A STRONG TITS ALTERNATIVE

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Abstract. We show that for every integer $d \in \mathbb{N}$, there is $N(d) \in \mathbb{N}$ such that if K is any field and F is a finite subset of $GL_d(K)$, which generates a non amenable subgroup, then $F^{N(d)}$ contains two elements, which freely generate a non abelian free subgroup. This improves the original statement of the Tits alternative. It also implies a growth gap and a co-growth gap for non-amenable linear groups, and has consequences about the girth and uniform expansion of small sets in finite subgroups of $GL_d(\mathbb{F}_q)$ as well as other diophantine properties of non-discrete subgroups of Lie groups.

1. Introduction

The goal of this paper is to show the following theorem and some consequences of it.

Theorem 1.1. For every $d \in \mathbb{N}$ there is $N(d) \in \mathbb{N}$ such that if K is any field and F a finite symmetric subset of $GL_d(K)$ containing 1, either $F^{N(d)}$ contains two elements which freely generate a non abelian free group, or the group generated by F is virtually solvable (i.e. contains a finite index solvable subgroup).

By $F^{N(d)} = F \cdot ... \cdot F$ we mean the set of elements which can be written as a product of at most $N(d)$ elements from F, and by symmetric we mean that if $f \in F$ then $f^{-1} \in F$. This statement is a strengthening of the classical Tits alternative [39], which asserts that any finitely generated subgroup $\langle F \rangle$ of $GL_d(K)$, where K is any field, either contains a non abelian free subgroup or contains a solvable subgroup of finite index. It also improves earlier strengthenings of the Tits alternative, due to Eskin-Mozes-Oh [18] (for free semigroups) and to T. Gelander and the author [14] (for free groups), which showed a statement of a similar form, except that the integer $N(d)$ depended on the group Γ generated by F (not on the generating set) but was not independent of the field of coefficients. Note that $N(d)$ cannot be bounded uniformly in d (see Remark 1.4).

The present paper essentially contains the geometric part of the proof of Theorem 1.1. The arithmetic part is the object of the paper [13]. The reader

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only interested in the GL_2 case can read a self-contained proof of Theorem 1.1 (both arithmetic and geometric parts) and its consequences in this special case in [12].

The novelty of the above statement resides precisely in the fact that the integer $N(d)$ can be taken to depend only on d and not on F nor $\langle F \rangle$. As such Theorem 1.1 is a statement of a different nature. What it really asserts is an inclusion of countably many algebraic varieties into another algebraic variety. Indeed, the condition on a k-tuple of matrices in say $GL_d(\mathbb{C})$ that they generate a virtually solvable group is an algebraic one (see Prop. 7.1 below). On the other hand to say that no two words of length at most $N(d)$ with letters in this k-tuple are generators of a free group is itself a countable union of algebraic conditions. This way of interpreting the result allows to derive, via an effective Nullstellensatz, several corollaries about the girth in finite simple groups of Lie type, as well as some diophantine properties of non-discrete subgroups of $GL_n(\mathbb{C})$, in the spirit of the works of Kaloshin-Rodnianski, Helfgott and Bourgain-Gamburd ([26], [22], [9], [10]).

Comments on the proof.

Tits' proof of his alternative consists of two parts. In a first arithmetic step, he exhibits a semisimple element of $\langle F \rangle$ which has some eigenvalue of absolute value $|\lambda| > 1$ for a clever choice of absolute value on K. Then in a second geometric step, he studies the action of $\langle F \rangle$ on the projective space $\mathbb{P}(k^n)$ under some suitably chosen linear representation, where k is the completion of K with respect to that absolute value. The free group is then obtained by building a so-called ping-pong pair acting on $\mathbb{P}(k^n)$ (see [39]).

The proof of Theorem 1.1 consists in reproducing Tits' proof almost word by word while making sure that each step can be done in a uniform way. The arithmetic step is much harder to perform, as we need a uniform gap $|\lambda| > 1 + \varepsilon$, where ε is allowed to depend on d only. This first arithmetic step is the content of the paper [13], which shows a height gap theorem for non amenable linear groups (see Theorem 3.3). The key idea there and also in the present paper is to introduce arithmetic heights in order to treat all absolute values of K on an equal footing. This first arithmetic step is needed only in characteristic zero. In a second arithmetic step, we find an absolute value for which the geometric conditions needed for the ping-pong to work are fulfilled. This is done in Section 6 by estimating the Arakelov heights of the characteristic subspaces of the matrices in F in terms of the normalized height $\widehat{h}(F)$ introduced in [13] and by making use of another result from [13] which says that $\widehat{h}(F)$ can be realized up to a multiplicative factor as the height of some conjugate of F inside $SL_d(\overline{\mathbb{Q}})$. Once the right absolute value has been found, the actual geometric construction of the ping-pong pair follows Tits' geometric step very closely (unlike the argument in [14]) ; the only notable difference is that our estimates need to be uniform over all local fields. This requires a bit of care and is performed in Sections 4 and 5.

Some consequences.

Theorem 1.1 admits several consequences about the structure of non-amenable linear groups. The first is a gap for the growth exponent, namely:

Corollary 1.2. (*Uniform exponential growth*) For every $d \in \mathbb{N}$, there exists a constant $\varepsilon = \varepsilon(d) > 0$ such that if K is any field and F is a finite subset of $GL_d(\mathbb{C})$ containing 1 and generating a non amenable subgroup, then for all $n \geq 1$

$$|F^n| \geq (1+\varepsilon)^n$$

Hence

$$\rho_F = \lim_{n\to+\infty} \frac{1}{n} \log |F^n| \geq \log(1+\varepsilon) > 0$$

Remark 1.3. It is possible that the assumption "non-amenable" in the above corollary can be replaced by "of exponential growth". However we observed in [11] that this would imply the Lehmer conjecture about the Malher measure of algebraic numbers. We also observed there that although every linear solvable group of exponential growth contains a free semigroup, no analog of Theorem 1.1 holds for solvable groups, namely one may find sets F_n in $GL_2(\mathbb{C})$ containing 1 and generating a solvable subgroup of exponential growth, such that no pair of elements in $(F_n)^n$ may generate a free semigroup.

Remark 1.4. Examples due to Grigorchuk and de la Harpe [21] (see also [3]) show that there is a sequence of groups Γ_n with finite generating set F_n which are virtually a direct product of finitely many copies of the free group F_2 such that $\rho_{F_n} \to 0$ as $n \to +\infty$. Those examples can be embedded in $SL_m(\mathbb{Z})$ for some possibly large $m = m(n)$. Therefore we must have $N(d) \to +\infty$ and $\varepsilon(d) \to 0$ as $d \to +\infty$ in Theorem 1.1 and Corollary 1.2.

The following corollary says that non-amenable linear groups have few relations: there is a co-growth gap.

Corollary 1.5. (*Co-growth gap*) For every $d, k \in \mathbb{N}$, there is $\varepsilon > 0$ such that if K is a field and $F = \{a_1, ..., a_k\}$ generates a non virtually solvable subgroup of $GL_d(K)$, then for every $n \in \mathbb{N}$, the proportion of relations $w(a_1, ..., a_k) = 1$ in the free group F_k of word length at most n among all elements in F_k of word length at most n is at most $\exp(-\varepsilon n)$.

Von Neumann showed that groups containing a free subgroup are non amenable, i.e. have a spectral gap in ℓ^2. The uniformity in Theorem 1.1 implies also a uniformity for the spectral gap (see [36] for this observation). More precisely:

Corollary 1.6. (*Uniform Spectral Gap in ℓ^2*) For every $d \in \mathbb{N}$, there is $\varepsilon = \varepsilon(d) > 0$ with the following property. If K is a field and F is a finite subset of $GL_d(K)$ containing the identity and generating a non amenable subgroup and if Γ is any countable subgroup of $GL_d(K)$ containing F and $f \in \ell^2(\Gamma)$, then there is $\sigma \in F$ such that

$$\sum_{x\in\Gamma} \left|f(\sigma^{-1}x) - f(x)\right|^2 \geq \varepsilon \cdot \sum_{x\in\Gamma} |f(x)|^2$$

In particular, if F in $GL_d(K)$ is a finite subset containing the identity and generating a non amenable subgroup, then for every finite subset A in $GL_2(K)$, we have $|FA| \geq (1+\varepsilon)|A|$.

This shows also that if μ is a uniform probability measure on a set F of cardinal k in $GL_d(K)$, then the *Kesten spectral radius* of μ (see [24]) is uniformly bounded away from 1 by a bound depending only on k and d. Hence the return probability of the simple random walk on the group $\langle F \rangle$ decays exponentially with an exponential rate depending only on k and d.

The uniformity in Theorem 1.1 allows to reduce mod p and we obtain a statement giving a lower bound on the girth of subgroups of GL_d in positive characteristic:

Corollary 1.7. (*Large girth*) Given $d, k \in \mathbb{N}$, there is $N_0, N \in \mathbb{N}$ and $\varepsilon_0, C > 0$ such that for every prime p and every field K of characteristic p and any finite k-element subset F generating a subgroup of $GL_d(K)$ which contains no solvable subgroup of index at most N, then F^{N_0} contains two elements a, b such that $w(a,b) \neq 1$ in $GL_d(K)$ for any non trivial word w in F_2 of length at most $f(p) = C \cdot (\log p)^{\varepsilon_0}$.

Corollary 1.8. (*Expansion of small sets*) There is $\varepsilon = \varepsilon(d) > 0$ such that given $k, N \in \mathbb{N}$, there is a constant $C_{k,N,d}$ such that for any field K of charateristic $p > 1$ and any subset F of $GL_d(K)$ with k elements generating a subgroup which has no solvable subgroup of index at most N, we have $\max_{f\in F} |A \triangle fA| \geq \varepsilon|A|$ for all subsets A in $GL_d(K)$ with $|A| \leq C_{k,N,d} \log\log\log p$.

It was conjectured in [20] that the statement of Corollary 1.7 holds for generating subsets F of $GL_2(\mathbb{F}_p)$ with $\varepsilon_0 = 1$. It was also proved there that a random k-regular Cayley graph of $GL_2(\mathbb{F}_p)$ has girth at least $(1 - o(1))\log_{k-1}(p)$.

In a similar fashion one can derive the following weak diophantine property for subgroups of $GL_d(\mathbb{C})$. Let d be some Riemannian distance on $GL_d(\mathbb{C})$.

Corollary 1.9. *(Weak diophantine condition)* Given $d \in \mathbb{N}$, there is $N_0 \in \mathbb{N}$ and $\varepsilon_1 > 0$ with the following property. For every finite set $F \subset GL_d(\mathbb{C})$ generating a non virtually solvable subgroup, there is $\delta_0(F) > 0$ such that for every $\delta \in (0, \delta_0)$ there are two short words $a, b \in F^{N_0}$ such that $d(w(a,b), 1) \geq \delta$ for every reduced word w in the free group F_2 with length $\ell(w)$ at most $(\log \delta^{-1})^{\varepsilon_1}$.

In [26] Kaloshin and Rodnianski proved that for $G = SU(2, \mathbb{R}) \leq GL_2(\mathbb{C})$ almost every pair $(a, b) \in G \times G$ satisfies $d(w(a,b), 1) \geq \exp(-C(a,b) \cdot \ell(w)^2)$ for all $w \in F_2 \backslash \{e\}$ and some constant $C(a,b) > 0$. Besides it is easy to see that if $a, b \in GL_2(\overline{\mathbb{Q}})$ then the pair (a, b) satisfies the stronger diophantine condition $d(w(a,b), 1) \geq \exp(-C(a,b) \cdot \ell(w))$. It is conjectured in [34] and [20], that this stronger condition also holds for almost every pair $(a, b) \in SU(2, \mathbb{R})$.

Our result also allows us to estimate the number of words of length $\leq n$ that fall in a shrinking neighborhood of 1 in $GL_d(\mathbb{C})$. More precisely,

Corollary 1.10. *(Weak equidistribution)* Given $d \in \mathbb{N}$, there are $\tau, \varepsilon_1, C > 0$ with the following property. For every $\{a, b\} \leq GL_d(\mathbb{C})$ which generates a non virtually solvable subgroup, there is $\delta_0(a,b) > 0$ such that for every $\delta \in (0, \delta_0)$ and every $n \leq C(\log \delta^{-1})^{\varepsilon_1}$, the proportion of elements w in the free group F_2 of word length n such that $d(w(a,b), 1) \leq \delta$ is at most $\exp(-\tau n)$.

In [19], Gamburd, Jacobson and Sarnak, showed for $G = SU(2, \mathbb{R})$ that if a pair $(a, b) \in G$ satisfies the conclusion of Corollary 1.10 with $\varepsilon_1 = 1$ and $C > C_0$ (for some explicit $C_0 > 0$) then (a, b) has a spectral gap on $\mathbb{L}^2(G)$. In [9], Bourgain and Gamburd showed that if a pair $(a, b) \in G$ satisfies the above condition with $\varepsilon_1 = 1$ and some $C = C(a, b) > 0$, then (a, b) has a spectral gap on $\mathbb{L}^2(G)$. This latter condition is automatically satisfied if (a, b) satisfies the stronger diophantine condition above, for instance if $(a, b) \in GL_2(\overline{\mathbb{Q}})$.

Remark 1.11. Corollaries 1.7, 1.9 and 1.10, are derived from Theorem 1.1 by using we use a standard version of the effective Nullstellensatz due to Masser and Wustholz (see [29]) after reformulating Theorem 1.1 in terms of inclusion of algebraic varieties. See Section 7.

Contents

2. Minimal norm and spectral radius formula

In this section, we recall results obtained in [13] about the spectral radius of a finite set of matrices. Given a local field k, we defined the standard norm $||\cdot||_k$ on k^d to be the canonical Euclidean (resp. Hermitian) norm if k is $\mathbb{R}$ (resp. $\mathbb{C}$) and the sup norm if k is ultrametric. This induces on operator norm on the space of $d\times d$ matrices $M_d(k)$, which we again denote by $||\cdot||_k$. Given a finite subset F of matrices in $M_d(k)$, we define its norm $||F||_k$ to be the maximal norm of any given element of F. We define the following quantities

$$E_k(F) = \inf_{g\in GL_d(\overline{k})} ||gFg^{-1}||_k$$

$$\Lambda_k(F) = \max\{|\lambda|_k, \lambda \text{ eigenvalue of some } f\in F\}$$

where $\overline{k}$ is an algebraic closure of k and $|\cdot|_k$ is the absolute value on k extended (uniquely) to $\overline{k}$. We also set the *spectral radius* of F to be:

$$R_k(F) = \lim_{n\to+\infty} ||F^n||_k^{\frac{1}{n}}$$

These quantities enjoy the following key properties.

