DENSE EMBEDDINGS OF SURFACE GROUPS

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Abstract. We discuss dense embeddings of surface groups and limit groups in topological groups. For instance, we give a characterization of the Lie groups which admit a dense faithfully embedded surface group, we prove that any connected semisimple Lie group contains a dense copy of any limit group, and we show that any compact topological group which contains a nonabelian free group contains also a surface group.

1. Introduction

Given a locally compact topological group G and an abstract group Γ, it is natural to ask whether Γ can be embedded densely in G. More generally, for a given G one would like to understand its dense subgroups, and for a given Γ one would like to know its possible completions G which are topological groups. These questions are more accessible in the case where G is a finite dimensional analytic Lie group over a local field. While discrete subgroups of Lie groups have been thoroughly studied for the last fifty years, very little is known about non-discrete, and in particular, dense[1] subgroups of Lie groups. A dense embedding of Γ in G may yield interesting data on Γ, G, and the spaces on which they act (c.f. [11, 14, 7, 10, 3, 4, 1]).

By a *surface group*, we mean the fundamental group of a closed oriented surface of genus at least 2. By a *free group*, we mean a free group on at least two generators. We obtain various results, all of which are proved by continuously deforming a given representation to a faithful one.

Our first result states that free groups and surface groups have the same compactifications within the category of topological groups.

Theorem 1.1. Let G be a compact group. Then the following two assertions are equivalent:

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[1]Note that when G is a connected simple Lie group, a generic subgroup with sufficiently many generators is either discrete or dense

- G contains a dense free subgroup of finite rank.
- G contains a dense surface group.

As a corollary we obtain that a compact group contains a surface group if and only if it contains a free group. It is sometime fairly easy to verify that a given compact group contains a free subgroup by means of probabilistic methods. However, we do not know a simple characterization of the compact groups containing a dense free subgroup of finite rank. It was shown in [4] that the profinite completion $\widehat{\Gamma}$ of a finitely generated linear group Γ contains a dense free subgroup of finite rank if and only if it is not virtually solvable (i.e. contains no solvable subgroup of finite index). However, there are examples of topologically finitely generated profinite groups that satisfy a law (hence admit no free subgroups) although they are not virtually solvable (cf. [8]). On the other hand, it is easy to verify that any connected second countable nonabelian compact group contains a dense free subgroup of rank 2 (see Proposition 8.2). Hence any such group contains also a dense surface group.

For locally compact groups, we obtain the following result:

Theorem 1.2. Let G be a locally compact group. Suppose that G contains a non-discrete free subgroup F of finite rank $r > 1$. Then G has a subgroup Γ containing F such that Γ is isomorphic to a surface group (of genus $2r$). In particular, if G has a dense free subgroup of finite rank, then it has a dense surface group.

Remark 1.3. As a corollary of Theorem 1.2 we obtain an elementary proof of a result from [6], that surface groups are primitive, i.e. admit faithful primitive permutation representations. Indeed, let Γ be a surface group and imbed Γ densely in $\mathrm{PSL}_2(\mathbb{Q}_p)$. Then $\Delta = \Gamma \cap \mathrm{PSL}_2(\mathbb{Z}_p)$ is a maximal subgroup of Γ which contains no nontrivial normal subgroup of Γ, and the action of Γ on Γ/Δ is primitive and faithful.

When G is a (non discrete) real Lie group with a countable number of connected components, it was shown in [3],[4] that G contains a finitely generated dense free group if and only if the connected component of the identity G° is not solvable and G/G° is finitely generated. We thus obtain:

Corollary 1.4. Let G be a non-discrete real Lie group. Then the following are equivalent:

- G contains a finitely generated dense free subgroup.
- G contains a dense surface group.
- G° is not solvable and G/G° is finitely generated.

One key property of surface groups which motivated this research is the fact that they are fully residually free. Recall that a finitely generated group Γ is *fully residually free* if for every finite set $K \subset \Gamma \setminus \{1\}$ there is a homomorphism $\phi : \Gamma \to F$ onto a free group F with $K \cap \mathrm{Ker}(\phi) = \emptyset$. In other words, Γ is fully residually free if any finite set can be separated through a surjective map onto a free group. It is a theorem of Sela [13] that a group is fully residually free if and only if it is a limit group. The fact that surface groups are fully residually free is due to Baumslag [2]. A group Γ is called *d-fully residually free* if any finite set can be separated trough a surjection on F_d. Note that if Γ is d-fully residually free then it is also k-fully residually free for any $2 \leq k < d$ (see Lemma 2.1 below).

For general fully residually free groups we prove the following:

Theorem 1.5. *Let G be a connected non-solvable Lie group. Then there is a number $d = d(G) < \dim(G)$ such that: if Γ is a d-fully residually free group, then there is a dense embedding $\Gamma \hookrightarrow G$.*

When G is topologically perfect, i.e. does not surject onto the circle, then we can take d to be the number of generators for the Lie algebra of G. Since any semisimple Lie algebra is generated by 2 elements, we obtain:

Theorem 1.6. *Any connected semisimple Lie group contains a dense copy of any nonabelian fully residually free group.*

Remark 1.7. A group Γ is *residually free* if for every $\gamma \in \Gamma \setminus \{1\}$ there is $\phi : \Gamma \to F_d$ with $\gamma \notin \mathrm{Ker}(\phi)$. For example, if Γ is a surface group then $\Gamma \times \Gamma$ is residually free. Since $\mathrm{PSL}_2(\mathbb{C})$ does not have subgroups isomorphic to $\Gamma \times \Gamma$ we observe that in Theorem 1.5 the condition "Γ is fully residually free" cannot be weakened to "Γ is residually free".

