# EQUIDISTRIBUTION OF DENSE SUBGROUPS ON NILPOTENT LIE GROUPS

EMMANUEL BREUILLARD

ABSTRACT. Let $\Gamma$ be a dense subgroup of a simply connected nilpotent Lie group $G$ generated by a finite symmetric set $S$. We consider the $n$-ball $S^n$ for the word metric induced by $S$ on $\Gamma$. We show that $S^n$ (with uniform measure) becomes equidistributed on $G$ with respect to the Haar measure as $n$ tends to infinity. We also prove the analogous result for random walk averages.

## 1. INTRODUCTION

In this paper we are concerned with equidistribution properties of dense subgroups of Lie groups. Let $G$ be a Lie group and $S = \{1, a_1, ..., a_s, a_1^{-1}, ..., a_s^{-1}\}$ a finite symmetric set of elements in $G$, which we assume to generate a dense subgroup $\Gamma$ in $G$. Let $S^n$ be the $n$-ball in the Cayley graph of $\Gamma$ induced by $S$, i.e. the set of elements of $\Gamma$ that can be written as a product of at most $n$ elements from $S$. We give ourselves two open subsets $U$ and $V$ in $G$ with Lebesgue-negligible boundary. Consider the ratios

$$R_n(U, V) = \frac{|S^n \cap U|}{|S^n \cap V|}$$

where $|A|$ denotes the cardinal of a set $A$. The question is to find out whether or not $R_n(U, V)$ converges and if it does to determine the measure $m$ on $G$ such that

$$\lim_{n \to +\infty} R_n(U, V) = \frac{m(U)}{m(V)} \tag{1}$$

Various kinds of averages are possible instead of the uniform averages over $S^n$ above. In particular one may consider random walk averages or averages over the abstract free group generated by $S$. Limits such as (1) and results of this kind are usually called *ratio limit theorems* (see [19], [16]).

Arnol'd and Krylov were among the first to consider such a question and they showed in [2] how to use the representation theory of $G$ to quickly give a positive answer to this problem when $G$ is a compact Lie group and $S$ is the set of free generators of a dense free subgroup of rank $s$ in $G$. Then (1) holds with $m$ the normalized Haar measure on $G$.

In this paper we prove that (1) holds with $m$ a Haar measure when $G$ is any closed subgroup of a simply connected nilpotent Lie group (Corollary 1.2 below). The case of semi-simple Lie groups (even $SL_2(\mathbb{R})$) remains an open question up to now. In [17], Kazhdan

*Date*: October 24, 2007.

considered the case when $G = Isom(\mathbb{R}^d)^\circ$ for $d = 2$ and obtained an analogous ratio limit theorem for random walk averages. His argument was corrected and the result extented by Guivarc'h in [16] (see also [7] for the precise asymptotics) but the case when $d > 2$ is a well-known open problem.

Let us fix some notation and then state the main result of this paper. For a group $G$, we let $vol_G$ be a left Haar measure on $G$. Recall that according to a theorem of Guivarc'h (see [15] and [8]), if $G$ is nilpotent, locally compact and compactly generated, then there is an integer $d(G) \in \mathbb{N}$ such that $vol_G(\Omega^n) \approx n^{d(G)}$ for any compact generating set $\Omega$. When $G$ is connected and simply connected, the integer $d(G)$ is given by the Bass-Guivarc'h formula (9) below. We show:

**Theorem 1.1.** *(Dense subgroups are equidistributed) Let $\Gamma$ be a finitely generated nilpotent group and $S = \{1, a_1, ..., a_s, a_1^{-1}, ..., a_s^{-1}\}$ a finite generating set. Let $G$ a closed subgroup of a simply connected nilpotent Lie group. Suppose $\phi : \Gamma \to G$ is a homomorphism with dense image. Then there is a positive constant $C > 0$ such that*

$$\lim_{n\to+\infty} \frac{|S^n \cap \phi^{-1}(B)|}{n^{d(\Gamma)-d(G)}} = C \cdot vol_G(B) \tag{2}$$

*for every bounded Borel subset $B \subset G$ with negligible boundary.*

**Corollary 1.2.** *If $G$ is a closed subgroup of a simply connected Lie group, then (1) holds for $m$ a Haar measure on $G$.*

This result can be seen as a generalization of the classical Weyl equidistribution ([23]) of multiples of an irrational number $\alpha$ modulo 1. Indeed, let $G = \mathbb{R}$, $\Gamma = \mathbb{Z}^2$, and $S = \{(0,0), (0, \pm 1), (\pm 1, 0)\}$, $\phi(x, y) = x + \alpha y$ and $B = (a, b) \subset [0, 1]$. Then (2) in this case translates as

$$\lim_{n\to+\infty} \frac{1}{2n} |\{k \in [-n, n], k\alpha \in (a, b) \bmod 1\ \}| = b - a.$$

For abelian $G$ the theorem can be deduced from Weyl's equidistribution. However, as we will see below, when $G$ is nilpotent, it requires different techniques.

Combining Alexopoulos' theorem [1] on the asymptotics of the return probability of random walks on finitely generated nilpotent groups with Theorem 1.1 allows to prove the analogous limit theorem for random walk averages, namely:

**Corollary 1.3.** *(Local Limit Theorem) Let $G$ be a simply connected nilpotent Lie group and $\mu$ a symmetric and finitely supported and probability measure on $G$ whose support generates a dense subgroup, then there is $c(\mu) > 0$ such that for any Borel set $B$ with Lebesgue-negligible boundary,*

$$\lim_{n\to+\infty} n^{-d(G)/2} \mu^{*n}(B) = c(\mu) \cdot vol_G(B) \tag{3}$$

*where $\mu^{*n}$ denotes the $n$-th fold convolution power of $\mu$.*

The proof of Theorem 1.1 makes use of two crucial ingredients. First, we need precise information on the shape of the $n$-balls $S^n$ in $\Gamma$, and this is essentially provided by Pansu's theorem from [20]. Second, we use a now well-known principle from ergodic theory (see [10], [18]) according to which the ergodic properties of the action of $\Gamma$ on a homogeneous space $N/M$ are dual to those of the action of $M$ on $N/\Gamma$. The unique ergodicity of unipotent flows on nilmanifolds, which is an "ancestor" of Ratner's theorem, allows then to reduce the equidistribution statement to a computation of the asymptotic volume of cosets of $M$ inside large balls in $N$. The goal of Section 2 is to prove this volume estimate (Proposition 2.14) and give some background on homogeneous quasi-norms on nilpotent Lie groups. In Section 3, we complete the proof of Theorem 1.1.

**N.B.: a.** We prove (see Theorem 3.1 and 3.5) that the limit (2) exists also for more general averages of the form $B(n) \cap \phi^{-1}(U)$ in place of $S^n \cap \phi^{-1}(U)$, where $B(n)$ is the $n$-ball for a quasi-norm on $\Gamma$, or any left-invariant coarsely geodesic distance.

**b.** The techniques of this paper allow to get uniformity of convergence in (2) (resp. (3)), when $B$ is allowed to vary among translates $xB$ such that the distance between $x$ and 1 is a $o(n)$ (resp. $o(\sqrt{n})$), and a uniform upper bound for the left hand side of (2) holds for all translates. But we will not need these refinements here.

**c.** Our equidistribution problem of a dense subgroup $\Gamma$ in a nilpotent Lie group $G$ can be phrased more generally as the question of whether $\Gamma$-orbits equidistribute in their closure on a homogeneous space $N/M$. Here we treated the case of $M$ normal in $N$ and a dense orbit. The general case, when $N$ is nilpotent, is slightly more involved but can be treated by similar methods.

**d.** In [9] we showed, in the case when $G$ is the Heisenberg group, that (3) holds for more general measures, namely any centered and compactly supported measure $\mu$ on $G$. The methods of [9] use representation theory and are of a very different nature from the proof displayed in the present paper.

## Contents

## 2. Homogeneous quasi-norms and volume of balls on nilpotent Lie groups

The main goal of this section is to prove the following result for a simply connected nilpotent Lie group $N$.

**Proposition 2.1.** *For any quasi-norm on $N$, the balls $(D_t)_{t>0}$, $D_t = \{|x| \leq t\}$, form a nicely growing family of subsets of $N$.*

Quasi-norms on $N$ and nicely growing subsets are defined below. This statement essentially means that given a left invariant metric on $N$, we can compute the asymptotics of the volume of balls for the induced metric on a closed connected subgroup. To prove it, we will need to introduce some background on nilpotent Lie groups. This is also the second goal of this section. In the next two paragraphs we deal exclusively with filtered vector spaces, while in the remaining four we apply the results to nilpotent Lie groups.

### 2.1. **Filtrations on vector spaces and exterior powers.**

On a nilpotent Lie algebra, there is a canonical filtration given by the descending central series. In this paragraph, we describe some properties of filtrations and associated degree functions in the more general context of vector spaces. This allows to attach a degree $\deg_V(W)$ to every vector subspace $W$, a notion that will be useful when proving Theorem 2.14. The content of this paragraph is probably well known but we couldn't find an adequate reference for it.