Lemma 2.1. (Spectral Radius Formula for F, [13], Lemma 2.1.)
(a) If k is ultrametric, then for any compact set F containing 1 in $M_d(k)$, there is a positive integer $q\le d^2$ such that $\Lambda_k(F^q) = E_k(F)^q$. In particular, $E_k(F) = R_k(F) = \max_{1\le q\le d^2}\Lambda_k(F^q)^{\frac{1}{q}}$.
(b) If k is archimedean, there is a constant $c = c(d)\in(0,1)$ such that for any compact set F in $M_d(k)$, there is a positive integer $q\le d^2$ such that $\Lambda_k(F^q)\ge c\cdot E_k(F)^q$. In particular, $c\cdot E_k(F)\le \max_{1\le q\le d^2}\Lambda_k(F^q)^{\frac{1}{q}}\le R_k(F)\le E_k(F)$.

Remark 2.2. This lemma expresses in a condensed form some ideas present in the proof of the main result of [18] by Eskin-Mozes-Oh. It is useful to produce elements with large eigenvalues in F^n for some small n.

We also record the following:

Lemma 2.3. ([13], Proposition 2.5.) Suppose k is archimedean (i.e. k is $\mathbb{R}$ or $\mathbb{C}$). Then for every $n \in \mathbb{N}$ and every compact subset F in $SL_d(k)$ containing 1, we have

$$E_k(F^n) \geq E_k(F)^{\sqrt{\frac{n}{8d}}}$$

3. Normalized height and Height gap

In this section we recall results obtained in [13] about heights. In [13], we introduced the notion of normalized height $\widehat{h}(F)$ of a finite subset of matrices F in $SL_d(\overline{\mathbb{Q}})$. A similar definition can be made over the algebraic closure $\overline{\mathbb{F}_p(t)}$ of $\mathbb{F}_p(t)$. Below we recall the relevant definitions and notations.

Let Ω be either $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p(t)}$ for some prime $p > 1$. By a *global field* K, we mean a field isomorphic to a finite algebraic extension of K_0, where either $K_0 = \mathbb{Q}$ or $K_0 = \mathbb{F}_p(t)$ for some prime $p > 1$. We denote by V_K the set of equivalence classes of non trivial absolute values on K. We make the following standard choice of representatives $|\cdot|_v$ for each v in V_K. Every $v \in V_K$ induces on K_0 an absolute value $v_0 \in V_{K_0}$. We first determine a set of representatives of V_{K_0}, then pick in each $v \in V_K$ the representative with that normalization. If $K_0 = \mathbb{Q}$ then any absolute value is equivalent to either the standard absolute value over $\mathbb{R}$ or the p-adic absolute value normalized so that $|p|_p = \frac{1}{p}$. These form our representatives. If $K_0 = \mathbb{F}_p(t)$, then every absolute value is equivalent to either $|\frac{P}{Q}|_0 = p^{\deg P - \deg Q}$ or $|\frac{P}{Q}|_\pi = p^{(v_\pi(Q) - v_\pi(P)) \cdot \deg \pi}$ where $\pi \in \mathbb{F}_p[t]$ is a monic irreducible polynomial and $v_\pi(P)$ is the valuation of π in the prime factor decomposition of $P \in \mathbb{F}_p[t]$. These form our representatives. For background on these issues see Weil's book [43].

Each $v \in V_K$ gives rise to a local field K_v which is the completion of K according to this absolute value. Let n_v be the dimension of K_v over the closure of K_0 in K_v. The product formula reads

$$\sum_{v \in V_K} n_v \log |x|_v = 0 \tag{1}$$

for every $x \in K$. We can now recall the definition of the standard Weil height of an algebraic number. Let $x \in K \backslash \{0\}$,

$$h(x) = \frac{1}{[K : K_0]} \sum_{v \in V_K} n_v \log^+ |x|_v,$$

where $\log^+ = \max\{\log, 0\}$.

In [13], we introduced the following heights for F a finite subset of $M_d(K) \backslash \{0\}$,

$$h(F) = \frac{1}{[K : K_0]} \sum_{v \in V_K} n_v \log^+ ||F||_v \tag{2}$$

We also defined the *normalized height of* F as

$$\widehat{h}(F) = \lim_{n \to +\infty} \frac{1}{n} h(F^n) = \frac{1}{[K:K_0]} \sum_{v \in V_K} n_v \log^+ R_v(F)$$

and the *minimal height of* F as

$$e(F) = \frac{1}{[K:K_0]} \sum_{v \in V_K} n_v \log^+ E_v(F)$$

where we have denoted by $E_v(F)$ (resp. $R_v(F)$) the quantity $E_{K_v}(F)$ (resp. $R_{K_v}(F)$) defined above. Observe that the height $h(F)$ depends on the choice of basis in K^d, while the normalized height $\widehat{h}(F)$ and minimal height $e(F)$ do not. We will often write $h = h_f + h_\infty$ to distinguish the finite part and the infinite part of the height in the obvious way.

In [13] we proved the following results:

Lemma 3.1. ([13] Proposition 2.18) There is a constant $c_1 = c_1(d) > 0$ such that for every finite subset F in $M_d(\Omega)$

$$e(F) \geq \widehat{h}(F) \geq e_f(F) + c_1 \cdot e_\infty(F) \min\{1, e_\infty(F)\}$$

It is easy to verify that $\widehat{h}(F) = 0$ if and only if $e(F) = 0$ if and only if $\langle F \rangle$ is virtually unipotent. Note in particular that if $char(\Omega) > 0$, there are no infinite places so the normalized height and the minimal height coincide, and

Lemma 3.2. If $char(\Omega) = p > 0$ then $\widehat{h}(F) = 0$ if and only if $\langle F \rangle$ finite.

Proof. The if part follows from the definition of $\widehat{h}(F)$. Suppose now that $\widehat{h}(F) = 0$. Then for every eigenvalue λ of an element $g \in F$, $h(\lambda) = 0$, hence λ is of finite order and belongs to $\overline{\mathbb{F}_p}$. Hence g also is of finite order as both the semisimple part g_s and the unipotent part g_u are of finite order. But Shur's theorem (see [16]) says that any finitely generated torsion linear group is finite. □

The main theorem of [13] is the following.

Theorem 3.3. (Height Gap, [13] Theorem 1.1) There is a constant $\varepsilon = \varepsilon(d) > 0$, such that if F is a finite subset of $SL_d(\overline{\mathbb{Q}})$ generating a non virtually solvable subgroup, then

$$\widehat{h}(F) > \varepsilon$$

Given a Chevalley group $\mathbb{G}$, there is a special choice of basis of the Lie algebra $\mathfrak{g} = Lie(\mathbb{G})$ which is made of weight vectors of a maximal split torus and defines a $\mathbb{Z}$-structure on $\mathbb{G}$ (see Steinberg's notes [38], and Paragraph 6.3 below). With respect to this basis and viewing $\mathbb{G}$ as a subgroup of $SL_d(\mathfrak{g})$ we may define the height $h(g)$ for any $g \in \mathbb{G}(\Omega)$ as in (2). We then have:

Theorem 3.4. ([13] Proposition 3.3) If $\mathbb{G}$ is a Chevalley group, then there is a constant $C = C(\mathbb{G}) > 0$ and a Zariski open subset $\mathcal{O} = \mathcal{O}(\mathbb{G})$ of $\mathbb{G} \times \mathbb{G}$ such that for any choice of Ω and for any pair $(a, b) \in \mathcal{O}(\Omega)$, there is $g \in \mathbb{G}(\Omega)$ such that, setting $F = \{a, b\}$,

$$h(gFg^{-1}) \leq C \cdot \widehat{h}(F) \tag{3}$$

Unlike Theorem 3.4, there is no analog of Theorem 3.3 for $\Omega = \overline{\mathbb{F}_p(t)}$. In [13], we proved Theorem 3.4 when $\Omega = \overline{\mathbb{Q}}$ because we were only concerned with characteristic zero. However the proof we gave works the same word by word in the positive characteristic case, and is even simpler since in that case there are no infinite places : in particular the additive constants C_∞ and C'_∞ that are obtained along the way vanish and the use of Theorem 3.3 to get rid of them is not needed (see [13]).

3.1. **Arakelov Height on Grassmannians.** Here we record some well-known facts about Arakelov heights. Let K be a global field. The Arakelov height on the projective space $\mathbb{P}(K^d)$ is defined as follows (see [6]) for $x = (x_1 : ... : x_d)$,

$$h_{Ar}(x) = \frac{1}{[K : K_0]} \sum_{v \in V_K} n_v \log ||x||_v$$

where $||x||_v$ is the standard norm on K_v^d as defined above. It is well defined thanks to the product formula (1) and always non-negative. This allows to define the height of a projective linear subspace of $\mathbb{P}(K^d)$. Indeed if $W \leq \mathbb{P}(K^d)$ is such then we set

$$h_{Ar}(W) = h_{Ar}(\Lambda^{\dim W} W)$$

where $\Lambda^{\dim W} W$ is the wedge product of W viewed as a projective point in the projective space $\mathbb{P}(\Lambda^{\dim W} K^d)$. By convention we set $h_{Ar}(\{0\}) = 0$. Recall that the following holds for two projective linear subspaces (see [6]) V and W

$$h_{Ar}(V) + h_{Ar}(W) \geq h_{Ar}(V + W) + h_{Ar}(V \cap W).$$

Moreover for every linear form f, seen as a point in the dual space $(K^d)^*$, $h_{Ar}(f)$ makes sense as we have (see [6]),

$$h_{Ar}(\ker f) = h_{Ar}(f)$$

and more generally, $h_{Ar}(W) = h_{Ar}(W^\perp)$, where $W^\perp$ is the orthogonal of W in $(K^d)^*$.

Also note that if $g \in SL_d(K)$ and W is a subspace of K^d, then

$$h_{Ar}(gW) \leq d \cdot h(g) + h_{Ar}(W)$$

where $h(g) = h(\{g\})$ as defined in the last paragraph. Note also that $h(g^{-1}) \leq (d-1)h(g)$.

Definition 3.5. Given $A \in SL_d(K)$, we will say that a vector subspace W (or its projectivization) is A**-admissible** if it is a sum of generalized eigenspaces of A. We also denote by W^c its complementary subspace, i.e. the sum of the remaining generalized eigenspaces.

Lemma 3.6. Let W be an A-admissible subspace. Then

$$h_{Ar}(W) \leq d^2 \cdot (2h(A) + \varepsilon_\Omega \log 2)$$

where $\varepsilon_\Omega = 0$ if $car(K) > 1$ and 1 if $car(K) = 0$.

Note that if $A, B \in M_d(K)$, then $h(A+B) \leq h(A) + h(B) + \varepsilon_\Omega \log 2$. Moreover, if α is an eigenvalue of A, then $h(\alpha) \leq h(A)$. Also $h_{Ar}(\ker A) \leq (rk(A)) \cdot h(A)$. Indeed $h_{Ar}(\ker A) = h_{Ar}((\ker A)^\perp) = h_{Ar}(ImA^t)$. But for $B \in M_d(K)$, $h_{Ar}(ImB) \leq (rk(B)) \cdot h(B)$, since we may choose a subset of the canonical basis, say $e_1, ..., e_k$ such that $Be_1, ..., Be_k$ generates ImB, and $||Be_1 \wedge ... \wedge Be_k||_v \leq ||B||_v^k$ where $k = rk(B)$. With these observations in hand we can prove Lemma 3.6.

Proof. We have $W = \bigoplus E_\alpha$ for some eigenvalues α of A, where E_α is the corresponding generalized eigenspace. Hence $h_{Ar}(W) \leq \sum h_{Ar}(E_\alpha)$. If $n_\alpha = \dim E_\alpha$, then $E_\alpha = \ker(A - \alpha)^{n_\alpha}$. Hence $h_{Ar}(E_\alpha) \leq d \cdot h((A-\alpha)^{n_\alpha}) = dn_\alpha \cdot (2h(A) + \varepsilon_\Omega \log 2)$. Hence the result. □

4. Proximality

In this paragraph we recall the well-known notion of a proximal element in $SL_d(k)$, where k is a local field, and we show some precise estimates as to how such elements act on the projective space $\mathbb{P}(k^d)$. The results of this paragraph are contained in Lemma 4.6 and Lemma 4.7 below.

A element $a \in SL_d(k)$ is said to be **proximal** if there is a unique (multiplicity one) eigenvalue of a with maximum modulus $\Lambda_k(a)$. We will also need to consider almost proximal elements where the eigenvalues which are larger than, say, some ω are much larger than all other eigenvalues.

Lemma 4.6 computes the rate of convergence to the attracting point of powers of a given proximal element a in terms of three quantities : its norm $||a||$, the modulus of its maximal eigenvalue $\Lambda_k(a)$ and the modulus of its second to maximal eigenvalue $\lambda_k(a)$. A similar estimate is given for an almost proximal element depending on the choice of the cursor ω. Lemma 4.7 is a converse statement originally used by Tits in the proof of his alternative which gives a sufficient condition for $a \in SL_d(k)$ to be proximal : it is as soon as a stabilizes some open subset where it contracts distances.

We had to be careful in those estimates, and they differ in some non insignificant ways from the estimates used in earlier works (as in [1][14]). In particular they are uniform over all ultrametric local fields. The multiplicative constants $C_{k,i}$'s that appears in the estimates always disappears when k is ultrametric. This will turn out to be crucial for us in the sequel.

4.1. **The Fubini-Study metric on $\mathbb{P}(k^d)$.** Let k be a local field and $\overline{k}$ an algebraic closure of k. Recall that we endow the projective space $\mathbb{P}(\overline{k}^d)$ with the standard (Fubini-Study) distance defined by

$$d([u],[v]) = \frac{||u \wedge v||}{||u|| \cdot ||v||} \tag{4}$$

for any $u, v \in \overline{k}^d \backslash \{0\}$ and $|| \cdot ||$ is the standard norm on $\overline{k}^d$ (i.e. Euclidean norm if k is archimedean and sup norm if k is non archimedean). To avoid heavy notation, we will denote by the same letter a non zero vector, or subspace of $\overline{k}^d$ and its projectivization in $\mathbb{P}(\overline{k}^d)$. This ambiguity should not lead to any serious confusion.

We denote by $\mathbb{K}_k$ the maximal compact subgroup of $SL_d(k)$ equal to $SO(d,\mathbb{R})$ if $k = \mathbb{R}$, $SU(d,\mathbb{R})$ if $k = \mathbb{C}$, and $SL_d(\mathcal{O}_k)$ is k is ultrametric. Its action on $\mathbb{P}(k^d)$ preserves d (in fact this characterizes d up to composition by some positive function).

Lemma 4.1. Let $h \in SL_d(k)$. Then $Lip(h) \leq (||h|| \cdot ||h^{-1}||)^2 \leq ||h||^{2d}$, where $Lip(h)$ is the smallest constant $L \geq 0$ such that $d(hx, hy) \leq L \cdot d(x,y)$ for all $x, y \in \mathbb{P}(k^d)$.

