for the growth of leaves in a Riemannian foliation on a compact manifold. In particular, Theorem 1.1 implies these results as well, see Section 8.3.

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2. Some preliminaries

2.1. **Dynamics of projective transformations.** For a more exhaustive and detailed study of the dynamical properties of projective transformations we refer the reader to [[6], Section 3] and [[7], Section 3].

Let k be a local field and $\|\cdot\|$ the standard norm on k^n, i.e. the standard Euclidean (resp. Hermitian) norm when k is $\mathbb{R}$ or $\mathbb{C}$ and $\|x\| = \max_{1\leq i\leq n} |x_i|$ where $x = \sum x_i e_i$ when k is non-Archimedean and $(e_1, \ldots, e_n)$ is the canonical basis of k^n. This induces an operator norm on $\mathrm{SL}_n(k)$. Consider the standard Cartan decomposition of $\mathrm{SL}_n(k)$,

$$\mathrm{SL}_n(k) = KAK$$

where K is $\mathrm{SO}_n(\mathbb{R}), \mathrm{SU}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathcal{O}_k)$ according to whether $k = \mathbb{R}, \mathbb{C}$ or is non-Archimedean, and $A = \{\mathrm{diag}(a_1, \ldots, a_n) : a_1 \geq \ldots \geq a_n > 0, \prod a_i = 1\}$ if k is Archimedean, and $A = \{\mathrm{diag}(\pi^{j_1}, \ldots, \pi^{j_n}) : j_i \in \mathbb{Z}, j_i \leq j_{i+1}, \sum j_i = 0\}$ if k is non-Archimedean with uniformizer π. Any element $g \in \mathrm{SL}_n(k)$ can be decomposed as a product $g = k_g a_g k'_g$, where $k_g, k'_g \in K$ and $a_g \in A$. The A–part a_g is uniquely determined by g, but k_g, k'_g are not. We will set

$$a_g = \mathrm{diag}(a_1(g), \ldots, a_n(g)).$$

Note that $a_1(g) = \|a(g)\| = \|g\|$. For $g \in \mathrm{SL}_n(k)$ we denote by $[g]$ the corresponding projective transformation $[g] \in \mathrm{PSL}_n(k)$. Similarly, for $v \in k^n$ we denote by $[v]$ the corresponding projective point, and for a linear subspace $H \leq k^n$ we let $[H]$ be the corresponding projective subspace.

The canonical norm on k^n induces the associated canonical norm on $\bigwedge^2 k^n$. We define the *standard metric* on $\mathbb{P}^{n-1}(k)$ by the formula

$$d\big([v],[w]\big) = \frac{\|v \wedge w\|}{\|v\| \cdot \|w\|}$$

This is well defined and satisfies the following properties:

(i) d is a distance on $\mathbb{P}^{n-1}(k)$ which induces the canonical topology inherited from the local field k.

(ii) d is an ultra–metric distance if k is non-Archimedean, i.e.

$$d\big([v],[w]\big) \leq \max\{d\big([v],[u]\big), d\big([u],[w]\big)\}$$

for any non-zero vectors u, v and w in k^n.

(iii) If f is a linear form $k^n \to k$, then for any non-zero vector $v \in k^n$,

$$d\big([v],[\ker f]\big) = \frac{|f(v)|}{\|f\| \cdot \|v\|} \tag{1}$$

(iv) Every projective transformation $[g] \in \mathrm{PSL}_n(k)$ is bi–Lipschitz on the entire projective space with Lipschitz constant $|\frac{a_1(g)}{a_n(g)}|^2 = \|g\|^2 \cdot \|g^{-1}\|^2$.

Definition 2.1. A projective transformation $[g] \in \mathrm{PGL}_n(k)$ is called *ϵ–contracting*, for some $\epsilon > 0$, if there is a projective hyperplane $[H]$, called a *repelling hyperplane*, and a projective point $[v]$, called an *attracting point* such that for all points $[p] \in \mathbb{P}^{n-1}(k)$,

$$d([p],[H]) \geq \epsilon \Rightarrow d([gp],[v]) \leq \epsilon.$$

An element $[g]$ is called *(r,ϵ)–proximal*, for $r > 2\epsilon$, if it is ϵ–contracting with respect to some $[H],[v]$ with $d([H],[v]) \geq r$. An element $[g]$ is called *ϵ–very contracting* (resp. *(r,ϵ)–very proximal*) if both $[g]$ and $[g^{-1}]$ are ϵ–contracting (resp. (r,ϵ)–proximal).

The following proposition summarizes the relations between contraction, Lipschitz constants and the ratio between the highest coefficients of a_g.

Proposition 2.2 (See Lemma 3.4 and 3.5 in [6] and Proposition 3.3 in [7]). *Let $\epsilon \in (0, \frac{1}{4}]$, $r \in (0,1]$. Let $g \in SL_n(k)$.*

1. *If $|a_2(g)/a_1(g)| \leq \epsilon$ then $[g]$ is ϵ/r^2–Lipschitz outside the r–neighborhood of the repelling hyperplane $[span\{k'^{-1}_g(e_i)\}_{i=1}^n]$.*
2. *If the restriction of $[g]$ to some open subset $O \subset \mathbb{P}^{n-1}(k)$ is ϵ–Lipschitz, then $|a_2(g)/a_1(g)| \leq \epsilon/2$.*
3. *If $|a_2(g)/a_1(g)| \leq \epsilon^2$ then $[g]$ is ϵ -contracting, and vice versa, if $[g]$ is ϵ–contracting, then $|a_2(g)/a_1(g)| \leq c\epsilon^2$ where c is some constant depending on k.*

Note that the attracting point and repelling hyperplane of a contracting or proximal element are not uniquely defined. In case g is semisimple, it is sometimes useful to choose them to be the span of relevant eigenvectors of g, while it is also possible to define them using the Cartan decomposition like in point (1) above. Very proximal elements are our tool to generate free subgroups via the following version of the classical ping-pong lemma:

Lemma 2.3 (The Ping–Pong Lemma). *Assume that x and y are (r,ϵ)–very proximal projective transformations of $\mathbb{P}^{n-1}(k)$ (for some $r > 2\epsilon$), and suppose that the distances between the attracting points of $x^{\pm 1}$ (resp. of $y^{\pm 1}$) and the repelling hyperplanes of $y^{\pm 1}$ (resp. of $x^{\pm 1}$) are at least r, then x and y are independent.*

2.2. **How to get out of proper subvarieties in bounded time.** Recall the following classical theorem (c.f. [21]):

Theorem 2.4 (Generalized Bezout theorem). *Let $\mathbb{K}$ be a field, and let $X_1, \ldots, X_s$ be pure dimensional algebraic subvarieties of $\mathbb{K}^n$. Denote by $Z_1, \ldots, Z_t$ the irreducible components of $X_1 \cap \ldots \cap X_s$. Then*

$$\sum_{i=1}^{t} deg(Z_i) \leq \prod_{j=1}^{s} deg(X_j).$$

For an algebraic variety X we will denote by $\chi(X)$ the sum of the degrees and dimensions of its irreducible components. The following lemma is a consequence of Theorem 2.4 (see Lemma 3.2 in [11] and its proof[1]).

Lemma 2.5. [11] *Given an integer χ there is $N = N(\chi)$ such that for any field K, any integer $d \geq 1$, any K–algebraic subvariety X in $GL_d(K)$ with $\chi(X) \leq \chi$ and any subset $\Sigma \subset GL_d(K)$ which contains the identity and generates a subgroup which is not contained in $X(K)$, we have $\Sigma^N \nsubseteq X(K)$.*

When X is given, we will sometimes abuse notations and write $N(X)$ for $N(\chi(X))$.

[1] In [11] it is assumed that Σ is finite, that the characteristic of the field is 0, and that the algebraic group $\mathbb{G}$ and the variety X are fixed, however the proof in [11] does not depend on these assumptions.

3. Reduction to the S–arithmetic setting

Here we reduce Theorem 1.1 to Theorem 1.2. Given a global field $\mathbb{K}$ and a finite set S of places of $\mathbb{K}$ including all the infinite ones, we denote by $\mathcal{O}_{\mathbb{K}}(S)$ the ring of S–integers in $\mathbb{K}$. The following lemma is well known:

Lemma 3.1. *Let Γ be a finitely generated linear group which is not virtually solvable. Then there is a global field $\mathbb{K}$, a finite set of places S of $\mathbb{K}$ and a representation $f : \Gamma' \to GL_d(\mathcal{O}_{\mathbb{K}}(S))$ of some finite index subgroup $\Gamma' \leq \Gamma$ whose image is Zariski dense in a simple $\mathbb{K}$–algebraic group.*

Proof. (Suggested to us by G.A. Margulis) In the proof of the classical Tits alternative [24], Tits produces a local field k and a homomorphism $\varphi : \Gamma \to \mathrm{GL}_n(k)$ such that $\varphi(\Gamma)$ contains two proximal elements $\varphi(x), \varphi(y)$ which are "playing ping–pong" on the projective space $\mathbb{P}^{n-1}(k)$ (i.e. satisfy the hypothesis of Lemma 2.3) and hence generate a free subgroup.

Let F be a global field whose completion is k, and let $\overline{F}$ be its integral closure in k, i.e. the field of all elements in k algebraic over F. Let $X = \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k))$ be the variety of representations of Γ into $\mathrm{GL}_n(k)$, realized as a subset of $\mathrm{GL}_n(k)^{d(\Gamma)}$ where $d(\Gamma)$ is the size of some finite generating set of Γ. Then X is an algebraic variety defined over F, and as follows from the implicit function theorem, the set $X(\overline{F})$ of $\overline{F}$ points is dense in $X(k)$ in the topology induced from $\mathrm{GL}_n(k)^{d(\Gamma)}$. Thus we can choose a deformation $\rho \in X(\overline{F})$ arbitrarily close to φ. Now if ρ is sufficiently close to φ, then $\rho(x)$ and $\rho(y)$ still play ping–pong on $\mathbb{P}^{n-1}(k)$, and this implies that $\rho(\Gamma)$ is not virtually solvable.

Let $\mathbb{K}$ be the field generated by the entries of $\rho(\Gamma)$. Since Γ is finitely generated, $\mathbb{K}$ is a global field. Let Γ' be a finite index subgroup of Γ such that $\rho(\Gamma')$ is Zariski connected. We then obtain the representation f and the group $\mathbb{G}$ by dividing by the solvable radical and projecting to a simple factor of the Zariski closure of $\rho(\Gamma')$. Note that as $\rho(\Gamma') \subset \mathrm{GL}_n(\mathbb{K})$ its Zariski closure and solvable radical are defined over $\mathbb{K}$. Therefore $f(\Gamma') \leq \mathbb{G}(\mathbb{K})$.

Finally, since Γ is finitely generated, there is a finite set of places S such that $f(\Gamma)$ lies in the S-arithmetic group $\mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$. $\square$

It is easy to check that if n is the index of Γ' inside Γ, then for any generating set $\Sigma \ni 1$ of Γ containing the identity, Σ^{2n+1} contains a generating set for Γ'. Hence Lemma 3.1 implies that Theorem 1.1 is a consequence of Theorem 1.2. The main part of this paper is therefore devoted to the proof of 1.2.

4. Minimal norm versus Maximal eigenvalue

In this section, we state and prove the Comparison Lemma, Lemma 4.2. Roughly speaking, this statement says that the minimal displacement of a compact subset of isometries of a symmetric space or an affine building is comparable to the minimal displacement of a single element belonging to some bounded power of the subset. When we came up with Lemma 4.2, we were strongly inspired by Proposition 8.5 in [11].

4.1. **Minimal norm, maximal eigenvalue, and the Comparison Lemma.** Let k be a local field with absolute value $|\cdot|_k$. It induces the standard norm on k^d which in turn gives rise to an operator norm $\|\cdot\|$ on $M_d(k)$. If k is not Archimedean, let $\mathcal{O}_k$ be its ring of integers and m_k the maximal ideal in $\mathcal{O}_k$. We note that $\|a\|_k \geq 1$ for all $a \in \mathrm{SL}_d(k)$. Let $\Lambda_k(a)$ be the maximum absolute value of all eigenvalues of a (recall that the absolute value has a unique extension to the algebraic closure of k). If $g \in \mathrm{SL}_d(k)$ we denote by a^g the conjugate gag^{-1}.

For a compact subset $Q \subset M_d(k)$ we denote:

$$\begin{aligned} \Lambda_k(Q) &= \max\{\Lambda_k(a) : a \in Q\} \\ \|Q\|_k &= \max\{\|a\|_k : a \in Q\} \\ \Delta_k(Q) &= \inf_{g \in \mathrm{SL}_d(k)} \|gQg^{-1}\|_k. \end{aligned}$$

Remark 4.1. One can define $\tilde{\Delta}_k$ by taking the infimum over $g \in \mathrm{PGL}_d(k)$. This has some advantages in the non-Archimedean case, e.g. $\tilde{\Delta}_k(Q) = 1$ whenever Q lies in a compact group. Moreover, the ratio between $\tilde{\Delta}_k$ and Δ_k is bounded since PSL_d has finite index in PGL_d. However, we found it more convenient for our purposes to use Δ_k as defined above.

In terms of the action of $\mathrm{SL}_d(k)$ on its symmetric space or affine building, $\log \Delta_k(Q)$ is, up to a multiplicative constant, the minimal displacement of Q, i.e. the smallest radius of a Q–orbit. When $Q = \{a\}$ is a single element, diagonalizable over k, we have $\Delta_k(\{a\}) = \Lambda_k(a)$. The following gives a similar relation when Q is an arbitrary compact subset. We denote by Q^i the set of all products of i, not necessarily different, elements of Q.

Lemma 4.2. ***(Norm–versus–Spectrum Comparison Lemma)*** *There exists a constant $c = c(d, k) > 0$ such that for any compact subset $Q \subset M_d(k)$ we have*

$$\Delta_k(Q)^i \geq \Lambda_k(Q^i) \geq c \cdot \Delta_k(Q)^i$$

for some $i \leq d^2$.