Let us end this introduction by remarking that all the results obtained in this paper are concerned with the existence of subgroups with certain desired properties. However we do not obtain concrete examples. In general, this problem seems much more difficult.

2. Eventually faithful homomorphisms and a lemma of G. Baumslag

Let $(\rho_n)_{n\geq 0}$ be a sequence of homomorphisms from a group H to a group G. We say that $(\rho_n)_{n\geq 0}$ is *eventually faithful* if for every $h \in H\setminus\{1\}$ there exists an integer $n_0 = n_0(h) \geq 0$ such that $\rho_n(h) \neq 1$ for all $n \geq n_0$.

Since any finitely generated free group can be embedded into F_2, the free group on two generators, it follows that a finitely generated group is fully residually free if and only if it admits an eventually faithful sequence of homomorphisms to F_2.

Let us recall the following lemma of G. Baumslag [2].

Lemma 2.1. (G. Baumslag) Let $u, a_1, ..., a_k$ be elements of a free group F. Assume that u does not commute with any of the a_i's. Then there exists $n_0 \geq 0$ such that for all integers $n_1, ..., n_k$ with $|n_i| \geq n_0$ we have

$$u^{n_1} a_1 u^{n_2} a_2 \cdot ... \cdot u^{n_k} a_k \neq 1$$

This lemma has the following corollaries. The first one below shows in particular that surface groups are fully residually free.

Corollary 2.2. Let $\Gamma = \Gamma_{2r}$ be the fundamental group of an orientable surface of genus $2r$ ($r \geq 1$). Let us write a presentation of Γ as

(1)
$$\Gamma = \langle a_i, a_i', b_i, b_i', 1 \leq i \leq r \ \mid [a_1, a_1'] \cdot ... \cdot [a_r, a_r'] \cdot [b_r', b_r] \cdot ... \cdot [b_1', b_1] = 1 \rangle$$

Now consider the automorphism σ of Γ that leaves the a_i's and a_i''s fixed while it sends every b_i to $\gamma b_i \gamma^{-1}$ and every b_i' to $\gamma b_i' \gamma^{-1}$ where $\gamma = [a_1, a_1'] \cdot ... \cdot [a_r, a_r']$. Finally let f be the surjective homomorphism from Γ to the free group F_{2r} with free generators $x_1, ..., x_r$ and $x_1', ..., x_r'$ defined by $f(a_i) = f(b_i) = x_i$, $f(a_i') = f(b_i') = x_i'$.

Then the sequence of maps $(f \circ \sigma^n)_{n \geq 0}$ is eventually faithful.

The maps σ and f have the following simple topological interpretation. In the above classical representation of Γ as the fundamental group of a surface, the relation gives the gluing instructions for forming a genus $2r$ surface from a $4r$-gon in the plane. This is demonstrated in Figure 1 for the case $r = 2$. The element γ correponds to the closed curve separating the surface into two equal parts. The map σ corresponds to a Dehn twist around γ. The map f is obtained by reflecting the surface across the separating curve γ. The image of this reflection is a surface of genus r with one boundary component, whose fundamental group is freely generated by $x_1, ..., x_r, x_1', ..., x_r'$.

Proof. Let $g \in \Gamma \backslash \{1\}$. The element g can be written in the form $g = w_1(a_i, a_i') \cdot w_2(b_i, b_i') \cdot .. \cdot w_{2p-1}(a_i, a_i') \cdot w_{2p}(b_i, b_i')$ where each w_i is a reduced word in $2r$ letters and the first and the last w_i may be trivial. Up to modifying the odd w_{2j-1}, we may assume that each even w_{2j} ($1 \leq j \leq p$) is such that $w_{2j}(b_i, b_i')$ is not a power of γ. Note that the centralizer of γ in Γ is the cyclic group generated by γ. Regrouping several w_j's into a longer word if necessary, unless g itself is a power of γ, we may also assume that each odd w_{2j-1} is such that $w_{2j-1}(a_i, a_i')$

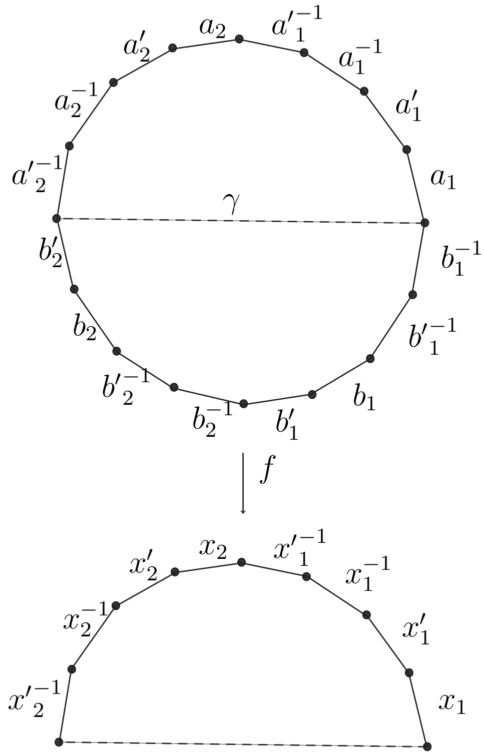

FIGURE 1. All curves are oriented counterclockwise. Fold the genus 4 surface across γ to obtain f.

is not a power of γ. Let $\overline{\gamma}$ be the image of γ under f. We have $f \circ \sigma^n(g) = w_1 \cdot \overline{\gamma}^n w_2 \overline{\gamma}^{-n} \cdot .. \cdot w_{2p-1} \cdot \overline{\gamma}^n w_{2p} \overline{\gamma}^{-n}$ where each $w_j = w_j(x_i, x_i')$. Since γ does not commute with any of the w_j. Lemma 2.1 implies that $f \circ \gamma^n$ is eventually faithful. $\square$

The next two corollaries are very simple applications of Lemma 2.1, and are only recorded here for further use.