2.1.1. *Filtrations and degree on $V$ and $\Lambda^* V$.* Let $V$ be a real vector space equipped with a filtration, i.e. a non-increasing finite sequence of vector subspaces $V = V_1 \supseteq V_2 \supseteq ... \supseteq V_r$.

**Proposition 2.2.** *Associated to this filtration is a function $\deg : V \to \mathbb{N}$ called the degree defined for $v \in V$ by $\deg(v) = \max_{i \geq 1}\{i, v \in V_i\}$. This degree function extends in a canonical way to the exterior power $\Lambda^* V$.*

To see this, one can for instance consider a basis $(e_1, ..., e_n)$ of $V$ which is adapted to the filtration $(V_i)_i$ in the sense that $V_i = span\{e_k | k = 1, ..., n, \deg(e_k) \geq i\}$. This basis gives rise to an associated basis for $\Lambda^* V$ given by the $e_I = e_{i_1} \wedge ... \wedge e_{i_k}$'s where $I = \{i_1, ..., i_k\}$ ranges over all subsets of $\{1, ..., n\}$. We can then define the degree of a basis element by

$$\deg(e_I) = \deg(e_{i_1}) + ... + \deg(e_{i_k}). \tag{4}$$

Subsequently, this defines a filtration on $\Lambda^* V$ by letting $\widehat{V_i} = span\{e_I | I \subseteq \{1, ..., n\}, \deg(e_I) \geq i\}$. In turn, we get a degree function on $\Lambda^* V$ extending the definition (4) by setting $\deg(\xi) = \max_{i \geq 1}\{i, \xi \in \widehat{V_i}\}$. It is just a matter of simple verification to check that the filtration $(\widehat{V_i})_i$ and the degree thus defined on $\Lambda^* V$ are independent of the choice of an adapted basis.

2.1.2. *Induced filtration on a subspace, degree of a subspace.* Let $W$ be a vector subspace of $V$.

**Proposition 2.3.** *The operations of restricting to a subspace $W$ and extending to the exterior power commute and give rise, after composition, to a uniquely defined degree function* $\deg : \Lambda^* W \to \mathbb{N}$.

*Proof.* Given an adapted basis of $W$ with respect to the filtration $(W \cap V_i)_i$, it is possible to complete this basis into a basis $(e_1, ..., e_n)$ of $V$ which is adapted to the original filtration $(V_i)_i$. So we easily check that

$$\widehat{W \cap V_i} = span\{e_I | I \subseteq J_W, \deg(e_I) \geq i\} = \Lambda^* W \cap \widehat{V_i}$$

where $J_W$ is the set of indices such that $W = span\{e_i, i \in J_W\}$. It follows that there is a unique notion of degree on $\Lambda^* W$ associated to the original degree on $V$. □

In particular, this allows to define the degree of a subspace $W$ by setting $\deg_V(W) = \deg(f_1 \wedge ... \wedge f_k)$, where $(f_1, ..., f_k)$ is any basis of $W$ (observe that deg is really defined on the projective space $\mathbb{P}(\Lambda^* V)$). If $(e_1, ..., e_k)$ is an adapted basis for the filtration $(W \cap V_i)_i$, then (4) yields

$$\deg_V(W) = \deg(e_1 \wedge ... \wedge e_k) = \sum_{i=1}^{k} \deg(e_i) = \sum_{j \geq 1} \dim(W \cap V_j). \tag{5}$$

Finally if $(\delta_t)_{t>0}$ is the one-parameter group of endomorphisms of $V$ defined by $\delta_t(e_i) = t^{\deg(e_i)} e_i$ for some adapted basis $(e_1, ..., e_n)$. We easily check the following:

**Proposition 2.4.** *The $\delta_t$'s extend canonically to $\Lambda^* V$ and for any non-zero $\xi \in \Lambda^* V$, $t^{-d}\delta_t(\xi)$ tends to a non-zero limit in $\Lambda^* V$ as $t \to 0$ if and only if $d = \deg(\xi)$.*

2.2. **Homogeneous quasi-norms on filtered vector spaces.** Let $V$ be a real vector space with a filtration $V = V_1 \supseteq V_2 \supseteq ... \supseteq V_r \supseteq V_{r+1} = \{0\}$. We say that $(\delta_t)_{t>0}$ is a *one-parameter group of dilations* associated to this filtration if the $\delta_t$'s are linear automorphisms of $V$ such that, for every $i = 1, ..., r$, the eigenspace $m_i = \{x \in V, \delta_t(x) = t^i x\}$ is independent of $t$ ($t \neq 1$) and is a (possibly trivial) supplementary vector subspace of $V_{i+1}$ inside $V_i$, i.e.

$$V_i = m_i \oplus V_{i+1}. \tag{6}$$

The $r$ subspaces $m_i$'s are determined by $(\delta_t)_{t>0}$ and vice-versa any choice of $r$ subspaces $m_i$'s verifying (6) determines a unique group of dilations. If $V = \bigoplus_{1 \leq i \leq r} m_i$ and $V = \bigoplus_{1 \leq i \leq r} m_i'$ are two choices of supplementary subspaces, then the associated one-parameter groups of dilations satisfy the relation $\delta_t' = \phi \circ \delta_t \circ \phi^{-1}$ where $\phi$ is the coordinate change from the first direct sum to the second. Also we have uniformly on bounded subsets of $V$:

$$\lim_{t \to +\infty} \delta_{\frac{1}{t}} \circ \phi \circ \delta_t = id. \tag{7}$$

We now introduce the following definition.

**Definition 2.5.** *A continuous function $|\cdot| : V \to \mathbb{R}_+$ is called a homogeneous quasi-norm associated to the dilations $(\delta_t)_t$, or simply a* ***quasi-norm****, if it satisfies the following properties:*

*(i)* $|x| = 0 \Leftrightarrow x = 0$.

*(ii)* $|\delta_t(x)| = t|x|$ *for all* $t > 0$.

Examples of quasi-norms are given by supremum quasi-norms of the type $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ where $\|\cdot\|_p$ are ordinary norms on the vector space $m_p$ and $\pi_p$ is the projection on $m_p$ according to the decomposition $V = \bigoplus_{1 \le i \le r} m_i$. More examples will be given below in the context of nilpotent Lie groups.

Clearly, two quasi-norms associated to the same one-parameter group of dilations are equivalent in the sense that $\frac{1}{c}|\cdot|_1 \le |\cdot|_2 \le c|\cdot|_1$ for some constant $c > 0$. Furthermore, using (7), we check the following:

**Proposition 2.6.** *If $|\cdot|'$ is a quasi-norm that is homogeneous with respect to the one-parameter group $(\delta'_t)_t$ associated to the decomposition $V = \bigoplus_{1 \le i \le r} m'_i$ then there is a unique quasi-norm $|\cdot|$ that is homogeneous with respect to $(\delta_t)_t$ such that*

$$|x| - |x|' = o(|x|)$$

*as $|x|$ is large. In fact $|x| = |\phi(x)|'$.*

As will be observed below in Section 2.6.2, to every reasonable left-invariant distance on a simply connected nilpotent Lie group $N$ is associated a unique quasi-norm that is asymptotic to it. In particular, every large ball for a left-invariant distance is well approximated by some quasi-norm ball. This fact makes the volume computations needed in our main theorem possible because, thanks to their scaling property, such computations are easy in the case of quasi-norm balls.

2.2.1. *Invariance under restriction to a subspace or projection to a quotient.* Let $|\cdot|$ be a homogeneous quasi-norm associated to some fixed one-parameter group of dilations $(\delta_t)_{t>0}$ and let $D_t := \{x \in V, |x| \le t\}$ be the corresponding quasi-norm ball. Let $W$ be a vector subspace of $V$ endowed with the induced filtration $(W \cap V_i)_i$ and let $(\delta_t^0)_t$ be some one-parameter group of dilations on $W$ with respect to that filtration. Although the restriction of $|\cdot|$ to $W$ may not be a quasi-norm on $W$, the following holds:

**Proposition 2.7.** *There exists a unique homogeneous quasi-norm $|\cdot|_0$ on $W$ such that $|x| - |x|_0 = o(|x|)$ if $x \in W$ and $|x|$ is large. In particular, if $D_t^0$ is the homogeneous quasi-norm ball for $|\cdot|_0$ on $W$, then there exists $\varepsilon_t > 0$, $\varepsilon_t \to 0$ as $t \to +\infty$, such that*

$$D^0_{t(1-\varepsilon_t)} \subset D_t \cap W \subset D^0_{t(1+\varepsilon_t)}.$$

Similarly, we can consider the quotient vector space $V/W$ endowed with the induced filtration $(V_i/V_i \cap W)_i$ and with some choice of a one-parameter group of dilations $(\overline{\delta}_t)_t$. Let $\pi : V \to V/W$ be the canonical projection and let $|y|_\pi := \inf\{|x|, \pi(x) = y\}$. Although $|\cdot|_\pi$ may not be a quasi-norm on $V/W$, the following holds:

**Proposition 2.8.** *There exists a unique homogeneous quasi-norm $|\cdot|_1$ on $V/W$ such that $|y|_\pi - |y|_1 = o(|y|_\pi)$ if $y \in V/W$ and $|y|_\pi$ is large. In particular, if $D_t^1$ is the homogeneous quasi-norm ball for $|\cdot|_1$ on $V/W$, then there exists $\varepsilon_t > 0$, $\varepsilon_t \to 0$ as $t \to +\infty$, such that*

$$D^1_{t(1-\varepsilon_t)} \subset \pi(D_t) \subset D^1_{t(1+\varepsilon_t)}.$$

We leave the proof of these propositions as an exercise.