Remark 4.3. The proof that we are about to give uses a compactness argument and hence is not effective. In [9] we will give an effective proof of 4.2. This relies on an effective proof of Wedderburn's theorem on the existence of idempotents in non-nilpotent subalgebras of matrices. Additionally, we will show in [9] that when k is non-Archimedean, by taking finite extensions, we can make c arbitrarily close to 1 (actually $c = (|\pi|_k)^{2d-1}$), and derive a strong uniformity result concerning the growth functions of linear groups.

We now proceed to the proof of Lemma 4.2. We start with the following classical statement:

Lemma 4.4. *Let R be a field or a finite ring and let $\mathcal{A} \leq M_d(R)$ be a subring and R–submodule. Suppose that $\mathcal{A}$ is spanned as an R–module by nilpotent matrices, then $\mathcal{A}$ is nilpotent, i.e. $\mathcal{A}^n = \{0\}$ for some $n \geq 1$.*

Proof. In the 0 characteristic case, the lemma follows easily from Engel's theorem using the fact that a matrix is nilpotent iff all its powers have 0 trace. The proof we give now works in arbitrary characteristic and was suggested to us by A. Salehi-Golsefidy.

The ring $\mathcal{A}$ is Artinian and therefore its Jacobson radical $J(\mathcal{A})$ is nilpotent. We will prove the lemma by showing that $\mathcal{A} = J(\mathcal{A})$. Let $\mathcal{B} = \mathcal{A}/J(\mathcal{A})$ and assume by way of contradiction that $\mathcal{B} \neq 0$. Now $\mathcal{B}$ is semisimple, hence by the Artin–Wedderburn theorem, $\mathcal{B} \cong \bigoplus M_{d_i}(\mathcal{D}_i)$, where the $\mathcal{D}_i$ are division rings. Since $\mathcal{A}$ is spanned by nilpotent elements, so is $\mathcal{B}$. This implies that the trace of any element in $M_{d_i}(\overline{k})$ is 0, and hence that $d_i = 0$. A contradiction. □

Note that an element $A \in M_d(k)$ is nilpotent iff $\Lambda_k(A) = 0$. The following generalizes this statement to compact subsets.

Lemma 4.5. *For a compact subset $Q \subset M_d(k)$ the following are equivalent:*
(i) Q generates a nilpotent subalgebra.
(ii) $\Delta_k(Q) = 0$.
(iii) $\Lambda_k(Q^i) = 0,\ \forall i \leq d^2$.

Proof. Let $\mathcal{A}$ be the algebra generated by Q.
$(i) \Rightarrow (ii)$: By Engel's theorem $\mathcal{A}$ and hence Q can be conjugated by a matrix in $\mathrm{SL}_d(k)$ into the algebra of upper triangular matrices with 0 diagonal. Conjugating further by some suitable diagonal element in $\mathrm{SL}_d(k)$ we can make the norm of Q arbitrarily small.
$(ii) \Rightarrow (iii)$: For any element $g \in M_d(k)$, $\|g\| \geq \Lambda_k(g)$, hence $\Delta_k(Q)^i \geq \Delta_k(Q^i) \geq \Lambda_k(Q^i)$.
$(iii) \Rightarrow (i)$: Take $q \leq d^2$ such that $\dim(\mathrm{span} \cup_{j=1}^{q} Q^j) = \dim(\mathrm{span} \cup_{j=1}^{q+1} Q^j)$, then $\cup_{j=1}^{q} Q^j$ spans $\mathcal{A}$. Since $\Lambda(Q^i) = 0$ for $i \leq d^2$, it consists of nilpotent elements; hence the implication follows from Lemma 4.4. □

Proof of Lemma 4.2. Suppose by contradiction that there is a sequence of compact sets $Q_1, Q_2, \ldots$ in $M_d(k)$ such that $\Lambda_k(Q_n^i) < \Delta_k(Q_n^i)/n,\ \forall i \leq d^2$. By replacing Q_n with a suitable conjugate of it, we may assume that $\|Q_n\|_k \leq 2\Delta_k(Q_n)$, and by normalizing we may assume that $\|Q_n\|_k = 1$. Let Q be a limit of Q_n with respect to the Hausdorff topology on $M_d(k)$. Then $\|Q\|_k = 1$, $\Delta_k(Q) \geq \frac{1}{2}$ since Δ_k is upper semi-continuous, and by continuity of Λ_k, $\Lambda_k(Q^i) = 0,\ \forall i \leq d^2$. This however contradicts Lemma 4.5. □

4.2. **Geometric interpretation of the Comparison Lemma.** For $g \in \mathrm{SL}_d(k)$ and x in the associated symmetric space (resp. affine building) X, we denote by $d_g(x) = d(g \cdot x, x)$ the displacement of g at x. Similarly, for a compact set $Q \subset \mathrm{SL}_d(k)$ we let $d_Q(x) = \max_{g \in Q} d_g(x)$. Finally, we consider the minimal displacement of g, or Q, namely $d_g := \inf_{x \in X} d_g(x)$ and $d_Q := \inf_{x \in X} d_Q(x)$.

Therefore, Lemma 4.2 implies the following geometric statement:

Corollary 4.6. *There is a universal constant $C = C(d) > 0$ such that for any compact set $Q \subset SL_d(k)$ there exists $g \in \cup_{1 \leq i \leq d^2} Q^i$ such that*

$$\frac{1}{\sqrt{d}} d_Q - C \leq d_g \leq d^2 \cdot d_Q$$

Proof. Clearly, if $g \in Q^i$, $d_g \leq d_{Q^i} \leq i \cdot d_Q$. Note that (see Lemma 5.3 below) for every $x \in X$, and $g \in \mathrm{SL}_d(k)$, we have $\log \|g\|_x \leq d_g(x) \leq \sqrt{d} \log \|g\|_x$ where $\|\cdot\|_x$ is the norm associated to the compact stabilizer of x in $\mathrm{SL}_d(k)$, and the log is taken in base $|\pi|_k^{-1}$ when k is non-Archimedean. Since the action of $\mathrm{SL}_d(k)$ on X is transitive in the Archimedean case and transitive on the cells in the non-Archimedean case, it follows that $\log \Delta_k(Q) \geq \frac{1}{\sqrt{d}}(d_Q - 2)$. On the other hand, $\log \Lambda_k(g) \leq d_g$ for all $g \in \mathrm{SL}_d(k)$, and by Lemma 4.2, there exists an $i \leq d^2$ and $g \in Q^i$ with $\Lambda_k(g) \geq c \cdot \Delta_k(Q)^i$. Hence $d_g \geq \log \Lambda_k(g) \geq \frac{i}{\sqrt{d}}(d_Q - 2) + \log c$. $\square$

5. Uniform linear growth of displacement functions

In this section we prove Theorem 1.4 and derive an arithmetic analog to Lemma 4.2 that will be crucial in the proof of Theorem 1.2.

Let $\mathbb{K}$ be a global field, S a finite set of places containing all infinite ones and $\mathcal{O}_{\mathbb{K}}(S)$ the ring of S–integers. For $v \in S$ we let $\mathbb{K}_v$ denote the completion of $\mathbb{K}$ with respect to v. Since v extends uniquely to any finite extension of $\mathbb{K}_v$ we will, abusing notations, denote by $|\cdot|_v$ also the absolute value on any such extension. Let $\mathbb{G} \leq \mathrm{SL}_d$ be a Zariski connected semisimple $\mathbb{K}$–algebraic group. Let

$$G = \prod_{v \in S} \mathbb{G}(\mathbb{K}_v) \leq H = \prod_{v \in S} \mathrm{SL}_d(\mathbb{K}_v).$$

The group of S–integers $\mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ is an S–arithmetic group. We will identify it with its diagonal embedding in G. This makes $\mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ a discrete subgroup of G. The Borel Harish-Chandra theorem says that it is a lattice in G, i.e. the quotient space $G/\mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ carries a finite G–invariant measure, and that if $\mathbb{G}$ is $\mathbb{K}$–anisotropic then $G/\mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ is compact. We will set $\Gamma = \mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$.

Consider $v \in S$ and set $G_v = \mathbb{G}(\mathbb{K}_v)$ and $H_v = \mathrm{SL}_d(\mathbb{K}_v)$. Let $K_v \leq \mathrm{SL}_d(\mathbb{K}_v)$ be the maximal compact subgroup corresponding to the standard norm on $\mathbb{K}_v^d$. Recall that for v Archimedean any maximal compact is conjugate to K_v in $\mathrm{SL}_d(\mathbb{K}_v)$, and for v non-Archimedean there are $d+1$ conjugacy classes. Let $X_v = \mathrm{SL}_d(\mathbb{K}_v)/K_v$ be the associated symmetric space or affine building, let A_v be a Cartan semigroup of $\mathrm{SL}_d(\mathbb{K}_v)$ corresponding to K_v with respect to a Cartan decomposition of $\mathrm{SL}_d(\mathbb{K}_v)$, and let $x_0 \in X_v$ be the point corresponding to K_v.

We also set the following notations. For $a \in \mathrm{SL}_d(\mathbb{K})$ let

$$\Lambda(a) = \max\{|\lambda|_v : \lambda \text{ is an eigenvalue of } a,\ v \in S\} = \max\{\Lambda_v(a) : v \in S\}.$$

For $v \in S$ let $\|\cdot\|_v$ be the standard operator norm on $\mathrm{SL}_d(\mathbb{K}_v)$, and for $g = (g_v)_{v \in S} \in H$ let

$$\|g\| = \max\{\|g_v\|_v : v \in S\}.$$

For a compact subset $Q \subset H$ we let

$$\|Q\| = \max_{a \in Q} \|a\|,\ \Delta(Q) = \inf_{h \in H} \|Q^h\| = \max_{v \in S} \Delta_{\mathbb{K}_v}(Q),\ \text{and } \Lambda(Q) = \max_{a \in Q} \Lambda(a) = \max_{v \in S} \Lambda_{\mathbb{K}_v}(Q).$$

5.1. **Restating Theorem 1.4.**

Definition 5.1. We will say that a subgroup of $\mathbb{G}$ is irreducible in $\mathbb{G}$ if it is not contained in a proper parabolic subgroup of $\mathbb{G}$.

Recall the following result of Mostow in the Archimedean case (c.f. [18] Theorem 3.7) and Landvogt in the non-Archimedean (c.f. [15]):

Theorem 5.2. *There exists a point $x_1 \in X_v$ when v is Archimedean (resp. a cell $\sigma_1 \subset X_v$ when v is non-Archimedean) such that the orbit $G_v \cdot x_0$ (resp. $\cup\{g \cdot \sigma : g \in G_v\}$) is convex and isometric to the symmetric space (resp. affine building) associated to G_v.*

Recall that the norm of a matrix in SL_d is comparable to the exponent of its displacement. More precisely:

Lemma 5.3. *For any $h \in SL_d(\mathbb{K}_v)$ we have:*

- $\|h\| \leq e^{d(h\cdot x_0, x_0)} \leq \|h\|^{\sqrt{d}}$.
- *If $x = g^{-1} \cdot x_0$ then $\|h^g\| \leq e^{d(h\cdot x, x)} \leq \|h^g\|^{\sqrt{d}}$.*

Proof. If $h = k_h a_h k'_h$ is a KAK expression for h then $\|h\| = \|a_h\|$ and $d(h \cdot x_0, x_0) = d(a_h \cdot x_0, x_0)$ hence its enough to prove the first inequality for elements in A, and for such elements it follows by a direct computation.

The second inequality is a direct consequence of the first one. □

Assume that $\mathbb{G}$ is isotropic over $\mathbb{K}$, i.e that Γ is a non-uniform lattice in G. Let $\pi : G \to G/\Gamma$ be the canonical projection, and for $g \in G$ denote

$$\|\pi(g)\| = \min_{\gamma\in\Gamma} \|g\gamma\|.$$

Note that the convex orbit of G from Theorem 5.2 may not pass through the origin x_0, however, since any two orbits of G are equidistant, in view of Lemma 5.3, Theorem 1.4 can be restated as follows:

Theorem 5.4. *There are positive constants C_1, C_2 such that for any finite subset Σ in Γ generating a subgroup whose Zariski closure is semisimple and irreducible in $\mathbb{G}$, we have $\forall g \in G$*

$$\|\Sigma^g\| \geq C_2 \|\pi(g)\|^{C_1}. \tag{2}$$

Remark 5.5. The statement of Theorem 5.4, as well as of Lemma 5.6 below, remains true without the assumption that the non-uniform lattice Γ is arithmetic. To see this one carries the same argument as below, using a variant of Corollary 8.16 from [19] instead of Lemma 5.7. Moreover, the constant C_1 can be taken to depend only on G and not on the choice of the lattice Γ.

5.2. **A quantitative Kazhdan–Margulis Theorem.** Let $\mathbb{K}, S, \mathbb{G}, G, \Gamma$ be as in the previous paragraph, in particular we assume that G/Γ is non-compact.

According to the Kazhdan–Margulis Theorem (see [19]), if $\|\pi(g)\|$ is large enough, then Γ^g contains a non-trivial unipotent close to the identity. The following quantitative version of this theorem was suggested to us by G.A. Margulis.

Lemma 5.6. *There are positive constants k_Γ, l_Γ such for any $g \in G$ the lattice $\Gamma^g = g\Gamma g^{-1}$ contains a non-trivial unipotent $u \in \Gamma^g$ with*

$$\|u - 1\| \leq l_\Gamma \|\pi(g)\|^{-k_\Gamma}.$$

Lemma 5.6 is proved along the same lines as the original Kazhdan-Margulis Theorem.

Lemma 5.7. *There is a positive constant ϵ_G such that if $u_1, \ldots, u_t$ are elements belonging to a non-uniform S–arithmetic subgroup of G and $\|u_i - 1\| \leq \epsilon_G$, $\forall i \leq t$, then the group $\langle u_1, \ldots, u_t \rangle$ is unipotent.*

Proof of Lemma 5.7. If ϵ_G is sufficiently small then by the Zassenhaus Lemma (c.f. [19] 8.8. and 8.17.) the u_i's generate a nilpotent group, and by [17] 4.21(A) the u_i are unipotent. The result follows since any nilpotent group which is generated by unipotent elements is unipotent. □

Proof of Lemma 5.6. We will first assume that $\mathrm{char}(\mathbb{K}) = 0$, and later indicate the changes to be made in the positive characteristic case.