Corollary 2.3. Let F be a free group of rank $n+1$ with free generators $x_1, ..., x_{n+1}$. Let F^- be the subgroup generated by $x_1, ..., x_n$. Suppose a and b are non commuting elements in F^-. Consider the automorphism σ of F defined by $\sigma(x_i) = x_i$ if $i \leq n$ and $\sigma(x_{n+1}) = bx_{n+1}b^{-1}$. Let f be the homomorphism of F into F^- that sends each x_i, $1 \leq i \leq n$, to itself and x_{n+1} to a. Then the sequence of homomorphisms $(f \circ \sigma^n)_{n \geq 0}$ is eventually faithful.

Remark 2.4. It follows from the definition that Lemma 2.1 remains true when the free group F is replaced by any nonabelian fully residually free group, and in particular by a surface group.

Corollary 2.5. Let $\Gamma_r = \langle a_i, a_i', 1 \leq i \leq r \mid [a_1, a_1'] \cdot ... \cdot [a_r, a_r'] = 1\rangle$ be a presentation of a surface group of genus r. Let F be a free group of rank $2r$ generated by x_i, x_i' for $1 \leq i \leq r$. For each integer $n \geq 0$ consider the homomorphism $\rho_n : F \to \Gamma_r$ given by $\rho_n(x_i) = a_i$ for $i \geq 2$, $\rho_n(x_i') = a_i'$ for $i \geq 1$ and $\rho_n(x_1) = a_1 \cdot (a_1')^n$. Then the sequence $(\rho_n)_{n\geq 1}$ is eventually faithful.

3. Proof of Theorem 1.1

Here we give a proof of Theorem 1.1. Let G be a compact group containing a dense free group F on r generators. We are going to show that G contains a surface group containing F. In order to do so, we first make sure that G contains a dense free group on an even number of generators. This is done, if r is odd, by enlarging F in the following way. Let $a_1, ..., a_r$ be generators of F and fix a and b in F two non commuting elements. Then let B be the closure in G of the cyclic group generated by b. Let F^+ be the abstract free group on $r+1$ generators $y_1, ..., y_{r+1}$. To every $\beta \in B$, we associate the following homomorphism $\rho_\beta : F^+ \to G$ that sends each y_i to a_i when $1 \leq i \leq r$ and sends y_{r+1} to $\beta a \beta^{-1}$. From Corollary 2.3, we know that the sequence of homomorphisms $(\rho_{b^n})_{n\geq 1}$ is eventually faithful. Let $w \in F^+ \backslash \{1\}$ and consider the set $O_w = \{\beta \in B, \rho_\beta(w) \neq 1\}$. Clearly O_w is open in B. It is also dense because for any $n_0 \geq 0$ the set $\{b^n, n \geq n_0\}$ is dense in B. Applying Baire's theorem, we obtain that $O := \cap_{w\in F^+\backslash\{1\}} O_w$ is dense in B and in particular non empty. Let $\beta_0 \in O$. The homomorphism ρ_{β_0} is faithful and $\rho_{\beta_0}(F^+)$ is an dense free subgroup of G of rank $r+1$.

We may thus assume that $r = 2k$. Let $x_1, x_1', ..., x_k, x_k'$ be the $2k$ free generators of F. Set $\gamma = [x_1, x_1'] \cdot ... \cdot [x_k, x_k']$. Let K be the closure in G of the cyclic group generated by γ. Keeping the same notations as in Corollary 2.2 for the presentation of the surface group Γ_{2r}, we define, for every $\alpha \in K$ a homomorphism $\sigma_\alpha : \Gamma_{2r} \to G$ by sending a_i to x_i, a_i' to x_i', b_i to $\alpha x_i \alpha^{-1}$ and b_i' to $\alpha x_i' \alpha^{-1}$. From Corollary 2.2, we know that the sequence of homomorphisms $(\sigma_{\gamma^n})_{n\geq 1}$ is eventually faithful. As above, let $w \in \Gamma_{2r}\backslash\{1\}$ and consider the set $U_w = \{\alpha \in K, \sigma_\alpha(w) \neq 1\}$. Clearly U_w is open in K. It is also dense, because for any $n_0 \geq 0$ the set $\{\gamma^n, n \geq n_0\}$ is dense in K. Applying Baire's theorem, we obtain that $U := \cap_{w\in F^+\backslash\{1\}} U_w$ is dense in K and in particular non empty. Let $\alpha_0 \in U$. Note that for every $\alpha \in K$ the image $\sigma_\alpha(\Gamma_{2r})$ is dense in G because it contains F as a subgroup. The homomorphism σ_{α_0} is faithful and $\sigma_{\alpha_0}(\Gamma_{2r})$ is a surface group densely embedded in G.