2.3. **Volume growth.** Let $N$ be a simply connected nilpotent Lie group and $(C^p(N))_{p=1,\ldots,r}$ its descending central series. The integer $r$ is the nilpotency length, that is the largest $r$ for which $C^r(N)$ is non trivial. We identify $N$ with its Lie algebra $\mathfrak{n}$ via the exponential map, which is a diffeomorphism. The Lie product is a polynomial function on $\mathfrak{n} \times \mathfrak{n}$ and this makes $N$ into a real algebraic group with a Zariski topology. By Theorem 2.1 of [21], a closed subgroup of $N$ is Zariski-dense if and only if it is co-compact. Furthermore, any closed subgroup $H$ of $N$ is contained in a unique minimal connected closed subgroup $\widetilde{H}$ of $N$. The subgroup $\widetilde{H}$ is Zariski closed, simply connected, and $\widetilde{H}/H$ is compact.

It was proved by Guivarc'h in [15] (and independently by Bass [5] in the special case of finitely generated nilpotent groups) that if $G$ is a closed co-compact subgroup of $N$ and $U$ be a compact generating neighborhood of the identity in $G$, then there are positive constants $C_1$ and $C_2$ such that for any positive integer $n$

$$C_1 \cdot n^{d(N)} \leq vol_G(U^n) \leq C_2 \cdot n^{d(N)} \tag{8}$$

where $d(N)$ is an integer called the *homogeneous dimension* of $N$ and is given by the Bass-Guivarc'h formula:

$$d(N) = \sum_{p\geq 1} \dim(C^p(N)) = \sum_{p\geq 1} p \cdot \dim(C^p(N)/C^{p+1}(N)). \tag{9}$$

For general $H$ as above we set $d(H) = d(\widetilde{H})$. For instance, if $\Gamma$ is any nilpotent group generated by a finite symmetric set $S$ then, we can view it, modulo its finite torsion group, as a lattice in a simply connected nilpotent Lie group $N$ according to a theorem of Malcev ([21] chp. 2). Hence $d(\Gamma) = \sum_{p\geq 1} p \cdot rk(C^p(\Gamma)/C^{p+1}(\Gamma))$ and by (8) the ball $S^n$ of radius $n$ in the word metric defined $S$ has, up to multiplicative constants, $n^{d(\Gamma)}$ elements.

The estimate (8) was later refined by Pansu who showed in [20] that $|S^n|/n^{d(\Gamma)}$ has a non-zero limit when $n \to +\infty$. In fact, Pansu's argument can be adapted (see [8] for details) to extend his result to all closed subgroups of $N$, namely:

**Theorem 2.9.** *Let $G$ be a closed subgroup of $N$ and $U$ be a compact generating neighborhood of the identity in $G$. Then there is a positive constant $AsVol(U) > 0$*

$$\lim_{n\to+\infty} \frac{vol_G(U^n)}{n^{d(G)}} = AsVol(U)\cdot$$

Note that the homogeneous dimension $d(N)$ coincides the degree $\deg_{\mathfrak{n}}(\mathfrak{n})$ defined in (5) where the filtration on $\mathfrak{n} = Lie(N)$ is given by the central descending series. If $M$ is a normal closed and connected subgroup of $N$, then we check that $\deg_{\mathfrak{n}}(\mathfrak{m}) = d(N) - d(N/M)$.

2.4. **Polynomials, dilations and quasi-norms.** We recall here a few well-known facts about the analysis on nilpotent Lie groups (see [13]).

2.4.1. *Degree of an element and a polynomial.* Let $N$ be a simply connected nilpotent Lie group, which we identify to its Lie algebra $\mathfrak{n}$ via the exponential map $exp : \mathfrak{n} \to N$. We say that a map $P : N \to \mathbb{R}$ is polynomial if $x \mapsto P(\exp(x))$ is a polynomial map on the real vector space $\mathfrak{n}$. The central descending series $(C^k(\mathfrak{n}))_{k\geq 1}$ gives a canonical filtration on $\mathfrak{n}$ and hence induces a degree function on $\mathfrak{n}$ (as in Prop. 2.2).

Let $(e_i)_{i=1,\dots,n}$ be an *adapted basis* of $\mathfrak{n}$, namely we assume that $C^i(\mathfrak{n}) = span\{e_k | \deg(e_k) \geq i\}$ for all $i \geq 1$. We define the *degree* of a monomial $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ to be $d(\alpha) := \alpha_1 \deg(e_1) + \dots + \alpha_n \deg(e_n)$ for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, and the degree of an arbitrary polynomial map to be the maximum degree of each of its monomials. This definition is easily seen to be independent of the choice of the adapted basis used to define it.

The coordinates of the product of two elements in the basis $(e_i)_{i=1,..,n}$ are obtained from the Campbell-Hausdorff formula as follows (see [13] p. 14):

$$(xy)_i = x_i + y_i + P_i(x,y) \tag{10}$$

where $P_i$ is a polynomial map on $\mathfrak{n} \times \mathfrak{n}$ of the following special type:

$$P_i(x,y) = \sum C_{\alpha,\beta} x^\alpha y^\beta$$

where $d(\alpha) + d(\beta) \leq \deg(e_i)$ and $d(\alpha) \geq 1, d(\beta) \geq 1$ and some constants $C_{\alpha,\beta}$.

2.4.2. *Associated graded algebra, dilations.* Let $(m_p)_{p\geq 1}$ be a collection of supplementary subspaces on $\mathfrak{n}$ as in (6) for $V_i = C^i(\mathfrak{n})$. If $x \in \mathfrak{n}$, we write $x = \sum_{p\geq 1} \pi_p(x)$ where $\pi_p(x)$ is the linear projection onto $m_p$. Let $(\delta_t)_{t>0}$ be the one-parameter group of dilations associated to the $m_p$'s. The dilations $\delta_t$ do not *a priori* preserve the Lie bracket on $\mathfrak{n}$. This is the case if and only if

$$[m_p, m_q] \subseteq m_{p+q} \tag{11}$$

for every $p$ and $q$. If (11) holds, we say that the $(m_p)_{p\geq 1}$ form a *gradation* of the Lie algebra $\mathfrak{n}$, and that $\mathfrak{n}$ is a *graded* Lie algebra and $N$ is called a Carnot group.

If (11) does not hold, we can nevertheless consider a new Lie algebra structure on the real vector space $\mathfrak{n}$ by setting $[x,y]_0 = \pi_{p+q}([x,y])$ if $x \in m_p$ and $y \in m_q$. This new structure $\mathfrak{n}_0$ is graded and the $(\delta_t)_{t>0}$ are automorphisms. We denote by $N_0$ the associated Lie group. In fact the original Lie bracket $[x,y]$ on $\mathfrak{n}$ can be deformed continuously to $[x,y]_0$ by setting $[x,y]_t = \delta_t([\delta_{\frac{1}{t}} x, \delta_{\frac{1}{t}} y])$ and letting $t \to 0$.

On the other hand, the *graded Lie algebra* associated with $\mathfrak{n}$ is by definition $gr(\mathfrak{n}) = \bigoplus_{p\geq 1} C^p(\mathfrak{n})/C^{p+1}(\mathfrak{n})$. endowed by the Lie bracket induced from the Lie bracket of $\mathfrak{n}$. The quotient map $m_p \to C^p(\mathfrak{n})/C^{p+1}(\mathfrak{n})$ gives rise to a linear isomorphism between $\mathfrak{n}$ and $gr(\mathfrak{n})$, which is a Lie algebra isomorphism between $\mathfrak{n}_0$ and $gr(\mathfrak{n})$. Hence graded Lie algebra structures induced by a choice of supplementary subspaces $(m_p)_{p\geq 1}$ as in (6) are all isomorphic to $gr(\mathfrak{n})$.

2.4.3. *Homogeneous quasi-norms on nilpotent Lie groups.* Let $|\cdot|$ be a homogeneous quasi-norm on $N$ with respect to some direct sum decomposition given by $(m_p)_p$'s. Let $(e_i)_i$ be an adapted basis of $\mathfrak{n}$.

**Proposition 2.10.** *There is a constant $C > 0$ such that*

*(a)* $|x_i| \leq C \cdot |x|^{\deg(e_i)}$ *if* $x = x_1 e_1 + \dots + x_n e_n$.

($b$) $|x^{-1}| \leq C \cdot |x|$.
($c$) $|xy| \leq C(|x| + |y| + 1)$.

It is straightforward to check ($a$) and ($b$). It can be a problem that the constant in ($c$) need not be 1. In fact this is why we use the word quasi-norm instead of just norm: we do not require the triangle inequality axiom to hold. However the following lemma of Guivarc'h is often a good enough remedy to this situation. Let $\|\cdot\|_p$ be an arbitrary norm on the vector space $m_p$.

