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Brownian Loops

Goal: We would like to understand the geometry of a space by running a random walk or brownian motion on it.

Pb: often the classical brownian motion goes too fast.

→ we want to slow down the motion, so that it has time to visit many parts of the space

Simplest example: ① $\mathbb{R}^d$ with brownian motion B_t

$$\frac{d(0,B_t)}{t} \to 0 \quad \Big) \leftarrow B_t \text{ goes slowly to } \infty,$$

in fact $\frac{d(0,B_t)}{\sqrt{t}} = \frac{\|B_t\|}{\sqrt{t}} \sim \sqrt{\sum_i^d B_i^2}$ $\qquad B_i \sim \mathcal{N}(0,1)$ IID gaussians

↓

has density

$r^{d-1} e^{-r^2/2}$

So this density yields d, the dimension.

② BM with a drift on $\mathbb{R}^d$ $\qquad W_t = B_t + t(c,0,\dots,0) \qquad c \neq 0$

$$\frac{d(0,W_t)}{t} \to c \neq 0 \qquad \text{so } W_t \text{ goes fast to } \infty.$$

$$\frac{d(0,W_t)-ct}{\sqrt{t}} = \frac{\left((B_t^1+ct)^2+\dots+{B_t^d}^2\right)^{1/2} - c\sqrt{t}}{\sqrt{t}} \sim c\sqrt{t}\left[\sqrt{1+2\frac{B_1^1}{c\sqrt{t}}+\frac{\|B_1\|^2}{c^2t}} - 1\right]$$

in law

B_t at time $t=1$: $\quad B_1 = (B_1^1,\dots,B_1^d)$

Hence

$$\frac{d(0, W_t) - ct}{\sqrt{t}} \sim B_1' \quad \text{in law as } t \to \infty$$

So we don't recover the dimension here!

And this phenomenon takes place in many other situations

What about Riemannian mflds of negative curvature?

Same phenomenon!

If B_t is BM on (X, d) = neg. curved mfld

we have $\dfrac{d(x_0, B_t)}{t} \to c > 0$ $\quad x_0 \in X$

and there are instances of the central limit theorem CLT:

$$\frac{d(x_0, B_t) - ct}{\sqrt{t}} \to \mathcal{N}(0, \sigma^2)$$

- For symmetric spaces : done in the 60's
- for surfaces : Pinsky '78
- for covering of cpct mflds : Ledrappier '95
- for hyperbolic groups. Bjorklund '08

→ we don't see the dimension of the space here!

Brownian loops - Bridges:

B_t on X, $0 \le t \le T$ $\quad 0 \in X$ starting point

You force B_T to be $= 0$

Claim 1: on $B_{\frac{T}{2}}$ we can see some geometry!

<u>Claim 2</u>: when $T \to \infty$ this new process has no drift!

↳ true in $\mathbb{R}^d$ even if we started with a BM with a drift.

<u>Doob Processes</u>:

M a mfld. $(B_t)_{t\geq 0}$ a Markov process with a generator $\frac{1}{2}L$ symmetric on $L^2(M, m)$

$$\begin{cases} P_t(x,dy) = p_t(x,y)\,dm(y) \\ p_t(x,y) = p_t(y,x) > 0 \end{cases}$$

Let h be a positive eigenfunction $\frac{1}{2}Lh = -\alpha h$ $\alpha \geq 0$

The <u>Doob h-process</u> is defined as

$$-\frac{1}{2}L^h f = \frac{1}{2h}L(hf) + \alpha f \qquad \frac{1}{2}L^h 1 = 0$$

and the associated Markov kernel is

$$P^h(x,dy) = e^{\alpha t} P_t(x,dy)\frac{h(y)}{h(x)}$$

→ symmetric on $L^2(h\,dm)$

We now assume that M is a Riemannian manifold and has finite volume dm And let $\phi \geq 0$ a function on M. ~~...~~