We now pass to the converse statement. Let Γ_r be a dense surface group of genus r in G. Keep the notations of Corollary 2.5 and let A be the closure in G of the cyclic group generated by a_1'. For every $\omega \in A$ let $\pi_\omega : F_{2r} \to G$ be the homomorphism that sends x_1 to $a_1\omega$, while for $i \geq 2$, x_i is sent to a_i and for $i \geq 1$, x_i' is sent to a_i'. Then, according to Corollary 2.5, the sequence $(\pi_{(a_1')^n})_{n\geq 1}$ is eventually faithful. A Baire argument similar to the one above show that π_{ω_0} is faithful for some $\omega_0 \in A$. It remains to check that $\pi_{\omega_0}(F_{2r})$ is dense in G. This is clear because it contains all a_i's for $i \geq 2$ and a_i' 's for $i \geq 1$. In particular π_{ω_0} contains a_1', hence the closure of $\pi_{\omega_0}(F_{2r})$ must contain ω_0. Therefore the closure of π_{ω_0} must contain a_1, and must be all of G. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Let us make the obvious remark that there are (non locally compact) Hausdorff non-discrete topological groups where the property of Theorem 1.2 does not hold. For instance consider the free group with the induced topology coming from a dense embedding inside a compact Lie group.

As in the compact case, we are first going to enlarge the free subgroup F (of rank r) to a bigger free subgroup of rank $2r$ by adding r free generators, then deform that free subgroup into a surface group. By the structure theory of locally compact groups (Van Dantzig's theorem, see [12]) G has an open subgroup H containing the connected component of the identity G° in such a way that H/G° is compact. Moreover there is a normal compact subgroup K of H such that H/K is a Lie group (see [12] Theorem 4.6). Up to changing H into another open subgroup of G we can assume that H/K is connected.

Let $x_1, ..., x_r$ be the r free generators of the non-discrete free subgroup F. Let U be a neighborhood of the identity in H/K chosen sufficiently small so that $ux_iu^{-1}x_i^{-1} \in U$ for any $u \in U$ and $i = 1, ..., r$ and so that any element in U^r lies in a 1-parameter subgroup of H/K. We are going to build elements $x_1', ..., x_r'$ in H that are in U modulo K and are such that, together with the x_i's they form $2r$ free generators of a free subgroup of G.

For this purpose, pick two non-commuting elements a and b in F that are in U modulo K. This is always possible because F is not discrete. We will separate two cases. Case (A) is when $F \cap K \neq \{1\}$. Case (B) is when $F \cap K = \{1\}$. In case (A) we can clearly assume that a and b belong to K (pick an element in $F \cap K$ and some suitable conjugate of it). Suppose that $x_1', ..., x_j'$ have been constructed. Define F_{r+j+1} to be an abstract free group on $r + j + 1$ generators $y_1, ..., y_r, y_1', ..., y_{j+1}'$, and let us find x_{j+1}'. In case (A), we set $B \leq K$ to be the closure of the cyclic group generated by b, while in case (B) we let B be the 1-parameter subgroup of H/K where b lies modulo K. For $\beta \in B$, let ρ_β be the map that sends each y_i to x_i, each y'_i to x'_i for $i \leq j$, y'_{j+1} to $\beta a \beta^{-1}$ in case (A), and y'_{j+1} to $\beta \overline{a} \beta^{-1}$ in case (B) where $\overline{a} = aK$. Corollary 2.3 and Baire's theorem ensure that the subset of those β for which ρ_β is faithful is Baire dense in B (in case (B) we use the analytic structure of H/K) hence nonempty. Fix such a $\beta_0 \in B$. In case (A) the desired new generator is $x'_{j+1} = \rho_{\beta_0}(y_{j+1})$. In case (B) define x'_{j+1} to be any preimage of $\rho_{\beta_0}(y_{j+1})$ in G.

Call F' this new free subgroup on $2r$ generators. Note that $F \leq F'$ and that in case (B) we have $F' \cap K = \{1\}$. Let us consider the product of commutators $\gamma = [x_1, x'_1] \cdot ... \cdot [x_r, x'_r]$. Let Γ_{2r} be a surface group given with the presentation written above in (1). Consider the centralizer $Z_G(\gamma)$ of γ in G. Given an element α in $Z_G(\gamma)$ we can define a representation $\rho_\alpha : \Gamma_{2r} \to G$ by setting $\rho_\alpha(a_i) = x_i$, $\rho_\alpha(a'_i) = x'_i$ and $\rho_\alpha(b_i) = \alpha x_i \alpha^{-1}$ and $\rho_\alpha(b'_i) = \alpha x'_i \alpha^{-1}$. Corollary 2.2 shows that the sequence $(\rho_{\gamma^n})_{n \geq 1}$ is eventually faithful. We will make use of the following lemmas:

Lemma 4.1. Let H be a locally compact group and K a normal compact subgroup such that H/K is a Lie group. Let $\{x(t)\}_t$ be a 1-parameter subgroup in H/K. Then $\{x(t)\}_t$ can be lifted to a 1-parameter subgroup $\{\widetilde{x}(t)\}_t$ in H such that $\pi(\widetilde{x}(t)) = x(t)$ where $\pi : H \to H/K$ is the quotient map.

Proof. See the end of Section 4.7 of [12]. □

Lemma 4.2. Let H be a locally compact group and $K \lhd H$ a compact normal subgroup such that H/K is connected. Then $H = Z_H(K)K$ where $Z_H(K)$ is the centralizer of K in H.

Proof. It follows from Lemma 4.1 that $H = H^\circ K$.