**Lemma 2.11.** *([15] lemme II.1) Up to rescaling each $\|\cdot\|_p$ into a proportional norm $\lambda_p \|\cdot\|_p$ ($\lambda_p > 0$) if necessary, the quasi-norm $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ satisfies $|xy| \leq |x| + |y| + c$ for some constant $c > 0$ and for all $x, y \in N$. Besides $N$ is graded with respect to $(\delta_t)_t$ if and only if $c = 0$.*

The proof is based on the Campbell-Hausdorff formula (10). Lemma 2.11 yields property ($c$) above and also is the key step to prove (8).

**Example 2.12.** *Note that one important class of quasi-norms consists of those of the form $|x| = d(e, x)$, where $d(x, y)$ is a Carnot-Carathéodory Finsler metric induced on a graded nilpotent Lie group by some ordinary norm on the vector subspace $m_1$.*

2.5. **Nicely growing subsets.** We define here *nicely growing subsets*. These are essentially Folner subsets with some extra properties that behave well under intersection with a connected subgroup. Our main result, Theorem 3 below, will hold for all such families of subsets.

Recall that $\deg_N$ is the degree function from Paragraph 2.1.2 defined for all vector subspaces of $\mathfrak{n} = Lie(N)$.

**Definition 2.13.** *We say that a family of measurable subsets $(A_t)_{t>0}$ of $N$ is **nicely growing** if it satisfies the following properties:*
*($i$) $(A_t)_{t>0}$ increases and exhausts $N$, i.e. $A_t \subseteq A_s$ if $t \leq s$ and $\bigcup_{t>0} A_t = N$.*
*($ii$) For every compact subset $K \subset N$, there exists a positive function $\varepsilon_t > 0$ with $\varepsilon_t \to 0$ as $t \to +\infty$ such that $A_{t(1-\varepsilon_t)} \subseteq KA_tK \subseteq A_{t(1+\varepsilon_t)}$ for all $t$ large enough.*
*($iii$) For any connected subgroup $M$ of $N$, there exists a constant $C(M) > 0$ such that*

$$\lim_{t \to +\infty} \frac{vol_M(A_t \cap M)}{t^{\deg_N(M)}} = C(M).$$

*($iv$) There is a constant $\alpha > 1$ such that $A_t A_t^{-1} \subseteq A_{\alpha t}$ for all $t > 1$.*
*Additionally, if $\Gamma$ is a finitely generated torsion free nilpotent group, then a family $(\Lambda_t)_{t>0}$ of subsets of $\Gamma$ is said to be **nicely growing** if there is a nicely growing family $(A_t)_{t>0}$ of subsets of the Malcev closure of $\Gamma$ such that $\Gamma \cap A_{t(1-\varepsilon_t)} \subseteq \Lambda_t \subseteq \Gamma \cap A_{t(1+\varepsilon_t)}$ for all $t > 0$ and some positive function $\varepsilon_t > 0$ with $\varepsilon_t \to 0$ as $t \to +\infty$.*

It will be convenient to broaden this definition a little bit by allowing different degree functions than $\deg_N$ in axiom ($iii$). By a degree function on $N$, we mean any map deg from the set of all connected subgroups of $N$ to $\mathbb{N}$ which is non-decreasing in the sense that $M_1 \subset M_2 \Rightarrow \deg M_1 \leq \deg M_2$. Then we can speak of a nicely growing family of subsets of $N$ *relative to* the degree function deg . By definition, such a family of subsets will satisfy all four axioms except that axiom $(iii)$ will now hold with $\deg(M)$ in place of $\deg_N(M)$. The notion of nicely growing sets is stable under intersection with a connected subgroup say $M$, but then the degree function remains $\deg_N$ .

Let us recall the statement of Proposition 2.1, which is our main goal here because it provides us with the many examples of nicely growing subsets.

**Proposition 2.14.** *For any quasi-norm on $N$, the balls $(D_t)_{t>0}$, $D_t = \{|x| \leq t\}$, form a nicely growing family of subsets of $N$.*

*Proof.* Property $(i)$ is obvious and property $(iv)$ follows from properties $(b)$ and $(c)$ of Prop. 2.10. As for property $(ii)$, it is a consequence of the following more general fact.

**Lemma 2.15.** *Let $|\cdot|$ be a quasi-norm on $N$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ and $C > 0$ such that if $|y| > C$ and $|x| \leq \delta|y|$ we have*

$$||xy| - |y|| \leq \varepsilon|y|.$$

*Proof.* Let $|\cdot|_0$ be the quasi-norm obtained from Guivarch's lemma (Lemma 2.11). Since any two quasi-norms are equivalent, $|x|_0 = o(|y|_0)$ and $|x| = o(|y|)$ are equivalent conditions. Using Prop. 2.10 $(b)$ and Lemma 2.11, we see that if $|x|_0 = o(|y|_0)$ then

$$||xy|_0 - |y|_0| = o(|y|_0). \tag{12}$$

Now write $||xy| - |y|| = \left|\left|\delta_{\frac{1}{|xy|_0}}(xy)\right| |xy|_0 - \left|\delta_{\frac{1}{|y|_0}}(y)\right| |y|_0\right|$. By (12), we get $||xy| - |y|| = \left|\left|\delta_{\frac{1}{|y|_0}}(xy)\right| |xy|_0 - \left|\delta_{\frac{1}{|y|_0}}(y)\right| |y|_0\right| + o(|y|)$. So it remains to show that $\delta_{\frac{1}{|y|_0}}(xy)\delta_{\frac{1}{|y|_0}}(y^{-1})$ tends to 1 as $|y|$ tends to infinity and $|x| = o(|y|)$. By compactness, we may even assume that $\delta_{\frac{1}{|y|_0}}(y) = z$ is fixed and, posing $t = |y|_0$, we have reduced to showing that $\delta_{\frac{1}{t}}(x\delta_t(z)) \to z$ as $t \to +\infty$ and $|x| = o(t)$. To do this we use formula (10) for the product in coordinates

$$\left[\delta_{\frac{1}{t}}(x\delta_t(z))\right]_i = \frac{x_i}{t^{\deg(e_i)}} + z_i + \sum C_{\alpha,\beta} x^\alpha \frac{z^\beta}{t^{\deg(e_i)-d(\beta)}}$$

with the additional constraints $d(\alpha)+d(\beta) \leq \deg(e_i)$ and $d(\alpha) \geq 1$, $d(\beta) \geq 1$. The condition $|x| = o(t)$ means that $x_i = o(t^{\deg(e_i)})$ for any index $i$. As $t$ tends to infinity, we indeed obtain the convergence of the above expression towards $z_i$. □

Observe that the analogous result holds when $xy$ is changed into $yx$ ($x \mapsto |x^{-1}|$ is another equivalent quasi-norm).

We now turn to the proof of property $(iii)$. This is where we will need the discussion on filtrations from the previous sections. Let us denote by $\mathfrak{m} = Lie(M)$ the Lie subalgebra of $\mathfrak{n} = Lie(N)$ corresponding to $M$. Let $d = \dim(M)$ and $(f_1, ..., f_d)$ a basis for the vector space $\mathfrak{m}$.

The orthogonal linear transformations (for some Euclidean norm $\|\cdot\|$ on $\mathfrak{n}$) act transitively on $Gr_d(\mathfrak{n})$, the Grassmannian variety of $d$-dimensional linear subspaces of $\mathfrak{n}$. Hence there is some orthogonal map $o_t$ such that $o_t^{-1}\delta_{\frac{1}{t}}$ fixes $\mathfrak{m}$. Let $vol_M$ be a Haar measure on $M$, which we identify with Lebesgue measure on $\mathfrak{m}$. Let $\alpha_t^{-1}$ be the abolute value of the determinant of the endomorphism induced on $\mathfrak{m}$ by $o_t^{-1}\delta_{\frac{1}{t}}$. We have

$$\alpha_t^{-1} = \frac{\left\|\delta_{\frac{1}{t}} f_1 \wedge ... \wedge \delta_{\frac{1}{t}} f_d\right\|}{\|f_1 \wedge ... \wedge f_d\|}$$

We can estimate the behavior of $\alpha_t$ when $t \to +\infty$. As follows from (5) above,

$$\deg(f_1 \wedge ... \wedge f_d) = \sum_{i \geq 1} \dim(\mathfrak{m} \cap C^i(\mathfrak{n})) = \deg_N(M)$$

Hence by Prop. 2.4 there is a non-zero $\xi \in \Lambda^d\mathfrak{n}$ such that

$$\lim_{t \to +\infty} t^{\deg_N(M)} \delta_{\frac{1}{t}} f_1 \wedge ... \wedge \delta_{\frac{1}{t}} f_d = \xi \tag{13}$$

Then we can define

$$c_M := \lim_{t \to +\infty} \frac{\alpha_t}{t^{\deg_N(M)}} = \frac{\|f_1 \wedge ... \wedge f_d\|}{\|\xi\|} > 0 \tag{14}$$

By (13) the subspaces $\delta_{\frac{1}{t}}\mathfrak{m}$ converge to a limit subspace $\mathfrak{m}_\infty$ in the Grassmannian variety. So we could choose $o_t$ so that $o_t$ converges to some $o$ as $t \to +\infty$. Then $\mathfrak{m}_\infty = o\mathfrak{m}$. Also observe that $\mathfrak{m}_\infty$ is invariant under the full one-parameter group of dilations $(\delta_t)_t$.