Proof. Writing h in Cartan's $\mathbb{K}_k A \mathbb{K}_k$ decomposition of $SL_d(k)$, one sees that we can assume that h is diagonal and we are thus reduced to a straightforward verification. □

Recall that if H is a hyperplane in k^d, and f a non zero linear form on k^d with kernel H, then if $u \in k^d \backslash \{0\}$, its distance to H is

$$d(u, H) = \frac{|f(u)|}{||f|| \cdot ||u||} \tag{5}$$

where $||f|| = \sup\{|f(x)|, ||x|| \leq 1, x \in k^d\}$. More generally, if V and W are two k-subspaces in direct sum, i.e. $V \oplus W = k^d$, then

$$d(V, W) = \frac{||\underline{v} \wedge \underline{w}||}{||\underline{v}|| \cdot ||\underline{w}||} \tag{6}$$

where $\underline{v} = v_1 \wedge ... \wedge v_l$ and $\underline{w} = w_1 \wedge ... \wedge w_{d-l}$ for any basis $(v_1, ..., v_l)$ of V and $(w_1, ..., w_{d-l})$ of W. In particular, when k is archimedean, two subspaces are orthogonal if and only if they are at distance 1. Let $(e_1, ..., e_d)$ be the canonical basis in k^d.

Lemma 4.2. Let f be a non-zero linear form on k^d and $H = \ker f$. Let V a k-subspace in k^d and V^* the orthogonal of V in the dual of k^d. Then for every $v \in V$,

$$d(v, H) = d(v, V \cap H) \cdot d(f, V^*) \tag{7}$$

Proof. Observe that as $\mathbb{K}_k$ permutes transitively the k-subspaces of given dimension and preserves d, we may assume that $V = \langle e_1, ..., e_p \rangle$ for some $p \in [0, d]$. Then we may write f in the dual canonical basis $f = \sum f_i e_i^* = f^< + f^>$ where $f^<$ is the part of the sum involving indices $i \leq k$ and $f^>$ the other part. Let $\underline{e}^> = e_{p+1}^* \wedge ... \wedge e_d^*$. Then $||f|| \cdot d(f, V^*) = ||f \wedge \underline{e}|| = ||f^< \wedge \underline{e}|| = ||f^<||$. On the other hand note that $f^<$ coincides with f on V. Hence for $v \in V$, $d(v, V \cap H) = \frac{|f(v)|}{||f^<|| \cdot ||v||}$. As $d(v, H) = \frac{|f(v)|}{||f|| \cdot ||v||}$, combining these relations we do obtain (7). □

Lemma 4.3. Let $V \oplus W = k^d$ and H a hyperplane in k^d with $V \nsubseteq H$. Let π be the linear projection onto V with kernel W. Then for every $u \in \mathbb{P}(k^d) \backslash W$ we have

$$d(\pi u, V \cap H) \geq d(u, W + V \cap H) \cdot d(V, W)$$

Proof. Write $u = \pi u + \pi u^\perp \in V \oplus W$. If $v_1, ..., v_{k-1}$ is a basis of $V \cap H$ and $w_1, ..., w_{d-k}$ a basis of W we set $\underline{v} = v_1 \wedge ... \wedge v_{k-1}$ and $\underline{w} = w_1 \wedge ... \wedge w_{d-k}$. We have $d(\pi u, V \cap H) \geq d(\pi u, W + V \cap H) = \frac{||\pi u \wedge \underline{v} \wedge \underline{w}||}{||\pi u|| \cdot ||\underline{v} \wedge \underline{w}||} = d(u, W + V \cap H) \cdot \frac{||u||}{||\pi u||}$. We may assume $u \notin V$, then on the other hand $d(V, W) \leq d(\pi u, \pi u^\perp) = \frac{||u \wedge \pi u^\perp||}{||\pi u|| \cdot ||\pi u^\perp||} \leq \frac{||u||}{||\pi u||}$. We are done. □

4.2. **Contraction properties of proximal and almost proximal elements.** For $a \in SL_d(k)$ we set E_λ its generalized eigenspace with eigenvalue λ. In this paragraph, we will assume that eigenvalues of a belong to k. We let $\Lambda_k(a) = \max\{|\mu|_k, \mu \text{ eigenvalue of } a\}$ and $\lambda_k(a)$ the modulus of the second heighest eigenvalue of a. An element $a \in SL_d(k)$ is said to be **proximal** if $\lambda_k(a) < \Lambda_k(a)$.

To deal with non proximal elements we introduce some positive real number $\omega > 0$, such that $\Lambda_k(a^{-1})^{-1} < \omega \leq \Lambda_k(a)$. We set $\Lambda_k^\omega(a) = \min\{|\mu|_k, \mu$ eigenvalue of $a, |\mu|_k \geq \omega\}$ and $\lambda_k^\omega(a) = \max\{|\mu|_k, \mu \text{ eigenvalue of } a, |\mu|_k < \omega\}$.

Lemma 4.4. Suppose $a \in SL_d(k)$ and let $A = \Lambda_k(a)\Lambda_k(a^{-1}) \geq 1$. For every $\varepsilon > 0$ there is $\eta = \eta(\varepsilon, d) > 0$ and ω such that

$$\Lambda_k(a^{-1})^{-1} < \omega \leq \Lambda_k(a)$$

and

$$A^\eta \cdot \left(\frac{\Lambda_k(a)}{\Lambda_k^\omega(a)}\right)^{\frac{1}{\varepsilon}} \leq \frac{\Lambda_k^\omega(a)}{\lambda_k^\omega(a)} \tag{8}$$

Proof. Let $\lambda_1, ..., \lambda_d$ be the eigenvalues of a ordered as $|\lambda_1|_k \geq ... \geq |\lambda_d|_k$. Let $\ell_i = \log \frac{|\lambda_i|}{|\lambda_{i+1}|} \geq 0$. Fix $\varepsilon > 0$ and take some $\eta > 0$. We claim that for η small enough, there exists $i_0 \in [0, d-2]$ such that $\ell_{i_0+1} - \eta \log A \geq \frac{1}{\varepsilon} \cdot (\ell_1 + ... + \ell_{i_0})$. Indeed, otherwise we would have $\ell_1 < \eta \log A$, $\ell_2 < \eta \log A + \frac{1}{\varepsilon}\ell_1$, etc, until we get $\log A = \ell_1 + ... + \ell_{d-1} \leq C(\varepsilon, d)\eta \log A$ for some computable constant $C(\varepsilon, d)$, a contradiction if η is smaller than say $\frac{1}{2C(\varepsilon,d)}$. Let $\omega = |\lambda_{i_0+1}|_k$. We are done. □

When ε is fixed and ω so given by Lemma 4.4, we will refer to a as being *almost proximal for* ω.

We will let H_a^ω be the vector subspace equal to the sum of the E_λ's for which $|\lambda|_k \leq \lambda_k^\omega(a)$. Similarly, we denote its complementary subspace by $V_a^\omega = \bigoplus E_\lambda$, the sum being over those λ's such that $|\lambda|_k \geq \Lambda_k^\omega(a)$. We let π_a^ω be the linear projection onto V_a^ω with kernel H_a^ω. We also set $l_\omega = \dim V_a^\omega$. If a is proximal, we will drop the superscript ω (and set it to be $\Lambda_k(a)$) and simply denote by V_a, H_a, and π_a the corresponding quantities.

Remark 4.5. Note that if $a \in GL_d(k)$, then its eigenvalues belong to the extension of k generated by all algebraic extensions of k in a given algebraic closure $\overline{k}$ of degree at most d (there are finitely many such). So this extension is also a local field. Hence up to passing to this finite extension one may always assume that the eigenvalues of a belong to k.

Lemma 4.6 below is the main result of this section. Its proof will occupy the subsequent two paragraphs. When k is archimedean, let $C_k = 2$ and $C_{k,1} = d$. When k is ultrametric set $C_k = C_{k,1} = 1$. Let also $p(d) = 10^d$ (these are only given as crude estimates, we made no attempt at finding sharp constants in this statement). Finally let $L_k^\omega(a) = 1$ if k is ultrametric while $L_k^\omega(a) = \frac{\Lambda_k^\omega(a)}{\Lambda_k^\omega(a) - \lambda_k^\omega(a)}$ if k is archimedean.

Lemma 4.6. Let $a \in SL_d(k)$ whose eigenvalues belong to k and assume ω is a real number such that $\Lambda_k(a^{-1})^{-1} < \omega \leq \Lambda_k(a)$. Let $l_\omega = \dim V_a^\omega$ and π_a^ω the projection on V_a^ω with kernel H_a^ω. Then for any $u \neq v \in \mathbb{P}(k^d)$, and any integer $n \in \mathbb{N}$

(9)

$$d(a^n u, \pi_a^\omega(a^n u)) \cdot d(u, H_a^\omega) \leq (C_k \cdot ||a||_k)^{p(d)} \cdot \left(C_{k,1}^{l_\omega} \cdot \left(\frac{\Lambda_k(a)}{\Lambda_k^\omega(a)} \right)^{l_\omega - 1} \cdot \frac{\lambda_k^\omega(a)}{\Lambda_k^\omega(a)} \right)^n$$

Furthermore

(10)

$$\frac{d(a^n u, a^n v) \cdot d(v, H_a^\omega) \cdot d(u, H_a^\omega)}{d(u,v)} \leq (C_k \cdot L_k^\omega(a) \cdot ||a||_k)^{p(d)} \cdot \left(C_{k,1}^{2l_\omega + 2} \cdot \left(\frac{\Lambda_k(a)}{\Lambda_k^\omega(a)} \right)^{2l_\omega - 1} \cdot \frac{\lambda_k(a)}{\Lambda_k(a)} \right)^n$$

Observe that (9) says nothing if the quantity inside the bracket is not < 1.

The following Tits Converse Lemma is useful when one needs to build an element x such that both x and x^{-1} are proximal.

Lemma 4.7. (Tits Converse Lemma [39]) Let $a \in SL_d(k)$. Assume that there exists a point $v \in \mathbb{P}(k^d)$ and an open neighborhood U of v such that $\overline{aU} \subset U$ and such that $Lip(a_{|U}) < 1$, where $Lip(a_{|U})$ is the smallest constant $L > 0$ such that $d(ax, ay) \leq L \cdot d(x, y)$ for every $x, y \in U$. Then a is proximal, $V_a \in U$ and $\frac{\lambda_k(a)}{\Lambda_k(a)} \leq Lip(a_{|U})$.

Proof. The compact subset $\overline{aU}$ is stable under a and on it a contracts distances. It follows immediately that all orbits $(a^n u)_{n\geq 0}$ converge to the unique fixed point v_a of a in $\overline{aU}$. Let α be the eigenvalue of a with eigenvector v_a. Let β be another eigenvalue of a (if α has multiplicity higher than 1, we may take $\beta = \alpha$). There is a non zero vector w such that $aw = \beta w + \kappa v_a$ for some $\kappa \in k$. Let $\varepsilon \in k\backslash\{0\}$ with $|\varepsilon|_k$ arbitrarily small. Then one computes from (4) $\lim_{|\varepsilon|\to 0} \frac{d(a(v_a+\varepsilon w), v_a)}{d(v_a+\varepsilon w, v_a)} = \frac{|\beta|}{|\alpha|}$. If $|\varepsilon|_k$ is small enough, $v_a + \varepsilon w \in U$ and thus $\frac{|\beta|}{|\alpha|} \leq Lip(a_{|U}) < 1$. We are done. □

4.3. **Four intermediary geometric lemmas.** In this paragraph, we state and prove four intermediary results needed in the proof of Lemma 4.6. Unless otherwise stated $a \in GL_d(k)$ and its eigenvalues belong to k.

Lemma 4.8. Let $a \in GL_d(k)$ and α an eigenvalue of a. Then there is some $h \in \mathbb{K}_k$ such that hah^{-1} is a lower triangular matrix with top left entry equal to α.

Proof. Since eigenvalues of a belong to k, a and hence also its transpose a^t are triangularizable over k, i.e. a^t stabilizes a full k-flag $\mathcal{F}$. We may also assume that $\mathcal{F}$ starts with the line kv, where v is an eigenvector of a^t with eigenvalue α. But full k-flags are conjugate under $GL_d(k)$. Hence $\mathcal{F} = g\mathcal{F}_0$ where $\mathcal{F}_0$ is the standard flag generated by the canonical basis of k^d and $ge_1 = v$. The Iwasawa decomposition reads $GL_d(k) = \mathbb{K}_k B_0$ where B_0 is the Borel stabilizing $\mathcal{F}_0$. Thus we may assume that $g \in \mathbb{K}_k$. Thus $g^{-1}a^t g$ stabilizes $\mathcal{F}_0$ and is upper triangular. Hence $h = g^t \in \mathbb{K}_k$ will do. □

Let $C_{k,1}$ be equal to d if k is archimedean and equal to 1 if k is ultrametric.

Lemma 4.9. Let $a \in GL_d(k)$. Then there exists an $h \in SL_d(k)$ such that $\|hah^{-1}\| \leq C_{k,1}\Lambda_k(a)$ and $\max\{\|h\|, ||h^{-1}||\} \leq \|a\|^{\frac{d-1}{2}}$.

Proof. By Lemma 4.8, one may assume that $a \in GL_d(k)$ is lower triangular. Let $h = t^{\frac{d+1}{2}} diag(t^{-1}, ..., t^{-d}) \in SL_d(k)$. Choose $t \in k$ such that $|t^{-1}|_k = ||a||_k$. Then $\max\{\|h\|, ||h^{-1}||\} \leq \|a\|^{\frac{d-1}{2}}$ and the off-diagonal coefficients of hah^{-1} are of modulus ≤ 1. As $||a|| \leq C_{k,1} \max |a_{ij}|$, we are done. □

Remark 4.10. Note that we also get $||\Lambda^2(hah^{-1})|| \leq C_{k,1}^4 \Lambda_k(a)\lambda_k(a)$.

Recall that C_k is 2 if k is archimedean and 1 if k is ultrametric.

Lemma 4.11. Let $a \in SL_d(k)$ and ω with $\Lambda_k(a^{-1})^{-1} < \omega \leq \Lambda_k(a)$. Let $l_\omega = \dim V_a^\omega$ and $L_k^\omega(a) = 1$ if k is ultrametric while $L_k^\omega(a) = \frac{\Lambda_k^\omega(a)}{\Lambda_k^\omega(a) - \lambda_k^\omega(a)}$ if k is archimedean. Then $d(V_a^\omega, H_a^\omega)^{-1} \leq (C_k \cdot L_k^\omega(a)||a||^l)^{(\binom{d}{l}-1)}$.

Proof. First let us assume a is proximal (and $\omega = \Lambda_k(a)$) with eigenvalue of maximal modulus α_1, and let $\alpha_2, ..., \alpha_d$ be the remaining eigenvalues. By Lemma 4.8, one may assume that $a \in SL_d(k)$ is lower triangular with α_1 in the upper left corner. Let $V_a \in k^d \backslash \{0\}$ be such that $aV_a = \alpha_1 V_a$. As $V_a \notin H_a = \langle e_2, ..., e_d \rangle$, we may assume that $V_a = (1, x_2, ..., x_d)$ in the canonical basis. Then $d(V_a, H_a) = 1/||V_a||$. Decomposing $aV_a = \alpha_1 V_a$ in coordinates, we obtain $(\alpha_1 - \alpha_2)v_2 = a_{21}$, $(\alpha_1 - \alpha_3)v_3 = a_{31} + a_{32}v_2$, etc. This allows to recursively estimate each v_i and at the end we get that $||V_a|| \leq (1 + L_k^2 \frac{||a||^2}{\Lambda_k(a)^2})^{(d-1)/2} \leq 2^{(d-1)/2} \cdot L_k^{d-1} \cdot \left(\frac{||a||}{\Lambda_k(a)}\right)^{d-1}$ when k is archimedean while when k is ultrametric $||V_a|| \leq L_k^{d-1} \left(\frac{||a||}{\Lambda_k(a)}\right)^{d-1}$.