Curvature: (g_{ij}) $\quad \mathrm{grad}\, f = \sum g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$

$$dm(x) = \sqrt{\det g(x)}\,dx$$

$$\operatorname{div} \xi = \sum \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, \xi\right)$$

$$L = \phi^{-2} \operatorname{div} \phi^2 \cdot \operatorname{grad}$$

Rk: • If $\phi = 1$ we recover the Laplace-Beltrami operator

• L is symmetric on $L^2(\phi^2 dm)$

It's straightforward to verify that:

$$\frac{1}{2} L^h(f) = \frac{1}{2} Lf + \langle \operatorname{grad} \log h, \operatorname{grad} \rangle$$

↳ this new h-process is symmetric on $L^2(\phi^2 h^2 dm)$

Lie group situation: (see book Coulhon - Saloff - Varopoulos)

Let G be a Lie group non unimodular

$(X_i)_{1 \le i \le d}$ a basis made of left invariant vector fields

↳ give a metric (Carnot - Caratheodory)

and a "subLaplacian" $\Delta = -\sum_{i=1}^{d} X_i^2$

↳ symmetric w.r.t. the right invariant Haar measure $d^r x$

$$\operatorname{grad} f = (X_1 f, \dots, X_d f)$$

$$(\Delta f, f) = \int \|\operatorname{grad} f\|^2 d^r x$$

Let $h = \sqrt{\mathrm{mod}}$ mod = modular function $d^l x = \mathrm{mod}(x)\, d^r x$

$\rightsquigarrow P_t^h, \Delta^h$

And $(\Delta^h f, f) = \int \|\mathrm{grad}\, f\|^2 \, d^{\ell}x$

$$P_t^h(x,y) = P_t^h(e, y^{-1}x)$$

$\rightsquigarrow$ B_t^h is symmetric ! $\rightsquigarrow$ has no drift.

Hence this is a typical example where we have changed a BM with drift (here b_t has a drift since G is not unimodular) into a new process with no drift by applying the Doob transform.

Brownian Bridge :

$(B_t)_t \qquad P_t(x,dy) = p_t(x,y)\, dm(y)$

Let $a, b \in M$, $T > 0$ $\qquad (B_t^T)_t$

Informally the brownian bridge $(B_t^T)_t$ is $\{B_t, 0 \le t \le T\}$ conditionned to the requirements $B_0 = a$, $B_T = b$.

Formally it is ~~the Markov process with transition~~ the process with finite dimensional distributions given by

$\left(B_{t_1}^T, B_{t_2}^T, \dots, B_{t_n}^T\right)$ has law

$$\frac{1}{p_T(a,b)} \cdot p_{t_1}(a,x_1)\, p_{t_2-t_1}(x_1,x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1},x_n)\, p_{T-t_n}(x_n,b)$$

Observe that $\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2} L p_t(x,y)$ (Heat equation)

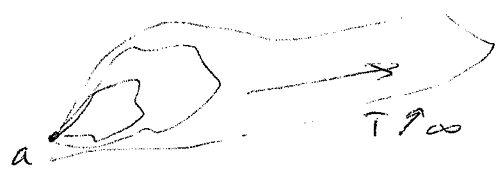
Hence $\left(\frac{\partial}{\partial t} + \frac{1}{2} L_x\right) p_{T-t}(x,y) = 0$

And B_t^T is the h-transform of the space-time process with transition probabilities $x \mapsto p_{T-t}(x,b)$ starting at a

Its generator is $\frac{1}{2} L + \langle \operatorname{grad} \log p_{T-t}(x,b), \operatorname{grad} \rangle$

Infinite brownian loop.