For any Zariski connected unipotent group U there is an element $g_U \in G$ such that the restriction of $\mathrm{Ad}(g_U)$ to the Lie algebra of U expands the norm of any element by at least a factor 4. Since the Grassmann manifolds are compact, it follows that there are finitely many elements $g_1, \ldots, g_k$, $g_i = g_{U_i}$ such that for any Lie subalgebra $\mathfrak{u}$, corresponding to some unipotent subgroup, there is $i \leq k$ such that the restriction of $\mathrm{Ad}(g_i)$ to $\mathfrak{u}$ expands the norm of any element by at least a factor 3. Now since the exponential map $\exp : \mathrm{Lie}(G) \to G$ is a diffeomorphism near the origin 0 of $\mathrm{Lie}(G)$ with differential 1 at 0 it follows that for some $\epsilon_1 > 0$, smaller than ϵ_G, we have:

$$\|g_i u g_i^{-1} - 1\| \geq 2\|u - 1\|, \ \forall\, u \in \exp(\mathfrak{u}) \text{ with } \|u - 1\| \leq \epsilon_1. \tag{3}$$

Fix

$$a = \max_{i \leq k} \|g_i^{\pm 1}\|.$$

and let

$$k_\Gamma = \log_a 2.$$

Fix $\epsilon_2 > 0$ smaller than ϵ_1, such that

$$B_{\epsilon_2}(1_G) \subset \cap_{i \leq k} (B_{\epsilon_1}(1_G))^{g_i^{\pm 1}}.$$

Since $M = G/\Gamma$ has finite volume the "ϵ_2–thick part"

$$M_{\geq \epsilon_2} := \{\pi(g) : g \in G, \text{ and } \Gamma^g \cap B_{\epsilon_2}(1_G) = \{1\}\}$$

is compact. Let

$$l_\Gamma = \sup_{\{g:\pi(g)\in M_{\geq\epsilon_2}\}} \big(\min_{\text{all unipotents } u\in\Gamma\setminus\{1\}} \|u^g - 1\|\big) \cdot \sup_{\{g:\pi(g)\in M_{\geq\epsilon_2}\}} (\|\pi(g)\|)^{k_\Gamma},$$

then the lemma holds for any g with $\pi(g) \in M_{\geq\epsilon_2}$.

Now suppose $\pi(g) \notin M_{\geq\epsilon_2}$. Then Γ^g has a non-trivial unipotent in $B_{\epsilon_2}(1_G)$. Let

$$b = \min\{\|u^g - 1\| : u \in \Gamma \setminus \{1\} \text{ unipotent}\}.$$

By Lemma 5.7 $\Gamma^g \cap B_{\epsilon_1}(1)$ is contained in Zariski connected unipotent group, and hence by (3) there is some g_{i_1} such that the conjugation by it increases the distance of any non-trivial element of this intersection by at least a factor of 2. After this conjugation, there might be some new unipotent elements in the ϵ_1–ball around 1_G, however, by the choice of ϵ_2 there are no new unipotents in the ϵ_2–ball. Therefore we can iterate this argument $\lceil \log_2 \frac{\epsilon_2}{b} \rceil$ times, and get a sequence $g_{i_1}, \ldots, g_{i_t}$, $t = \lceil \log_2 \frac{\epsilon_2}{b} \rceil$, such that $\Gamma^{g_{i_t}\cdot\ldots\cdot g_{i_1} g}$ intersect $B_{\epsilon_2}(1_G)$ trivially. It follows that $\pi(g_{i_t} \cdot \ldots \cdot g_{i_1} g) \in M_{\geq\epsilon_2}$, and hence, if $u \in \Gamma^{g_{i_t}\cdot\ldots\cdot g_{i_1} g}$ is a non-trivial unipotent closest to 1_G

$$\|\pi(g_{i_t} \cdot \ldots \cdot g_{i_1} g)\|^{k_\Gamma} \cdot \|u - 1\| \leq l_\Gamma.$$

Since $\|u - 1\| \geq 2^t b$, and since all the g_i's have norm at most a, the result follows.

Let us now explain the required modifications in the proof for the positive characteristic case. For the positive characteristic version of the Kazhdan–Margulis theorem see [20]. In the positive characteristic case, the unipotent group provided by Lemma 5.7 is not Zariski connected, in fact it is finite. However, it was shown by Borel and Tits [4] that for any unipotent group U there is a canonical parabolic group $P(U)$ which contains the normalizer of U and contains U in its unipotent radical. The unipotent radical of a parabolic subgroup is Zariski connected, and pro-p. Using the KP decomposition where P is a minimal parabolic and K is a maximal compact, it is easy to show that there is some compact set C such that for any parabolic subgroup there is $g \in C$ such that conjugation by g expends the norm of each element in the unipotent radical of the parabolic by at least 4, and one can carry out the same argument as above. □

5.3. **Proof of Theorem 5.4.** Let $\Gamma, \mathbb{G}$, G, k_Γ and l_Γ be as in the previous paragraph. Clearly, the following claim implies Theorem 5.4:

Claim 5.8. There is a constant N, depending only on $\mathbb{G}$, such that $\forall g \in G$

$$l_\Gamma \|\pi(g)\|^{-k_\Gamma} \|\Sigma^g\|^{2N} \geq \epsilon_G. \tag{4}$$

Proof. Assume first that $\mathrm{char}(\mathbb{K}) = 0$ and let $N = d^2$. Suppose by way of contradiction that the lemma is false, and let $u \in \Gamma^g \setminus \{1\}$ be a unipotent element as in Lemma 5.6 with

$$\|u - 1\| \leq l_\Gamma \|\pi(g)\|^{-k_\Gamma}.$$

Then it follows that for any word W in the elements of Σ^g of length at most d^2 we have $\|u^W - 1\| \leq \epsilon_G$. Let $\mathcal{U}_i$ be the Zariski closure of the group generated by $\{u^W : W \text{ is a word}$ in the elements of Σ^g of length $\leq i\}$. Then by Lemma 5.7, $\mathcal{U}_i$ is a unipotent group, hence is

Zariski connected. Therefore, for some $i_0 < \dim(\mathbb{G}) \leq d^2$ we have $\mathcal{U}_{i_0} = \mathcal{U}_{i_0+1}$, and hence $\mathcal{U}_{i_0}$ is normalized by Σ^g. But this implies that Σ^g is contained in some proper parabolic subgroup, a contradiction to the assumption that Σ generates an irreducible subgroup. Hence the claim is proved.

We now give an alternative proof which holds in arbitrary characteristic. Let U be a maximal unipotent subgroup of $\mathbb{G}$. For any $u \in U \setminus \{1\}$ let $Y_u = \{h \in \mathbb{G} : u^h \in U\}$. Then Y_u is a proper algebraic subset of $\mathbb{G}$, and one easily sees that $\chi(Y_u)$ is bounded independently of u, by some χ say. Now if $\mathbb{E}$ is an irreducible subgroup of $\mathbb{G}$, i.e. not contained in a proper parabolic subgroup, then $\cap_{h\in\mathbb{E}} U^h$ is trivial. It follows that $\mathbb{E} \subsetneq Y_u$ for any $u \in U \setminus \{1\}$. Thus Lemma 2.5 yields a constant $N = N(\chi)$ such that some word W of length at most N in Σ^g satisfies $\|u^W - 1\| > \epsilon_G$, where u is the element in (5.3). For otherwise, $\langle u^W : W \text{ is a word of length } \leq N \text{ in } \Sigma^g\rangle$ would be a unipotent group and hence some conjugate of it would lie in U, and since the corresponding conjugate of Σ^g generates a group whose Zariski closure $\mathbb{E}$ is irreducible in $\mathbb{G}$, this contradicts the property of $N(\chi)$. It follows that equation (4) holds with $N = N(\chi)$. $\square$

5.4. **An S–arithmetic version of the Comparison Lemma.** Let $\mathbb{K}, S, \mathbb{G}, G, \Gamma, d$ be as in the beginning of this Section (we do not assume that $\mathbb{G}$ is isotropic over $\mathbb{K}$). The goal of the remaining part of this section is to prove the following arithmetic version of Lemma 4.2.

Proposition 5.9. *For some constant r, depending only on $\mathbb{G}$, $\mathbb{K}$, and S, we have that for any finite subset $\Sigma \subset \Gamma = \mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ ($\Sigma \ni id$) generating a subgroup whose Zariski closure $\mathbb{F}$ is irreducible in $\mathbb{G}$, there is an element $\gamma \in \Gamma$ such that $\|\Sigma^\gamma\| \leq \Lambda(\Sigma^r)$.*

Remark 5.10. In the proof of Theorem 1.2 in the next section we will apply Proposition 5.9 only in the case where $\mathbb{G} = \mathrm{SL}_d$ and $\Gamma = \mathrm{SL}_d(\mathcal{O}_{\mathbb{K}}(S))$.

Note that if $\alpha \in \mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$, $\Lambda(\alpha) = 1$ if and only if all the eigenvalues of α are roots of unity, i.e. if and only if α has finite order. Moreover, there is a positive constant $\tau > 1$ such that if $\alpha \in \mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ has $\Lambda(\alpha) > 1$ then $\Lambda(\alpha) \geq \tau$. This follows from the fact that $\mathcal{O}_{\mathbb{K}}(S)$ embeds discretely in $\prod_{v\in S} \mathbb{K}_v$. Moreover, the Zariski closure Y of the set of torsion elements in Γ is a proper algebraic subvariety of $\mathbb{G}$ (there is an upper bound of the order of torsion elements in Γ, see Proposition 2.5. [24]). Hence Lemma 2.5 implies that $\Sigma^{\overline{r}}$ contains a non torsion element, where $\overline{r}$ is some integer independent of Σ. This shows that $\Lambda(\Sigma^n) \geq \tau$ for all $n \geq \overline{r}$. We can therefore reformulate the Comparison Lemma (4.2) as follows, omitting the multiplicative constant.

Lemma 5.11. *For some constant r', depending only on $\mathbb{G}$, $\mathbb{K}$, and S, we have that for any finite subset $\Sigma \subset \Gamma = \mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$ ($\Sigma \ni id$) generating a subgroup whose Zariski closure $\mathbb{F}$ is irreducible in $\mathbb{G}$, there is an element $h \in H$ such that $\|\Sigma^h\| \leq \Lambda(\Sigma^{r'})$.*

In order to derive Proposition 5.9 from Lemma 5.11 we will first replace the conjugating element $h \in H$ by an element $g \in G$ (of course this step is unnecessary when $G = H$ which is the situation in the proof of Theorem 1.2). The second part of the proof which consists in replacing g by some $\gamma \in \Gamma$ relies on Theorem 5.4.

5.4.1. *Step 1: Projection to a homogeneous subspace.* By Theorem 5.2 we may identify the symmetric space (resp. affine building) of G_v with a convex subset C of X_v of the form $G_v \cdot x_1$ (resp. $G_v \cdot \sigma_1$) for some point x_1 (resp. some cell $\sigma_1 \ni x_1$) in X_v.

Since X_v is a CAT(0) space, the projection to the nearest point $P_C : X_v \to C$ is 1–Lipschitz. Let $h \in \mathrm{SL}_d(\mathbb{K}_v)$ be the element from Lemma 5.11, let $x = P_C(h \cdot x_0)$ and let $g_v \in G_v$ be an element such that $g_v \cdot x = x_1$ (resp. $g_v \cdot x \in \sigma_1$). In any case, we have $d(x_1, g_v \cdot x) \leq 1$. Since $\Sigma \subset G_v$, it preserves C and since P_C is 1–Lipschitz we have $d_\Sigma(x) \leq d_\Sigma(h \cdot x_0)$, where $d_\Sigma(x) = \max_{\gamma \in \Sigma} d(x, \gamma \cdot x)$. We get $d_\Sigma(g_v^{-1} \cdot x_1) \leq d_\Sigma(h \cdot x_0) + 2$, and finally we obtain:

$$d_\Sigma(g_v^{-1} \cdot x_0) \leq d_\Sigma(h \cdot x_0) + 2 + 2d(x_0, x_1).$$

With Lemma 5.3 we can translate this to: $\|\Sigma^{g_v}\| \leq \|\Sigma^h\|^b$ for some constant $b > 0$. Repeating this argument for every $v \in S$, we get from Lemma 5.11:

Corollary 5.12. *For some constant r'' (independent of Σ) we have*

$$\|\Sigma^g\| \leq \Lambda(\Sigma^{r''}).$$

5.4.2. *Step 2: Finding a relatively close point in a given Γ–orbit.* We will now explain how to replace $g = (g_v) \in G$ by some $\gamma \in \Gamma$ and obtain the proof of Proposition 5.9.

Assume first that $\mathbb{G}$ is $\mathbb{K}$–anisotropic, i.e. that G/Γ is compact. Let Ω be a fixed bounded fundamental domain for Γ in G and let $\gamma \in \Gamma$ be the unique element such that $g \in \Omega\gamma$. Write

$$c = \max\{\|f\| : f \in \Omega \cup \Omega^{-1}\},$$

then Theorem 5.9 holds with $r = r''(1 + 2\log_\tau c)$.