We now explain how to reduce the general case to the proximal case. Let $(v_1, ..., v_l)$ and $(w_1, ..., w_{d-l})$ be respective basis of V_a^ω and H_a^ω. Let $\underline{v} = v_1 \wedge ... \wedge v_l$ and $\underline{w} = w_1 \wedge ... \wedge w_{d-l}$. From (6) we have $d(V_a^\omega, H_a^\omega) = \frac{||\underline{v} \wedge \underline{w}||}{||\underline{v}|| \cdot ||\underline{w}||}$. The canonical map $\Lambda^l k^d \times \Lambda^{d-l} k^d \to k$ establishes an isomorphism between $\Lambda^{d-l} k^d$ and the dual of $\Lambda^l k^d$. Under this identification $\underline{w}$ is a linear form on $\Lambda^l k^d$ and formulae (6) and (5) coincide, i.e. $d(V_a^\omega, H_a^\omega) = d(\Lambda^l V_a^\omega, \ker \underline{w})$. On the other hand $\Lambda^l a$ is proximal on $\Lambda^l k^d$ with $V_{\Lambda^l a} = \Lambda^l V_a^\omega$ and $H_{\Lambda^l a} = \ker \underline{w}$. Hence by the above $d(\Lambda^l V_a, \ker \underline{w})^{-1} \leq (C_k \cdot L_k^\omega(\Lambda^l a) \cdot \frac{||\Lambda^l a||}{\Lambda_k(\Lambda^l a)})^{D-1}$ where $D = \dim \Lambda^l k^d = \binom{d}{l}$ and the result follows as $\Lambda_k(\Lambda^l a) \geq 1$. $\square$

Recall that $C_{k,1}$ is d if k is archimedean and 1 if k is ultrametric.

Lemma 4.12. Let $a \in SL_d(k)$ with $\Lambda_k(a^{-1})^{-1} < \omega \leq \Lambda_k(a)$ and $l_\omega = \dim V_a^\omega$. Set $V = \langle e_2, ..., e_{l_\omega} \rangle$, $H = \langle e_{l_\omega+1}, ..., e_d \rangle$. There exists $h \in SL_d(k)$ with $hV_a^\omega = V$, $hH_a^\omega = H$ such that if we set $\widetilde{a} = hah^{-1}$ then $\left\|\widetilde{a}_{|H}\right\| \leq C_{k,1}\lambda_k^\omega(a)$ and $\left\|\widetilde{a}_{|V}\right\| \leq C_{k,1}\Lambda_k(a)$ and $\|h^{-1}\| \leq \left(\frac{\sqrt{C_{k,1}}}{d(V_a^\omega, H_a^\omega)}\right)^{\frac{d(d+1)}{2}} \|a\|^{\frac{d-1}{2}}$.

Proof. First note that applying Lemma 4.8 we can assume that a is lower triangular and that $H_a^\omega = H$. Then observe that for any subspace F of k^d, one may find a basis $f_1, ..., f_p$ of F such that $||f_1 \wedge ... \wedge f_p|| = 1$ and $||f_i|| = 1$ for each $i = 1, ..., p$. Choose such a basis, say $v_1, ..., v_l$ of V_a^ω and, for $\mu \in k$ to be defined later, denote by $h_1 \in GL_d(k)$ the map $h_1 v_i = e_i$ if $i < l$, $h_1 v_l = \mu e_l$ and $h_1 e_i = e_i$ for $i > l$. Then compute $h_1^{-1}e_1 \wedge ... \wedge h_1^{-1}e_d = \det(h_1^{-1})\underline{e} =$

$\mu^{-1}\underline{v} \wedge \underline{w}$ where $\underline{e} = e_1 \wedge ... \wedge e_d$, $\underline{v} = v_1 \wedge ... \wedge v_l$ and $\underline{w} = e_{l+1} \wedge ... \wedge e_d$. Now choose $\mu \in k$ so that $\det(h_1) = 1$, then $|\mu|_k = ||\underline{v} \wedge \underline{w}|| = d(V_a^\omega, H_a^\omega)$. Then $||h_1^{-1}|| \leq |\mu^{-1}|_k$ when k is ultrametric and $||h_1^{-1}|| \leq \sqrt{d}|\mu^{-1}|_k$ when k is archimedean.

So $h_1 a h_1^{-1}$ stabilizes V and H. Now applying Lemma 4.9 on V and on H, we can find $h_0 \in SL_d(k)$, stabilizing V and H such that $||h_0 h_1 a h_1^{-1} h_{0|V}^{-1}|| \leq C_{k,1}\Lambda_k(a)$ and $||h_0 h_1 a h_1^{-1} h_{0|H}^{-1}|| \leq C_{k,1}\lambda_k^\omega(a)$ and $||h_0^{-1}|| \leq ||h_1 a h_1^{-1}||^{\frac{d+1}{2}}$. Set $h = h_0 h_1$ we are done. $\square$

4.4. **Proof of Lemma 4.6.** For $l \in [1, d-1]$ set as above $V = \langle e_2, ..., e_l \rangle$ and $H = \langle e_{l+1}, ..., e_d \rangle$. Let $b \in SL_d(k)$ be such that $bV = V$ and $bH = H$ and let π_b be the linear projection onto V with kernel H. We first claim that for every $u \in \mathbb{P}(k^d)$

$$d(u, H) \cdot d(bu, \pi_b(bu)) \leq ||b_{|H}|| \cdot ||(b_{|V})^{-1}|| \tag{11}$$

Indeed, note that $d(u, H) = \frac{||u \wedge \underline{w}||}{||u||} \leq \frac{||\pi_b(u)||}{||u||}$ where $\underline{w} = e_{l+1} \wedge ... \wedge e_d$ and writing $u = \pi_b(u) + \pi_b(u)^\perp$ we have $d(bu, \pi_b(bu)) = \frac{||b_{|H}\pi_b(u)^\perp \wedge \pi_b(u)||}{||bu||} \leq \frac{||b_{|H}|| \cdot ||u||}{||b\pi_b(u)||}$. Combining both inequalities we get (11).

Now we claim that for any $u \neq v \in \mathbb{P}(k^d)$ we claim that:

(12)
$$\frac{d(u, H) \cdot d(v, H) \cdot d(bu, bv)}{d(u, v)} \leq \max\{||\Lambda^2 b_{|V}||, ||\Lambda^2 b_{|H}||, ||b_{|H}|| \cdot ||b_{|V}||\} \cdot ||(b_{|V})^{-1}||^2$$

Indeed, using Cartan's $\mathbb{K}_k A \mathbb{K}_k$ decomposition on V and H separately, we may assume that b is diagonal $diag(\alpha_1, ..., \alpha_l)$. Then write $bu \wedge bv = bu_{|V} \wedge bv_{|V} + bu_{|H} \wedge bv_{|V} + bu_{|V} \wedge bv_{|H} + bu_{|H} \wedge bv_{|H}$. Since $bu_{|H} \wedge bv_{|V} + bu_{|V} \wedge bv_{|H} = \sum_{1 \leq i \leq l} \alpha_i e_i \wedge (u_i v_{|H} - v_i u_{|H})$ we get $||bu \wedge bv|| \leq \max\{||\Lambda^2 b_{|V}||, ||\Lambda^2 b_{|H}||, ||b_{|H}|| \cdot ||b_{|V}||\} \cdot ||u \wedge v||$. On the other hand $||bu|| \geq \frac{||u_{|V}||}{||b_{|V}^{-1}||}$ (resp. $||bv|| \geq \frac{||v_{|V}||}{||b_{|V}^{-1}||}$) and $d(u, H) \leq \frac{||u_{|V}||}{||u||}$ (resp.$d(v, H) \leq \frac{||v_{|V}||}{||v||}$). This shows (12).

We now prove (9) and (10). For $n \in \mathbb{N}$ and $a \in SL_d(k)$, we may apply Lemma 4.12 and Remark 4.10 to a and get $h \in SL_d(k)$ with $\widetilde{a} = hah^{-1} \in SL_d(k)$ satisfying $\widetilde{a}V = V$ (resp. $\widetilde{a}H = H$) and $||\widetilde{a}_{|H}|| \leq C_{k,1}\lambda_k^\omega(a)$ (resp. $||\widetilde{a}_{|V}|| \leq C_{k,1}\Lambda_k(a)$ and $||\Lambda^2\widetilde{a}_{|V}|| \leq C_{k,1}^4 \Lambda_k(a)\lambda_k(a)$). Note that $\left\|\widetilde{a}_{|V}^{-1}\right\| \leq \frac{1}{\det \widetilde{a}_{|V}} \left\|\widetilde{a}_{|V}\right\|^{l_\omega - 1} \leq \frac{1}{\Lambda_k^\omega(a)} \cdot \left(C_{k,1} \cdot \frac{\Lambda_k(a)}{\Lambda_k^\omega(a)}\right)^{l_\omega - 1}$. Let $b = (\widetilde{a})^n$. Then (11) and (12) translate as

$$d(u, H_a^\omega) \cdot d(a^n u, \pi_a^\omega(a^n u)) \leq Lip(h^{-1})^2 \cdot \left(C_{k,1}^{l_\omega} \cdot \left(\frac{\Lambda_k(a)}{\Lambda_k^\omega(a)}\right)^{l_\omega - 1} \cdot \frac{\lambda_k^\omega(a)}{\Lambda_k^\omega(a)}\right)^n$$

$$d(u,H_a^\omega)\cdot d(v,H_a^\omega)\cdot d(a^nu,a^nv)\leq Lip(h^{-1})^3Lip(h)\cdot\left(C_{k,1}^{2l_\omega+2}\cdot\left(\frac{\Lambda_k(a)}{\Lambda_k^\omega(a)}\right)^{2l_\omega-1}\cdot\frac{\lambda_k(a)}{\Lambda_k^\omega(a)}\right)^n$$

Recall that by Lemma 4.1, $Lip(h)$ and $Lip(h^{-1})$ are at most $||h^{-1}||^{2d}$. From Lemma 4.11 we have $d(V_a^\omega,H_a^\omega)^{-1}\leq\left(C_k\cdot L_k(a)\cdot||a||^l\right)^{\left(\binom{d}{l}-1\right)}$. Then from Lemma 4.12

$$||h^{-1}||\leq\left(\sqrt{C_{k,1}}\cdot(L_k(a)C_k)^{\left(\binom{d}{l}-1\right)}\right)^{\frac{d(d+1)}{2}}\|a\|^{\frac{d-1}{2}+l\left(\binom{d}{l}-1\right)\frac{d(d+1)}{2}}$$

Note that, when k is archimedean, inequality (9) is trivial if $\lambda_k^\omega(a)\geq\frac{1}{2}\Lambda_k^\omega(a)$. Therefore we may assume in the archimedean case that $L_k(a)\leq 2$. As $8d(\frac{d-1}{2}+l\left(\binom{d}{l}-1\right)\frac{d(d+1)}{2})\leq 10^d$ estimating the constant we do indeed obtain (10) and (9).

5. Ping-Pong

In this technical section, we work with a fixed local field k and we explain how to construct two short words x and y with letters in some finite set F in $SL_d(k)$ such that x and y form a ping-pong pair and thus generate a free subgroup. The goal of this introductory paragraph is to give a list of several conditions of geometric nature (i) to (vi) on F and state two lemmas, Lemma 5.1 and 5.3 below, which assert precisely that these conditions are sufficient to construct the ping-pong pair. These two statements are the only ones which will be used in further sections.

As in Tits [39], the construction of the ping-pong pair follows two steps. First, starting from a proximal element a lying in F or in a bounded power of F, we need to build a short word with letters in F, say x, such that both x and x^{-1} are proximal elements (Lemma 5.1). Second, we need to find a conjugate of it, say $y=cxc^{-1}$ such that x and y together play ping-pong (Lemma 5.3).

The construction presented here follows verbatim that of Tits. But while Tits needed only asymptotic statements which held for sufficienlty high powers of group elements, no matter how high, we need to have control on the length of the words. We thus have to give a quantified version of Tits' argument and give precise estimates at each step. More importantly, while Tits did not need to care about the choice of a distance on $\mathbb{P}(k^d)$ (any one inside the "admissible" class he defined was good for his purposes), it is crucial for us that we work with the Fubini-Study distance introduced in Section 4. The reason is that all constants then disappear and are equal to 1 for all ultrametric local fields, hence giving to us the possibility to bound the length of the generators of the free subgroup **independently of the choice of the local field.**

Let $(k_i)_{1\leq i\leq 4}$ be four positive integers and $\varepsilon_0, T_0, T_1, T_2 > 0$ be positive real numbers. Let $\varepsilon > 0$ with $\varepsilon \leq \varepsilon_0/12d^2$. Let k_0 be a local field. Suppose $F \subset SL_d(k_0)$ is a finite set containing 1. All eigenvalues and eigenspaces of elements in the group generated by F are defined over a fixed finite extension of k_0 of degree at most $d!$. Let k be this extension. For a subspace V in k^d we denote by $V^\perp$ its orthogonal in the dual space of k^d. We say that a non-trivial subspace W of k^d is *a-admissible* for $a \in SL_d(k_0)$ if it is a sum of generalized eigenspaces of a. We also denote by W^c its complementary, i.e. the sum of the remaining generalized eigenspaces, so that $k^d = W \oplus W^c$.

List of Conditions for ping-pong (i)-(vi):
Let $a \in F^{k_1}$, $b \in F^{k_2}$, $t \in F^{k_3}$. Assume

(**i**) a is proximal

(**ii**)

$$||F||_k > C_{k,1}^{2d} \tag{13}$$

(**iii**)

$$\left(\frac{\Lambda_k(a)}{\lambda_k(a)}\right)^{\frac{1}{\varepsilon_0}} \geq \Lambda_k(a) \geq ||F||_k^{T_1} \tag{14}$$

(**iv**) For any a-admissible subspace W (see Def. 3.5) we have

$$d({}^t b^{\pm 1} \cdot H_a^\perp, W^\perp) > ||F||_k^{-T_0} \tag{15}$$

(**v**) For any a-admissible subspace W we have

$$d(tV_a, W^c + W \cap b^{-1}H_a) \;\;\geq\;\; ||F||_k^{-T_0} \tag{16}$$

$$d(t^{-1}V_a, W^c + W \cap bH_a) \;\;\geq\;\; ||F||_k^{-T_0} \tag{17}$$

Note that condition (15) on b implies that $W^c + W_1 \cap b^{\pm 1}H_a$ are hyperplanes, so these distances are computable via (5).