Let $\rho : H \to Aut(K)$ that sends $h \in H$ to the automorphism $i(h)$ of K given by the conjugation by h. Then $\ker \rho = Z_H(K)$. We need to show that $\rho(H) = \rho(K)$ and for this it is clearly enough to prove that $\rho(H^\circ)$ is contained in $Inn(K)$, the group of inner automorphisms of K. This is a consequence of the following lemma:

Lemma 4.3. Let K be a compact group. Then the connected component of the identity in $Aut(K)$ is contained in $Inn(K)$.

Proof. First assume that K is a Lie group. Then $K \simeq D \times T \times S$ where S is semisimple, T a torus and D is a finite group. As is well-known $Inn(S)$ is of finite index in $Aut(S)$ and $Aut(T)$ is discrete. It follows easily that $Aut(K)^\circ \leq Inn(S)$. Now we pass to the general case.

According to Peter-Weyl's theorem, K has a descending chain of compact normal subgroups $C_1 \supset C_2 \supset C_3 \supset \dots$ such that any open neighborhood of the identity contains all but a finite number of the subgroups $\{C_i\}$, and the quotient K/C_i is always a Lie group. By pulling back a small identity neighborhood from K/C_i to K we obtain an open set $U_i \subset K$ containing C_i such that any subgroup of K inside U_i is in fact contained in C_i. Therefore by connectivity every automorphism in $Aut(K)^\circ$ preserves C_i. This yields a map

$$Aut(K)^\circ \to Aut(K/C_i)^\circ \leq Inn(K/C_i).$$

Let $\phi \in Aut(K)^\circ$. For each i pick an element $h_i \in K$ such that $\phi(g)C_i = h_i g h_i^{-1} C_i$ for all $g \in K$. Since the subgroups $\{C_i\}$ become arbitrarily small it follows that $\mathrm{i}(h_i) \to \phi$ in the compact-open topology on $Aut(K)$. Since $Inn(K)$ is a closed subgroup of $Aut(K)$ it follows that ϕ is an inner automorphism. This completes the proof of Lemmas 4.2 and 4.3. $\square$

Let us go back to the proof of Theorem 1.2. Suppose we are in case (A). Then $\gamma \in K$ and let A be the closure in K of the cyclic group generated by γ. Then $A \leq Z_G(\gamma)$ and, as follows from Corollary 2.2, if w is a non trivial element in Γ_{2r} then $\{\alpha \in A, \rho_\alpha(w) \neq 1\}$ is an open and dense subset of A. By Baire's theorem there is an $\alpha_0 \in A$ such that ρ_{α_0} is faithful. Its image contains F and is isomorphic to the surface group Γ_{2r}. We are done.

Finally suppose that we are in case (B). Then $F' \cap K = \{1\}$ and the sequence $(\pi \circ \rho_{\gamma^n})_{n \geq 1}$ is eventually faithful, where $\pi : H \to H/K$ is the projection map. Let $\pi(\gamma) = \beta(1)$ where $\{\beta(t)\}_t$ is a 1-parameter subgroup of H/K. The centralizer $Z_H(K)$ is closed in H, hence locally compact, and by Lemma 4.2, $H/K \cong Z_H(K)/K \cap Z_H(K)$. Lemma 4.1 ensures that $\{\beta(t)\}_t$ can be lifted to a 1-parameter subgroup $\{c(t)\}_t$ in $Z_H(K)$, i.e. $\beta(t) = c(t)K$. We now claim that $\{c(t)\}_t \leq Z_G(\gamma)$.

Indeed, by Lemma 4.2, we can write $\gamma = ck$ where $c \in Z_H(K)$ and $k \in K$. However $\pi(\gamma) = \beta(1)$, hence $c(1)K = \gamma K$, hence $c(1)k' = c$ for some $k' \in K$. But for each t, $c(t)$ commutes with $c(1)$ and with k', hence it commutes with c. As $c(t) \in Z_H(K)$ it must also commute with k, hence with γ. This proves the claim.

As a consequence, we obtain a one parameter family of representations $\rho_t : \Gamma_{2r} \to G$ by setting $\rho_t := \rho_{c(t)}$. Again $\{t \in \mathbb{R}, \pi \circ \rho_t(w) \neq 1\}$ is open because $t \mapsto \rho_t(w)$ is continuous from $\mathbb{R}$ to G. By Corollary 2.2, the sequence $\pi \circ \rho_n = \pi \circ \rho_{\gamma^n}$ is eventually faithful. This implies that the analytic map $t \mapsto \pi \circ \rho_t(w)$ from $\mathbb{R}$ to the Lie group H/K is not constant. Therefore the set $\{t \in \mathbb{R}, \pi \circ \rho_t(w) \neq 1\}$ is dense. Again, by Baire's theorem there must be a $t_0 \in \mathbb{R}$ such that $\pi \circ \rho_{t_0}$ is faithful. Then ρ_{t_0} is also faithful and its image is a subgroup of G isomorphic to Γ_{2r} containing F. This ends the proof of Theorem 1.2.

5. The analytic structure of $\mathrm{Hom}(\Gamma, G)$

In this section we will recall some facts about the structure of $\mathrm{Hom}(\Gamma, G)$ as an analytic variety, where Γ is a finitely generated group and G is a Lie group.