Next assume that $\mathbb{G}$ is $\mathbb{K}$–isotropic. By equation (4)

$$\|\pi(g)\| \leq \big(l_\Gamma \frac{\|\Sigma^g\|^{2N}}{\epsilon_G}\big)^{1/k_\Gamma} \leq \big(l_\Gamma \frac{\Lambda(\Sigma^{r''})^{2N}}{\epsilon_G}\big)^{1/k_\Gamma}.$$

Which means that for some $\gamma \in \Gamma$

$$\|g\gamma^{-1}\| \leq \big(l_\Gamma \frac{\Lambda(\Sigma^{r''})^{2N}}{\epsilon_G}\big)^{1/k_\Gamma} \leq \Lambda(\Sigma^{r''})^{\frac{2N}{k_\Gamma} + \frac{1}{k_\Gamma}\log_\tau \frac{l_\Gamma}{\epsilon_G}}.$$

and therefore

$$\|\Sigma^\gamma\| = \|\Sigma^{\gamma g^{-1} g}\| \leq \|\Sigma^g\| \|g\gamma^{-1}\| \|\gamma g^{-1}\| \leq \Lambda(\Sigma^r)$$

for some computable constant r. □

6. Construction of the ping-pong players

In this section we will construct two bounded words in the alphabet Σ that will play ping-pong on some projective space and hence will be independent. This will prove Theorem 1.2. Since the detailed proof below is somewhat technical we refer the reader to [8] for an outline of the main ideas.

Let $\Sigma \subset \mathrm{SL}_d(\mathcal{O}_{\mathbb{K}}(S))$ be as in the statement of Theorem 1.2. Inconsistently with the previous section we will denote by $\mathbb{G}$ the Zariski closure of $\langle \Sigma \rangle$ in SL_d, and $\Gamma = \mathbb{G}(\mathcal{O}_{\mathbb{K}}(S))$. Then $\mathbb{G}$ is a semisimple irreducible subgroup in SL_d and Γ is an S–arithmetic subgroup of $\mathbb{G}$. We let $G = \prod_{v \in S} \mathbb{G}(K_v)$ and identify Γ via the diagonal embedding with the corresponding S–arithmetic lattice in G.

In this section, whenever we say that some quantity is a *constant*, we mean that it may depend only on d, $\mathbb{K}$ and S.

The following proposition will allow us to assume that Σ is finite, hence compact. Let $s \in \mathbb{N}$ be a constant. We will specify some condition on s in Paragraph 6.4 (Step (4)), for the moment we only require it to be at least r, the constant from Proposition 5.9.

Proposition 6.1. *There is a constant f, such that for any subset $1 \in \Sigma \subset \Gamma$ which generates a Zariski dense subgroup of $\mathbb{G}$, there is a subset Σ' of Σ^f of cardinality* $\dim(\mathbb{G})$ *such that:*

1. *The Zariski closure $\overline{\langle \Sigma' \rangle}^Z$ of the group generated by Σ' equals $\mathbb{G}$, and*
2. *$(\Sigma')^s$ consists of semisimple elements.*

Recall the following fact:

Lemma 6.2. ***(see Borel*** [3]***)*** *Let $\mathbb{G}$ be a connected semisimple algebraic group, and $k \geq 2$ an integer. If W is a non-trivial word in the free group F_k, then the corresponding map $W : \mathbb{G}^k \to \mathbb{G}$ is dominant.*

Proof of Proposition 6.1. Let $k = \dim(\mathbb{G})$, let $W_1, \ldots, W_t$ be all the reduced words in F_k of length $\leq s$, and consider the map $w : \mathbb{G}^k \to \mathbb{G}^t$ defined by substitution in $(W_1, \ldots, W_t)$. Let $\Phi \subset \mathbb{G}$ be a Zariski open subset which consists of semisimple elements. We shall construct inductively elements $\sigma_i,\ i = 1, \ldots$ in a bounded power of Σ which $\forall i$ satisfy:

- There are some $g_{i+1}, \ldots, g_k \in \mathbb{G}$ such that $w(\sigma_1, \ldots, \sigma_i, g_{i+1}, \ldots, g_k) \in \Phi^t$.
- $\dim(\overline{\langle \sigma_1, \ldots \sigma_i \rangle}^Z) \geq i$.

In order to construct σ_1 choose some $(g_1, g_2, \ldots, g_k) \in w^{-1}(\Phi^t)$, which is non-empty by Lemma 6.2 , and define

$$V_1 = \{g \in \mathbb{G} : w(g, g_2, \ldots, g_k) \in \mathbb{G}^t \setminus \Phi^t\}.$$

As noted before Lemma 5.11, the Zariski closure X of the elements in Γ whose projection to one of the factors of $\mathbb{G}$ is torsion is a proper subvariety of $\mathbb{G}$. Let N_1 be the constant obtained from Lemma 2.5 applied to $X \cup V_1$ and take $\sigma_1 \in \Sigma^{N_1} \setminus (X \cup V_1)$. It is straightforward to check that $\chi(V_1)$ (i.e. the sum of the degrees and dimensions of the irreducible components of V_1) can be bounded independently of the choice of $(g_2, \dots, g_k)$ and hence that N_1 can be taken to be a constant. Finally, since σ_1 has infinite order $\overline{\langle\sigma_1\rangle}^Z$ has positive dimension.

To explain the i'th step let us suppose that $\sigma_1, \dots, \sigma_{i-1}$ were already constructed. Since $\sigma_1, \dots, \sigma_{i-1}$ are assumed to satisfy the requirements above, we can chose some new $g_{i+1}, \dots, g_k \in \mathbb{G}$ for which the algebraic set

$$V_i = \{g \in \mathbb{G} : w(\sigma_1 \dots, \sigma_{i-1}, g, g_{i+1}, \dots, g_k) \in \mathbb{G}^t \setminus \Phi^t\}$$

is proper. Additionally, the Zariski connected group $\mathbb{G}_i = (\overline{\langle\sigma_1, \dots, \sigma_{i-1}\rangle}^Z)^\circ$ cannot be proper normal since by the properties of σ_1 it projects non-trivially to each simple factor of the semisimple group $\mathbb{G}$. If $\mathbb{G}_i = \mathbb{G}$ take $\sigma_i = 1$ and otherwise take $\delta_i \in \Sigma \setminus N_{\mathbb{G}}(\mathbb{G}_i)$, let N_i be the constant obtained from Lemma 2.5 applied to $V_i \cup \delta V_i$, chose $\sigma_i' \in \Sigma^{N_i} \setminus (V_i \cup \delta_i V_i)$, and set $\sigma_i = \sigma_i'$ if $\sigma_i' \notin N_{\mathbb{G}}(\mathbb{G}_i)$ and $\sigma_i = \delta_i^{-1}\sigma_i'$ otherwise. Again N_i can be taken to be a constant (independent of the previous choice of $\sigma_j,\ j < i$, the choice of $g_j,\ j > i$ and the choice of δ_i, since $\chi(V_i \cup \delta_i V_i)$ too can be bounded by a constant). Finally, since σ_i does not normalize $\mathbb{G}_i$, $\dim(\overline{\langle\sigma_1, \dots, \sigma_i\rangle}^Z > \dim(\overline{\langle\sigma_1, \dots, \sigma_{i-1}\rangle}^Z$. □

We will therefore assume that Σ itself is finite and Σ^s consists of semisimple elements (where $s \geq r$). Applying Proposition 5.9 we see that up to changing Σ into Σ^γ for some $\gamma \in \Gamma$, we may assume that $\Lambda(A_0) \geq \|\Sigma\|$ for some $A_0 \in \Sigma^r$.

We will now fix once and for all a place $v \in S$ for which $\Lambda_v(A_0) = \Lambda(A_0)$. The local field $\mathbb{K}_v$ has only finitely many extensions of degree at most $d!$. Let $\tilde{\mathbb{K}}_v$ be their compositum, then any semisimple element in $\mathrm{SL}_d(\mathbb{K}_v)$ is diagonalizable in $\mathrm{SL}_d(\tilde{\mathbb{K}}_v)$. Similarly, let $\tilde{\mathbb{K}}$ be the splitting field of A_0, and let $\tilde{S}$ be the set of all places of $\tilde{\mathbb{K}}$ extending elements of S.

By passing to a suitable wedge power representation $V = \Lambda^i\mathbb{K}^d$ for some i, $1 \leq i \leq d-1$, we may assume that A_0 has a unique eigenvalue $\alpha_1(A)$ of maximal v–absolute value and that the ratio between $\alpha_1(A_0)$ and the second largest eigenvalue $\alpha_2(A_0)$ satisfies

$$\Lambda(A_0)^d \geq \Big|\frac{\alpha_1(A_0)}{\alpha_2(A_0)}\Big|_v \geq \Lambda(A_0)^{\frac{1}{d}} \geq \tau^{1/d},$$

where τ is the constant introduced in the proof of Proposition 5.9. Note that the norm of a matrix in a wedge power representation such as V is bounded by its original norm to the power d. Thus, we have

$$|\alpha_1(A_0)/\alpha_2(A_0)|_v^{d^2} \geq \|\Sigma\|_{End(V)}. \tag{5}$$

We will set $n = dimV$ the dimension of the new representation. Note that $n \leq 2^d$. Note also that in the canonical basis of the wedge power space, the matrix elements from Σ (viewed as matrices in $\mathrm{SL}_n(\mathbb{K})$) are still in $\mathcal{O}_{\mathbb{K}}(S)$. Finally observe that V may not be $\mathbb{G}$–irreducible. This is not a fundamental problem. However to keep exposition as simple as possible we will assume throughout that $V = \mathbb{K}_v^n$ is an irreducible $\mathbb{G}$–space with A_0 and Σ with matrix coefficients in $\mathcal{O}_{\mathbb{K}}(S)$ and satisfying the two inequalities above. At the end we will indicate the changes to be made to accomodate with the fact that $\Lambda^i\mathbb{K}^d$ is not irreducible in general.

Working with the corresponding projective representation over $\tilde{\mathbb{K}}_v$ we will now produce two ping–pong players in four steps. In the first we will construct a proximal element, in the second a very contracting one and in the third a very proximal one. Then we will find a suitable conjugate of the very proximal element and obtain in this way a second ping–pong partner.

6.1. **Step 1.** We set $r_0 = rd^2$. Let $\{\hat{u}_i\}$ be a basis of $\tilde{\mathbb{K}}_v^n$ consisting of normalized eigenvectors of A_0 with corresponding eigenvalues $\{\alpha_i\}$, such that whenever $\alpha_i = \alpha_j$ the vectors $\hat{u}_i$ and $\hat{u}_j$ are orthogonal[2], and let $\hat{u}_i^{\perp}$ denote the hyperplane spanned by $\{\hat{u}_j : j \neq i\}$.

Lemma 6.3. *For some constant $r_1 \in \mathbb{N}$, depending only on Γ,*

$$d(\hat{u}_i, \hat{u}_i^{\perp}) \geq |\frac{\alpha_1}{\alpha_2}|_v^{-r_1}$$

for $i = 1, \ldots n$.

Proof. First note that since $|\alpha_i - \alpha_j|_w \leq 2\Lambda(A_0)$ for any $w \in \tilde{S}$ and $|\alpha_i - \alpha_j|_w \leq 1$ for any $w \notin \tilde{S}$, it follows from the product formula that if $\alpha_i \neq \alpha_j$ then

$$|\alpha_i - \alpha_j| \geq (2\Lambda(A_0))^{-|\tilde{S}|} \geq \Lambda(A_0)^{-|\tilde{S}|(1+\log_\tau 2)} \geq |\frac{\alpha_1}{\alpha_2}|_v^{-d|\tilde{S}|(1+\log_\tau 2)} = |\frac{\alpha_1}{\alpha_2}|_v^{-t_0}$$

where $t_0 = d|\tilde{S}|(1 + \log_\tau 2)$. Note also that $|\tilde{S}| \leq d!|S|$.

Next, observe that it is enough to show that for some constant r_1',

$$d(\overrightarrow{u_i}, \mathrm{span}\{\hat{u}_j : \alpha_j \neq \alpha_i\}) \geq |\frac{\alpha_1}{\alpha_2}|_v^{-r_1'}$$

for any i and any unit vector $\overrightarrow{u}_i \in \mathrm{span}\{\hat{u}_j : \alpha_j = \alpha_i\}$. This in turn will follow from the next claim which we will prove by induction on k:

Claim. For any k there is a positive constant t_k such that if $\overrightarrow{u} \in \mathrm{span}\{\hat{u}_j : j \in I, \alpha_j \neq \alpha_i\}$ where I is a set of indices with $\dim(\mathrm{span}\{\hat{u}_j : j \in I, \alpha_j \neq \alpha_i\}) = k$ then $\|\overrightarrow{u}_i - \overrightarrow{u}\| \geq |\frac{\alpha_1}{\alpha_2}|_v^{-t_k}$ for any unit vector $\overrightarrow{u}_i \in \mathrm{span}\{\hat{u}_j : \alpha_j = \alpha_i\}$.

For $k = 1$ we can write $\overrightarrow{u} = \lambda\hat{u}_j$, so

$$A_0(\overrightarrow{u}_i - \lambda\hat{u}_j) = (\alpha_i - \alpha_j)\overrightarrow{u}_i + \alpha_j(\overrightarrow{u}_i - \lambda\hat{u}_j)$$

i.e.

$$(A_0 - \alpha_j)(\overrightarrow{u}_i - \lambda\hat{u}_j) = (\alpha_i - \alpha_j)\overrightarrow{u}_i,$$

which implies that (recall $r_0 = rd^2$)

$$\|\overrightarrow{u}_i - \lambda\hat{u}_j\|_v \geq \frac{|\alpha_i - \alpha_j|_v}{\|A_0\|_v + |\alpha_j|_v} \geq |\frac{\alpha_1}{\alpha_2}|_v^{-t_0-(r_0+d\log_\tau 2)} := |\frac{\alpha_1}{\alpha_2}|_v^{-t_1}.$$

[2]In the non-Archimedean case this is simply taken to mean that $\|\hat{u}_i - \hat{u}_j\| = 1$.