**Lemma 2.16.** *The subspace $\mathfrak{m}_\infty$ is a Lie subalgebra of $\mathfrak{n}_0$.*

*Proof.* We need to show that if $x, y \in \mathfrak{m}_\infty$, then $x \cdot y \in \mathfrak{m}_\infty$, where $x \cdot y$ is the product in $\mathfrak{n}_0$. Let $x_t, y_t$ in $\mathfrak{m}$ be such that $x = \lim_{t \to +\infty} \delta_{\frac{1}{t}}(x_t)$ and $y = \lim_{t \to +\infty} \delta_{\frac{1}{t}}(y_t)$. Then by (10) we have

$$\left[\delta_{\frac{1}{t}}(x_t y_t) - \delta_{\frac{1}{t}}(x_t) \cdot \delta_{\frac{1}{t}}(y_t)\right]_i = \frac{1}{t^{\deg(e_i)}} \sum_{d_\alpha + d_\beta < \deg(e_i)} C_{\alpha,\beta} x_t^\alpha y_t^\beta.$$

By Prop. 2.10 (a) this expression is a $O(\frac{1}{t})$. Hence $x \cdot y = \lim_{t \to +\infty} \delta_{\frac{1}{t}}(x_t y_t)$ , i.e. $x \cdot y \in \mathfrak{m}_\infty$. □

Let $vol_{M_\infty}$ be a Haar measure on $M_\infty$, again identified with Lebesgue measure on $\mathfrak{m}_\infty$. Fixing the Haar measure on $N$, the choice of a Euclidean norm on $\mathfrak{n}$ specifies a normalization for $vol_M$ and $vol_{M_\infty}$. Then we have:

**Lemma 2.17.** *We have the following weak convergence of measures:*

$$\lim_{t \to +\infty} \frac{1}{t^{\deg_N(\mathfrak{m})}} \left(\delta_{\frac{1}{t}}\right)_* vol_M = c_M \cdot vol_{M_\infty}$$

*Proof.* By (14) we have

$$\lim_{t\to+\infty}\frac{1}{t^{\deg_N(\mathfrak{m})}}\left(\delta_{\frac{1}{t}}\right)_* vol_M=\lim_{t\to+\infty}\frac{\alpha_t}{t^{\deg_N(\mathfrak{m})}}(o_t)_* vol_M=c_M\cdot vol_{M_\infty}$$

□

By Lemma 2.17 applied to $D_1$, we get the desired convergence, namely

$$\lim_{t\to+\infty}\frac{vol_M(M\cap D_t)}{t^{\deg_N(\mathfrak{m})}}=c_M\cdot vol_{M_\infty}(D_1)$$

after we check that $vol_{M_\infty}(\partial D_1)=0$. This is clear however since, using the invariance of $M_\infty$ under $(\delta_t)_{t>0}$, the function $vol_{M_\infty}(D_t)=|Jac((\delta_t)_{|M_\infty})|\cdot vol_{M_\infty}(D_1)$ is continuous as a function of $t$. This ends the proof of Proposition 2.14. □

2.6. **Other examples of nicely growing subsets.** Other types of balls give rise to nicely growing subsets besides quasi-norm balls. We show here two more examples: balls obtained by considering exponential coordinates of the second kind, and more importantly $\rho$-balls for any "reasonable" left-invariant distance $\rho$ on $N$.

2.6.1. *Balls in privileged coordinates.* Let $(e_i)_{1\le i\le n}$ be an adapted basis as in Paragraph 2.4.1, and $(x_i)_i$ be the associated exponential coordinates of the first kind, i.e. $x=x_1e_1+\ldots+x_ne_n$. For $i=1,...,n$, let $P_i$ be a polynomial map on $\mathfrak{n}$ of total (homogeneous) degree $\le d_i=\deg(e_i)$, with $P_i(0)=0$. The map $P_i$ can be split in two parts, $P_i(x)=L_i(x)+M_i(x)$ where $L_i(x)$ is a linear form on $\mathfrak{n}$ depending only on those coordinates $x_j$ such that $d_j=d_i$ and where $M_i(x)$ is a polynomial depending only on those $x_j$'s with $d_j\le d_i-1$. Assume further that the $L_i$'s, $i=1,...,n$, are linearly independent. Then, following [6], we call *privileged coordinates* any choice of coordinates on $\mathfrak{n}$ that is obtained from the $x_i$'s by a coordinate change $\phi: x\mapsto x'$ of the form

$$x_i'=P_i(x) \tag{15}$$

For instance, writing $x$ in the associated exponential coordinates of the second kind, i.e.

$$x=\exp(x_1'e_1)\cdot\ldots\cdot\exp(x_n'e_n)$$

it is easy to check that (15) holds and we even have $L_i(x)=x_i$. We have:

**Proposition 2.18.** *Let $|\cdot|$ be a homogeneous quasi-norm on $N$ and $\phi: x\mapsto x'$ a privileged coordinate change. Then there exists a unique homogeneous quasi-norm $|\cdot|'$ such that $|\phi(x)|=|x|'+o(|x|')$. In particular the balls $\{|\phi(x)|\le t\}$ form a nicely growing family if $t$ tends to $+\infty$.*

*Proof.* Let us write $\phi(x)=\psi(x)+\eta(x)$ where $(\psi(x))_i$ is the component of $P_i(x)$ that is homogeneous of homogeneous degree $d_i$. Then clearly, the map $x\mapsto|\psi(x)|$ is a homogeneous quasi-norm (property $(i)$ of Def. 2.5 follows from the linear independence of the $L_i$'s). Therefore, after composing by $\psi^{-1}$, we may assume that $\psi(x)=x$, i.e. $\phi(x)=x+\eta(x)$ where each $(\eta(x))_i$ is a polynomial map of degree $\le d_i-1$. We wish to show that $|\phi(x)| = |x| + o(|x|)$. Since $|\eta(x)| = o(|x|)$ the proof is a straightforward copy of that of Lemma 2.15.

Given $\varepsilon > 0$, we thus get $D_{t(1-\varepsilon)} \subset \{|\phi(x)| \leq t\} \subset D_{t(1+\varepsilon)}$, for all large $t$, where $D_t = \{|x| \leq t\}$. By Prop. 2.14, this clearly implies all of the four defining properties for nicely growing subsets. □

2.6.2. *Balls for left-invariant metrics.* Following [8], we say that a distance function $\rho$ on $N$ is a *periodic metric* if it is left-invariant under some co-compact subgroup of $N$ and is asymptotically geodesic, namely for every $\varepsilon > 0$ there is $s > 0$ such that $\forall x, y \in N$, one can find points $x_1 = x$, $x_2, ..., x_n = y$ in $N$ such that $\rho(x, y) \geq (1 - \varepsilon) \sum_{i=1}^{n-1} \rho(x_i, x_{i+1})$ and $\rho(x_i, x_{i+1}) \leq s$ for each $i$. This means that we require a kind of weak existence of geodesic axiom. Examples of such metrics are given by left-invariant Riemmanian or sub-Riemmanian metrics on $N$, and also by word metrics induced by a compact generating set of $N$ (see [8]).

In [20], Pansu associates to any such $\rho$ a Carnot-Carathéodory metric $d_\rho$ on $N$ in the following way. Let $(\delta_t)_t$ be a one-parameter group of (linear) dilations on $\mathfrak{n}$ and $\mathfrak{n} = \oplus_p m_p$ the eigenspace decomposition, with $\pi_1$ the projection to $m_1$. This yields a graded structure $\mathfrak{n}_0$ on $\mathfrak{n}$ as defined in 2.4.2. Let $E_s$ is the closed convex hull of all $\pi_1(x)/\rho(e, x)$ with $x \in N$ and $\rho(e, x) > s$ and $E = \bigcap_{s>0} E_s$. Pansu shows that $E$ is a compact symmetric subset of $m_1$ with non-empty interior. Hence it is the unit ball of some norm $\|\cdot\|_\rho$ on $m_1$. In turn, this norm defines a Carnot-Carathéodory Finsler distance $d_\rho$ on $N_0$ by setting $d_\rho(x, y) = \inf\{L(\gamma), \gamma \text{ horizontal path from } x \text{ to } y\}$, where horizontal means almost everywhere tangent to a $\mathfrak{n}_0$-left translate of $m_1$ and $L(\gamma)$ is the length of $\gamma$ measured according to $\|\cdot\|_\rho$. Pansu's main result reads (see [20], or [8] for a proof):.