Lemma 5.1. There is $\tau_1(d,\varepsilon) \in \mathbb{N}$ and $\tau_3 = \tau_3(d,k_1,k_2,k_3,\varepsilon_0,\varepsilon,T_0,T_1) \in \mathbb{N}$ such that if $T_1 \geq \tau_1$ and $T_3 \geq \tau_3$, there is $l = l(d,k_1,k_2,k_3,\varepsilon_0,\varepsilon,T_0,T_1,T_3) \in \mathbb{N}$ such that for some $l_0, l_1 \in [0,l]$ the element $x = a^{l_0}ba^{-l_1}t$ is proximal as well as x^{-1} and $\Lambda_k(x) \geq \Lambda_k(a)^{T_3}\lambda_k(x)$ and $\Lambda_k(x^{-1}) \geq \Lambda_k(a)^{T_3}\lambda_k(x^{-1})$ and $d(V_x, H_x) \geq \Lambda_k(a)^{-2T_3}$ and $d(V_{x^{-1}}, H_{x^{-1}}) \geq \Lambda_k(a)^{-2T_3}$.

We let $k_4 = 2k_1 l + k_2 + k_3$ so that $x \in F^{k_4}$.

Remark 5.2. As Y. Benoist observed in [4] (see also J-F. Quint [32]) it is possible to construct Zariski-dense semi-groups, say in $SL_3(\mathbb{Q}_p)$ which are made exclusively of proximal elements whose inverses are not proximal. Hence our method does not allow in general (the SL_2 case is fine however) to construct the generators of a free subgroup as positive words in F.

Assume T_1 and T_3 satisfy the assumptions of Lemma 5.1 let x be the element we get. Assume that there is $c \in F^{k_5}$ such that

(**vi**)

$$d(c^{\pm 1}(V_x \cup V_{x^{-1}}), H_x \cup H_{x^{-1}}) \geq \frac{1}{||F||_k^{T_2}} \tag{18}$$

Lemma 5.3. Then there is $l_2 = l_2(d, (k_i)_{1\leq i\leq 5}, \varepsilon, \varepsilon_0, (T_i)_{0\leq i\leq 3}) \in \mathbb{N}$ such that for every $n \geq l_2$, x^n and $y = cx^nc^{-1}$ play ping-pong on $\mathbb{P}(k^d)$ and generate a free subgroup of $SL_d(k)$.

5.1. **Cayley-Hamilton trick.** In [39] Tits used the fact that if $a_0 \in GL_d(k)$ has all eigenvalues of the same modulus and if a vector v lies far from a hyperplane H then for a set of positive density of $n \in \mathbb{N}$ the vectors $a^n \cdot v$ lie far from H. In [18], Eskin-Mozes-Oh made clever use of the Cayley-Hamilton theorem in order to show a statement of a similar nature which also gives a bound on the smallest appropriate n. The following lemma is a reformulation of the same trick.

Recall that $C_{k,2}$ is $d2^d$ when k is archimedean and 1 when k is ultrametric. The following lemma, which we will use in the proof of Claim 0 below, expresses the same idea.

Lemma 5.4. Let $a_0 \in GL_d(k)$, let H be a hyperplane in k^d and let $v \in \mathbb{P}(k^d)$. Then there is some integer $j_0 \in [1, d-1]$ such that

$$d(a_0^{j_0}v, H) \geq \frac{1}{C_{k,2}} \cdot \left(\frac{\Lambda_k(a_0)}{||a_0||_k}\right)^{j_0} \cdot \frac{|\det a_0|_k}{\Lambda_k(a_0)^d} \cdot d(v, H) \tag{19}$$

Proof. Let $\Lambda \in k$ such that $|\Lambda| = \Lambda_k(a_0)$ and set $\widetilde{a_0} = \frac{a_0}{\Lambda}$. According to the Cayley-Hamilton theorem, there are coefficients $(c_j)_{1\leq j\leq d-1}$ in k such that $\sum_{j=1}^{d-1} c_j \widetilde{a_0}^j = \det \widetilde{a_0}$. Moreover $|c_j|_k \leq \binom{d}{j} \leq 2^d$ when k is archimedean, when $|c_j|_k \leq 1$ when k is ultrametric. Let f be a linear form on k^d with $||f||_k = 1$ and $\ker f = H$. There must exist some $j_0 \in [1, d-1]$ such that $|c_{j_0} f(\widetilde{a_0}^{j_0}v)|_k \geq \frac{|\det \widetilde{a_0}|_k}{C_{k,1}} \cdot |f(v)|_k$ where $C_{k,1}$ is d if k is archimedean and 1 if k is ultrametric. Hence $|f(a_0^{j_0}v)|_k \geq \frac{1}{C_{k,2}} \cdot \Lambda_k(a_0)^{j_0} \cdot \frac{|\det a_0|_k}{\Lambda_k(a_0)^d} \cdot |f(v)|_k$ and (19) follows. $\square$

5.2. **Proof of Lemma 5.1.** Recall that a is proximal but a^{-1} may not be. However, as we have fixed $\varepsilon > 0$, Lemma 4.4 gives us some ω for which a^{-1} is almost proximal. Let $\alpha = \frac{\lambda_k^\omega(a^{-1})}{\Lambda_k^\omega(a^{-1})}$. It also give $\eta = \eta(d, \varepsilon) > 0$. Recall that ε_0, T_0 and T_1 are defined in (13) to (16). We assume here that $T_1 \geq \tau_1 := \max\{2/\eta, 3/\eta\varepsilon, 4/\varepsilon_0\}$ and $\varepsilon \leq \varepsilon_0/12d^2$ and let $\varepsilon_1 = \frac{\varepsilon_0}{4}$.

Claim 0: There is $n_0 = n_0(k_1, T_0, T_1, \varepsilon, d) \in \mathbb{N}$ such that for all $n \geq n_0$ there exists $j_0(n), j_1(n) \in [1, d-1]$ such that

$$\min\left\{d(a^{-nj_0(n)}t^{-1}V_a, bH_a), d(a^{-nj_1(n)}tV_a, b^{-1}H_a)\right\} \geq \frac{||F||_k^{-r}}{C_{k,2} \cdot (C_{k,1} \cdot \alpha^{-\varepsilon})^{dn} \cdot C_k^{p(d)+1}}$$

where $r = 2T_0 + k_1(2p(d) + d(d-1))$ and $p(d) = 10^d$.

Claim 1: Let $\varepsilon_1 = \frac{\varepsilon_0}{4}$. There exists $n_1 = n_1(k_1, k_2, k_3, T_0, T_1, \varepsilon, \varepsilon_0, d) \in \mathbb{N}$ such that for every $n \geq n_1$ the ball $B_n = B(V_a, \Lambda_k(a)^{-\varepsilon_1 n})$ (resp. $B'_n = B(t^{-1}V_a, \Lambda_k(a)^{-\varepsilon_1 n})$ is mapped into $B_n^- = B(V_a, \Lambda_k(a)^{-2\varepsilon_1 n})$ (resp. $B_n'^- = B(t^{-1}V_a, \Lambda_k(a)^{-2\varepsilon_1 n})$) by $x_n = a^{nj_0(n)}ba^{-nj_1(n)}t$ (resp. x_n^{-1}).

Claim 2: Under the assumptions of Claim 1, there is $n_4 \in \mathbb{N}$ depending only on $k_1, k_2, k_3, T_0, T_1, \varepsilon, \varepsilon_0$ and d such that for any $n \geq n_4$ we also have $Lip(x_{n|B_n}) \leq \Lambda_k(a)^{-\varepsilon_1 n}$ (resp. $Lip(x^{-1}_{n|B_n^-}) \leq \Lambda_k(a)^{-\varepsilon_1 n}$).

The proofs of these claims are straightforward once we have at our disposal the Lemmas proved in Section 4 and in particular Lemma 4.6. Nevertheless we provide full details in the next paragraph below.

With these claims in hands we can quickly prove Lemma 5.1. Indeed let $n = T_3/\varepsilon_1$. If $T_3 \geq \varepsilon_1 \cdot \max\{n_0, n_1, n_4\}$ we get by Claim 2 and 3 that x_n sends B_n into itself and x_n^{-1} sends B_n^- into itself, while the Lipschitz constants are $\leq \Lambda_k(a)^{-\varepsilon_1 n}$. We are thus in a position to apply Tits Converse Lemma, Lemma 4.7, which says that x_n and x_n^{-1} are proximal and satisfy $\frac{\lambda_k(x_n)}{\Lambda_k(x_n)}, \frac{\lambda_k(x_n^{-1})}{\Lambda_k(x_n^{-1})} \leq \Lambda_k(a)^{-T_3}$. Finally by Claim 2, x_n maps B_n into the smaller ball B_n^-, which must then contain V_{x_n} while B_n cannot intersect H_{x_n}. It follows that $d(V_{x_n}, H_{x_n}) \geq d(B_n, (B_n^-)^c)$. But we see that in both the archimedean and the ultrametric case:

$$d(B_n, (B_n^-)^c) \geq \frac{1}{C_k}\Lambda_k(a)^{-T_3} \geq \Lambda_k(a)^{-2T_3}$$

as soon as $\Lambda_k(a)^{-T_3} < 1/C_k$, which holds if $T_3 \geq 1$ for instance. A similar argument takes place for x_n^{-1}. This ends the proof of Lemma 5.1.

5.2.1. *Proof of Claim 0.* Let $W = V^\omega_{a^{-1}}$ and hence $W^c = H^\omega_{a^{-1}}$ and π the projection on W with kernel W^c. Recall that (14) gives

$$\alpha \leq \Lambda_k(a)^{-\eta} \leq ||F||^{-\eta T_1} \tag{20}$$

When k is archimedean this together with (13) forces $L_k^\omega(a^{-1}) \leq 2$ if $\eta T_1 \geq 2$. Indeed, we have $\alpha \leq ||F||^{-\eta T_1} \leq \frac{1}{2}$ i.e. $\lambda_k^\omega(a^{-1}) \leq \frac{1}{2}\Lambda_k^\omega(a^{-1})$.

We do the calculation for $u = tV_a$, keeping in mind that an entirely analogous calculation can be done for $u^- = t^{-1}V_a$ at the same time at each step. Let $n \in \mathbb{N}$ be arbitrary.

Since as $d(u, W^c) \geq d(u_0, W^c + W \cap b^{-1}H_a) \geq ||F||^{-T_0}$ we can combine Lemmas 4.4, 4.6 (a) and (16) to get

$$d(a^{-n}u, \pi(a^{-n}u)) \leq (C_k \cdot ||a^{-1}||_k)^{p(d)} \cdot \left(C_{k,1}^{l_{\omega_1}} \cdot \left(\frac{\Lambda_k(a^{-1})}{\Lambda_k^{\omega}(a^{-1})} \right)^{l_\omega - 1} \cdot \frac{\lambda_k^{\omega}(a^{-1})}{\Lambda_k^{\omega}(a^{-1})} \right)^n \cdot d(u, W^c)^{-1} \tag{21}$$

$$\leq C_k^{p(d)} \cdot ||F||_k^{T_0 + dk_1 p(d)} \cdot \left(C_{k,1}^d \cdot \alpha^{1-d\varepsilon} \right)^n$$

On the other hand according to Lemma 4.3,

$$d(\pi u, W \cap b^{-1}H_a) \geq d(u, W^c + W \cap b^{-1}H_a) \cdot d(W, W^c) \tag{22}$$

But by Lemma 4.11

$$d(W, W^c)^{-1} \leq (C_k^2 \cdot ||a^{-1}||^{l_\omega})^{(\binom{d}{l_\omega}-1)} \leq (C_k \cdot ||a^{-1}||_k)^{p(d)} \tag{23}$$

because when k is archimedean $L_k^{\omega}(a^{-1}) \leq 2$ as explained above. Hence (22) (23) and (16) give

$$d(\pi u, W \cap b^{-1}H_a)^{-1} \leq C_k^{p(d)} \cdot ||F||_k^{T_0 + dk_1 p(d)} \tag{24}$$

We may now apply Lemma 5.4 to $a_0 = a^{-n}$ restricted to W. We find $j_1 \in [1, d-1]$ such that

$$d(a^{-nj_1}\pi u, W \cap b^{-1}H_a)^{-1} \leq C_{k,2} \cdot \left(\frac{||a^{-1}||_k}{\Lambda_k(a^{-1})} \right)^{nj_1} \cdot \left(\frac{\Lambda_k(a^{-1})}{\Lambda_k^{\omega}(a^{-1})} \right)^{\ell_\omega n} \cdot d(\pi u, W \cap b^{-1}H_a)^{-1} \tag{25}$$

But Lemma 4.9 applied to a^{-1} gives an $h \in SL_d(k)$ such that $||ha^{-1}h^{-1}|| \leq C_{k,1} \cdot \Lambda_k(a^{-1})$ and $\max\{||h||, ||h^{-1}||\} \leq ||a^{-1}||^{\frac{d-1}{2}}$. Hence $||a^{-n}||_k \leq ||h|| \cdot ||h^{-1}|| \cdot (C_{k,1} \cdot \Lambda_k(a^{-1}))^n$ and $\frac{||a^{-n}||_k}{\Lambda_k(a^{-n})} \leq ||a^{-1}||^{d-1} \cdot C_{k,1}^n$. Thus combining (24) and (25) and bearing in mind that

$$\frac{\Lambda_k(a^{-1})}{\Lambda_k^{\omega}(a^{-1})} \leq \alpha^{-\varepsilon} \tag{26}$$

(this is 8), we get

$$d(a^{-nj_1}\pi u, W \cap b^{-1}H_a)^{-1} \leq C_{k,2} \cdot (C_{k,1}\alpha^{-\varepsilon})^{nd} \cdot C_k^{p(d)} \cdot ||F||_k^{T_0 + dk_1 p(d) + k_1 d^2(d-1)} \tag{27}$$

Compare (21) and (28). When k is ultrametric

$$\begin{aligned} d(a^{-n}u, \pi(a^{-n}u)) &\leq ||F||_k^{T_0 + dk_1 p(d)} \alpha^{n(1-d\varepsilon)} \\ &< ||F||_k^{-T_0 - dk_1 p(d) - k_1 d^2(d-1)} \alpha^{\varepsilon nd} \\ &< d(a^{-nj_1}\pi u, W \cap b^{-1}H_a) \end{aligned}$$

as soon as $\alpha^{n(1-2d\varepsilon)} < ||F||_k^{-r}$ where $r = r(T_0, d) := 2T_0 + dk_1(2p(d) + d(d-1))$. As $\alpha^{-1} \geq \Lambda_k(a)^\eta$ by Lemma 4.4, this happens as soon as

$$n > \frac{r}{T_1\eta(1-\varepsilon d)}$$

Similarly, if k is archimedean, $d(a^{-n}u, \pi(a^{-n}u)) \leq \frac{1}{2}d(a^{-nj_1}\pi u, W \cap b^{-1}H_a)$ as soon as $n > n_0(T_0, T_1, \varepsilon, d)$ for some computable constant n_0. Finally whether k is archimedean or ultrametric we get:

(28)

$$d(a^{-nj_1}u, W \cap b^{-1}H_a) \geq C_{k,2}^{-1} \cdot (C_{k,1}^{-1} \cdot \alpha^\varepsilon)^{dn} \cdot C_k^{-p(d)-1} \cdot ||F||_k^{-T_0-k_1p(d)-k_1d^2(d-1)}.$$

Finally applying Lemma 4.2 and (15) we get

$$\begin{aligned} d(a^{-nj_1}u, b^{-1}H_a) &\geq d(a^{-nj_1}u, W \cap b^{-1}H_a) \cdot d({}^tb^{-1} \cdot H_a^\perp, W^\perp) \\ &\geq C_{k,2}^{-1} \cdot (C_{k,1}^{-1} \cdot \alpha^\varepsilon)^{dn} \cdot C_k^{-p(d)-1} \cdot ||F||_k^{-r} \end{aligned}$$

This ends the proof of Claim 0.