Consider first the case that Γ is isomorphic to a free group F_k with free basis $e_1, \ldots, e_k$. A homomorphism $\sigma \in \mathrm{Hom}(F_k, G)$ is determined by $\sigma(e_1), \ldots, \sigma(e_k)$ and hence we have an identification of $\mathrm{Hom}(F_k, G)$ with the analytic manifold $G^k = G \times \cdots \times G$. Given an element $\gamma = e_{i_1} \ldots e_{i_l} \in F_k$ we consider the analytic map

$$P_\gamma : G^k \to G, \quad P_\gamma(A_1, \ldots, A_k) = A_{i_1} \cdots A_{i_l}.$$

The set $P_\gamma^{-1}(\mathrm{Id}_G) = \{\rho \in \mathrm{Hom}(F_k, G) : \gamma \in \mathrm{Ker}(\rho)\}$ is a closed analytic subvariety of $\mathrm{Hom}(F_k, G)$. Recall the following basic result (see [5]):

Theorem 5.1. *Let G be a connected non–solvable Lie group. Then the set of faithful homomorphisms $\sigma : F_k \to G$ is dense and has full Haar measure in* $\mathrm{Hom}(F_k, G) \cong G^k$.

We include a proof for the convenience of the reader.

Proof. By definition $\cup_{\gamma \in F_k \setminus \mathrm{Id}_{F_k}} P_\gamma^{-1}(\mathrm{Id}_G)$ is the complement of the set of faithful representations. The claim follows from the Baire category theorem if $P_\gamma^{-1}(\mathrm{Id}_G)$ is nowhere dense and of 0 measure for all $\gamma \neq \mathrm{Id}_\Gamma$. Since $P_\gamma^{-1}(\mathrm{Id}_G)$ is an analytic subvariety of G^k, it is either nowhere dense and of 0 measure or contains G^k since G is connected. To show that the later case cannot occur, it suffices to find one faithful representation, i.e. it suffices to find a nonabelian free subgroup of G. The existence of a free subgroup in G follows for example from the Tits alternative [16], or more simply from the fact that G contains a subgroup locally isomorphic to either $\mathrm{PSL}_2(\mathbb{R})$ or to $\mathrm{PSO}(3)$ and each of these groups contains a free subgroup. □

Now let Γ be a general finitely generated group. To a given presentation $\Gamma = \langle \gamma_1, \ldots, \gamma_k \mid \{R_i\}_{i \in I} \rangle$ of Γ, we associate the surjection $\pi : F_k \to \Gamma$ defined by $\pi(e_j) = \gamma_j$. The homomorphism π induces an injective map

$$\pi^* : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(F_k, G)$$

and its image coincides with $\cap_{i \in I} P_{R_i}^{-1}(\mathrm{Id}_G)$. Hence, we can identify $\mathrm{Hom}(\Gamma, G)$ with an analytic subvariety of G^k. (In fact, the induced structure of $\mathrm{Hom}(\Gamma, G)$ as an analytic variety does not depend on the presentation of Γ. We will not use this.) An important observation is that for all $\gamma \in \Gamma$ the map $P^{\Gamma}_{\gamma} : \mathrm{Hom}(\Gamma, G) \to G$ given by $P^{\Gamma}_{\gamma}(\rho) = \rho(\gamma)$ is analytic. Moreover, if $[\gamma] \in F_k$ is an element representing γ then we have $P^{\Gamma}_{\gamma} = P_{[\gamma]}|_{\mathrm{Hom}(\Gamma,G)}$. This is why in the sequel we will simplify notation and write $P^{\Gamma}_{\gamma} = P_{\gamma}$.

An important fact for our considerations is that analytic subvarieties admit locally finite stratifications with smooth strata. The following crucial result is due to Whitney, Thom and Lojasiewicz. We refer to Kaloshin [9] for its proof.

Proposition 5.2. *Let V be an analytic subvariety of an analytic manifold M. Then there is a locally finite decomposition $V = \cup V_i$, where V_i are connected analytic submanifolds of M.*

The statement of Proposition 5.2 is much weaker than, and follows directly from, [9, Theorem 1]. We have chosen this simplified statement to avoid recalling the more subtle properties of stratifications. Despite this, we will refer to the submanifolds V_i as the *strata* of V.

6. Dense subgroups of connected Lie groups

This section will establish some properties of dense subgroups of connected Lie group. We begin by recalling some results from [3]. We then determine the number $d(G)$ of Theorem 1.5, and finally study the structure of the set $\mathcal{D}(F_k, G)$ of dense representation of the free group F_k in G.

A Lie group H is called topologically perfect if its commutator group is dense. Recall the following theorem from [3] (see also [7]):

Theorem 6.1. *Let H be a connected topologically perfect Lie group. Assume that the Lie algebra Lie(H) is generated (as a Lie algebra) by $d = d(H)$ elements. Then there is an identity neighborhood $U \subset H$, and a proper exponential algebraic subvariety $R \subset U^d$, such that $\langle h_1, \dots, h_d\rangle$ is dense in H for any $(h_1, \dots, h_d) \in U^d \setminus R$.*

In case G is topologically perfect we can define the constant $d(G)$ Theorem 1.5 to be the minimal number of generators for $\mathrm{Lie}(G)$. As a consequence of Theorem 6.1 we get:

Corollary 6.2. *Let G be a connected topologically perfect Lie group. Then $\mathcal{D}(\Gamma, G)$ is open in* $\mathrm{Hom}(\Gamma, G)$.

Proof. If $\rho_0 \in \mathrm{Hom}(\Gamma, G)$ is a representation of Γ in G with dense image, then for some $\gamma_1, \dots, \gamma_d \in \Gamma$, where $d = d(G)$, we have $(\rho_0(\gamma_1), \dots, \rho_0(\gamma_d)) \in$

$U^d \setminus R$. But then $(\rho(\gamma_1), \dots, \rho(\gamma_d)) \in U^d \setminus R$ for any ρ sufficiently close to ρ_0 in $\mathrm{Hom}(\Gamma, G)$. By Theorem 6.1 any such ρ has a dense image. □

We now define $d(G)$ for a general connected Lie group G.

