Random walks and geometry: some related results. Orsay, January 2009

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1 Lecture 1: invariance under quasi-isometry

The aim of these lectures is to present some results (and problems) concerning random walks on groups that are closely related to the geometry of Cayley graphs. Only a very incomplete selection of topics on this subject will be discussed.

Random walks on non-commutative groups go back a long way, at the very least to works of Poincaré and Borel in the early twenty century. Both were interested in card shuffling. The second edition of Poincaré treaty on probability contains a small section on card shuffling and Borel wrote a book with Chéron on card games including several sections on shuffling.

As is well-known, Polya introduced the problem of transience/recurrence for the simple random walk on the square lattice $\mathbb{Z}^d$ and solved it by proving a local limit theorem to the effect that simple random walk on the square lattice has a return probability at even time that is asymptotic to $c(d)n^{-d/2}$ (recurence, i.e., the property that random walk returns infinitely often to its starting point, is equivalent to $\sum_n P_e(X_n = e) = \infty$).

In his 1957 Cornell Ph.D. Thesis, Harry Kesten explicitely studied some properties of random walks driven by arbitrary probability measures on arbitrary finitely generated groups. Among other things, he proved around that time that a finitely generated group G is non-amenable (does not admit an invariant mean) if and only if for one/any symmetric measure with generating support the probability of return at time n decays exponentially fast with n. Note that, in fact, Kesten's argument is via the Følner characterization of amenability in terms of isoperimetry. In many ways, Kesten's result serves as a model of the possible connections between random walk and geometry.

From Kesten's work emerged the conjectured that the only (finitely generated) groups that admits a (non-degenerate) recurrent random walk are those

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which contain $\{e\}, \mathbb{Z}$ or $\mathbb{Z}^2$ as subgroups of finite index. Partial results were obtained (e.g., for linear groups) during the seventies but the positive solution of this conjecture had to wait until the eighties and is based on Gromov's theorem about the structure of groups of polynomial volume growth and Varopoulos's work relating the probability of return to volume growth. It is worth noting that the recent new proof of Gromov's polynomial growth theorem by Bruce Kleiner brings that theorem much closer to the theory of random walk. Kleiner's proof is based on a number of ideas but one of the central one is that, on a group of polynomial growth (taken in the weakest possible sense), the space of harmonic functions with growth bounded by r^α (for any fixed $\alpha > 0$) is (non-trivial for α large enough and) finite dimensional. Here, harmonic functions are with respect to a fixed symmetric probability measure whose support is a finite generating set. The proof of this finite dimensionality result under the required weak hypothesis follows closely an argument developped by Colding and Minicozzi in a different context and based on the doubling property and Poincaré inequality.

Below is a short list of basic problems about random walk that seem to have strong connections to geometry. Let G be a finitely generated group, S a symmetric generating set, $|x|$ the word length of x in the generating set S (i.e., the minimal number of generators necessary to write x as a product of generators). Let p be a probability measure on G, symmetric (i.e., $p(x) = p(x^{-1})$), finitely supported and with support that generates G. If ξ_i, $i = 1, 2, \dots$, are indepedent G-random variables of law p, set $X_n = \xi_0\xi_1 \dots \xi_n$. This is the random walk on G driven by p, started at ξ_0. For $\xi_0 = e$, its law is the n-th convolution power $p^{(n)}$ of p and the probability of return is $P(X_n = e|\xi_0 = e) = p^{(n)}(e)$. When p is symmetric as assumed here, $p^{(2n)}(e)$ is a decreasing function of n. In fact, $p^{(2n)}(e) = \|f\|_2^2$ where $\|p\|^2 = \sum_{x\in G} |p(x)|_2^2$ denotes the l^2-norm of p (viewed as a function on G).

1. Understand the behavior of $p^{(2n)}(e)$.
2. Understand for which $x \in G$, $p^{(2n)}(x) \geq \epsilon p^{(2n)}(e)$ (for some fixed $\epsilon > 0$). In this precise form, this is very difficult!
3. Understand the behavior of $E(|X_n|^\alpha)^{1/\alpha}$ (it should not depend on $\alpha \in [1, \infty)$).
4. Let N_n be the number of point visited by the random walk up to time n. Understand the behavior of $E(e^{-sN_n})$.
5. Let $\theta_n(x)$ be the number of visits to x up to time n. Understand some functions of these random variables.