5.2.2. *Proof of Claim 1.* First recall as in Claim 0 that $L_k^\omega(a^{-1}) \leq 2$ when k is archimedean (since $\eta T_1 \geq 2$, which we assume). We give the proof for x_n and B_n keeping in mind that the same arguments are being performed at the same time and at each step for x_n^{-1} and B_n^-.

We first justify the following:

Claim 1.1: There is $m_0 = m_0(d, \varepsilon, k_3, T_0, T_1) \in \mathbb{N}$ such that for $n \geq m_0$ and $u \in B(V_a, \alpha^{3d\varepsilon n})$

$$d(tu, W^c) \geq \frac{1}{C_k}d(tV_a, W^c) \geq \frac{1}{C_k||F||_k^{T_0}} \tag{29}$$

Indeed, the second inequality is just (16), while to get the first, it is enough that $d(tu, tV_a) < \frac{1}{C_k||F||_k^{T_0}} \leq \frac{1}{C_k}d(tV_a, W^c)$ (recall that C_k is 2 is k is archimedean and 1 if k is ultrametric). But

$$d(tu, tV_a) \leq Lip(t) \cdot d(u, V_a) \leq ||F||^{2dk_3}\alpha^{3d\varepsilon n}$$

Thus the existence of m_0 follows from (20) (13) and (14). Hence (29).

Claim 1.2.: For some $n_2 = n_2(\varepsilon, d, T_0, T_1, k_1, k_3) \in \mathbb{N}$ and all $n \geq n_2$ we have

$$\begin{aligned} d(a^{-nj_1(n)}tu, b^{-1}H_a) &\geq \frac{1}{C_k}d(a^{-nj_1(n)}tV_a, b^{-1}H_a) \\ &\geq C_{k,2}^{-1} \cdot (C_{k,1}^{-1} \cdot \alpha^\varepsilon)^{dn} \cdot C_k^{-p(d)-2} \cdot ||F||_k^{-r} \end{aligned} \tag{30}$$

for all $u \in B(V_a, \alpha^{3d\varepsilon n})$.

Proof of Claim 1.2.: Indeed, to show this it is enough that

$$d(a^{-nj_1(n)}tu, a^{-nj_1(n)}tV_a) < \frac{1}{C_k}d(a^{-nj_1(n)}tu, b^{-1}H_a), \tag{31}$$

which by Claim 0 reduces to show

$$d(a^{-nj_1(n)}tu, a^{-nj_1(n)}tV_a) < C_{k,2}^{-1} \cdot (C_{k,1}^{-1} \cdot \alpha^{\varepsilon})^{dn} \cdot C_k^{-p(d)-2} \cdot ||F||_k^{-r} \tag{32}$$

But bearing in mind (26) Lemma 4.6 (10), we have for $n \geq m_0$

$$\begin{aligned}
\frac{d(a^{-nj_1(n)}tu, a^{-nj_1(n)}tV_a)}{d(u, V_a)} &\leq Lip(t) \cdot \frac{d(a^{-nj_1(n)}tu, a^{-nj_1(n)}tV_a)}{d(tu, tV_a)} \\
&\leq Lip(t) \cdot (C_k^2 \cdot ||a^{-1}||)^p \cdot \frac{\left(C_{k,1}^{2d+2}\alpha^{-\varepsilon(2d-1)}\right)^n}{d(tu, W^c) \cdot d(tV_a, W^c)} \\
&\leq C_k^{2p+1} \cdot ||F||_k^{k_1dp+k_32d+2T_0} \cdot \left(C_{k,1}^{2d+2}\alpha^{-\varepsilon(2d-1)}\right)^n
\end{aligned}$$

Hence we get (32) as soon as

$$C_k^{3p+3} \cdot C_{k,2} \cdot ||F||_k^{k_1dp+k_32d+2T_0+r} \cdot \left(C_{k,1}^{3d+2}\alpha^{-\varepsilon(3d-1)}\right)^n \cdot d(u, V_a) < 1$$

Since $u \in B(V_a, \alpha^{3d\varepsilon n})$ this holds as soon as

$$C_k^{3p+3} \cdot C_{k,2} \cdot ||F||_k^{k_1dp+k_32d+2T_0+r} \cdot \left(C_{k,1}^{3d+2}\alpha^{\varepsilon}\right)^n < 1 \tag{33}$$

Since $\alpha^{\varepsilon} \leq ||F||_k^{-\eta\varepsilon T_1}$ by (20) and $C_{k,1}^{3d+2} \leq ||F||_k^2$ by (13) while we assumed $\eta\varepsilon T_1 \geq 3$, we get the existence of $n_2 = n_2(\varepsilon, d, T_0, T_1, k_1, k_3) \in \mathbb{N}$ for which (33) holds for $n \geq n_1$. Hence (30) holds and Claim 1.2. is proved.

With (30) in hand we can apply Lemma 4.6 (9) to positive powers of a this time and get:

Claim 1.3.: Suppose $\varepsilon_0 \geq 12\varepsilon d^2$ and fix $\varepsilon_1 = \varepsilon_0/4$. There is $n_3 \in \mathbb{N}$ depending on $\varepsilon, \varepsilon_0, d, T_0, T_1, k_1, k_2, k_3$ such that for $n \geq n_3$ and $u \in B(V_a, \Lambda_k(a)^{-\varepsilon_1 n})$ we have for $x_n = a^{nj_0(n)}ba^{-nj_1(n)}t$

$$d(x_n u, V_a) < \Lambda_k(a)^{-\varepsilon_1 n}$$

Proof of Claim 1.3.: First note that $\Lambda_k(a)^{-\varepsilon_1} \leq \alpha^{3d\varepsilon}$ because $\alpha^{-1} \leq \Lambda_k(a)\Lambda_k(a^{-1}) \leq \Lambda_k(a)^d$ and $\varepsilon_1 = \varepsilon_0/4 \geq 3\varepsilon d^2$. Lemma 4.6 (9) translates as

$$\begin{aligned}
d(a^{nj_0(n)}ba^{-nj_1(n)}tu, V_a) &\leq (C_k \cdot ||a||_k)^{p(d)} \cdot \left(C_{k,1}^4 \cdot \frac{\lambda_k(a)}{\Lambda_k(a)}\right)^n \cdot d(ba^{-nj_1(n)}tu, H_a)^{-1} \\
&\leq C_k^p \cdot ||F||_k^{k_1p(d)+2dk_2} \cdot \left(\frac{C_{k,1}^4}{\Lambda_k(a)^{\varepsilon_0}}\right)^n \cdot d(a^{-nj_1(n)}tu, b^{-1}H_a)^{-1} \\
&\leq C_k^{2p+2}C_{k,2} \cdot ||F||_k^{k_1p(d)+2dk_2+r} \cdot \left(\frac{C_{k,1}^{d+4} \cdot \alpha^{-\varepsilon d}}{\Lambda_k(a)^{\varepsilon_0}}\right)^n
\end{aligned}$$

where we have used successively (14) and Claim 1.2. Now

$$\frac{C_{k,1}^{d+4} \cdot \alpha^{-\varepsilon d}}{\Lambda_k(a)^{\varepsilon_0}} \cdot \Lambda_k(a)^{\varepsilon_1} \leq \frac{C_{k,1}^{d+4}}{\Lambda_k(a)^{\varepsilon_0-\varepsilon_1-\varepsilon d^2}} \leq \frac{C_{k,1}^{d+4}}{\Lambda_k(a)^{\varepsilon_0/2}}$$

because $\alpha^{-1} \leq \Lambda_k(a)\Lambda_k(a^{-1}) \leq \Lambda_k(a)^d$ and we have assumed $\varepsilon_0 \geq 4\varepsilon d^2$ and $\varepsilon_0 = 4\varepsilon_1$. Then the existence of n_3 follows from (13) and (14). Thus Claim 1.3. is proved.

Working out the same three claims for x_n^{-1} and B'_n in place of x_n and B_n we get Claim 1.

5.2.3. *Proof of Claim 2.* We apply Lemma 4.6 (10) to $a^{nj_0(n)}$ and points $ba^{-nj_1(n)}tu$ and $ba^{-nj_1(n)}tv$ for $u, v \in B_n$. Recall that $L_k(a) \leq 2$ when k is archimedean as $\frac{\lambda_k(a)}{\Lambda_k(a)} \leq ||F||^{-T_1\varepsilon_0} \leq \frac{1}{2}$ by (13) and (14) and since $T_1\varepsilon_0 \geq 1$. We get

$$\frac{d(x_n u, x_n v)}{d(u,v)} \leq (C_k^2 ||a||_k)^{p(d)} \cdot \left(\frac{C_{k,1}^4 \lambda_k(a)}{\Lambda_k(a)} \right)^n \cdot d(ba^{-nj_1(n)}tu, H_a)^{-1} \cdot d(ba^{-nj_1(n)}tv, H_a)^{-1}$$

Since $\Lambda_k(a)^{-\varepsilon_1} \leq \alpha^{3d\varepsilon}$ Claim 1.2. applies and we get

$$\begin{aligned} \frac{d(x_n u, x_n v)}{d(u,v)} &\leq Lip(b^{-1})^2 \cdot ||a||_k{}^{p(d)} \cdot \left(\frac{C_{k,1}^4 \lambda_k(a)}{\Lambda_k(a)} \right)^n \cdot C_{k,2}^2 \cdot (C_{k,1} \cdot \alpha^{-\varepsilon})^{2dn} \cdot C_k^{4p(d)+4} \cdot ||F||_k^{2r} \\ &\leq C_k^{4p+4} C_{k,2}^2 \cdot ||F||_k^{4dk_2+2r+k_1p(d)} \cdot \left(C_{k,1}^{4+2d} \frac{\alpha^{-2d\varepsilon}}{\Lambda_k(a)^{\varepsilon_0}} \right)^n \end{aligned}$$

But

$$\frac{\alpha^{-2d\varepsilon}}{\Lambda_k(a)^{\varepsilon_0}} \leq \frac{1}{\Lambda_k(a)^{2\varepsilon_1}}$$

Hence for some computable n_4, for all $n \geq n_4$ and $u, v \in B_n$

$$\frac{d(x_n u, x_n v)}{d(u,v)} \leq \Lambda_k(a)^{-\varepsilon_1 n}$$

A similar argument proves the claim about x_n^{-1} and B'_n. Thus Claim 2 is proved.

5.3. **Proof of Lemma 5.3.** Let $n, k \in \mathbb{N}$, $k = T_2 + 3dk_5$. Let $B_k(x) = B(V_x, \Lambda_k(a)^{-kT_3})$ and $B_k^-(x) = B(V_{x^{-1}}, \Lambda_k(a)^{-kT_3})$. Similarly, let $B_k(c) = B(cV_x, \Lambda_k(a)^{-kT_3})$ and $B_k^-(c) = B(cV_{x^{-1}}, \Lambda_k(a)^{-kT_3})$. Note that $d(u, c^{\pm 1}V_x) < \frac{1}{C_k} d(c^{\pm 1}V_x, H_x)$ implies $d(u, H_x) \geq \frac{1}{C_k} d(c^{\pm 1}V_x, H_x)$. Hence if $u \in B_k(c)$, then as $k \geq T_2$, $\Lambda_k(a)^{kT_3} > C_k ||F||_k^{T_2}$ and

$$d(u, cV_x) \leq \Lambda_k(a)^{-kT_3} < \frac{1}{C_k} d(cV_x, H_x)$$

Hence

$$d(u, H_x) \geq \frac{1}{C_k ||F||_k^{T_2}} \geq \frac{1}{||F||_k^{T_2+1}}$$

Similarly if $u \in B_k(c)$ then $d(u, H_{x^{-1}}) \geq ||F||_k^{-T_2-1}$ and if $u \in B_k^-(c)$, then $d(u, H_x \cup H_{x^{-1}}) \geq ||F||_k^{-T_2-1}$. Finally:

$$d(B_k^-(c) \cup B_k(c), H_x \cup H_{x^{-1}}) \geq ||F||_k^{-T_2-1} \tag{34}$$

Similarly we check that

$$d(B_k^-(x) \cup B_k(x), cH_x \cup cH_{x^{-1}}) \geq ||F||_k^{-T_2-2dk_5-1}$$

We know that for each $n \geq 1$, x^n maps B_k into itself and x^{-n} maps B_k^- into itself. Similarly we check that cx^nc^{-1} maps $B_k(c)$ into itself and $cx^{-n}c^{-1}$ maps $B_k^-(c)$ into itself.

We now check that x^n maps $B_k^-(c) \cup B_k(c)$ into B_k and x^{-n} maps $B_k^-(c) \cup B_k(c)$ into B_k^-. From Lemma 5.1, we have $\frac{\lambda_k(x)}{\Lambda_k(x)} \leq \Lambda_k(a)^{-T_3}$. By Lemma 4.6 (9) applied to x, for $u \in \mathbb{P}(k^d)$,

$$\begin{aligned} d(x^n u, V_x) \cdot d(u, H_x) &\leq (C_k \cdot ||x||_k)^{p(d)} \cdot \left(C_{k,1} \cdot \frac{\lambda_k(x)}{\Lambda_k(x)} \right)^n \\ &\leq ||F||_k^{p(d)(1+ld+k_2+k_3)} \cdot \Lambda_k(a)^{-\frac{T_3 n}{2}} \end{aligned}$$

Hence if $u \in B_k(c) \cup B_k^-(c)$, then $d(u, H_x) \geq ||F||_k^{-T_2-1}$ by (34) and

$$\begin{aligned} d(x^n u, V_x) &\leq ||F||_k^{p(d)(1+ld+k_2+k_3)+T_2+1} \cdot \Lambda_k(a)^{-\frac{T_3 n}{2}} \\ &\leq \Lambda_k(a)^{-kT_3} \end{aligned}$$

as soon as $n \geq n_5 = n_5(l, d, (k_i)_i, (T_i)_i, k)$. Similarly

$$\begin{aligned} d(x^{-n} u, V_{x^{-1}}) \cdot d(u, H_{x^{-1}}) &\leq (C_k \cdot ||x^{-1}||_k)^{p(d)} \cdot \left(C_{k,1} \cdot \frac{\lambda_k(x^{-1})}{\Lambda_k(x^{-1})} \right)^n \\ &\leq ||F||_k^{p(d)d(1+ld+k_2+k_3)} \cdot \Lambda_k(a)^{-\frac{T_3 n}{2}} \end{aligned}$$

and hence $d(x^{-n}u, V_{x^{-1}}) \leq \Lambda_k(a)^{-kT_3}$ if $n \geq n_6 = n_6(l, d, (k_i)_i, (T_i)_i)$.