Def: the infinite brownian loop from a to a is the limit (when it exists) of B_t^T, $0 \le t \le T$ when $T \nearrow \infty$ in law



Theorem (Anker, Jealin, Bougerol)

IBL

The infinite brownian loop from a to a exists iff

$$\forall x \quad \lim_{t \to \infty} \frac{p_t(x,a)}{p_t(a,a)} = \phi_0(x) \quad \text{exists}$$

In that case: ϕ_0 is C^2, $(L + 2\lambda_0)\phi_0 \equiv 0$ where $L\phi_0 = \lambda_0 \phi_0$

λ_0 = bottom of the spectrum of L, $\phi_0 > 0$

And the IBL is the ϕ_0 process of B_t

It was conjectured by E.B. Davies that ϕ_0 always exists.
Gady Kozma has an example of a mfld where ϕ_0 does not exist.

Rk: If $(L + 2\lambda_0)\phi_0 = 0$, $\phi_0 > 0$ has a unique solution
then it holds (ie the limit exists)

Rk: when there are many solutions to $(L+2\lambda_0)\phi_0 = 0$, $\phi_0 > 0$,
this is interesting because this special solution is so to speak
chosen by the BM.

Rk: On model mflds, for instance "harmonic mflds" it exists.

Rk: In very few cases do we know how to answer the question
of whether or not this limit exists. Even for Lie groups.
Even for hyperbolic mflds.

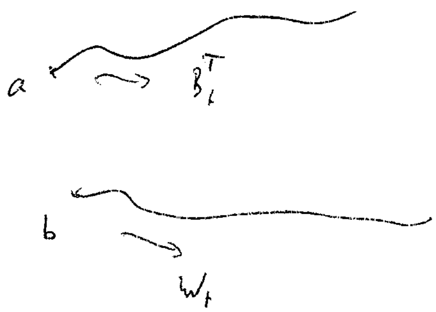
Rk: Our new process, the IBL has no spectral gap: it has a
positive harmonic function. In some sense we have killed
the drift: it's likely that $\frac{d(o, B_t^\infty)}{t} \to 0$

Notation: $(B_t^\infty)_{t>0} = $ IBL

Rk: The bridges of $(B_t)_{t>0}$ and $(B_t^\infty)_{t>0}$ are the same.

Suppose that $\dfrac{p_t(x,y)}{p_t(x_0,x_0)} \longrightarrow \phi(x,y) \quad \forall x_0, x, y$

Look at $(B_t^T,\ 0 \le t \le T)$ and $(W_t^T = B_{T-t}^T,\ t \ge 0)$



Then:

$$(B_t^T, W_t^T) \xrightarrow{T \uparrow \infty} (B_t^\infty, W_t^\infty)$$

2 processes that may not be independent!

this is what happens on a tree.

<u>Rmk</u>: If G is discrete amenable and μ is symmetric, generating then $\frac{\mu^n(x)}{\mu^n(y)} \longrightarrow 1 \quad \forall x, y$ (result of Avez 1972)

A consequence is that the associated ϕ_0-process is the process itself (since $\phi_0 \equiv 1$).

However Kaimanovich and Vershik studied walks on such amenable groups as the wreath product $\mathbb{Z}^3 \ltimes \bigoplus_{i \in \mathbb{Z}^3} (\mathbb{Z}/2\mathbb{Z})$: and showed that even symmetric walks have a drift.

$\rightsquigarrow$ Hence the ϕ_0-process does not always kill the drift.

Central limit theorem for Brownian bridge and IBL.

From the analytic point of view IBL is easier to deal with than the bridge. We would like to prove a result of this kind:

"<u>Principle</u>": For the IBL we would like to find the limit law of the (renormalized) distance process, i.e.

$$\frac{1}{\sqrt{T}}\, d(0, B^{\infty}_{tT}) \xrightarrow[T \nearrow \infty]{} Z_t \qquad 0 \le t \le 1$$

<u>Rk</u>: For a symmetric walk on a grp we always have

$$\frac{d(0, B_n)}{n} \rightarrow c \geq 0$$

When $c = 0$, is $d(0, B_n)$ always roughly $\sqrt{n}$?