Now suppose $k > 1$. We can write $\overrightarrow{u} = \sum \lambda_j \overrightarrow{u}_j$ where the $\overrightarrow{u}_j$'s are normalized eigenvectors of different eigenvalues. Abusing indices, we will assume that $\overrightarrow{u}_j$ corresponds to the eigenvalue α_j. Now

$$A_0(\overrightarrow{u}_i - \sum \lambda_j \overrightarrow{u}_j) = \alpha_i(\overrightarrow{u}_i - \sum \lambda_j \overrightarrow{u}_j) + \sum_j (\alpha_i - \alpha_j)\lambda_j \overrightarrow{u}_j,$$

therefore

$$(A_0 - \alpha_i)(\overrightarrow{u}_i - \sum \lambda_j \overrightarrow{u}_j) = \sum_j (\alpha_i - \alpha_j)\lambda_j \overrightarrow{u}_j.$$

Note that we may assume that $\|\overrightarrow{u}\|_v \geq 1/2$, for otherwise the statement is obvious, and hence for some j_0, $|\lambda_{j_0}|_v \geq 1/(2n)$ and by the induction hypothesis

$$\|\sum_j (\alpha_i - \alpha_j)\lambda_j \overrightarrow{u}_j\| \geq |\lambda_{j_0}|_v |\frac{\alpha_1}{\alpha_2}|_v^{-t_0-t_{k-1}} \geq \frac{1}{2n}|\frac{\alpha_1}{\alpha_2}|_v^{-t_0-t_{k-1}}.$$

It follows that

$$\|\overrightarrow{u}_i - \sum \lambda_j \overrightarrow{u}_j\|_v \geq \frac{1}{2n}|\frac{\alpha_1}{\alpha_2}|_v^{-t_0-t_{k-1}} \frac{1}{\|A_0\|_v + |\alpha_i|_v} \geq |\frac{\alpha_1}{\alpha_2}|_v^{d\log_\tau \frac{1}{2n} - t_0 - t_{k-1} - (r_0 + d\log_\tau 2)} := |\frac{\alpha_1}{\alpha_2}|_v^{-t_k}.$$

□

As a consequence we obtain that for some constant r_2, depending only on Γ, which we may take $\geq r_1$, we have:

Corollary 6.4. *There is a matrix $D \in SL_n(\tilde{\mathbb{K}}_v)$ such that:*
- $\|D\|^2, \|D^{-1}\|^2 \leq |\frac{\alpha_1}{\alpha_2}|_v^{r_2}$, *and*
- $A_0^D = DA_0D^{-1}$ *is diagonal.*

Proof. Let D be the matrix defined by the condition $D(\hat{u}_i) = e_i$, $i = 1, \ldots, n$. Clearly $|\det(D^{-1})|_v \leq 1$. Since $D^{-1} = \det(D^{-1})\mathrm{Adj}(D)$ and since $\|\mathrm{Adj}(D)\| \leq n!\|D\|^{n-1}$ it is enough to prove that $\|D\| \leq |\frac{\alpha_1}{\alpha_2}|_v^{r_2'}$.

Let $\hat{u}$ be a unit vector, and write $\hat{u} = \sum \lambda_i e_i$. Then for some i_0 we have $|\lambda_{i_0}|_v \geq 1/n$. Since $D^{-1}(\hat{u}) = \sum \lambda_i \hat{u}_i$, it follows from the previous lemma that

$$\|D^{-1}(\hat{u})\| = \|\sum \lambda_i \hat{u}_i\| = \|\lambda_{i_0}\hat{u}_{i_0} + \sum_{j \neq i_0} \lambda_j \hat{u}_j\| \geq \frac{1}{n}|\frac{\alpha_1}{\alpha_2}|_v^{-r_1} \geq |\frac{\alpha_1}{\alpha_2}|_v^{-r_2'},$$

i.e. $\|D\| \leq |\frac{\alpha_1}{\alpha_2}|_v^{r_2'}$. □

We derive the following proposition and thus conclude the first step in our construction of ping–pong players:

Proposition 6.5 (The proximal element A_1). *Whenever $r_3 \geq 8r_2$, the element $A_1 = A_0^{r_3}$ is $(|\frac{\alpha_1}{\alpha_2}|_v^{-r_1}, |\frac{\alpha_1}{\alpha_2}|_v^{-(r_3/2-2r_2)})$–proximal with attracting point $[\hat{u}_1]$ and repelling hyperplane $[\hat{u}_1^\perp] = [span(\hat{u}_2, \ldots, \hat{u}_n)]$.*

Proof. The diagonal matrix $DA_0^{r_3}D^{-1}$ is obviously $|\frac{\alpha_1}{\alpha_2}|_v^{-r_3/2}$–contracting with attracting point $[e_1]$ and repelling hyperplane $[\mathrm{span}(e_2,\ldots,e_n)]$. Since $\|D\|,\|D^{-1}\| \le |\frac{\alpha_1}{\alpha_2}|_v^{r_2}$, D is $|\frac{\alpha_1}{\alpha_2}|_v^{2r_2}$ bi-Lipschitz. It follows that $A_0^{r_3}$ is $|\frac{\alpha_1}{\alpha_2}|_v^{-(r_3/2-2r_2)}$–contracting. Finally, Lemma 6.3 implies that $d([\hat{u}_1],[\hat{u}_1^{\perp}]) \ge |\frac{\alpha_1}{\alpha_2}|_v^{-r_1}$ □

6.2. **Step 2.** Our next goal is to build a very contracting element out of the matrix A_1. To achieve this, we will find some bounded word B_1 in Σ which will be in "general position" with respect to A_1. Then $A_2 = A_1^{r_7}B_1A_1^{-r_7}$ will be our candidate. In this process we will "lose" the information we have on the position of the repelling neighborhoods. However we will still have a good control on the positions of the attracting points of A_2 and A_2^{-1}, a control which will turn crucial in the following step when producing a very proximal element A_3. The key idea is that while B_1 sends the eigen-directions of A_1 away from the eigen-hyperplanes of A_1, we can estimate this quantitatively by giving an explicit lower bound. In order to formulate a precise statement, we will need to introduce another basis of eigenvectors for A_1.

Lemma 6.6. *For each $k \le n$ there is an eigenvector $\overrightarrow{u}_k \in \overline{\mathbb{K}}^n$ for A_0 with corresponding eigenvalue α_k whose coordinates are $\tilde{S}$–integers and whose w–norm is at most $|\alpha_1/\alpha_2|_v^{r_4}$ for any $w \in \tilde{S}$, where r_4 is some constant depending only on r_0, d and the size of S.*

Proof. Recall from inequality (5) that for each $w \in S$ we have $\|A_0\|_w \le |\alpha_1/\alpha_2|_v^{r_0}$ (where $r_0 = rd^2$). Suppose that α_i has multiplicity k, say $\alpha_i = \alpha_{i+1} = \ldots = \alpha_{i+k-1}$, then we can pick k indices between 1 and n such that the $(n-k)\times(n-k)$ matrix obtained by restricting $A_0 - \alpha_i$ to the remaining indices is invertible. We can then define $\overrightarrow{u_{i+j}}$, $j \le k-1$ to be the eigenvector of $\alpha_{i+j} = \alpha_i$ whose entries corresponding to the chosen k indices are all 0 except the $(j-1)$'th one which equals the determinant of the $(n-k)\times(n-k)$ submatrix. Solving the corresponding linear equation, it is easy to verify that these vectors satisfy the requirement with respect to some bounded constant r_4. □

In analogy to our previous notations, we will denote by $\overrightarrow{u_i}^{\perp}$ the span of the $\overrightarrow{u_j}$'s, $j \ne i$. Note that since α_1 has multiplicity one, we have $[\hat{u}_1] = [\overrightarrow{u_1}]$ and $[\hat{u}_1^{\perp}] = [\overrightarrow{u_1}^{\perp}]$.

Definition 6.7. Let N be an integer and $v_1,\ldots,v_n \in \overline{\mathbb{K}}^n$ a basis. We will say that a matrix $C \in \mathrm{SL}_n(\overline{\mathbb{K}})$ is in *N–general position* with respect to $\{v_1,\ldots,v_n\}$ if

- for any $1 \le i,j \le n$, not necessarily distinct, both vectors Cv_i and $C^{-1}v_i$ do not lie in the hyperplane spanned by $\{v_k\}_{k\ne j}$, and
- for any n integers $1 \le i_1 < \ldots < i_n \le N$ and any $1 \le j \le n$ the vectors $C^{i_1}v_j,\ldots,C^{i_n}v_j$ are linearly independent.

For a fixed N, the varieties

$$X(N,v_1,\ldots,v_n) = \{g \in \mathrm{SL}_n(\overline{\mathbb{K}}) : g \text{ is } \textit{not} \text{ in } N\text{–general position w.r.t. } \{v_i\}_{i=1}^n\}$$

are all conjugate inside $\mathrm{SL}_n(\overline{\mathbb{K}})$. Since $\mathbb{G}$ is Zariski connected and irreducible, one can derive that $X(N,v_1,\ldots,v_n)\cap\mathbb{G}$ is a proper subvariety of $\mathbb{G}$. Hence by Lemma 2.5 for any N there is a constant $m_2(N)$ such that for any set Ω which generates a Zariski dense subgroup of $\mathbb{G}$, and any basis $\{v_i\}_{i=1}^n$ of $\mathbb{K}^n$, there is an element in $\Omega^{m_2(N)}$ which is in N–general position with respect to $\{v_i\}_{i=1}^n$. In particular we may find $B_1 \in \Sigma^{m_2}$ (with $m_2 = m_2(2n-1)$) which is in $(2n-1)$–general position with respect to $\{\overrightarrow{u}_i\}_{i=1}^n$.

In the proof of Proposition 6.10 we will make use of the following lemma only for $i = n$ and $j = 1$.

Lemma 6.8. *For some positive bounded constant r_5 we have*

$$d((B_1^{\pm 1}) \cdot [\overrightarrow{u_i}], [\overrightarrow{u_j^{\perp}}]) > |\frac{\alpha_1}{\alpha_2}|_v^{-r_5},$$

for any $i, j \leq n$.

Proof. For each $w \in \tilde{S}$, the w–absolute values of the coordinates of $B_1(\overrightarrow{u}_i)$ are at most $|\alpha_1/\alpha_2|_v^{m_2 r_0 + r_4}$. Consider the determinant

$$\mathcal{D}_{\pm 1} = \det(B^{\pm 1}(\overrightarrow{u}_i), \overrightarrow{u}_1, \ldots, \overrightarrow{u}_{j-1}, \overrightarrow{u}_{j+1}, \ldots, \overrightarrow{u}_n).$$

This is again an $\tilde{S}$–integer and its w–absolute value is at most $|\alpha_1/\alpha_2|_v^{m_2 r_0 + n r_4}$. Since B_1 is in general position with respect to $\{\overrightarrow{u}_i\}_{i=1}^n$ we have $\mathcal{D}_{\pm 1} \neq 0$. By the product formula $\prod_{\text{all places}} |\mathcal{D}_{\pm 1}|_w = 1$ and hence $\prod_{w \in \tilde{S}} |\mathcal{D}_{\pm 1}|_w \geq 1$. It follows that

$$|\mathcal{D}_{\pm 1}|_v \geq |\alpha_1/\alpha_2|_v^{-(m_2 r_0 + n r_4)|\tilde{S}|}.$$

Now since all the vectors involved in this determinant have v–norm at most $|\alpha_1/\alpha_2|_v^{m_2 r_0 + r_4}$, the distance between each of them to the hyperplane spanned by the others is at least

$$\frac{|\mathcal{D}_{\pm 1}|_v}{|\alpha_1/\alpha_2|_v^{(m_2 r_0 + r_4)(n-1)}} \geq |\alpha_1/\alpha_2|_v^{-(m_2 r_0 + n r_4)(|\tilde{S}| + n - 1)}.$$

The lemma follows. ☐

We will also need the following:

Lemma 6.9. *There exists some $\epsilon = \epsilon(n)$, such that if $d = diag(d_1, \ldots, d_n) \in SL_n(\tilde{\mathbb{K}}_v)$ is a diagonal matrix with $d_1 \geq d_2 \geq \ldots \geq d_n$, then $[d]$ is 2–Lipschitz on the ϵ–ball around $[e_1]$.*

Proof. The lemma follows by a direct simple computation. In the non-Archimedean case a diagonal matrix is 1–Lipschitz on the open unit ball around $[e_1]$. In the Archimedean case the same is true for the metric which is induced on $\mathbb{P}(\tilde{\mathbb{K}}_v^n)$ from the L^∞ norm on $\tilde{\mathbb{K}}_v^n$. Since the renormalization map from the euclidean unit sphere to the L^∞ unit sphere is C^1 around e_1 with differential 1 at e_1 it has a bi-Lipschitz constant arbitrarily close to 1 in a small neighborhood of e_1. The result follows. ☐

We are now able to formulate:

Proposition 6.10 (The very contracting element A_2)**.** *For any $r_6 \in \mathbb{N}$, there exists $r_7 \in \mathbb{N}$ such that the element $A_2 = A_1^{r_7} B_1 A_1^{-r_7}$ is $|\frac{\alpha_1}{\alpha_2}|_v^{-r_6}$ very contracting, with both attracting points (of the element and its inverse) lying in the $|\frac{\alpha_1}{\alpha_2}|_v^{-r_6}$ ball around $[\hat{u}_1]$.*

The proof of Proposition 6.10 relies on Proposition 2.2, as well as the last two lemmas:

Proof of Proposition 6.10. Let $r_7 \in \mathbb{N}$ be arbitrary, to be determined later. By the previous lemma, the diagonal matrix $DA_1^{-r_7}D^{-1}$ is 2–Lipschitz on the on the $\epsilon(n)$–ball around $[e_n] = D[\hat{u}_n]$. By Corollary 6.4 $\|D^{\pm 1}\|^2 \leq |\alpha_1/\alpha_2|_v^{r_2}$ which implies that $D^{\pm 1}$ are $|\alpha_1/\alpha_2|_v^{2r_2}$ Lipschitz (on the entire projective space, see Section 2 (iv)). It follows that $A_1^{-r_7}$ is $2|\alpha_1/\alpha_2|_v^{4r_2}$ Lipschitz on the $\epsilon \cdot |\alpha_1/\alpha_2|_v^{-2r_2}$ ball around $[\hat{u}_n]$, or in other words, that $A_1^{-r_7}$ is $|\alpha_1/\alpha_2|_v^{d\log_\tau 2+4r_2}$ Lipschitz on the $|\alpha_1/\alpha_2|_v^{d\log_\tau \epsilon-2r_2}$ ball around $[\hat{u}_n]$.