**Theorem 2.19.** *([20]) For any periodic metric $\rho$ on $N$,*

$$\lim_{x\to\infty} \frac{\rho(e, x)}{d_\rho(e, x)} = 1 \tag{16}$$

**Corollary 2.20.** *The balls $B_\rho(t) = \{x \in N, \rho(e, x) \leq t\}$ form a family of nicely growing subsets of $N$.*

*Proof.* Clearly $x \mapsto d_\rho(e, x)$ is a homogeneous quasi-norm, so the Corollary follows from Prop. 2.14. □

When proving Theorem 1.1 we will apply this result to the case when $\rho(x, y) = d(\gamma_x, \gamma_y)$, where $d$ is any word metric on a lattice $\Gamma$ in $N$, and $x \in \gamma_x F$ for some fixed compact fundamental domain $F$ of $\Gamma$ in $N$.

## 3. Equidistribution

In this section, we prove Theorem 1.1 via the following version of it:

**Theorem 3.1.** *(Dense subgroups are equidistributed) Let $G$ be a closed subgroup of a simply connected nilpotent Lie group and $\Gamma$ a finitely generated torsion-free nilpotent group. Let*

*$\phi : \Gamma \to G$ be a homomorphism with dense image. Suppose that $(\Lambda_t)_{t>0}$ is a nicely growing family of subsets of $\Gamma$ (see Definition 2.13). Then there is a positive constant $C_1 > 0$ depending on $(\Lambda_t)_{t>0}$ and on the choice of a Haar measure $vol_G$ on $G$, such that for any bounded Borel subset $B \subset G$ with negligible boundary we have*

$$\frac{\#\{\gamma \in \Gamma, \gamma \in \Lambda_T, \phi(\gamma) \in B\}}{T^{d(\Gamma)-d(G)}} \underset{T\to+\infty}{\to} C_1 \cdot vol_G(B) \tag{17}$$

*where $d(\Gamma)$ and $d(G)$ are the integers defined in 2.3.*

Recall that by Malcev's theory (see [21] Theorem 2.18), every finitely generated torsion-free nilpotent group embeds in a simply connected nilpotent Lie group, its Malcev closure. Let $N$ be the Malcev closure of $\Gamma$. Let $(D_t)_{t>0}$ be the family of nicely growing subsets of $N$ such that $\Gamma \cap D_{t(1-\varepsilon_t)} \subseteq \Lambda_t \subseteq \Gamma \cap D_{t(1+\varepsilon_t)}$. First observe that it is enough to prove the theorem for sets $\Lambda_t$ of the form $\Lambda_t = \Gamma \cap D_t$.

Before starting the proof of Theorem 3.1, let us briefly explain how one can also reduce to the case when $G$ is connected. Recall that since $G$ is closed in a simply connected nilpotent Lie group, it is co-compact in the simply connected subgroup $\widetilde{G}$ (its Zariski-closure) defined in 2.3. By Malcev's rigidity (see [21] Theorem 2.11) $\phi$ extends to an epimorphism $\phi : N \to \widetilde{G}$, which gives rise to an isomorphism $N/N_0 \overset{\simeq}{\to} \widetilde{G}/G^\circ$, where $N_0 = \widetilde{\Gamma^\circ}$ is normal in $N$ and contains $M = \ker\phi$, and $\Gamma^\circ = \Gamma \cap \phi^{-1}(G^\circ)$. For every bounded Borel subset of $G$, the set $\{\gamma \in \Gamma, \gamma \in D_t, \phi(\gamma) \in B\}$ can be split into a finite number of translates of $\{\gamma \in \Gamma^\circ, \gamma \in \gamma_i D_t, \phi(\gamma) \in B_i\}$ where $\phi(\gamma_i^{-1})B_i \subset B$ and $\gamma_i \in \Gamma$ and $B_i \subset G^\circ$. Since $G^\circ$ is simply connected and $\Gamma^\circ$ dense in it, we may just as well work with these groups. However $D_t \cap N_0$ is a nicely growing family of subsets of $N_0$ only relative to the degree function $\deg_N$ and not relative to $\deg_{N_0}$ (see the remarks below Definition 2.13). Nevertheless, we show below that if $D_t$ is a nicely growing family of subsets of $N$ with respect to an arbitrary degree function, then under the hypothesis and notation of Theorem 3.1

$$\frac{\#\{\gamma \in \Gamma, \gamma \in D_t, \phi(\gamma) \in B\}}{vol_M(M \cap D_t)} \underset{t\to+\infty}{\to} C_1 \cdot vol_G(B) \tag{18}$$

We will thus assume below that $G$ is connected and simply connected.

3.1. **Unique ergodicity and counting.** By Malcev's rigidity ([21] Theorem 2.11), $\phi$ extends to an epimorphism $\phi : N \to G$. Let $M = \ker\phi$. We want to find the asymptotics of the number of points of $\Gamma \cap D_t$ which lie in $\phi^{-1}(B)$, the inverse image of the bounded Borel subset $B$ by the map $\phi$. Hence we are dealing with a counting problem, which we will treat via ergodic theory. The use of ergodic theory to solve counting problems is now standard (see for instance [12], [11] and [4] for a survey of these techniques) and what we are going to present here is yet another illustration of these ideas.

It is a fairly general principle in ergodic theory that the ergodic properties of the action of a closed subgroup $H_1$ of a group $H$ on the homogeneous space $H/H_2$ can be deduced from the ergodic properties of the action of the closed subgroup $H_2$ on $H/H_1$ and vice-versa. Here we will deduce the equidistribution of $\Gamma$ in $G \simeq N/M$ from the equidistribution of an $M$-orbit on the nilmanifold $N/\Gamma$. In order to do this, we first recall the following well-known theorem (see [22] Theorems 3.6 and 3.8, and also [3] Lemma 5.1):

**Theorem 3.2.** *(Unique ergodicity criterion for nilflows) Let $\Gamma$ be a co-compact lattice in a simply connected nilpotent Lie group $N$ and let $M$ be a closed subgroup of $N$. The following are equivalent:*
$(i)$ *The subset $M\Gamma$ is dense in $N$.*
$(ii)$ *$M$ acts ergodically on $N/\Gamma$.*
$(iii)$ *The $M$-action on $N/\Gamma$ is uniquely ergodic.*

Since $\Gamma$ is dense in $G \simeq N/M$, Theorem 3.2 implies that the action of $M$ on $N/\Gamma$ is uniquely ergodic, i.e. that the normalized Haar measure $\nu$ on $N/\Gamma$ is the only $M$-invariant probability measure on $N/\Gamma$. In order to translate the counting problem into an equidistribution question, we introduce the following counting function for $x \in N$,

$$F_t^B(x) = \#\left\{\gamma \in \Gamma, x\gamma \in D_t, \phi(x\gamma) \in B\right\}$$

Note that $F_t^B$ is $\Gamma$-invariant on the right hand side, hence it really defines a measurable function on the nilmanifold $N/\Gamma$. Note further that the quantity we are interested in is precisely $F_t^B(e) = \#\left\{\gamma \in \Gamma, \gamma \in D_t, \phi(\gamma) \in B\right\}$, and our goal (i.e. (18)) is to prove the convergence of $F_t^B(e)/V_t$, where $V_t = vol_M(M \cap D_t)$. The main step is to prove weak convergence ((24) below) of the functions $F_t^B(x)/V_t$.

3.2. **Proof of Theorems 3.1 and 1.1.** Since the Haar measure $\nu$ on $N/\Gamma$ is already normalized, the choice of a Haar measure on $G$, denoted by $vol_G$, determines uniquely a Haar measure on the kernel $M$, which we denote by $vol_M$. For every $g \in N$ we denote by $\nu_t^g$ the image under $\pi : N \to N/\Gamma$ of the uniform probability measure supported on $g^{-1}D_t \cap M$, i.e.

$$\nu_t^g = \pi_*\left(\frac{1_{D_t \cap gM}(gy)}{vol_M(g^{-1}D_t \cap M)} vol_M(dy)\right) \tag{19}$$

We let $\psi$ be a continuous function on $N/\Gamma$ and we consider the scalar product

$$\left\langle F_t^B, \psi\right\rangle = \int_{N/\Gamma} \sum_{\gamma\in\Gamma} 1_{x\gamma\in D_t} 1_{\phi(x\gamma)\in B}\psi(\overline{x})\nu(d\overline{x}) = \int_N 1_{D_t\cap\phi^{-1}(B)}(x)\psi(\overline{x})vol_N(dx)$$

Decomposing the Haar measure on $N$ along the fibers of the projection $\phi : N \to N/M$, we obtain,

$$\begin{aligned}\left\langle F_t^B, \psi\right\rangle &= \int_G 1_B(g) \int_M 1_{D_t}(gy)\psi(\overline{gy})vol_M(dy)vol_G(dg) \\ &= \int_B vol_M(g^{-1}D_t \cap M)\left(\int_{N/\Gamma} \psi(gz)\nu_t^g(dz)\right) vol_G(dg)\end{aligned} \tag{20}$$

In order to go further, we need the following proposition, which is the consequence of the unique ergodicity of the $M$-action on $N/\Gamma$.