Answer is no in general and A. Erschler constructed counterexamples.

what about the bridge? $\leadsto$ it may not have the same normalization as the process itself:

A Erschler showed examples where although $d(0, B_n) \sim \sqrt{n}$

for the bridge $d(0, B^T_{\frac{T}{2}}) \sim T^{1/3}$

Rk: For the "principle" above there are very few results:

⊛ On nilpotent grps if B_n is a random walk (centered), then $\frac{d(0, B_{tn})}{\sqrt{n}} \xrightarrow[n \uparrow \infty]{}$ some known process.

and in that case one can show that also for the bridge

$$\frac{d(0, B^n_{tn/2})}{\sqrt{n}} \xrightarrow[n \uparrow \infty]{} \text{corresponding bridge}$$

⊛ on the affine group: $\{ax+b\}$ → work of Grincevicius 1972.

⊛ semisimple groups

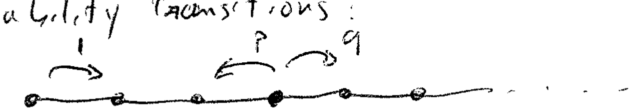
⊛ free groups.

Computations for a simple random walk on a tree

d-Regular tree

Fact: the law of the bridge B^N_n ($B^N_0 = 0$, $B^N_N = 0$, N even) is just the uniform distribution on all loops going from 0 to 0.

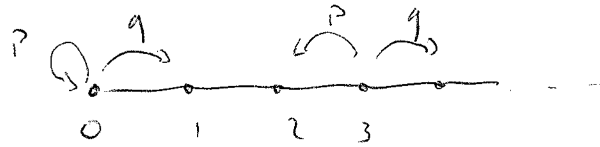
easier pb: study the law of $d(0, B^N_n)$: this is the bridge of the 1-dimensional Markov process on the integers $\mathbb{N} = \{0, 1, 2, \dots\}$ with probability transitions:



with $p = 1/d$

$q = \frac{d-1}{d}$

It is only a small perturbation of the following Markov process:



__But__: the bridges are very different!

__In the second case__: Bridge $(p,q) \underset{\text{in law}}{\sim}$ Bridge $(1/2, 1/2)$ = Bridge of 1-dim'l BM on $\mathbb{Z}$.

~~But this would give rise to~~ $\cancel{d(0, B^n_{n/2}) \approx \sqrt{n}}$

~~However on the free group~~ $\cancel{d(0, B^n_{n/2}) \approx \ln^{3/2}}$ ...

__In the first case__, if $p < q$ (ie $(p,q) \neq (1/2, 1/2)$)

then we have the following convergence in law:

(for the bridge of the process on $\mathbb{N}$ described above, corresponding to $d(0, B^N_n)$ on the free)

__Theorem__: As $N \uparrow \infty$:

$$(\alpha, \tau_1, \tau_2, \dots, \tau_\alpha, 0, \dots 0), \ (\beta, \sigma_1, \sigma_2, \dots, \sigma_\beta, 0, \dots 0)$$

α: exponential law of parameter $1 - 1/2p$

τ_i: return times for the simple $(\frac{1}{2}, \frac{1}{2})$-RW on $\mathbb{Z}$

σ_i: same

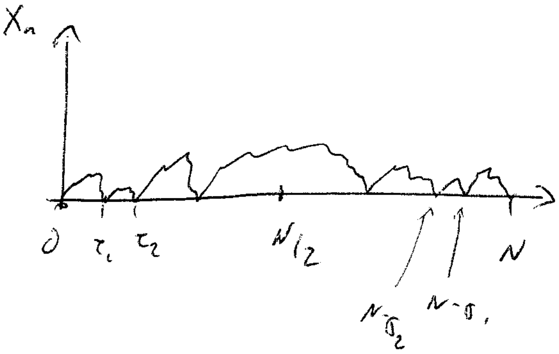
+ they become independent.

where: $\tau_1 = \inf\{n > 0 \ X_n = 0\}$, $\tau_2 = \inf\{n > \tau_1, X_n = 0\}$ ...