Question 1 is in the realm of what is call local limit theorems. Classically, in $\mathbb{Z}$ or $\mathbb{Z}^d$, a limit theorem for sums $S_n = \sum_1^n \xi_i$ of centered i.i.d. random variables asserts the existence of a (scalling) sequence a_n (e.g., $a_n = \sqrt{n}$ if ξ_1 has a second moment) such that S_n/a_n converges in law to a non-trivial (e.g., Gaussian with non-zero variance) random variable. A local theorem describes the asymptotic behavior of $P(S_n = x)$ as n tends to infinity for some range of

x (e.g., for fixed x). In the classical case, one expects to obtain true asymptotic results and the proofs are often based on the Fourier transform. The heuristic argument behind such results (say, in the second moment case) is that random walk converges to Brownian motion and Brownian motion has a scaling property (namely, $t^{-1/2}X_{ts} = X_s$ in law), the two being somewhat inter-related. This heuristic is relevent to the other problems mentioned above as well, with various level of technical difficulties.

However, the existence of an underlying limiting process (such as Brownin motion) is not an absolutely necessary prerequisite for thinking about a local limit theorem. For random walk on an abstract group G, one does not expect to be able to describe a limiting process in general (where would this process live is itself a good question: some asymptotic cone...) and it is a bit unclear what one means by scaling property. Perhaps because of this, it is important to lower the standards and think of question 1 in rougher terms that strict asymptotics: We will say that two decreasing functions f, g are $f \simeq g$ if there are constants $c, C \in (0, \infty)$ such that $cf(Ct) \leq g(t) \leq Cf(ct)$. We will be interested in the $\simeq$-behavior of $n \mapsto p^{(2n)}(e)$.

We now state a first pair of theorems. They are taken from joint work with Christophe Pittet.

Theorem 1.1. *Let G be a finitely generated group with symmetric finite generating set S. Let p, q be two symmetric measures on G with $\sum_x |x|^2 p(x) < \infty$ and q finitely supported with generating support. Then there exists $A, C \in (0, \infty)$ such that*

$$q^{(4An)}(e) \leq Cp^{(2n)}(e).$$

Recall that a map $\phi : X_1 \to X_2$ bewteen two metric spaces is a quasi-isometry if there is a constant C such that: (a) any point in X_2 is at distance at most C of the image $\phi(X_1)$; (b) for all points $x, y \in X_1$, $C^{-1}d_1(x, y) - C \leq d_2(\phi(x), \phi(y)) \leq Cd_1(x, y) + C$.

Theorem 1.2. *For $i = 1, 2$, let (G_i, S_i, p_i) be two finitely generated groups, each equipped with a symmetric finite generating set and a symmetric finitely supported measure with generating support. Assume that the Cayley graphs (G_1, S_1) and (G_2, S_2) are quasi-isometric. Then $p_1^{(2n)}(e) \simeq p_2^{(2n)}(e)$.*

Some ideas for the proof. We assume $G_1 = G_2 = G$ as in Theorem 1.1. To any symmetric measure p one associate the Dirichlet form

$$\mathcal{E}_p(f, f) = (1/2) \sum_{x,y} |f(xy) - f(x)|^2 p(y) = \langle (I - P)f, f \rangle.$$

where $Pf = f * p$, f finitely supported (say). If p has finite second moment and q has finite generating support, then it easily follows from a simple telescopic sum argument and the Cauchy-Schwarz inequality that

$$\mathcal{E}_p(f, f) \leq C(q) \left(\sum_x |x|^2 p(x) \right) \mathcal{E}_q(f, f)$$

(write any y in the support of p using the generators given by the support of q, which we can assume equal to S. The constant $C(q)$ is $1/\min_S\{q\}$).

To capture the main idea, think of the case where G is a finite group (perhaps very large!). Then a simple spectral argument gives $|G|p^{(n)}(e) = \sum \beta_i(p)^n$. where the $\beta_i(p)$ are the (real) eigenvalues of the (self-adjoint) operator P. Here $|G|$ is the order of G. It is a crucial observation that, because of the group structure, the eigenvectors are not needed in this formula. Because $p^{(2n+1)}(e) \geq 0$, one gets

$$\sum_i |\beta_i(p)|^{2n+2} \leq 2 \sum_{\beta_i > 0} \beta_i(p)^{2n}.$$

Setting A to be the smallest integer greater than $C(q)\sum_x |x|^2 p(x)$, the inequality between Dirichlet forms and the min-max formula for eigenvalues give

$$1 - \beta_i(p) \leq A(1 - \beta_i(q)).$$

That is,

$$\beta_i(q) \leq 1 - A^{-1}(1 - \beta_i(p)).$$

This can be used to compare the spectral formulas for $q^{(4An+2)}(e)$ and $p^{(2n)}(e)$ excepts for the part coming from the eigenvalues of q that are less than $1/2$. Namely,

$$\begin{aligned}
|G|q^{(4An+2)}(e) &\leq 2\left(|G|(1/2)^{4An} + \sum_{\beta_i(q)>1/2} \beta_i(q)^{4An}\right) \\
&\leq 2\left(|G|(1/2)^{4An} + \sum_i \beta(p)^{2n}\right) \\
&\leq 2|G|\left((1/2)^{4An} + p^{(2n)(e)}\right).
\end{aligned}$$