Now since $\|B_1^{\pm 1}\|_v \leq |\alpha_1/\alpha_2|^{m_2r_0}$, the matrices $B_1^{\pm 1}$ are $|\alpha_1/\alpha_2|^{2m_2r_0}$ Lipschitz on the projective space, and hence the matrices $B_1^{\pm 1}A_1^{-r_7}$ are $|\alpha_1/\alpha_2|_v^{d\log_\tau 2+4r_2+2m_2r_0}$ Lipschitz on the $|\alpha_1/\alpha_2|_v^{d\log_\tau \epsilon-2r_2}$ ball around $[\hat{u}_n]$.

Take

$$c^* = \max\{2r_2 - d\log_\tau \epsilon,\ d\log_\tau 2 + 4r_2 + 2m_2r_0 + 2r_7\},$$

then the $|\alpha_1/\alpha_2|_v^{-c^*}$–ball Ω around $[\hat{u}_n]$ is mapped under $B_1A_1^{-r_7}$ (resp. under $B_1^{-1}A_1^{-r_7}$) into the $|\alpha_1/\alpha_2|_v^{-2r_7}$–ball around $B_1[\hat{u}_n]$ (resp. around $B_1^{-1}[\hat{u}_n]$). By Lemma 6.8

$$d(B_1^{\pm 1}[\hat{u}_n], [\hat{u}_1^{\perp}]) \geq |\alpha_1/\alpha_2|_v^{-r_5}.$$

Note that without loss of generality we can set $[\hat{u}_n]$ to be equal to $[\overrightarrow{u_n}]$. Also we may assume that $|\alpha_1/\alpha_2|_v^{-r_5} < 1/\sqrt{2}$ and that $r_7 \geq r_5$. Therefore, $B_1A_1^{-r_7}\Omega$ and $B_1^{-1}A_1^{-r_7}\Omega$ lie outside the $|\alpha_1/\alpha_2|_v^{-2r_5}$ neighborhood of $[\hat{u}_1^{\perp}]$. It follows that both sets $DB_1^{\pm 1}A_1^{-r_7}\Omega$ lie outside the $|\alpha_1/\alpha_2|_v^{-r_5-2r_2}$ neighborhood of $D[\hat{u}_1^{\perp}] = [\mathrm{span}\{e_2, \ldots, e_n\}]$. By Proposition 2.2 (1) applied to the diagonal matrix $DA_1^{r_7}D^{-1}$, it is $|\alpha_1/\alpha_2|_v^{-r_7+2(r_5+2r_2)}$–Lipschitz outside the $|\alpha_1/\alpha_2|_v^{-r_5-2r_2}$ neighborhood of $[\mathrm{span}\{e_2, \ldots, e_n\}]$, and hence $A_1^{r_7}D^{-1}$ is $|\alpha_1/\alpha_2|_v^{-r_7+2(r_5+3r_2)}$-Lipschitz there. Thus $A_1^{r_7}B_1^{\pm 1}A_1^{-r_7} = (A_1^{r_7}D^{-1})D(B_1^{\pm 1}A_1^{-r_7})$ are both

$$|\alpha_1/\alpha_2|_v^{-r_7+c^{**}} - \text{Lipschitz}$$

on Ω, where we have set

$$c^{**} = 2(r_5 + 3r_2) + (d\log_\tau 2 + 4r_2 + 2m_2r_0) + 2r_2.$$

It follows from parts (2) and (3) of Proposition 2.2 that the elements $A_1^{r_7}B_1^{\pm}A_1^{-r_7}$ are both $|\alpha_1/\alpha_2|_v^{\frac{1}{2}[-r_7+c^{**}]}$ contracting. Thus taking

$$r_7 \geq 2r_6 + c^{**}$$

we guarantee that $A_1^{r_7}B_1A_1^{-r_7}$ is $|\alpha_1/\alpha_2|_v^{-r_6}$ very contracting.

Now suppose further that

$$r_7 \geq 2\max\{2r_6, c^*\} + c^{**},$$

then our elements $A_1^{r_7}B_1^{\pm 1}A_1^{-r_7}$ are $|\alpha_1/\alpha_2|_v^{-\max\{2r_6,c^*\}}$–very contracting. Moreover Ω is a $|\alpha_1/\alpha_2|_v^{-c^*}$–ball, hence contains a point p^+ (resp. a point p^-) that is at least $|\alpha_1/\alpha_2|_v^{-c^*}$-away from the repelling hyperplane of $A_1^{r_7}B_1A_1^{-r_7}$ (resp. of $A_1^{r_7}B_1^{-1}A_1^{-r_7}$). It follows that

$A_1^{r_7} B_1^{\pm} A_1^{-r_7}$ maps the points $p^{\pm}$ respectively into the $|\alpha_1/\alpha_2|_v^{-2r_6}$–ball around the corresponding attracting points $t^{\pm}$ of $A_1^{r_7} B_1^{\pm 1} A_1^{-r_7}$, i.e.

$$d(A_1^{r_7} B_1^{\pm 1} A_1^{-r_7}(p^{\pm}), t^{\pm}) \leq |\alpha_1/\alpha_2|_v^{-2r_6}.$$

Additionally the element $A_1^{r_7}$ is $|\alpha_1/\alpha_2|_v^{-r_7/2+4r_2}$–contracting with attracting point $[\hat{u}_1]$ and repelling hyperplane $[\hat{u}_1^{\perp}]$, and since the point $B_1[\hat{u}_n]$ lies outside the $|\alpha_1/\alpha_2|_v^{-r_5}$ neighborhood of $[\hat{u}_1^{\perp}]$, assuming further that $r_7 \geq 2r_5 + 8r_2$, we get that this point is mapped under $A_1^{r_7}$ to the $|\alpha_1/\alpha_2|_v^{-r_7/2+4r_2}$–ball around $[\hat{u}_1]$. We conclude that $[\hat{u}_n] \in \Omega$ is mapped under $A_1^{r_7} B_1 A_1^{-r_7}$ into the $|\alpha_1/\alpha_2|_v^{-r_7/2+4r_2}$–ball around $[\hat{u}_1]$.

Finally since $A_1^{r_7} B_1 A_1^{-r_7}$ is $|\alpha_1/\alpha_2|_v^{-r_7+c^{**}}$ Lipschitz on Ω, we get that

$$\begin{aligned} d(t^+, [\hat{u}_1]) &\leq d(t^+, A_1^{r_7} B_1 A_1^{-r_7} p^+) + \operatorname{diam}(A_1^{r_7} B_1 A_1^{-r_7}\Omega) + d(A_1^{r_7} B_1 A_1^{-r_7}[\hat{u}_n], [\hat{u}_1]) \\ &\leq |\alpha_1/\alpha_2|_v^{-2r_6} + 2|\alpha_1/\alpha_2|_v^{-r_7+c^{**}-c^*} + |\alpha_1/\alpha_2|_v^{-r_7/2+4r_2}. \end{aligned}$$

By choosing r_7 sufficiently large, we can make the last quantity smaller the $|\alpha_1/\alpha_2|_v^{-r_6}$, that is

$$d(t^+, [\hat{u}_1]) \leq |\alpha_1/\alpha_2|_v^{-r_6}.$$

The same computation with t^-, p^- replacing t^+, p^+ gives $d(t^-, [\hat{u}_1]) \leq |\alpha_1/\alpha_2|_v^{-r_6}$. This finishes the proof of the proposition. □

6.3. **Step 3.** Our next step is to use $A_2 = A_1^{r_7} B_1 A_1^{-r_7}$ to build a very proximal element. Note that we haven't specified any condition on the constants r_6, r_7 from Lemma 6.10 yet. We will show that for some suitable $k \leq 2n-1$ the matrix $B_1^k A_2$ is very proximal.

Let $\overrightarrow{u}_1 \in \mathbb{K}_v \cdot \hat{u}_1$ be an eigenvector of A_1 corresponding to α_1 as in Lemma 6.6, i.e. the coordinates of $\overrightarrow{u}_1$ are $\tilde{S}$-integers, and $|\overrightarrow{u}_1|_w \leq |\alpha_1/\alpha_2|_v^{r_4}$, for any $w \in \tilde{S}$.

For any $k \leq 2n-1$ we have

$$\|B_1^k(\overrightarrow{u}_1)\|_w \leq \|B_1\|_w^{2n-1} \|\overrightarrow{u}_1\|_w \leq |\alpha_1/\alpha_2|_v^{(2n-1)m_2 r_0 + r_4},$$

for any $w \in \tilde{S}$, while for any $w \notin \tilde{S}$ the w-norm of this vector is ≤ 1. It follows that for any $1 \leq k_1 < \ldots < k_n \leq 2n-1$ we have

$$|\det(B_1^{k_1}(\overrightarrow{u}_1), \ldots, B_1^{k_n}(\overrightarrow{u}_1))|_w \leq |\alpha_1/\alpha_2|_v^{((2n-1)m_2 r_0 + r_4)n}$$

for any $w \in \tilde{S}$, and

$$|\det(B^{k_1}(\overrightarrow{u}_1), \ldots, B^{k_n}(\overrightarrow{u}_1)|_w \leq 1$$

for any $w \notin \tilde{S}$. Since B_1 is in $(2n-1)$–general position with respect to the $\{\overrightarrow{u}_i\}$'s, this determinant is not zero, and hence by the product formula

$$\prod_{\text{over all places}} |\det(B_1^{k_1}(\overrightarrow{u}_1), \ldots, B_1^{k_n}(\overrightarrow{u}_1)|_w = 1,$$

which implies that

$$\prod_{w \in \tilde{S}} |\det(B_1^{k_1}(\overrightarrow{u}_1), \ldots, B_1^{k_n}(\overrightarrow{u}_1))|_w \geq 1.$$

We conclude:

Corollary 6.11. *For any $w \in \tilde{S}$, and in particular for $w = v$*

$$|det(B_1^{k_1}(\overrightarrow{u}_1), \ldots, B_1^{k_n}(\overrightarrow{u}_1))|_w \geq |\alpha_1/\alpha_2|_v^{-((2n-1)m_2r_0+r_4)n|\tilde{S}|}.$$

We will need also the following:

Lemma 6.12. *Suppose that $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_n$ are any n vectors in $\tilde{\mathbb{K}}_v^n$ satisfying*

- $\|\overrightarrow{v}_i\|_v \leq t^{c'}$, $\forall i \leq n$, *and*
- $|det(\overrightarrow{v}_1, \ldots, \overrightarrow{v}_n)|_v \geq t^{-c''}$,

for some $c', c'' \in \mathbb{N}$ and $t > 0$.

Then for any hyperplane $H \subset \tilde{\mathbb{K}}_v^n$ there is $i \leq n$ such that $d([\overrightarrow{v}_i], [H]) \geq \frac{1}{\lambda_1\lambda_{n-1}} t^{-c''-(n-1)c'}$ in the $\tilde{\mathbb{K}}_v$ projective space, where λ_k is the volume of the k–dimensional unit ball (in particular $\lambda_k = 1$ in the non-Archimedean case).

Proof. Let f be a linear form such that $\|f\| = 1$ and $H = \ker(f)$, then the volume of $\{x \in \tilde{\mathbb{K}}_v^n : |f(x)|_v \leq |a|_v, \|x\|_v \leq |b|_v\}$ is bounded above by $\lambda_1\lambda_{n-1}|a|_v|b|_v^{n-1}$ – the volume of a "cylinder" with base radius $|b|_v$ and "height" $2|a|_v$, for any $a, b \in \tilde{\mathbb{K}}_v$. Since $d([x], [H]) = \frac{|f(x)|_v}{\|x\|_v}$, we get the desired conclusion by comparing this volume to the determinant of the $\overrightarrow{v}_i$'s. □

Setting $c' = (2n-1)m_2r_0 + r_4$ and $c'' = ((2n-1)m_2r_0 + r_4)n|\tilde{S}|$ we get some constant[3] r_8, such that whenever $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_n$ are as in Lemma 6.12 with $t = |\alpha_1/\alpha_2|_v$ and $[H]$ is some projective hyperplane, there is one $[\overrightarrow{v}_i]$ at distance at least $|\alpha_1/\alpha_2|_v^{-r_8}$ from $[H]$, in particular:

Lemma 6.13. *For any $1 \leq k_1 < k_2 < \ldots < k_n \leq 2n-1$ and any hyperplane $H \subset \tilde{\mathbb{K}}_v^n$ there exists $i \leq n$ such that*

$$d([B_1^{\pm k_i}\hat{u}_1], [H]) \geq |\alpha_1/\alpha_2|_v^{-r_8}.$$

By the pigeonhole principle, we conclude:

Corollary 6.14. *For any two hyperplanes $H_1, H_2 \subset \tilde{\mathbb{K}}_v^n$ there is some $k \leq 2n-1$ such that we have simultaneously*

$$d([B_1^k\hat{u}_1], [H_1]) \geq |\alpha_1/\alpha_2|_v^{-r_8}, d([B_1^{-k}\hat{u}_1], [H_2]) \geq |\alpha_1/\alpha_2|_v^{-r_8}.$$

Now let $[H^+], [H^-]$ be the repelling hyperplanes for the $|\alpha_1/\alpha_2|_v^{-r_6}$–very contracting element $A_2 = A_1^{r_7}B_1A_1^{-r_7}$ and its inverse, and take the corresponding k in Corollary 6.14. Recall that the attracting points t^+, t^- of $A_2^{\pm 1}$ are both at distance at most $|\alpha_1/\alpha_2|_v^{-r_6}$ from $[\hat{u}_1]$. We thus obtain:

Proposition 6.15 (The very proximal element X)**.** *Assume that $r_8 > 2(2n-1)r_0$, then the element $X = B_1^kA_2$ is (ρ, δ)–very proximal with*

$$\rho = |\alpha_1/\alpha_2|_v^{-2m_2kr_0}(|\alpha_1/\alpha_2|_v^{-r_8} - |\alpha_1/\alpha_2|_v^{-r_6+4m_2kr_0}), \text{ and } \delta = |\alpha_1/\alpha_2|_v^{-r_6+4m_2kr_0}$$

[3]Note that we can fix r_8 before determining r_6, r_7.

and with repelling hyperplanes

$$[H_X^+] = [H^+],\ [H_X^-] = B_1^k[H^-]$$

and attracting points

$$[t_X^+] = B_1^k t^+,\ [t_X^-] = t^-.$$

Proof. Since $\|B_1^{\pm 1}\|_v \leq |\alpha_1/\alpha_2|_v^{m_2 r_0}$, B_1 is $|\alpha_1/\alpha_2|_v^{4m_2 r_0}$ bi-Lipschitz on the entire projective space. This implies that $X = B_1^k A_2$ is $|\alpha_1/\alpha_2|_v^{-r_6+4m_2kr_0}$ very contracting with the specified attracting points and repelling hyperplanes, and that

$$d(B_1^k(t^+), [H^+]) \geq d(B_1^k[\hat{u}_1], [H^+]) - d(B_1^k[\hat{u}_1], B_1^k(t^+)) \geq |\alpha_1/\alpha_2|_v^{-r_8} - |\alpha_1/\alpha_2|_v^{-r_6+4km_2r_0},$$

and

$$d(t^-, B_1^k[H^-]) \geq \|B_1^{\pm k}\|_v^{-4} d(B_1^{-k}(t^-), [H^-]) \geq |\alpha_1/\alpha_2|_v^{-4m_2kr_0}(|\alpha_1/\alpha_2|_v^{-r_8} - |\alpha_1/\alpha_2|_v^{-r_6+4km_2r_0})$$

$\square$

Taking $r_6 >> r_8$ sufficiently large (after choosing r_8 sufficiently large) we may assume that:

$$\rho = |\alpha_1/\alpha_2|_v^{-r_8-2m_2kr_0}(1 - |\alpha_1/\alpha_2|_v^{r_8-r_6+6m_2kr_0}) \geq \frac{1}{2}|\alpha_1/\alpha_2|_v^{-2r_8}.$$

Set

$$r_9 = 2r_8 + d\log_\tau 2,\ r_{10} = r_6 - 4m_2kr_0,$$

Then we get that $X = B_1^k A_1^{r_7} B_1 A_1^{-r_7}$ is $(|\alpha_1/\alpha_2|_v^{-r_9}, |\alpha_1/\alpha_2|_v^{-r_{10}})$–very proximal. The matrix X is our first ping–pong player.

6.4. **Step 4.** The last step of the proof consists in finding a second ping–pong partner Y by conjugating X by a suitable bounded word in the alphabet Σ. This is performed in quite the same way as in Step 3, so we only sketch the proof here.

Note first that X is a word in Σ of length at most $2(2n-1)m_2 + 2r_7r_3r$. Therefore, by requiring s from Proposition 6.1 to be at least this constant, we can assume that X is semisimple. Let $[\hat{v}_1]$ (resp. $[\hat{v}_n]$) be the eigendirection of the maximal (resp. minimal) eigenvalue of X.

Let B_2 be a word in Σ which is in $(2n-1)^2$–general position with respect to the eigenvectors of X (chosen as in Lemma 6.6). Again by Lemma 2.5 and the discussion following Definition 6.7, we may find B_2 as a word of length $\leq m_2((2n-1)^2)$. We can then apply the same pigeonhole argument as in Corollary 6.14 and obtain an index $k' \leq (2n-1)^2$ such that $B_2^{k'}[\hat{v}_1]$ and $B_2^{k'}[\hat{v}_n]$ are both far away from the repelling hyperplanes $[H_X^+]$ of X and $[H_X^-]$ of X^{-1} (i.e. $|\alpha_1/\alpha_2|_v^{-r_{11}}$–apart for some other constant r_{11}).

Setting $Y = B_2^{k'} X B_2^{-k'}$, we see that some bounded power $Y^{r_{12}}$ of Y is very proximal with attracting and repelling points $B_2^{k'}[\hat{v}_1]$ and $B_2^{k'}[\hat{v}_n]$ and repelling hyperplanes $B_2^{k'}[H_X^+]$ and $B_2^{k'}[H_X^-]$. Since those points are away from the repelling hyperplanes of X (or any power of X), we conclude that, after taking a larger power $r_{13} \geq r_{12}$ if necessary, Y^t and X^t play ping–pong, and hence independent, for any $t \geq r_{13}$.

Remark 6.16. As mentioned at the beginning of Section 6 we assumed throughout that the representation space V was irreducible. Lemma 6.6 as well as the rest of the argument above, relies on the assumption that the entries of the elements of Σ viewed as matrices acting on V are S–arithmetic. However, in general our wedge representation V might be reducible, and we have to replace it with some irreducible subquotient where this assumption may not hold. In order to cope with this problem we argue as follows. We change the representation space from V to an irreducible subquotient V_0/W where V_0, W are invariant subspaces of V. One can carry out the proof of Lemma 6.6 in V and first treat the eigenvectors in W, then those in $V_0 \setminus W$ and finally take the projections of those to V_0/W. This would yield an analogous statement for V_0/W which is sufficient for the whole argument. Note also that in characteristic zero, as $\mathbb{G}$ is semisimple, V is completely reducible so our irreducible representation is a sub-representation of the wedge power, rather than a subquotient, and hence, in this case, we may take V_0 instead of V without further changes.

This completes the proof of Theorem 1.2. $\square$

7. A Zariski dense free subgroup in characteristic zero

We will now give two stronger versions of Theorem 1.1 which are useful for applications. Since all the applications we have in mind are for fields of characteristic zero, we allow ourselves to make this restriction, although we believe that it is unnecessary.

Theorem 7.1. *Let K be a field of characteristic zero, $\mathbb{H}$ a Zariski connected semisimple K–group and $\Gamma \leq H = \mathbb{H}(K)$ a finitely generated Zariski dense subgroup. Then there is a constant $m_1 = m_1(\Gamma)$ such that for any symmetric generating set $\Sigma \ni 1$ of Γ, Σ^{m_1} contains two independent elements which generate a Zariski dense subgroup of $\mathbb{H}$.*

Remark 7.2. The proof of Theorem 7.1 also shows that Theorem 1.2 remains true, in characteristic zero, with the stronger conclusion that the independent elements generate a Zariski dense subgroup of $\mathbb{G}$.

In order to obtain Theorem 7.1 one needs to slightly modify the argument of Section 6 in a few places. We will now indicate these modifications. For the sake of simplicity, let us assume that $\mathbb{H}$ is simple.

It is well known that $\mathbb{H}$ admits two conjugate elements which generate a Zariski dense subgroup. Indeed, one can take a regular unipotent in $\mathbb{H}$ and a conjugate lying in an opposite parabolic (these unipotent elements can be taken in $\mathbb{H}(\tilde{K})$, where $\tilde{K}$ is a finite extension of K over which $\mathbb{H}$ is isotropic). Let $\mathcal{A}$ be the subalgebra spanned by $\mathrm{Ad}(\mathbb{H})$ in $\mathrm{End}(\mathfrak{h})$ where $\mathfrak{h}$ denotes the Lie algebra of $\mathbb{H}$, set

$$F = \{(g,h) \in \mathbb{H} \times \mathbb{H} : \mathrm{Ad}(g) \text{ and } \mathrm{Ad}(h) \text{ do not generate the algebra } \mathcal{A}\},$$

and $E = \{(g,h) \in \mathbb{H} \times \mathbb{H} : (g, hgh^{-1}) \in F\}$. It follows the algebraic variety E is proper in $\mathbb{H} \times \mathbb{H}$. Let $E_1 = \{g \in \mathbb{H} : (g,h) \in E \ \forall h \in \mathbb{H}\}$, and for $g \in \mathbb{H}$ let $E_2(g) = \{h \in \mathbb{H} : (g,h) \in E\}$. Then E_1 is a proper subvariety of $\mathbb{H}$ and one easily checks that $\chi(E_2(g))$ is bounded independently of g.

Note that in Theorem 7.1, we did not assume that the field is a global field. In order to take care of this issue we will use the specialization map introduced in Lemma 3.1. Without loss of generality, we may pass to a subgroup of finite index in Γ. We will need the following lemma.

Lemma 7.3. *Let $f : \Gamma \mapsto \mathbb{G}$ be the specialization map from Lemma 3.1. Then the subgroup $\Delta = \{(\gamma, f(\gamma)) \in \mathbb{H} \times \mathbb{G} \ / \gamma \in \Gamma\}$ is not contained in any algebraic subset of the form $(V \times \mathbb{G}) \cup (\mathbb{H} \times W)$, where V and W are proper closed subvarieties of $\mathbb{H}$ and $\mathbb{G}$ respectively.*

Proof. This is obvious since Γ is Zariski dense in $\mathbb{H}$ and $f(\Gamma)$ is Zariski dense in $\mathbb{G}$. □

When pursueing the argument of Section 6, we need to specify conditions on the elements of the generating set. These conditions are set on elements of $f(\Gamma) \in \mathbb{G}$. We will now introduce new algebraic conditions directly on the elements of $\Gamma \in \mathbb{H}$. Combining Lemma 2.5 with Lemma 7.3 we see that given a set of non-trivial algebraic conditions in $\mathbb{H}$ and another such set in $\mathbb{G}$, there is an integer N such that, for every generating set Σ of Γ, there is a point $\gamma \in \Sigma^N$ such that $\gamma \in \mathbb{H}$ and $f(\gamma) \in \mathbb{G}$ do not satisfy those conditions. This will be used repeatedly below. When no confusion may arise we will often say for instance that some element $A \in \Gamma$ acts proximally when we really mean that $f(A) \in \mathbb{G}$ acts proximally on the representation variety used in Section 6.

The first modification needed in the argument of Section 6 is in Proposition 6.15 when we construct the very proximal element X. Instead of using the same element B_1 which was used in the construction of the very contracting element, we should use an element B_1' which satisfies

- $f(B_1')$ is in $(2n-1)$–general position with respect to $\{\overrightarrow{u_i}\}_{i-1}^n$ (like $f(B_1)$), and
- $(B_1')^k \notin E_1 A_2^{-1}, \ \forall k \leq 2n-1$.

Note that the choice of B_1' depends on A_2, however, since $\chi(E_1 A_2^{-1})$ is independent of A_2 we can find B_1' in a fixed power of our generating set Σ. Retrospectively we should also take the constant r_6 big enough so that the very contracting element A_2 constructed in Proposition 6.15 will have sufficiently small attracting and repelling neighborhoods (i.e. that $|\alpha_1/\alpha_2|^{-r_6}$ will be small enough) so that the element $X = (B_1')^k A_2$ (where k is some integer $\leq 2n-1$) becomes very proximal. Additionally, we have to take the constant s in Proposition 6.1 sufficiently large to guarantee that $f(X)$ is still semisimple.

The second change one has to do is in Step (4) when choosing the appropriate conjugation of X. By the choice of B_1' we know that $X \notin E_1$. We take B_2' which satsfies:

- $f(B_2')$ is in $(2n-1)^2$–general position with respect to the eigenvectors of $f(X)$ (again chosen as in Lemma 6.6), and
- $(B_2')^k \notin E_2(X), \ \forall k \leq (2n-1)^2$.

Then, as in the previous section, if $Y = (B_2')^{k'} X (B_2')^{-k'}$ for some appropriate $k' \leq (2n-1)^2$ then X^t and Y^t are independent for any $t \geq r_{13}'$ for some constant r_{13}'.

Finally, since F is an algebraic subvariety of $\mathbb{H} \times \mathbb{H}$ and $\chi(Fx)$ is bounded independently of $x \in \mathbb{H} \times \mathbb{H}$, we may apply Lemma 2.5 to the set $\{(X,Y)\}$ and the variety $F \cdot (X^{-r_{13}'}, Y^{-r_{13}'})$.

Since $\{(X,Y)\}$ is not in F, it follows that $\{(X,Y)\}$ generates a group not contained in $F\cdot(X^{-r'_{13}},Y^{-r'_{13}})$. Hence Lemma 2.5 yields some t with $r'_{13}\le t\le r'_{13}+N(\chi(F))$ such that $(X^t,Y^t)\notin F$. Now since X has infinite order, the Zariski connected group $(\overline{\langle X^t,Y^t\rangle}^Z)^\circ$ has positive dimension, and since it is normalized by X^t and Y^t, while $\mathrm{span}\{\mathrm{Ad}(X^t),\mathrm{Ad}(Y^t)\}=\mathcal{A}$ it follows that $(\overline{\langle X^t,Y^t\rangle}^Z)^\circ$ is normal in $\mathbb{H}$. Since $\mathbb{H}$ is assumed to be simple we derive that $\langle X^t,Y^t\rangle$ is Zariski dense. □

Theorem 7.4. *Let $\mathbb{G}$ be a semisimple algebraic group defined over a field K of characteristic zero, Γ a finitely generated Zariski dense subgroup of $\mathbb{G}(K)$, and $V\subset\mathbb{G}\times\mathbb{G}$ a proper algebraic subvariety. Then there is a constant $m=m(\Gamma,V)$ such that for any generating set $\Sigma\ni 1$ of Γ, Σ^m contains a pair of independent elements x,y with $(x,y)\notin V$.*

Proof. The subset Σ^{m_1} contains a pair $\{A,B\}$ of independent elements for some constant $m_1=m_1(\Gamma)$ given by Theorem 7.1. This pair generates a Zariski dense subgroup of Γ. Hence $(1,A)$, $(1,B)$, $(A,1)$ and $(B,1)$ together generate a Zariski dense subgroup of $\mathbb{G}\times\mathbb{G}$. The set $V'=V\cup\{(x,y)|[x,y]=1\}$ is a proper closed algebraic subset of $\mathbb{G}\times\mathbb{G}$. By Lemma 2.5, there exists another constant $m_2=m_2(V)$ such that some word of length at most m_2 in those four generators lies outside V'. This word has the form $(W_1(A,B),W_2(A,B))$ where W_1,W_2 are bounded words in A and B that do not commute as words in the free group. It follows that they generate a free subgroup, hence form a pair of independent elements in $\Sigma^{m_1m_2}$. □

8. Some applications

In this section we draw some consequences of our main result.