For a connected abelian Lie group A, define $\mathrm{rank}(A)$ as the dimension of the tensor with $\mathbb{R}$, i.e. if $\mathbb{T}$ is the one dimensional torus and $A = \mathbb{T}^j \times \mathbb{R}^k$ then $\mathrm{rank}(A) := k$. It is easy to see that a generic tuple (in the sense of the Baire category theorem or measure theory) of $k+1$ elements generate a dense subgroup in A. E.g. if A is compact then a generic element generates a dense cyclic subgroup. In general, $k+1$ elements $a_1, \dots, a_{k+1} \in A$ generate a dense subgroup if and only if the projections of the first k elements $a_1, \dots, a_k$ to the second factor $\mathbb{R}^k$ form a basis, and, after identifying the compact quotient $A/\langle a_1, \dots, a_k \rangle$ with $\mathbb{T}^{j+k}$, the $j+k$ coordinates of the projection of the last element a_{k+1} to this torus are independent. For a connected abelian Lie group A, we thus define $d(A) = \mathrm{rank}(A) + 1$.

Let now G be a general connected Lie group. Set $G_0 = G$ and define inductively G_i to be the closure of the derived group $[G_{i-1}, G_{i-1}]$. The decreasing sequence G_i must stabilize after finitely many steps m to a group $H = G_m$, and H has the property that its commutator is dense, i.e. it is topologically perfect. The general case is reduced to the abelian and topologically perfect cases using the following:

Lemma 6.3. *A subgroup D of G is dense if and only if*

(1) *its image in G/G_2 is dense in G/G_2, and*
(2) *its intersection with H is dense in H.*

Proof. If D is dense then (1) follows immediately. Moreover the commutator group $[D, D]$ is clearly dense in G_1, and by a simple induction, the m^{th} commutator of D is dense in $H = G_m$.

The other direction would follow if we show that (1) implies that the image of G in G/H is dense in G/H. Now G/H is solvable and in a connected solvable Lie group B, a subgroup is dense if and only if its image in B/B_2 (modulo the second closed commutator) is dense in B/B_2. To see this, note that the commutator of a connected solvable Lie group is nilpotent, and that a subgroup of a nilpotent group is dense if and only if it is dense modulo the first commutator. □

We define the number $d(G)$ as follows[2]:

$$d(G) = \max\{d(G_0/G_1), d(G_1/G_2), d(H)\},$$

where $d(H)$ is the minimal number of generators for the Lie algebra of H, and $d(G_i/G_{i+1}) = \mathrm{rank}(G_i/G_{i+1}) + 1$.

[2]In case G/H is nilpotent, one can take $d(G) = \max\{d(G_0/G_1), d(H)\}$.

Consider the following subsets of $\mathrm{Hom}(\Gamma, G)$:

$$\mathcal{D}_H(\Gamma, G) := \{\rho \in \mathrm{Hom}(\Gamma, G) : \overline{\rho(\Gamma) \cap H} = H\}, \text{ and}$$

$$\mathcal{D}_{G/G_2}(\Gamma, G) := \{\rho \in \mathrm{Hom}(\Gamma, G) : \overline{\rho(\Gamma)G_2} = G\}.$$

By Lemma 6.3 we have:

$$\mathcal{D}(\Gamma, G) = \mathcal{D}_H(\Gamma, G) \cap \mathcal{D}_{G/G_2}(\Gamma, G).$$

Moreover, for free groups we have:

Lemma 6.4. *Suppose that $k \geq d(G)$, then*

- *The set $\mathcal{D}_H(F_k, G)$ is open in* $\mathrm{Hom}(F_k, G)$.
- *The set $\mathcal{D}_{G/G_2}(F_k, G)$ is the complement of a countable union of proper closed analytic subvarieties of* $\mathrm{Hom}(F_k, G)$*, in particular, it is of second category.*

Proof. The first claim follows from Theorem 6.1; if $\rho_0 \in \mathcal{D}_H(F_k, G)$, then $\rho(F_k)_m$ is dense in $G_m = H$. Hence there are $d(H)$ words involving commutators of length m in k letters, such that when applying them to the image (under ρ_0) of the generators of F_k (and think of them as the coordinates of a point in $G^{d(H)}$) one gets a point in $U^{d(H)} \setminus R \subset H^{d(H)}$. Clearly if ρ is sufficiently close to ρ_0 then the same words applied to the ρ image of the generators still yield a point in $U^{d(H)} \setminus R$. By Theorem 6.1, $\rho(F_k, G) \cap H$ is dense in H.

To see the second claim note that a subgroup of G/G_2 is dense in G/G_2 if its image in G/G_1 is dense in G/G_1 and its intersection with G_1 projects to a dense subgroup of G_1/G_2. Both conditions are generic in the sense that their complements are a countable union of proper analytic closed subvarieties: there are $d(G_0/G_1)$ words with k letters which generically generate a dense subgroup in the quotient G_0/G_1, and $d(G_1/G_2)$ words involving commutators of the k letters which generically generate a dense subgroup in the quotient G_1/G_2. The former assertion is clear, while the latter is a little harder to see and we leave it to the reader as an exercise: in fact, if $x_1, ..., x_k$ are generic elements of G then the commutators $[x_1, x_i]$ for $2 \leq i \leq k$ form a basis of G_1/G_2 and together with $\prod_{i=2}^{k}[x_1^i, x_i]$ they generate a dense subgroup of G_1/G_2. $\square$

7. Proof of Theorem 1.5

We are now in a position to complete the proof of Theorem 1.5. We begin by fixing once and for all a relatively compact open neighborhood $B \subset \mathrm{Hom}(\Gamma, G)$ of the trivial homomorphism and let $V_1, \ldots, V_s$ be finitely many strata of $\mathrm{Hom}(\Gamma, G)$ covering B.