Applications of this argument to card shuffling problems is in my work with Persi Diaconis. The application of this idea in the context of finitely generated groups is more technical but the same idea can be made to work. $\square$

2 Lecture 2: Some local limit results

This lecture is centered around the following theorems relating the volume growth to the probability of return. For a Cayley graph (G, S) (S is a symmetric finite generating set of G), we let V be the volume growth function. Hence, $V(n)$ is the number of group elements that can be written as a product of at most n generators.

Theorem 2.1. *Let $w(t) = \inf\{n : V(n) > t\}$ and define $\psi(t)$ by*

$$t = \int_1^{1/\psi(t)} w^2(s)\frac{ds}{s}.$$

Then for any symmetric measure p with generating support, there is a constant $C \in (0,\infty)$ such that

$$p^{(n)}(e) \le C\psi(n/C).$$

In particular,

- $\forall\, n,\ V(n) \ge cn^D$ *implies* $\forall\, n,\ p^{(n)}(e) \le Cn^{-D/2}$
- $\forall\, n,\ V(n) \ge ce^{cn^\alpha}$ *implies* $\forall\, n,\ , p^{(n)}(e) \le C \exp\left(-C^{-1}n^{\alpha/(\alpha+2)}\right)$.

This theorem is essentially due to Varopoulos (e.g., the specific cases mentioned). The general result in terms of ψ is an improvement due to works by Coulhon and myself. The first special case together with Gromov's theorem on groups of polynomial volume growth solves Kesten conjecture (the only recurrent finitely generated groups are the finite extensions of $\mathbb{Z}^d$ with $d = 0, 1, 2$).

In the theorem above, the fact that we are dealing with a group is crucial. For simple random walk on a general graph (of uniformly bounded degree), a volume lower bound of the form $\forall\, x, n,\ \ V(x,n) \ge cn^D$ only implies

$$P_x(X_n = x) \le Cn^{-D/(D+1)},$$

and there are graphs where this is sharp (results of Barlow, Coulhon, Grigor'yan, Kumagai).

I now describe a proof of the second special case due to Waldek Hebisch. With a little additional work, the same argument appplies to the case of $V(n) \ge cn^D$.

Proof of the $V(n) \ge ce^{cn^\alpha}$ case. We can assume without loss of generality that $p^2(x) \ge \epsilon$ for $x \in S$ (this is not entirely obvious but I skeep the details). Recall that P is the operator $f \mapsto f * p$ on $l^2(G)$, which is a selfadjoint operator. Using spectral theory, it is easy to check that

$$\|(I - P^2)^{1/2}P^m\|_{2\to 2} \le m^{-1/2}.$$

Set $\nabla f(x) = \max_{y\in S} |f(xy) - f(x)|$. Obviously,

$$|p^{(2n+m)}(x) - p^{(2n+m)}(e)| \le |x| \|\nabla p^{(2n+m)}\|_\infty$$

We claim that, for any two symmetric functions $f, h \in l^2(G)$,

$$\|\nabla(f * h)\|_\infty \le \|f\|_2 \|\nabla h\|_2.$$

This simple convolution inequality is the only place where the group structure is used in this proof. Moreover,

$$\|\nabla h\|_2^2 \le (2/\epsilon)\mathcal{E}_{p^2}(h,h) = (2/\epsilon)\|(I - P^2)^{1/2}h\|_2^2.$$

Applying this with $f = p^{(n)}$, $h = p^{(n+m)}$ gives

$$\begin{aligned}\|\nabla p^{(2n+m)}\|_\infty &\le \sqrt{2/\epsilon}\,\|p^{(n)}\|_2 \|(I - P^2)^{1/2}P^m p^{(n)}\|_2 \le \sqrt{2/\epsilon m}\,\|p^{(n)}\|_2^2 \\ &\le \sqrt{2/\epsilon m}\,\|p^{(2n)}(e).\end{aligned}$$