8.1. **Uniform non-amenability and a uniform Cheeger constant.** Recall that a group is called amenable if the regular representation admits almost invariant vectors. It follows that if a non-amenable group Γ is generated by a finite set Σ then there is a positive constant $\epsilon(\Sigma)$ such that for any $f\in L^2(\Gamma)$ there is some $\sigma\in\Sigma$ for which $\|\rho(\sigma)(f)-f\|\ge\epsilon(\Sigma)\|f\|$, where ρ denotes the left regular representation, i.e. $\rho(\gamma)(f)(x):=f(\gamma^{-1}x)$. Such an $\epsilon(\Sigma)$ is called a *Kazhdan constant for* (Σ,ρ). A finitely generated group Γ is said to be *uniformly non-amenable* if there is a positive Kazhdan constant $\epsilon=\epsilon(\Gamma)>0$ for the regular representation which is independent of the generating set Σ, i.e. if there is $\epsilon>0$ such that for any generating set Σ of Γ and any $f\in L^2(\Gamma)$ there is $\sigma\in\Sigma$ for which $\|\rho(\sigma)(f)-f\|\ge\epsilon\|f\|$. It was observed by Y. Shalom [22] that Theorem 1.1 implies:

Theorem 8.1. *A finitely generated non-amenable linear group is uniformly non-amenable.*

Proof. The proof is an elaboration of the original proof by Von-Neumann that a group which contains a non-abelian free subgroup is non-amenable.

Let Γ be a non-amenable linear group, and let $m=m(\Gamma)$ be the constant from Theorem 1.1. Let Σ be a generating set of Γ and let $x,y\in(\Sigma\cup\Sigma^{-1}\cup 1)^m$ be two independent elements. Denote by $F_2 = \langle x, y\rangle$ the corresponding free subgroup. Choose a complete set $\{c_i\}$ of right coset representatives for F_2 in Γ, and write

$$L^2(\Gamma) = \bigoplus L^2(F_2 c_i).$$

Let $f \in L^2(\Gamma)$, and let f_i denote the restriction of f to $F_2 c_i$. Let τ_0 be the Kazhdan constant for $(\rho_{F_2}, \{x, y\})$ then for any i either x or y moves f_i by at least $\tau_0\|f_i\|$. Let

$$f_x = \sum_{\|\rho(x)f_i - f_i\| \geq \tau_0\|f_i\|} f_i, \text{ and } f_y = \sum_{\|\rho(y)f_i - f_i\| \geq \tau_0\|f_i\|} f_i$$

Then either $\|f_x\| \geq \|f\|/\sqrt{2}$ or $\|f_y\| \geq \|f\|/\sqrt{2}$. Without loss of generality let us assume that $\|f_x\| \geq \|f\|/\sqrt{2}$. It follows that

$$\|\rho(x)f - f\| \geq \|\rho(x)f_x - f_x\| \geq \tau_0\|f_x\| \geq \frac{\tau_0}{\sqrt{2}}\|f\|.$$

Now write $x = \sigma_1^{\epsilon_1} \cdot \ldots \cdot \sigma_m^{\epsilon_m}$ where $\sigma_i \in \Sigma \cup \{1\}$ and $\epsilon_i = \pm 1$. Then by the triangle inequality, if we let $\sigma_0 = 1$ and $\epsilon_0 = 1$

$$\|\rho(x)f - f\| \leq \sum_{i=1}^{m} \|\rho(\sigma_0^{\epsilon_0} \cdot \ldots \cdot \sigma_i^{\epsilon_i})f - \rho(\sigma_0^{\epsilon_0} \cdot \ldots \cdot \sigma_{i-1}^{\epsilon_{i-1}})f\| = \sum_{i=1}^{m} \|\rho(\sigma_i)f - f\|,$$

and hence, for some i we have $\|\rho(\sigma_i)f - f\| \geq \frac{\tau_0}{m\sqrt{2}}\|f\|$. □

Note that usually such groups do not admit a uniform Kazhdan constant for arbitrary unitary representation, even if they have property (T) (see [13]).

By considering f to be a characteristic function of a finite subset of Γ and applying Theorem 8.1 we obtain the following useful result. We denote by $|A|$ the number of elements in A and by $\triangle$ the operator of symmetric difference between sets.

Theorem 8.2. *Let Γ be a finitely generated non-virtually solvable linear group. Then there is a positive constant $b = b(\Gamma)$ such that for any generating set Σ (not necessarily finite or symmetric) of Γ, and any finite subset $A \subset \Gamma$ there is $\sigma \in \Sigma$ such that:*

$$\frac{|\sigma \cdot A \triangle A|}{|A|} \geq b.$$

Consider a graph X and a finite subset $A \subset X$. The *boundary* of A is the set ∂A of all vertices in A which have at least one neighbor outside A. The *Cheeger constant* $\mathcal{C}(X)$ is defined by

$$\mathcal{C}(X) = \inf \frac{|\partial A|}{|A|},$$

where A runs over all finite subsets of $\mathrm{vert}(X)$ when X is infinite, and over all subsets of size at most $|\mathrm{vert}(X)|/2$ when X is finite. For a group Γ and a finite generating set Σ we denote by $\mathcal{C}(\Gamma, \Sigma)$ the Cheeger constant of the Cayley graph of Γ with respect to Σ, and by $\mathcal{C}(\Gamma)$ the *uniform Cheeger constant* of Γ:

$$\mathcal{C}(\Gamma) := \inf\{\mathcal{C}(\Gamma, \Sigma) : \Sigma \text{ is a finite generating set }\}.$$

In some places (c.f. [1],[22]) a group is called uniformly non-amenable if it has a positive uniform Cheeger constant. Clearly our definition of uniform non-amenability implies this one[4]:

Corollary 8.3. *A finitely generated non-virtually solvable linear group has a positive uniform Cheeger constant.*

When $\mathcal{C}(X) > \epsilon$ the graph X is said to be *ϵ–expander*. Hence Corollary 8.3 can be reformulated as follows:

Corollary 8.4. *The family of all Cayley graphs corresponding to finite generating sets of a given non-virtually solvable linear group Γ form a family of δ–expanders for some constant $\delta = \delta(\Gamma) > 0$.*

8.2. **Growth.** In [11], Eskin Mozes and Oh proved that any finitely generated non-virtually solvable linear group has a uniform exponential growth by showing that some bounded words in the generators generate a free semigroup. As a consequence of Theorem 1.1 (more precisely of Theorem 8.2) we obtain:

Theorem 8.5. *Let Γ be a finitely generated non-virtually solvable linear group. Then there is a constant $\lambda = \lambda(\Gamma) > 1$ such that if Σ is any finite generating set of Γ, then $|\Sigma^n| \geq |\Sigma|\lambda^{n-1}$, $\forall n \in \mathbb{N}$.*

Since the proof is straightforward, we will omit it. One can actually take $\lambda = 1 + \frac{b}{2}$ where b is the constant from Theorem 8.2.

Remark 8.6. Theorem 8.5 improves Eskin-Mozes-Oh theorem in several aspects:

- Unlike the situation in [11], the generating set Σ in Theorem 8.5 is not assumed to be symmetric, so 8.5 gives the uniform exponential growth for semigroups rather than just for groups.
- We didn't make the assumption from [11] that the characteristic of the field is 0.
- The estimate on the growth that we obtain is sharper; In particular if the generating set is bigger the growth is faster. This sharper estimate is important for applications,

As another consequence we obtain that also the spheres have uniform exponential growth, and moreover the size of each sphere is at least $\frac{b}{2}$ times the size of the corresponding ball. If $\Sigma \ni 1$ is a generating set for Γ the sphere $S(n, \Sigma)$ corresponding to Σ is the set of all elements in Γ of distance exactly n from 1 in the Cayley graph, i.e. $S(n, \Sigma) = \Sigma^n \setminus \Sigma^{n-1}$.

Corollary 8.7. *Let Γ, Σ and $\lambda = 1 + \frac{b}{2}$ be as in Theorem 8.5. Then*

$$|S(n, \Sigma)| \geq \frac{b}{2}|\Sigma^{n-1}| \geq \frac{b}{2}|\Sigma|(1 + \frac{b}{2})^{n-2}.$$

One can derive many other variants of these results from Theorem 8.2. Here is another example:

[4] It is still not known whether these two definitions are equivalent for a general finitely generated group.

Exercise 8.1. *Let Γ, Σ and λ be as above. There is a sequence $\{\sigma_i\}_{i\in\mathbb{N}}$ of elements of Σ such that for any $n \in \mathbb{N}$*

$$|\{\prod_{1\le i_1<\ldots<i_k\le n} \sigma_{i_1}\cdot\ldots\cdot\sigma_{i_k}\}| \ge \lambda^{n-1}.$$

8.3. **Dense free subgroups, amenable actions and growth of leaves.** Theorem 1.1 implies the following result from [6] which answered a question of Carrière and Ghys [10]:

Theorem 8.8. *Let G be a connected semisimple Lie group and $\Gamma \le G$ a dense subgroup. Then Γ contains a dense free subgroup of rank 2.*

Proof. Let us assume for simplicity that G is simple. The adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is irreducible and by Burnside's theorem its image spans $\mathrm{End}(\mathfrak{g})$. It is well known that $\mathrm{End}(\mathfrak{g})$ is generated by two elements and that these elements can be chosen in $\mathrm{Ad}(G)$. Since $\mathrm{End}(\mathfrak{g})$ is finite dimensional it follows that the set

$$V = \{(g_1, g_2) \in G\times G : \mathrm{Ad}(g_1) \text{ and } \mathrm{Ad}(g_2) \text{ generate } \mathrm{End}(\mathfrak{g})\}$$

is Zariski open $G \times G$. By Theorem 7.4 and the remark following it there is a constant $m = m(\Gamma, V)$ such that if $\Sigma \ni 1$ is any generating set for Γ then Σ^m contains independent elements x, y with $(x, y) \notin V$. Let $U \subset G$ be a Zassenhaus neighborhood (c.f. [19] 8.16), and let Ω be an identity neighborhood with $\Omega^m \subset U$. Take $\Sigma = \Gamma\cap\Omega$. Since G is connected and Γ is dense, Σ generates Γ, and therefore Σ^m contains x, y independent with $(x, y) \notin V$ according to Theorem 7.4. Now the connected component of the identity in $\overline{\langle x, y\rangle}$ is normalized by x, y and as $(x, y) \notin V$ it is normal in G, and by simplicity of G it is either 1 or G. In other words $\langle x, y\rangle$ is either discrete or dense. However $\langle x, y\rangle$ is free and hence non-nilpotent and since $x, y \in U$ it follows from Zassenhaus' theorem that $\langle x, y\rangle$ is not discrete. □

Recall that one of the main motivation to prove Theorem 8.8 was the Connes–Sullivan conjecture which was first proved by Zimmer:

Corollary 8.9 (Zimmer [25])**.** *Let Γ be a countable subgroup of a real Lie group G. Then the action of Γ on G by left translations is amenable iff the connected component of the identity of the closure of Γ is solvable.*

This corollary follows from Theorem 8.8 by the observation of Carrière and Ghys that a non-discrete free subgroup of G cannot act amenably (see [6],[7] for more details and stronger results).

Another motivation was the result about the polynomial–exponential dichotomy for the growth of leaves in Riemannian foliations which was conjectured by Carrière:

Theorem 8.10 ([7])**.** *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold. Then either the growth of any leaf in $\mathcal{F}$ is polynomial, or the growth of a generic leaf is exponential.*

Theorem 8.10 can be considered as a foliated version of the well known conjecture according to which the growth of the universal cover of any compact Riemannian manifold is either polynomial or exponential. The proof of 8.10 relies on the following strengthening of Theorem 8.8 as well as some special argument for solvable groups (see [7] for more details):

Theorem 8.11. *([6]) Let G be a connected semisimple Lie group, $\Gamma \leq G$ a dense subgroup and $\Omega_1, \ldots, \Omega_n$ some n open sets in G. Then one can pick $x_i \in \Gamma \cap \Omega_i$, $i = 1, \ldots, n$ which are independent, i.e. generate a free group of rank n.*

In [6] Theorem 8.11 was the main result and Theorem 8.8 followed as a consequence. Let us show that conversely it is possible to derive Theorem 8.11 from Theorem 8.8. This way, Theorem 8.11 will appear as a mere consequence of the main theorem of the present paper, namely Theorem 1.1.

Proof that Theorem 8.8 implies Theorem 8.11. Let $F_2 \leq \Gamma$ be a free subgroup of rank 2 which is dense in G, and let F_n be a subgroup of index $n-1$ in F_2. Then F_n is a free group of rank n which is still dense in G. We will pick the x_i in F_n inductively as follows. Suppose we picked already $x_1, \ldots, x_{i-1}$. Since F_n is dense and Ω_i is open, F_n is generated by $F_n \cap \Omega_i$. It follows that we can pick $x_i \in F_n \cap \Omega_i$ such that the abelianization of $\langle x_1, \ldots, x_i \rangle$ has rank i; Indeed, look at the tensor of the abelanization of F_n with $\mathbb{Q}$ and pick x_i in the generating set $F_n \cap \Omega_i$ which is not in the $(i-1)$–dimensional $\mathbb{Q}$–subspace spanned by the images of $x_1, \ldots, x_{i-1}$. It follows that $\langle x_1, \ldots, x_i \rangle$ is a free group whose minimal number of generators is exactly i. Since a free group is Hopfian it follows that $x_1, \ldots, x_i$ are independent. $\square$

We refer the reader to [7] for an extension of Theorems 8.8 and 8.11 to a more general setup.

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