**Proposition 3.3.** *The following weak convergence of probability measures on $N/\Gamma$ holds uniformly when $g$ varies in compact subsets of $N$.*

$$\lim_{t\to+\infty} \nu_t^g = \nu$$

*Proof.* Let $\nu_\infty$ be a weak limit of $\nu_t^g$ as $t \to +\infty$ and $g$ converges to some element in $N$. Since the $M$-action is uniquely ergodic on $N/\Gamma$ by Theorem 3.2, it is enough to show that $\nu_\infty$ is invariant under $M$. Let $\mu_t^g$ be the probability measure on $N$ such that $\nu_t^g = \pi_*(\mu_t^g)$ as defined in (19). The map $\pi_* : \mathcal{P}(N) \to \mathcal{P}(N/\Gamma)$ between spaces of probability measures is an $N$-equivariant contraction for the total variation norm, hence to show that $\nu_\infty$ is invariant under $M$, it is enough to prove the following lemma.

**Lemma 3.4.** *For any $h \in M$, the following convergence holds uniformly in $g$ as $g$ varies in compact subsets of $N$*

$$\lim_{t\to+\infty} \|\delta_h * \mu_t^g - \mu_t^g\| = 0$$

*Proof.* Since $h \in M$, the measures $\mu_t^g$ and $\delta_h * \mu_t^g$ are supported on $M$ and are absolutely continuous with respect to the Haar measure $vol_M$. Hence the total variation norm is simply the $\mathbb{L}^1$ norm. So

$$\|\delta_h * \mu_t^g - \mu_t^g\| = \frac{vol_M\left(M \cap (h^{-1}g^{-1}D_t \Delta g^{-1}D_t)\right)}{vol_M(M \cap g^{-1}D_t)} \tag{21}$$

where $\Delta$ is the symmetric difference operator. Note that combining both properties $(ii)$ and $(iii)$ of Definition 2.13, the following convergence holds uniformly when $g$ varies in compact subsets of $N$.

$$\lim_{t\to+\infty} \frac{vol_M(g^{-1}D_t \cap M)}{t^{\deg(M)}} = C(M) \tag{22}$$

Similarly, by property $(ii)$, if $g$ lies in a compact set, there will be some positive function $\varepsilon_t > 0$ with $\varepsilon_t \to 0$ such that we can write $h^{-1}g^{-1}D_t \Delta g^{-1}D_t \subseteq D_{t+t\varepsilon_t} \backslash D_{t-t\varepsilon_t}$, But by property $(iii)$ we can conclude that

$$\lim_{t\to+\infty} \frac{vol_M\left((D_{t+t\varepsilon_t} \backslash D_{t-t\varepsilon_t}) \cap M\right)}{vol_M(M \cap D_t)} = 0 \tag{23}$$

Combining (21) with (22) and (23) we are done. □

□

Let us resume the proof of Theorem 3.1. Since $B$ is bounded and $g \in B$ in the integral (20), when $t$ tends to $+\infty$ we obtain the weak convergence:

$$\lim_{t\to+\infty} \frac{\left\langle F_t^B, \psi \right\rangle}{vol_M(D_t \cap M)} = vol_G(B) \cdot \int_{N/\Gamma} \psi(z) d\nu(z) \tag{24}$$

In order to get the desired asymptotics for $F_t^B(e)$, we need to compare it to $\langle F_t^B, \psi\rangle$ where $\psi$ is chosen to better and better approximate the Dirac distribution $\delta_e$. For every sufficiently small neighborhood of the identity $U$ in $N$, which is homeomorphic to $U$ via the covering $\pi$, we may consider a bump function $\psi$ supported on $\pi(U)$ (i.e. a non-negative continuous function with total sum equal to 1). Then $F_t^B(e) \leq F_{t+o(t)}^{\phi(U)B}(g)$ for any $g \in U$ by property $(ii)$ of nicely growing subsets. Note also that $vol_G(\phi(U)B)$ gets closer and closer to $vol_G(B)$ as $U$ tends to the identity. It follows that $F_t^B(e) \leq \left\langle F_{t+o(t)}^{\phi(U)B}, \psi\right\rangle$ and from property $(iii)$ of nicely growing subsets, applying (24), we obtain that

$$\overline{\lim_{t\to+\infty}} \frac{F_t^B(e)}{vol_M(D_t \cap M)} \leq vol_G(B) \tag{25}$$

The lower bound is only slightly more delicate. We have $F_t^B(e) \geq F_t^{\mathring{B}}(e)$ where $\mathring{B}$ is the interior of $B$. Since $B$ has negligible boundary, for any sufficiently small neighborhood $V$ of the identity in $G$, there exists an open subset $B_V$ of $B$ such that $VB_V \subset B$ and $vol_G(B_V) \to vol_G(B)$ as $V$ narrows to the identity.

Now let $U$ be a neighborhood of the identity in $N$ so small that $\phi(U^{-1}) \subset V$. Then for any $g \in U$, $F_t^B(e) \geq F_{t-o(t)}^{B_V}(g)$ for any $g \in U$. It follows that $F_t^B(e) \geq \left\langle F_{t-o(t)}^{B_V}, \psi\right\rangle$ for any bump function supported on $\pi(U)$. Again from property $(iii)$ of nicely growing subsets, applying (24), we obtain

$$\underline{\lim}_{t\to+\infty} \frac{F_t^B(e)}{vol_M(D_t \cap M)} \geq vol_G(B).$$

We have proved

$$\lim_{t\to+\infty} \frac{F_t^B(e)}{vol_M(D_t \cap M)} = vol_G(B)$$

which, together with the volume estimate given by property $(iii)$ of Definition 2.13, ends the proof of Theorem 3.1.

*Proof of Theorem 1.1.* If $T$ is the torsion subgroup of $\Gamma$, the map $\phi$ factors through $T$ to a quotient map $\overline{\phi} : \Gamma/T \to G$. Since $T$ is finite, there is an integer $c > 0$ such that $S^{n-c} \cdot T \subset S^n$. It follows that $|S^n \cap \phi^{-1}(B)| \sim |T| \cdot |\overline{S}^n \cap \overline{\phi}^{-1}(B)|$ where $\overline{S}$ is the image of $S$ in $\Gamma/T$. Moreover the $\overline{S}^n$'s is a family of nicely growing subsets of $\Gamma/T$ according to Corollary 2.20. Therefore Theorem 1.1 follows from Theorem 3.1. □

*Proof of Corollary 1.2.* Apply Theorem 1.1 to $B = U$ and $B = V$ with $\phi = id$ and take the ratio. □

We finally state one last result which generalizes Theorem 3.1 in an obvious way (we assume here $G$ simply connected) and whose proof we only sketch because it is entirely analogous to the proof of Theorem 3.1 we just gave and presents no additional difficulties. Recall from Section 2.5 Lemma 2.16 that $M_\infty = \lim_{t\to+\infty} \delta_{\frac{1}{t}} M$ is a connected subgroup of $N_0$. Let $c_M > 0$ be the constant from Lemma 2.17. We have:

**Theorem 3.5.** *For any bounded Borel subset $B$ of $G$ with $vol_G(\partial B) = 0$ we have the following weak convergence of measures on $N$,*

$$\lim_{T\to+\infty} \frac{1}{T^{d(\Gamma)-d(G)}} \sum_{\gamma\in\phi^{-1}(B)} \mathbf{\Delta}_{\delta_{\frac{1}{T}}(\gamma)} = c_M \cdot vol_G(B) \cdot vol_{M_\infty}$$

*where $\mathbf{\Delta}_x$ is the Dirac mass at $x$.*

*Proof sketch.* Let $f \geq 0$ be a compactly supported function on $N$, let $h = \mathbf{1}_B$ and set $F_t(x) = \sum_{\gamma\in\Gamma} f(\delta_{\frac{1}{T}}(x\gamma))h(\phi(x\gamma))$ for $x \in N$. Let $V_t = \int_M f(\delta_{\frac{1}{T}}(m))dvol_M(m)$. By Lemma 2.17, $V_t/t^{\deg_N(\mathfrak{m})} \to c_M \cdot \int f dvol_{M_\infty}$. When this is positive, the unique ergodicity of $M$ on $N/\Gamma$ yields as above the weak convergence $\langle F_t^B, \psi\rangle / V_t \to vol_G(B) \cdot \int_{N/\Gamma} \psi d\nu$. The point-wise convergence is derived in a similar way. $\square$

3.3. **Local limit theorem for random walk averages.** In this last paragraph we prove Corollary 1.3. The proof makes use of the results obtained by Varopoulos and Alexopoulos about convolution powers of measures on nilpotent groups. We first explain some terminology and refer the reader to [1] for the details.