We check that similarly, cx^nc^{-1} maps $B_k^- \cup B_k$into $B_k(c)$ and $cx^{-n}c^{-1}$ maps $B_k^- \cup B_k$ into $B_k^-(c)$ as soon as n is larger that some fixed number depending only on the data $(l, d, (k_i)_i, (T_i)_i)$.

Finally we check that all balls B_k^-, B_k $B_k^-(c)$, $B_k(c)$ are disjoint, since $d(V_x, V_{x^{-1}}) \geq d(V_x, H_x) \geq \Lambda_k(a)^{-2T_3}$ and $d(V_x, cV_{x^{-1}}) \geq d(H_x, cV_{x^{-1}}) \geq ||F||^{-T_2} \geq \Lambda_k(a)^{-k}$ and similarly $d(cV_x, V_{x^{-1}}) \geq \Lambda_k(a)^{-k}$ and $d(cV_x, cV_{x^{-1}}) \geq ||F||^{-2dk_5} \Lambda_k(a)^{-2T_3} \geq \Lambda_k(a)^{-(k-1)T_3}$.

It follows that x^n and cx^nc^{-1} play ping-pong on $\mathbb{P}(k^d)$, hence generate a free subgroup. This ends the proof of Lemma 5.3.

6. Height bounds and proof of Theorem 1.1

6.1. A Product formula for subspaces. In this paragraph we define the *adelic distance* $\delta(V;W)$ between two projective subspaces and we give a *product formula* (36) relating it to the Arakelov heights of V, W and $V+W$.

In Paragraph 4.1 we recalled the Fubini-Study metric on $\mathbb{P}(k^d)$, where k is a local field. In particular, we had formula (6), which gives the distance between two projective linear subspaces. If K is a global field with prime field K_0 and V and W are disjoint projective linear subspace of $\mathbb{P}(K^d)$, we can put together the local distances (i.e. at each place of K) in a way similar to the way the height of an algebraic number is defined. Namely we set:

$$\delta(V;W) = \frac{1}{[K:K_0]} \sum_{v \in V_K} n_v \cdot \log \frac{1}{d_v(V,W)} \tag{35}$$

where $d_v(\cdot,\cdot)$ is the Fubini-Study metric on $\mathbb{P}(K_v^d)$. Each term in this sum is non negative. In fact, we see from (6) that $\delta(V;W)$ is linked to the Arakelov heights (see Paragraph 3.1) in the following simple way:

$$\delta(V;W) = h_{Ar}(V) + h_{Ar}(W) - h_{Ar}(V+W) \leq h_{Ar}(V) + h_{Ar}(W) \tag{36}$$

This can be seen as a *product formula for subspaces*, since when V and W are points in $\mathbb{P}^1(\overline{\mathbb{Q}})$ it reduces to the classical product formula on $\overline{\mathbb{Q}}$.

Note moreover that we can similarly define $\delta(V^\perp, W^\perp)$ just as $\delta(V,W)$ in the projective space of the dual vector space $(K^d)^*$. Since $h_{Ar}(V) = h_{Ar}(V^\perp)$ (see [6]), we also have

$$\delta(V^\perp, W^\perp) \leq h_{Ar}(V) + h_{Ar}(W)$$

We will often denote by $\delta_v(V;W)$ the term of the sum in (35) corresponding to the place v, so that

$$\delta(V;W) = \frac{1}{[K:K_0]} \sum_{v \in V_K} n_v \cdot \delta_v(V;W)$$

6.2. The Eskin-Mozes-Oh Escape Lemma. In this paragraph we recall a crucial Lemma due Eskin-Mozes-Oh, which allows to "escape from algebraic subvarieties in bounded time".

Recall Bezout's theorem about the intersection of finitely many algebraic subvarieties (see for instance [35]), namely:

Theorem 6.1 (Generalized Bezout theorem)**.** *Let K be a field, and let $Y_1, \ldots, Y_p$ be pure dimensional algebraic subvarieties of K^n. Denote by $W_1, \ldots, W_q$ the irreducible components of $Y_1 \cap \ldots \cap Y_p$. Then*

$$\sum_{i=1}^{q} deg(W_i) \leq \prod_{j=1}^{p} deg(Y_j).$$

Let K be a field and let X be an algebraic variety over K. We set $s(X)$ to be the sum of the degree and the dimension of each of its geometrically irreducible components. The following result was shown in [18], Lemma 3.2:

Lemma 6.2. [18] *Given an integer $m \geq 1$ there is $N = N(m)$ such that for any field K, any integer $d \geq 1$, any K–algebraic subvariety X in $GL_d(K)$ with $s(X) \leq m$ and any (not necessarily symmetric) subset $F \subset GL_d(K)$ which contains the identity and generates a subgroup which is not contained in $X(K)$, we have $F^N \nsubseteq X(K)$.*

6.3. **Irreducible representations of Chevalley groups.** In this paragraph we define the linear irreducible representations (ρ_α, E_α) which are the possible candidates for the projective representation where we will play ping-pong. We also set a particular basis in each E_α, which we use to define the height $h(\rho_\alpha(g))$ and then show Lemma 6.3.

Let $\mathbb{G}$ be a Chevalley group of adjoint type and $\mathfrak{g}$ its Lie algebra with $\mathbb{Z}$-structure $\mathfrak{g}_\mathbb{Z}$. Let T be a maximal torus and $\mathfrak{t}$ the corresponding Cartan subalgebra in $\mathfrak{g}$. Let Λ_R be the lattice of roots in the dual of $\mathfrak{g}$ which we identify with the space $X(T)$ of characters of T. Let Λ_W be the lattice of weights. We fix a set of positive roots Φ^+ and inside a base of simple roots Π. Since $\mathbb{G}$ is of adjoint type, to every dominant weight $\lambda \in \Lambda_R$, there correspond a finite dimensional absolutely irreducible representation E of $\mathbb{G}$. Let $\{\pi_\alpha\}_{\alpha\in\Pi} \subset \Lambda_W$ be the fundamental weights. For each $\alpha \in \Pi$, there is a smallest integer $k_\alpha \in \mathbb{N}$ such that $k_\alpha \pi_\alpha \in \Lambda_R$. Let $\chi_\alpha = k_\alpha \pi_\alpha$ be the corresponding dominant weight and (ρ_α, E_α) the corresponding absolutely irreducible representation of $\mathbb{G}$.

For background on Chevalley groups and their representations, see Steinberg's notes [38]. Let also (ρ_0, E_0) be the adjoint representation. According to [38] Section 2 Theorem 2, given an absolutely irreducible representation (ρ, E) of $\mathbb{G}$, one may find in each E a lattice Λ invariant under the action of $\rho(\mathbb{G}(\mathbb{Z}))$ and a basis of Λ which is made of weight vectors. Let us choose this basis. It defines a standard norm $||\cdot||_k$ on $\Lambda \otimes_\mathbb{Z} k$ and it also defines a height h in $SL(E_\alpha)$ as in Section 3. Recall that Ω is either $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}(t)}$ and $\varepsilon_\Omega = 1$ in the first case 0 otherwise. We have:

Lemma 6.3. There exists a constant $C_0 > 0$ such that for every finite subset $F \in \mathbb{G}(\Omega)$ and every $\alpha \in \Pi$, $h(\rho_\alpha(F)) \leq C_0 \cdot (h(\rho_0(F)) + \varepsilon_\Omega)$.

Let χ_ρ be the heighest weight of ρ, which belongs to the root lattice. Let L be the maximal coefficient appearing in the decomposition of χ_ρ as a sum of simple roots (let L_0 the corresponding integer for $\rho_0 = Ad$). Let M be the smallest positive integer such that $M\chi_\rho \geq \alpha$ for every $\alpha \in \Pi$ (for the order defined by Π). Then Lemma 6.3 follows from:

Lemma 6.4. For every local field k, there is a constant $c_0 = c_0(\rho, k) > 0$ such that for every $g \in \mathbb{G}(k)$, we have

$$\frac{1}{c_0}||Ad(g)||_k^{\frac{1}{L_0 M}} \leq ||\rho(g)||_k \leq c_0||Ad(g)||_k^L \tag{37}$$

and $c_0 = 1$ unless k is Archimedean.

Proof. Let $\mathbb{K}_k = \mathbb{G}(\mathcal{O}_k)$ when k is ultrametric. Then $\rho(\mathbb{K}_k)$ and $Ad(\mathbb{K}_k)$ preserve the norm. By Cartan's $\mathbb{K}_k T \mathbb{K}_k$ decomposition, it suffices to prove the inequalities for g in the maximal torus T. But then $||\rho(g)||_k = |\chi_\rho(g)|_k$ and $\max\{|\alpha(g)|_v, \alpha \in \Pi\}^{\frac{1}{M}} \leq |\chi_\rho(g)|_k \leq \max\{|\alpha(g)|_v, \alpha \in \Pi\}^L$. And $\max\{|\alpha(g)|_v, \alpha \in \Pi\} \leq ||Ad(g)||_k \leq \max\{|\alpha(g)|_v, \alpha \in \Pi\}^{L_0}$. Hence (37).

When k is Archimedean, $\mathbb{K}_k$ stabilizes another norm $||\cdot||_{k,new}$ on $\Lambda \otimes_{\mathbb{Z}} k$ (resp. $\Lambda_R \otimes_{\mathbb{Z}} k$). For this new norm the same argument gives (37). Since the two norms are equivalent, this gives us the constant c_0. □

6.4. **Combined adelic distance.** In this paragraph, we define the combined adelic distance $\delta(F) = \delta^1(F) + \delta^2(F)$ of all adelic distances $\delta(V;W)$ where V and W range over the relevant projective subspaces involved in the ping-pong conditions from Section 5.

Let K be a global field. Let $(q_i)_{1 \leq i \leq 5}$ be five positive integers. Given $a \in \mathbb{G}(K)$ and $\alpha \in \Pi \cup \{0\}$, let $\mathcal{B}_{a,\alpha}$ be the set of elements $b \in \mathbb{G}(K)$ such that ${}^t\rho_\alpha(b)(V^c)^\perp \nsubseteq W^\perp$ and ${}^t\rho_\alpha(b^{-1})(V^c)^\perp \nsubseteq W^\perp$ for every $(V, W) \in \mathcal{A}_\alpha(a)$, where $\mathcal{A}_\alpha(a)$ is the set of couples (V, W) of $\rho_\alpha(a)$-admissible (see def. 3.5) non-trivial linear subspaces of E_α such that $\dim(V) = 1$. Given $a, b \in \mathbb{G}(K)$ with $b \in \mathcal{B}_{a,\alpha}$, let $\mathcal{T}_{a,b,\alpha}$ be the set of elements $t \in \mathbb{G}(K)$ such that $\rho_\alpha(t)V \nsubseteq W^c + W \cap b^{-1}V^c$ and $\rho_\alpha(t^{-1})V \nsubseteq W^c + W \cap bV^c$ for every $V, W \in \mathcal{A}_\alpha(a)$ (note that since $b \in \mathcal{B}_{a,\alpha}$, $W^c + W \cap b^{-1}V^c$ and $W^c + W \cap bV^c$ are hyperplanes).

Recall from Paragraph 6.2 that given an algebraic variety $\mathcal{Z}$ over the algebraically closed field Ω, we denote by $s(\mathcal{Z})$ the sum of the dimension and degree of its irreducible components. Given two non-trivial subspaces V and W in E_α the set of all $g \in GL(E_\alpha)$ such that $gW \subset V$ or $g^{-1}W \subset V$ is a Zariski closed subset $\mathcal{Z}_{V,W}$ of $GL(E_\alpha)$. Moreover $s(\mathcal{Z}_{V,W})$ is bounded independently on V and W since the one can pass from one $\mathcal{Z}_{V,W}$ to the other by multiplying on the left and right by some automorphism in $GL(E_\alpha)$. From these remarks and Lemma 6.2 we obtain:

Lemma 6.5. There is a positive integer q_0 such that for any field K and any finite subset F of $\mathbb{G}(K)$ containing 1 and generating a Zariski-dense subgroup, any $\alpha \in \Pi \cup \{0\}$ and any $a \in \mathbb{G}(K)$ and $b \in \mathcal{B}_{a,\alpha}$, the set F^{q_0} intersects $\mathcal{B}_{a,\alpha}$ non trivially and the set F^{q_0} intersects $\mathcal{T}_{a,b,\alpha}$ non trivially.

We now fix the values of q_2, q_3 and q_5 to be equal to this q_0. The values of q_1 and q_4 will be specified later. Let $\mathcal{Q}_\alpha$ be the set of 3-tuples (a, b, t)

such that $a \in F^{q_1}$, $b \in F^{q_2} \cap \mathcal{B}_{a,\alpha}$, and $t \in F^{q_3} \cap \mathcal{T}_{a,b,\alpha}$. Let $\mathcal{R}_\alpha$ be the set of couples (x,c) such that $x \in F^{q_4}$, $c \in F^{q_5} \cap \mathcal{B}_{x,\alpha}$. Lemma 6.5 ensures that if F generates a Zariski-dense subgroup, then for any $a \in F^{q_1}$ there are b,t such that $(a,b,t) \in \mathcal{Q}_\alpha$ and also for any $x \in F^{q_4}$ there is c such that $(x,c) \in \mathcal{R}_\alpha$. Now define for any finite symmetric subset F in $\mathbb{G}(K)$, and $i = 1,2$

$$\delta^i(F) = \sum_{\alpha \in \Pi \cup \{0\}} \delta^i_\alpha(F)$$

and

$$\begin{aligned} \delta^1_\alpha(F) &= \sum_{(a,b,t) \in \mathcal{Q}_\alpha} \delta_{\alpha,(a,b,t)}(F) \\ \delta^2_\alpha(F) &= \sum_{(x,c) \in \mathcal{R}_\alpha} \delta_{\alpha,(x,c)}(F) \end{aligned}$$

and

$$\begin{aligned} \delta_{\alpha,(a,b,t)}(F) &= \sum_{(V,W) \in \mathcal{A}_\alpha(a)} \delta({}^t\rho_\alpha(b)(V^c)^\perp; W^\perp) + \delta({}^t\rho_\alpha(b^{-1})(V^c)^\perp; W^\perp) + \\ &\quad \delta(\rho_\alpha(t)V; W^c + W \cap \rho_\alpha(b^{-1})V^c) + \delta(\rho_\alpha(t^{-1})V; W^c + W \cap \rho_\alpha(b)V^c) \end{aligned}$$

$$\delta_{\alpha,(x,c)}(F) = \sum_{(V,W) \in \mathcal{A}_\alpha(x)} \delta(\rho_\alpha(c)V; W) + \delta(\rho_\alpha(c^{-1})V; W)$$

6.5. **Height bounds for subspace separation.** In this paragraph, applying the results of Paragraphs 3.1 and 6.1, we obtain (38) and (39) which give bounds for the combined adelic distances $\delta^i(F)$ in terms of the height $h(F)$ and the number of elements in F only.