We have thus proved that

$$|p^{(2n+m)}(x) - p^{(2n+m)}(e)| \leq C|x|m^{-1/2}p^{(2n)}(e)$$

with $C = \sqrt{2/\epsilon}$. Set $r(n,m) = (2C)^{-1}m^{1/2}\frac{p^{(2n+m)}(e)}{p^{(2n)}(e)}$. Then we have

$$p^{(2n+m)}(x) \geq \frac{1}{2}p^{(2n+m)}(e)$$

whenever $|x| \leq r(n,m)$. It immediately follows that

$$p^{(2n+m)}(e) \leq V(r(n,m))^{-1}.$$

This is a crucial estimate of the probability of return in terms of the volume also we still need to work a bit to get the desired result.

For any pair n, m with $n = km$, either for all $1 \leq i \leq k$

$$p^{(2n+2mi)}(e)/p^{(2n+2m(i-1))}(e) \leq 1/2$$

and then $p^{(4n)}(e) \leq 2^{-n/m}p^{(2n)}(e)$, or there is an $i \in \{1, \ldots, k\}$ such that

$$p^{(2n+2mi)}(e)/p^{(2n+2m(i-1))}(e) > 1/2$$

and then $p^{(4n)}(e) \leq p^{(2n+2mi)}(e) \leq V(m^{1/2}/2C)^{-1}$. If we choose m so that $m^{\alpha/2} \simeq n/m$, that is, $m \simeq n^{2/(\alpha+2)}$, we obtain that

$$p^{(4n)}(e) \leq \exp\left(-cn^{\alpha/(\alpha+2)}\right)$$

as desired. □

Remark 2.2. Suppose G is such that $V(n) \geq c\exp(cn^{\alpha})$ as above and that we know that $p^{(n)}(e) \geq \exp(-Cn^{\gamma})$. Taking $m = n^{1-\beta}$ with $\beta \in (0,\gamma)$, we see that the case when for all $1 \leq i \leq k$, $p^{(2n+2mi)}(e)/p^{(2n+2m(i-1))}(e) \leq 1/2$ can be ruled out since it contradicts the assumption $p^{(n)}(e) \geq \exp(-Cn^{\gamma})$. Hence, for each n there exists $t \in (2n, 4n)$ such that $r(2t, 2n^{1-\beta}) \simeq n^{(1-\beta)/2}$. In particular, this implies

$$\forall\, |x| \leq c\tau^{(1-\beta)/2},\;\; p^{2\tau}(x) \geq (1/2)p^{(\tau)}(e),$$

with $\tau = 2(t + n^{1-\beta})$.

For polycyclic groups with exponential volume growth, we have indeed (see below) $p^{(2n)}(e) \geq \exp(-Cn^{1/3})$. Fixing an arbitrarily small $\epsilon > 0$, we conclude that for any n there is a time $\tau = \tau(n) \simeq n$ such that

$$\forall\, |x| \leq c\tau^{(1-\epsilon)/3},\;\; p^{(2\tau)}(x) \geq (1/2)p^{(2\tau)}(e).$$

See Problem 2 in the First lecture. This is far from optimal since we must have

$$p^{(2n)}(x)/p^{(2n)}(e) \to 0 \text{ for some } |x| \simeq \tau^{(1+\epsilon)/3}$$

because $p(G) = 1$. The only earlier results of this type (outside the polynomial growth case) are for explicit examples in the work of D. Revel.

The following important results is based on the structure theory of discrete subgroups of connected Lie groups and results of Kesten, Varopoulos and Alexopoulos.

Theorem 2.3. *Assume that the finitely generated group G is a discrete subgroup of a connected Lie group. Let p be a symmetric finitely supported mesure with generating support. Then exactly one of the following three alternatives occurs:*

1. $p^{(2n)}(e) \simeq e^{-n}$ *(the non-amenale case)*
2. $p^{(2n)}(e) \simeq \exp(-n^{1/3})$ *(the virtually polycyclic+exponential volume growth case)*
3. $p^{(2n)}(e) \simeq n^{-D/2}$ *for some integer D (the virtually nilpotent case).*

Note that the second case corresponds to exponential volume growth, i.e., $\alpha = 1$ in Theorem 2.1(second special case) and that $\alpha/(\alpha + 2) = 1/3$. Hence Theorem 2.1 is sharp in this case.