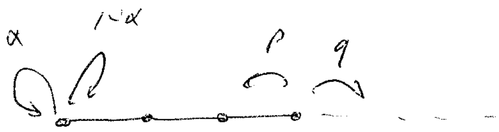
$\alpha = \sup\{n, \tau_n \leq N/2\}$

$\sigma_1 = \inf\{n, X_{N-n} = 0\}$, $\sigma_2 = \inf\{n > \sigma_2, X_{N-n} = 0\}$, ...

$\beta = \sup\{n, \sigma_n \leq N/2\}$



In *general*: if you take



hit 1

then you have a *phase transition*:

$$\frac{d(0, b^T_{T/2})}{\sqrt{T}}$$

$\alpha < q + \sqrt{pq} \;\longrightarrow\; 3/2$ — exponents in the density of hit law

$\alpha = q + \sqrt{pq} \;\longrightarrow\; 1/2$ — $n^{(1)}, e^{-n/2}$.

$\alpha > q + \sqrt{pq} \;\longrightarrow\;$ process is >0 recurrent

$X_n \to$ inv. measure.

Corollary: the bridge (for the simple RW on the tree) of the distance converges to bridge of 3-Bessel process $d(0, b_t)$ (three dim BM).

Case of semisimple Lie groups

trivial computation : hyperbolic space dim 3

① $\frac{1}{2}\Delta = \frac{1}{2}\frac{d^2}{dr^2} + \coth r \frac{d}{dr}$

② $\phi_0(x) = \frac{r}{\sinh(r)}$ $\quad r = d(0,x)$

much harder in general

generator of ϕ_0- process

$\frac{1}{2}\Delta - \langle \text{grad} \log \phi_0, \text{grad} \rangle$

$\Rightarrow \quad \frac{1}{2}\Delta^{\phi} = \underbrace{\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{r}\, d/dr}_{\text{generator of 3-dim Bessel}}$

semisimple groups: G/K non compact

B_t , Δ , $\lambda_0 \neq 0$

$$\begin{cases} \dfrac{d(0,B_t)}{t} \to c > 0 \\ \dfrac{d(0,B_t) - ct}{\sqrt{t}} \to \mathcal{N}(0,\sigma^2) \end{cases}$$

ϕ_0- process $\quad \Delta^{\phi} \neq \Delta$

Radial coordinates : $G/K \simeq K/M \times A^+$

K/M: "angle" $\qquad A^+$: radial part

$$\begin{cases} x = (\theta, X) \\ d(0,x) = \|X\| \end{cases}$$

A : euclidean plane $\quad A \simeq \mathbb{R}^d$

Σ = set of positive roots , multiplicity m_α

$A^+ = \{ x \in A = \mathbb{R}^d , \quad \alpha(x) > 0 \quad \forall \alpha \in \Sigma \}$

radial part of Laplacian

$$\frac{1}{2} \mathrm{Rad}(\Delta) = \frac{1}{2} \Delta_A + \nabla_A \log \delta^{1/2} \nabla_A$$

self adjoint on $L^2(\delta(x)dx)$

$$\delta(x) = \prod_{\alpha \in \Sigma} \sinh^{m_\alpha} \langle \alpha, x \rangle$$

$$\lambda_0 = \frac{1}{2} \left\| \frac{1}{2} \sum m_\alpha \alpha \right\|^2$$

eigenfunction of Δ on G/K with eigenvalue λ_0

$$x \longmapsto e^{-\rho H(xk^{-1})}$$

$$G = KAN$$
$$g = k\, e^{H(g)} n$$

In this case
$$\frac{p_t(e,x)}{p_t(e,e)} \xrightarrow[t \to \infty]{} \phi_0(x)$$

because, as we have seen earlier, we automatically have convergence if there is only one >0 solution to the differential equation: here this solution is unique and given by the Harish-Chandra function.