Let $\mu = \sum_{s\in S} \mu(s)\delta_s$ be a symmetric probability measure on $G$, whose finite support $S$ generates a dense subgroup $\Gamma$ of $G$. Let $\phi : N \to G \simeq N/M$ be the map given by Malcev's rigidity as before. We now define the sub-Laplacian associated to $\mu$ on $N$, by setting

$$L_\mu = \sum_{i=n_1+1}^{n_1+n_2} b_i X_i + \frac{1}{2} \sum_{1\leq i,j\leq n_1} a_{ij} X_i X_j$$

where the $X_i$'s are $N$-left invariant vector fields on $N$ that form an adapted basis for the Lie algebra $\mathfrak{n}$ (i.e. $\mathfrak{n} = \oplus_{p\geq1} m_p$ and $C^q(\mathfrak{n}) = \oplus_{p\geq q} m_p$ where $C^q(\mathfrak{n}) = span\{X_i, d_i \geq q\}$ and $n_p = \dim m_p$). We let $(\delta_t)_{t>0}$ be the corresponding one-parameter group of dilations. The coefficients are defined by

$$\begin{aligned} a_{ij} &= \int x_i x_j d\mu(x) \text{ if } 1 \leq i,j \leq n_1 \\ b_i &= \int x_i d\mu(x) \text{ if } n_1 < i \leq n_1 + n_2 \end{aligned}$$

By definition, the heat kernel $(p_t)_{t>0}$ associated to $\mu$ on $G$ is the solution to the heat equation for $L_\mu$ that is

$$\frac{\partial p_t}{\partial t} = L_\mu p_t \tag{26}$$

Since $\Gamma$ is dense in $G$, the matrix $(a_{ij})_{1\leq i,j\leq n_1}$ is non-degenerate and equation (26) has a unique solution $(p_t)_{t>0}$ such that $p_t$ is smooth and positive everywhere with total integral over $N$ equal to 1. We will also need to consider the "heat kernel at infinity" $(p_t^0)_{t>0}$ defined by the same equation, except that the $X_i$'s are replaced by the corresponding left-invariant vector fields for the graded group structure $N_0$ associated to the group of dilations $(\delta_t)_{t>0}$ (see 2.4.2) The $(p_t^0)_{t>0}$ enjoy the scaling property

$$t^{\frac{d(\Gamma)}{2}} p_t^0(\delta_{\sqrt{t}}(x)) = p_1^0(x). \tag{27}$$

Let $|\cdot|$ be a quasi-norm on $N$. We have:

**Theorem 3.6.** *([1], Corollary 1.19) For every $\alpha$, $0 < \alpha < \frac{1}{2}$, there is $c_\alpha > 0$ such that $\forall n \in \mathbb{N}$, $\forall \gamma \in \Gamma$,*

$$n^{\frac{d(\Gamma)}{2}} |\mu^{*n}(\gamma) - p_n(\gamma)| \leq \frac{c_\alpha}{n^\alpha} \exp\left(\frac{-|\gamma|^2}{c_\alpha \cdot n}\right) \tag{28}$$

**Theorem 3.7.** *([1], Theorem 6.6) $\exists c > 0$, $\forall n \in \mathbb{N}$, $\forall x$*

$$n^{\frac{d(\Gamma)}{2}} \left|p_n(x) - p_n^0(x)\right| \leq \frac{c}{\sqrt{n}} \tag{29}$$

**Theorem 3.8.** *([1], Corollary 6.5) Let $d_i = (X_i)$. $\exists c > 0$, $\forall k \in \mathbb{N}$, $\forall i_1, ..., i_k \geq 0$, $\forall t > 0$,*

$$|X_{i_1} \cdot ... \cdot X_{i_k} p_t(x)| \leq \frac{c}{t^{(d(\Gamma)+d_{i_1}+...+d_{i_k})/2}} \exp\left(\frac{-|x|^2}{c \cdot t}\right) \tag{30}$$

*Proof of Corollary 1.3.* We have $\mu^{*n}(B) = \sum_{\gamma \in \phi^{-1}(B)} \mu^{*n}(\gamma)$. Fix $\alpha \in (0, \frac{1}{2})$ and let $\varepsilon > 0$ with $\varepsilon \cdot (d(\Gamma) - d(G)) < \alpha$. Then combining (28) and (30) we get $n^{\frac{d(G)}{2}} \sum_{|\gamma| > n^{\frac{1}{2}+\varepsilon}} \mu^{*n}(\gamma) \ll n^{\frac{d(G)}{2}} \sum_{k > n^{\frac{1}{2}+\varepsilon}} k^{d(\Gamma)} \cdot \exp(\frac{-k^2}{cn})$ which tends to 0 as $n$ tends to $+\infty$. The same holds for $p_n(\gamma)$ or $p_n^0(\gamma)$ in place of $\mu^{*n}(\gamma)$. So when computing $n^{\frac{d(G)}{2}} \mu^{*n}(B)$, we may as well restrict to counting points $\gamma \in B$ with $|\gamma| \leq n^{\frac{1}{2}+\varepsilon}$. According to Theorem 3.1 $\#\{\gamma \in \Gamma, |\gamma| \leq n, \gamma \in \phi^{-1}(B)\} = O(n^{d(\Gamma)-d(G)})$, hence combining this estimate with (28) we get,

$$n^{\frac{d(G)}{2}} \mu^{*n}(B) = \sum_{|\gamma| \leq n^{\frac{1}{2}+\varepsilon}, \gamma \in \phi^{-1}(B)} n^{\frac{d(G)}{2}} p_n(\gamma) + o(1) = \sum_{\gamma \in \phi^{-1}(B)} n^{\frac{d(G)}{2}} p_n^0(\gamma) + o(1)$$

By the scaling property (27) $n^{\frac{d(G)}{2}} p_n^0(\gamma) = n^{-\frac{d(\Gamma)-d(G)}{2}} p_1^0(\delta_{\frac{1}{\sqrt{n}}}(\gamma))$. Then by Theorem 3.5

$$n^{\frac{d(G)}{2}} \mu^{*n}(B) = \frac{1}{n^{\frac{d(\Gamma)-d(G)}{2}}} \sum_{\gamma \in \phi^{-1}(B)} p_1^0(\delta_{\frac{1}{\sqrt{n}}}(\gamma)) + o(1) = c_M \cdot vol_G(B) \cdot \int_{M_\infty} p_1^0 dvol_{M_\infty} + o(1)$$

□

## References


[1] G. Alexopoulos, *Random walks on discrete groups of polynomial volume growth*, Annals of Proba. vol **30**, n. 2, (2002), p. 723-801.

[2] V.I. Arnol'd and A.L. Krylov, *Uniform distribution of points on a sphere and certain ergodic properties of solutions of linear ordinary differential equations in a complex domain*, Dokl. Akad. Nauk SSSR **148**, (1963) p. 9–12.

[3] L. Auslander and J. Brezin, *Uniform distribution in solvmanifolds*, Advances in Math. **7**, 111–144 (1971).

[4] M. Babillot, *Points entiers et groupes discrets, de l'analyse aux systèmes dynamiques*, in Panoramas et Synthèses SMF Monographs, **13**, "Rigidité, groupe fondamental et dynamique", 1-119, (2002).

[5] H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. (3) **25** (1972), 603–614.

[6] A. Bellaiche, *The tangent space in sub-Riemannian geometry*, in Sub-Riemannian Geometry, edited by A. Bellaiche and J-J. Risler,1–78, Progr. Math., **144**, Birkhauser, (1996).

[7] E. Breuillard, *Random walks on Lie groups*, preprint, www.math.polytechnique/˜breuilla/part0gb.pdf

[8] E. Breuillard, *Geometry of locally compact groups with polynomial growth and shape of large balls*, preprint 2007.

[9] E. Breuillard, *Local limit theorems and equidistribution of random walks on the Heisenberg group*, Geom. Funct. Anal (GAFA), **15** (2005) no 1, 49p.

[10] W. Duke, N. Rudnick, P. Sarnak, *Density of integer points on affine homogeneous varieties*, Duke Math. J. **71** (1993), no. 1, 143–179.

[11] A. Eskin, *Counting problems and semisimple groups*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 539–552.

[12] A. Eskin, S. Mozes, N. Shah, *Unipotent flows and counting lattice points on homogeneous varieties*, Ann. of Math. (2) **143** (1996), no. 2, 253–299.

[13] R. W. Goodman, *Nilpotent Lie groups: structure and applications to analysis*, LNM **562**, Springer-Verlag (1976).

[14] M. Gromov, *Carnot-Carathéodory spaces seen from within*, in Sub-Riemannian Geometry, edited by A. Bellaiche and J-J. Risler, 79-323, Birkauser (1996).

[15] Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Sc. Math. France **101**, (1973), p. 353-379.

[16] Y. Guivarc'h, *Equirépartition dans les espaces homogènes,* Théorie ergodique (Actes Journées Ergodiques, Rennes, 1973/1974), pp. 131–142. Lecture Notes in Math., Vol. 532, Springer, Berlin, (1976).

[17] D.A. Kazhdan, *Uniform distribution on a plane*, Trudy Moskov. Mat. Ob. **14**, (1965) , p. 299-305.

[18] F. Ledrappier, *Ergodic Properties of some linear actions*, Journal of Mathematical Sciences, Vol. **105**, No. 2, 2001

[19] E. Le Page, *Théorèmes quotients pour certaines marches aléatoires*, Comptes Rendus Acad. Sc., **279**, série A, n. 2, (1974).

[20] P. Pansu, *Croissance des boules et des géodésiques fermées dans les nilvariétés*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 415–445.

[21] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer Verlag (1972).

[22] A. Starkov, *Dynamical systems on homogeneous spaces*, Transl. of Math. Mono., **190**, AMS Providence, xvi+243 pp (2000).

[23] H. Weyl, *Über die gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313-352.

Emmanuel Breuillard, Ecole Polytechnique, 91128 Palaiseau, France
*E-mail address*: emmanuel.breuillard@math.polytechnique.fr