Finally, we quote the following example. This and more general results of this type are due to Anna Erschler. Consider the wreath product

$$\mathbb{Z}^d \wr \mathbb{Z} = \mathbb{Z}^d \ltimes (\oplus_{k\in\mathbb{Z}^d}\mathbb{Z}_k)$$

where the action of $\mathbb{Z}^d$ is by translation on the indices. For any symmetric probability measure p with finite generating support, we have

$$p^{(2n)}(e) \simeq \exp\left(-n^{d/(d+2)}(\log n)^{2/(d+2)}\right).$$

This shows that many different behaviors are possible on solvable groups with exponential volume growth.

3 Lecture 3: Groups that are spectrally polynomially bounded

The content of this lecture is part of recent unpublished joint work with Indira Chatterji and Christophe Pittet.

Let p be a symmetric probability measure on a finitely generated group G. Let

$$\rho(p) = \|P\|_{2\to 2} = \lim p^{(2n)}(e)^{1/(2n)} \leq 1$$

be the spectral radius of $P : f \mapsto f * p$. We say that p is admissible if it is symmetric with finite generating support.

We introduce the following class $\mathcal{P}$ of groups we call spectrally polynomially bounded groups. A group is in $\mathcal{P}$ if there exists a $D = D(G)$ such that for any admissible p there exists $c(p)$ for which

$$\forall\, n,\;\; \psi_p(n) = \rho(p)^{-2n} p^{(2n)}(e) \geq c(p)n^{-D}.$$

Groups of polynomial volume growth are in $\mathcal{P}$ (Gromov/Varopoulos) as well as free groups on k generators (Lalley).

Problem 3.1. Does $SL_n(\mathbb{Z})$ belong to $\mathcal{P}$?

We will see that the answer is yes of $n = 2$ whereas it is open for $n \geq 3$.

Let us note that it is known that the $\simeq$-behavior of ψ_p is not necessarily the same for all admissible measures p on a group. Donald Cartwright observed that on $\mathbb{Z}^5 * \mathbb{Z}^5$ there are two admissible measures p, q with the same support and such that

$$\psi_p(n) \simeq n^{-3/2}, \;\; \psi_q(n) \simeq n^{-5/2}.$$

Since hyperbolic groups are not so different from free groups, it is natural to ask:

Problem 3.2. Are all hyperbolic groups in $\mathcal{P}$?

The answer is yes. See below.

Recall that a (finitely generated) group has property (RD) if there are constants C, D such that

$$\|P\|_{2\to 2} \leq Cr^D \|p\|_2$$

for any symmetric finitely supported measure p with support in the ball of radius r in G. This inequality gives a control of the operator norm of convolution by p in terms of a quantity much easier to understand, namely, the l^2-norm of p (viewed as a function). We call D the (RD)-constant.

Theorem 3.3. *Groups with property (RD) are in $\mathcal{P}$.*

This is essentially obvious but very useful. For any admissible p, we have $\|P^n\|_{2\to 2} = \rho(p)^n$, $\|p^{(n)}\|_2 = p^{(2n)}(e)^{1/2}$ and $p^{(n)}$ is supported in the ball of radius n. Hence, applying property (RD) to $p^{(n)}$ yields exactly $\psi_p(n) \geq C^{-2}n^{-2D}$ (D being the (RD)-constant).

The list of groups with property (RD) includes: free groups (Haagerup), groups with polynomial volume growth (trivial), $SL_2(\mathbb{Z})$, free products of groups with property (RD) (e.g., $\mathbb{Z}^k * \mathbb{Z}^m$), hyperbolic groups (de la Harpe), cocompact lattices in $SL_3(\mathbb{R})$ (V. Lafforgue). A conjecture of A. Valette is that cocompact lattices in any simple Lie group have property *RD*. The webpage of Indira Chatterji is a good starting point for more information about property (RD).

Property (RD) passes to subgroups (for discrete groups). It follows that $SL_3(\mathbb{Z})$ does not have property RD because it contains copies of Sol (an amenable group with exponential volume growth cannot have property (RD)).

I finish with a natural question: Does there exists a finitely generated group G with two symmetric proability measures p, q with finite generating support and such that $\psi_p(n) \geq cn^{-\alpha}$ for some c, α but $n^\beta \psi_q(n) \to 0$ for all $\beta > 0$. An excellent candidate pointed out to me by W. Woess is $S * S$ where S is any amenable f.g. group with exponential volume growth. That is because such a group always carry a probability p such that $\psi_p(n) \sim cn^{-3/2}$. See Woess' book.

The following reference list is incomplete but contains pointers to the literature. Interesting surveys and material are on E. Breuillard wepage.

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