$$\phi_0(x) = \int_K e^{-\rho H(xk^{-1})} dk$$

The radial part of the ϕ_0-Laplacian is:

$$\frac{1}{2} \mathrm{Rad}(\Delta^{\phi_0}) = \frac{1}{2} \Delta_A + \nabla_A \log \phi_0 \delta^{1/2} \nabla_A$$

given below

Thm ("principle") $\quad \dfrac{d(0, B_{tT})}{\sqrt{T}} \xrightarrow[T \to \infty]{} Z_t = $ Bessel ρ-process

The case of complex groups: $m_\alpha = 2 \quad \forall \alpha$

so $\quad \phi_0 = \delta^{-1/2} \pi \quad$ where $\underbrace{\pi = \prod_{\alpha \in \Sigma} \langle \alpha, x \rangle}$

harmonic function on A^+

Hence $\quad \frac{1}{2} \operatorname{Rad} \Delta^\infty = \frac{1}{2} \Delta_A + \nabla_A \log \pi \, \nabla_A$

this is the usual BM on A^+ killed at the boundary conditionned to stay alive $\Rightarrow$ the "principle" is clear

$$d(0, B_t^\infty) = \|\operatorname{Rad} W_t\| = \|W_t\|$$) $\leftarrow$ p-dim Bessel

usual BM on $\mathbb{R}^p$

$p = \dim A + 2 \#$ of indivisible ≥ 0 roots (= $\dim G/K$ in the cplx case)

Rank one situation: $\mathbb{H}^n$

Look at B_t on $AN \qquad G = KAN$

$\quad\quad \parallel$

$\mathbb{R}_+^* \times \mathbb{R}^d$

generator $\quad X^2 + \sum_{i=1}^d X_i^2 \quad \to \quad$ IBL ~~B_t~~ $\sim \left(e^{B_t}, \int e^{B_s} dW_s \right)$

1-dim'l BM $\qquad$ d dim BM

indept

~~[illegible]~~ $G = SO(n+1,1)$

<u>Claim</u>: $\left\{ \frac{1}{\sqrt{T}} d(0, \tilde{B}_{tT}),\ 0 \le t \le 1 \right\} \xrightarrow[T \nearrow \infty]{} \text{Bessel } 3$

if $n=2$ this is trivial by the above computation (because $SO(3,1)$ is complex)

$$\left(e^{B_{tT}}, \int_0^{tT} e^{B_s} dW_s \right) \underset{\substack{\uparrow \\ \text{scaling}}}{\overset{\text{law}}{\sim}} \left(e^{\sqrt{T} B_t}, \int_0^t e^{\sqrt{T} B_s} \sqrt{T}\, dW_s \right)$$

$$\Big\downarrow \ \frac{1}{\sqrt{T}} \log$$

$$\left(B_t, \frac{1}{\sqrt{T}} \log \sqrt{T} + \frac{1}{\sqrt{T}} \log \left| \int_0^t e^{\sqrt{T} B_s} dW_s \right| \right)$$

$$\downarrow \qquad \swarrow$$

$$\left(B_t, \max_{0 \le s \le t} B_s \right)$$

But on $\mathbb{R}_+^* \times \mathbb{R}^d$ $\quad \cosh \frac{d(0,(a,b))}{2} = \sqrt{\left(\cosh \frac{a}{2}\right)^2 + \frac{1}{4} e^{-a} \|b\|^2}$

at long distance $\quad d(0,(a,b)) \sim \max(a, 2\|b\| - a)$

$$\rightsquigarrow \quad \lim \frac{1}{\sqrt{T}} d(0, \tilde{B}_T) \sim 2 \max_{0 \le s \le t} B_s - B_t \sim \text{Bessel } 3$$

$\Uparrow$ applying Pitman's thm