The group Γ is, as assumed, d-fully residually free for $d = d(G)$ and hence it is generated by $k \geq d$ elements and there is a sequence of surjective homomorphisms $\phi_i : \Gamma \to F_d$ such that for every $\gamma \in \Gamma \setminus \mathrm{Id}_\Gamma$ there is i_γ with $\gamma \notin \mathrm{Ker}(\phi_i)$ for all $i \geq i_\gamma$. The assumption that Γ is not abelian implies that $d \geq 2$. The homomorphisms ϕ_i induce analytic maps $\phi_i^* : \mathrm{Hom}(F_d, G) \to \mathrm{Hom}(\Gamma, G)$ and we can choose a sequence $(\sigma_i) \subset \mathrm{Hom}(F_d, G)$ of faithful representations with dense image which are close enough to the trivial homomorphism so that $\phi_i^*(\sigma_i) \in B$ for all i. Up to passing to a subsequence and relabelling, we may assume that $\phi_i^*(\sigma_i) \in V_1 \subset \mathrm{Hom}(\Gamma, G)$ for all i.

Given $\gamma \in \Gamma \setminus \mathrm{Id}_\Gamma$ we deduce from the connectivity of V_1, using analytic continuation and the implicit functions theorem, that either $V_1 \subset \{P_\gamma^{-1}(\mathrm{Id}_G)\}$ or $V_1 \cap \{P_\gamma^{-1}(\mathrm{Id}_G)\}$ is nowhere dense in V_1. The former case cannot occur since by construction we have $P_\gamma(\phi_i^*(\sigma_i)) = \sigma_i(\phi_i(\gamma)) \neq \mathrm{Id}_G$ for all i sufficiently large. In particular, Baire's category theorem implies that the set $\mathcal{F}(\Gamma, G) \cap V_1$ of all faithful $\rho \in V_1$ is of second category in V_1.

By construction the image of $\phi_1^*(\sigma_1)$ coincides with the image of σ_1 and hence is dense. This implies that the open subset $\mathcal{D}_H(\Gamma, G) \cap V_1$ of V_1 is nonempty.

Additionally it implies that the set $\mathcal{D}_{G/G_2}(\Gamma, G) \cap V_1$ is non-empty, and since it is the complement of a countable union of proper closed analytic subvarieties we conclude, again by analyticity and the implicit functions theorem, that all these varieties are proper, and that $\mathcal{D}_{G/G_2}(\Gamma, G) \cap V_1$ is also of second category. Hence

$$\mathcal{F}(\Gamma, G) \cap \mathcal{D}(\Gamma, G) \cap V_1 = \mathcal{F}(\Gamma, G) \cap \mathcal{D}_{G/G_2}(\Gamma, G) \cap \mathcal{D}_H(\Gamma, G) \cap V_1 \neq \emptyset,$$

being the intersection of the second category subset

$$\mathcal{F}(\Gamma, G) \cap \mathcal{D}_{G/G_2}(\Gamma, G) \cap V_1$$

with a nonempty open subset $\mathcal{D}_H(\Gamma, G) \cap V_1$ of V_1. □

8. Some remarks on connected compact groups

When G is a connected compact Lie group then $d(G) = 1$ if G is abelian and $d(G) = 2$ if it is not. For compact semisimple Lie groups one can easily deduce the following Lemma from Theorem 6.1. The general case follows by an simple argument similar to the one given in the proof of Lemma 6.4.

Lemma 8.1. Let G be a connected compact Lie group. Then the set $\mathcal{D}(F_2, G)$ is of full Haar measure and Baire dense in $\mathrm{Hom}(F_2, G) \cong G \times G$.

When G is nonabelian, Theorem 5.1 says that a generic pair $(a, b) \in G \times G$ also generates a free group. We shall now generalize this result to an arbitrary compact connected group.

Proposition 8.2. Let G be a second countable connected compact nonabelian group. Then there exists a subset $\mathcal{O}$ in $G \times G$, which is both of second Baire category and of full Haar measure, such that any pair (a, b) in $\mathcal{O}$ generates a dense free subgroup in G.

Proof. By the Peter-Weyl theorem (c.f. [12]) there is a decreasing sequence of normal compact subgroups $K_n \lhd G$ such that each quotient G/K_n is a connected compact Lie group and $\bigcap_{n\geq 1} K_n = \{1\}$. Let $\mathcal{D}_n$ be the set of all pairs (a, b) in $G \times G$ that generate a dense subgroup in the quotient G/K_n. Clearly $\mathcal{D}(F_2, G) = \bigcap \mathcal{D}_n$ and by Lemma 8.1 $\mathcal{D}_n$ is of second Baire category and of full Haar measure.

The analogous assertion for $\mathcal{F}(F_2, G)$ follows easily from Theorem 5.1 because one of the quotients G/K_n is nonabelian. $\square$

Corollary 8.3. *Any connected second countable nonabelian compact group contains a dense surface group of genus 2.*

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