Namely, if $r = rank(\mathbb{G})$ and $D = (r+1)\max_{\alpha \in \Pi \cup \{0\}} 24 \cdot 4^{d_\alpha} d_\alpha^2$ with $d_\alpha = \dim E_\alpha$, we have for every $(a,b,t) \in \mathcal{Q}_\alpha(F)$,

$$\begin{aligned} \delta_{\alpha,(a,b,t)}(F) &\leq \sum_{V,W \in \mathcal{A}_\alpha(a)} d_\alpha^2 \cdot (h(\rho_\alpha(b)) + h(\rho_\alpha(t))) + 4(h_{Ar}(W) + h_{Ar}(V)) + \\ &\quad +2(h_{Ar}(W^c) + h_{Ar}(V^c)) \\ &\leq h(\rho_\alpha(F^{q_0})) + 12 \cdot 4^{d_\alpha} \cdot \max\{h_{Ar}(W), W \text{ } a\text{-admissible}\} \\ &\leq 24 \cdot 4^{d_\alpha} d_\alpha^2 \cdot (q_0 + q_1) h(\rho_\alpha(F)) + 12 \cdot 4^{d_\alpha} d_\alpha^2 \cdot \varepsilon_\Omega \log 2 \end{aligned}$$

Hence using Lemma 6.3,

$$\begin{aligned} \delta^1(F) &\leq D|F|^{q_1+2q_0}(q_0+q_1) \cdot h(\rho_\alpha(F)) + D \cdot \varepsilon_\Omega \log 2 \\ &\leq D|F|^{q_1+2q_0}(q_0+q_1) C_0 \cdot (h(Ad(F)) + \varepsilon_\Omega) \end{aligned}$$

Note that if $char(\Omega) = 0$, then by the Height Gap Theorem 3.3, we have $h(Ad(F)) \geq \widehat{h}(Ad(F)) \geq g > 0$ where g is the gap. So at any case for all characteristic,

$$\delta^1(F) \leq D_1 \left(\frac{|F|}{5}\right)^{q_1+2q_0} h(Ad(F)), \tag{38}$$

where $D_1 = D5^{q_1+2q_0}(q_0+q_1)C_0(1+g^{-1})$. Similarly one obtains

$$\delta^2(F) \leq D_2 \left(\frac{|F|}{5}\right)^{q_4+q_0} h(Ad(F)), \tag{39}$$

where $D_2 = D5^{q_4+q_0}(q_0+q_4)C_0(1+g^{-1})$.

6.6. **Proof of Theorem 1.1.** The proof is done in three steps. First we reduce to the situation when F generates a Zariski-dense subgroup in $\mathbb{G}(\Omega)$ where $\mathbb{G}$ is a simple Chevalley group of adjoint type to be chosen among a finite list of such. Second we show that we may assume that $F = \{1, X, X^{-1}, Y, Y^{-1}\}$, i.e. F is a symmetric set with 4 elements plus the identity. And finally, in the third and most difficult step, we check that there exists a place v of the field K of coefficients for which the sufficient conditions (i) to (vi) stated in Section 5 are fulfilled with some explicit choice of constants depending only on $\mathbb{G}$, and thus yield the desired ping-pong pair.

Remark 6.6. It is not clear whether or not the assumption F symmetric is a necessary condition in Theorem 1.1. Our proof however requires this assumption (see Remark 5.2). If one needs only a free semi-group instead of a free group, then it is not necessary.

6.6.1. *Preliminary reductions.*
In this paragraph, we prove the first two steps, Claims 1 and 2. Before that note that we may assume that the field K is either $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p(t)}$, i.e. that $K = \Omega$. This follows from Proposition 7.3 below when K has characteristic 0 and from [14] Lemma 3.1 and the remark following it in general.

So we assume $K = \Omega$. We have:

Claim 1: In Theorem 1.1, we may assume that F generates a Zariski-dense subgroup in $\mathbb{G}(\Omega)$ where $\mathbb{G}$ is a simple Chevalley group of adjoint type.

Proof. Since F generates a non virtually solvable subgroup $\langle F \rangle$, the connected component $\mathbb{G}^0$ of the Zariski-closure $\mathbb{G}$ of $\langle F \rangle$ is not solvable. Moding out by the solvable radical of $\mathbb{G}^0$, which is a normal subgroup of $\mathbb{G}$, we see that we can assume that $\mathbb{G}^0$ is a non trivial semisimple algebraic group. We let $\mathbb{G}$ act on $\mathbb{G}^0$ by conjugation we obtain a homomorphism of $\mathbb{G}$ in $Aut(\mathbb{G}^0_{ad})$ where $\mathbb{G}^0_{ad}$ is the adjoint group of $\mathbb{G}^0$ whose image contains $\mathbb{G}^0_{ad}$. However by [7] IV.14.9. $Aut(\mathbb{G}^0_{ad})/\mathbb{G}^0_{ad}$ is a subgroup of the automorphisms of the Dynkin diagram of $\mathbb{G}$. In particular it is a finite group whose order is bounded in terms of $\dim \mathbb{G}$ only, hence in terms of d only. Recall (see for instance [13] Lemma 4.6.),

Lemma 6.7. Let F be a finite subset of a group Γ containing 1. Assume that the elements of F (together with their inverses) generate Γ. Let Γ_0 be a subgroup of index k in Γ. Then F^{2k+1} contains a generating set of Γ_0.

Applying this lemma, we may therefore assume that $\mathbb{G} = \mathbb{G}_{ad}^0$ is a semisimple algebraic group of adjoint type. Further projecting to one of the simple factors, we may assume that $\mathbb{G}$ is a simple algebraic group of adjoint type over Ω. As Ω is algebraically closed, $\mathbb{G}(\Omega)$ is the group of Ω-points of a Chevalley group (see [38]). $\square$

Let $\mathcal{O}$ be the Zariski-open subset of $\mathbb{G} \times \mathbb{G}$ obtained in Theorem 3.4.

Claim 2: In Theorem 1.1, we may assume that $F = \{1, X, X^{-1}, Y, Y^{-1}\}$ for some $(X, Y) \in \mathcal{O}(\Omega)$.

Proof. This claim was already proven in Proposition 4.14 of [13] in the special case of characteristic 0 making key use of Jordan's theorem about finite subgroups of $GL_n(\mathbb{C})$. This argument fails in positive characteristic so we now give a different (and more involved) argument. Let $\mathbb{G}$ be a simple Chevalley group. Following an idea used in [14] Section 7, we have:

Lemma 6.8. Then there is a proper closed subvariety $\mathcal{W}$ of $\mathbb{G} \times \mathbb{G}$ such that, for any choice of Ω, every pair $(x, y) \notin \mathcal{W}(\Omega)$ with x of infinite order generates a Zariski-dense subgroup of $\mathbb{G}$.

Proof. Let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$ (see [7] I.3.5). Let $\mathcal{W}$ be the subset of pairs (x, y) in $\mathbb{G}(\Omega) \times \mathbb{G}(\Omega)$ such that the associative subalgebra of $End(\mathfrak{g})$ generated by $Ad(x)$ and $Ad(y)$ is proper. Note that $\mathcal{W}$ is a closed algebraic subset with equations over $\mathbb{Z}$. It is also proper because one can construct pairs (x, y) for which the group they generate acts irreducibly on $\mathfrak{g}$ (see for instance [8] VIII. 2. ex.8. and [2] §3). Suppose $(x, y) \notin \mathcal{W}(\Omega)$ and x has infinite order. Let $\mathbb{H}$ be the Zariski closed subgroup generated by x and y. Then $\dim \mathbb{H} \geq 1$ and the Lie algebra of $\mathbb{H}$ is non trivial and invariant under $Ad(x)$ and $Ad(y)$, hence equal to $\mathfrak{g}$. By [7] I.3.6 we conclude that $\mathbb{H} = \mathbb{G}$. $\square$

In order to apply this lemma, we show:

Lemma 6.9. There is a constant $N = N(d) \in \mathbb{N}$ such that, for any choice of Ω, if F is a finite symmetric subset of $\mathbb{G}(\Omega)$ containing 1 and generating a Zariski dense subgroup, one may find a subset F_0 of F^N such that for all integers $n \geq 1$ the subset F_0^n is made only of elements of infinite order and the subgroup generated by F_0 and F_0^{-1} is Zariski dense in $\mathbb{G}$.

Before going into the proof of Lemma 6.9 let us explain how we deduce Claim 2 from this.

Proof of Claim 2. By Lemma 6.9, we can replace F by F_0. Now according to Lemma 6.2 applied to $\mathbb{G} \times \mathbb{G}$ and $F_0 \times F_0$ there is a constant $M \in \mathbb{N}$ depending only on $\mathcal{W}$ and $\mathcal{O}$, hence on d only, such that F_0^M contains a pair (x, y) such that $(x, y) \in \mathcal{O}$ and $(x, y) \notin \mathcal{W}$. By Lemma 6.9, x has infinite order, hence by Lemma 6.8, x and y generate a Zariski dense subgroup of $\mathbb{G}$, and Claim 2 is proved.

Proof of Lemma 6.9. Let $d_0 = \dim \mathbb{G}$. We have:

Lemma 6.10. There is a constant $N_0 = N_0(d_0) \in \mathbb{N}$ and $k \le d_0$ elements $\alpha_1, ..., \alpha_k$ in F^{N_0} of infinite order and such that the connected components C_i of the Zariski closures of each cyclic subgroup generated by each α_i together generate $\mathbb{G}$ as an algebraic group.

Proof. First we check that there is some α of infinite order in a bounded power of F, say F^{N_1}. This follows from Theorem 3.3 in characteristic 0 (see Corollary 1.2). In positive characteristic it follows directly from the fact that $\langle F \rangle$ is finite as soon as $\widehat{h}(F) = 0$ (Lemma 3.2) and Lemma 2.1 (a) which says that $F^{d_0^2}$ already contains an element with eigenvalue of absolute value > 1.

Let $\langle \alpha \rangle$ be the cyclic group generated by α and C_1 the connected component of its Zariski closure. Then $\dim C_1 = 1$. Set $\alpha_1 = \alpha$. Suppose $j \ge 1$ and we have built $\alpha_1, ..., \alpha_j$ and let C_i be the connected component of the Zariski closure of $\langle \alpha_i \rangle$ and $\mathbb{H}_i$ the algebraic subgroup generated by all C_m for $1 \le m \le i$. We show by induction that $\alpha_i = w_{i-1} \alpha w_{i-1}^{-1}$ for some $w_{i-1} \in F^{i-1}$ and $\dim \mathbb{H}_i \ge i$. If $\mathbb{H}_j \ne \mathbb{G}$, as $\mathbb{G}$ is simple and $\langle F \rangle$ Zariski dense, there must exist some $\beta_j \in F$ such that $\beta_j \mathbb{H}_j \beta_j^{-1} \ne \mathbb{H}_j$, hence some $i \le j$ such that $\beta_j C_i \beta_j^{-1}$ is not contained in $\mathbb{H}_j$. Let $\alpha_{j+1} = \beta_j \alpha_i \beta_j^{-1}$, i.e. $w_j = \beta_j w_{i-1} \in F^j$. Then $C_{j+1} = \beta_j C_i \beta_j^{-1}$ and $\dim \mathbb{H}_{j+1} \ge \dim \mathbb{H}_j + 1$. $\square$

We look at $\mathbb{G}$ viewed inside $SL(\mathfrak{g})$ via the adjoint representation. We know from Theorem 3.3 and Lemma 2.1 that either there is a non archimedean place v of Ω for which $\Lambda_v(F^{d_0^2}) > 1$ or there is an archimedean place for which $\Lambda_v(F^{d_0^2}) > 1 + \varepsilon$ where ε is the Height Gap. Let $f \in F^{d_0^2}$ be such that $\Lambda_v(f) = \Lambda_v(F^{d_0^2})$. At any case, in one of boundedly many irreducible representations of $\mathbb{G}$ over the local field K_v, f acts as a proximal transformation with a contracting eigenvalue and its action on the associated projective space $\mathbb{P}(K_v^D)$ is described by Lemma 4.6. Let v_f be its attracting point and H_f be the repelling hyperplane. By Lemma 2.1, we may conjugate F inside $GL_D(K_v)$ so that $||F||_v$ is less than say $\Lambda_v(f)^{c_0}$ where c_0 is some constant depending only on d_0. Up to changing f into f^{c_0} we may assume that $||F||_v \le \Lambda_v(f)$.

Let $\alpha_1, ..., \alpha_k$ be the elements from Lemma 6.10. According to Lemma 5.4, there is some $n_i \in [1, d_0]$ such that $d(\alpha_i^{n_i} v_f, H_f)^{-1}$ is bounded above by some bounded power of $||F||_v$. If follows from Lemma 4.6 that there is compact subset C of the projective space $\mathbb{P}(K_v^D)$ which is the complement of some neighborhood of H_f, such that after replacing f by some bounded power of it if necessary, the elements f, $\alpha_1^{n_1} f$, ..., $\alpha_k^{n_k} f$ are all proximal, send C inside itself, and have a Lipschitz constant < 1 on C. Let $F_0 = \{f, \alpha_1^{n_1} f, ..., \alpha_k^{n_k} f\}$. We check that F_0 satisfies the desired conditions. It lies in a bounded power of F, every positive word with letters in F_0 preserves C and is proximal by Tits' converse Lemma 4.7, hence of infinite order. Finally the group $\langle F_0 \rangle$ generated by F_0 contains each $\langle \alpha_i^{n_i} \rangle$, hence its Zariski closure contains the connected component C_i of the cyclic group $\langle \alpha_i \rangle$. Since the C_i's generate $\mathbb{G}$ as an algebraic group by Lemma 6.10, we get that $\langle F_0 \rangle$ is Zariski dense, and this ends the proof of Lemma 6.9. □

6.6.2. *End of the proof of Theorem 1.1.*

So from now on we assume that $\mathbb{G}$ is a simple Chevalley group of adjoint type over Ω viewed as embedded inside $SL(\mathfrak{g})$ ($\mathfrak{g} = Lie(\mathbb{G})$) where it acts via the adjoint representation. We also assume that $F = \{1, X, X^{-1}, Y, Y^{-1}\}$ generates a Zariski-dense subgroup of $\mathbb{G}$ and (X, Y) lies in the Zariski-open subset $\mathcal{O}$ defined in Theorem 3.4.

Constants.

We now define or recall our constants. All these constants depend only on $\mathbb{G}$ (equivalently only on $\dim \mathbb{G}$) and not on the field of coefficients we choose. And this is all that matters, so the reader may freely ignore their precise definition, all the more so since we did not try at all to give the best constants we could. However there dependence and order in which they are defined are important in the logic of the proof.

Recall that the constant $C_{k,1}$ from Section 5 was defined to be 1 if k is ultrametric and equal to the dimension of the vector space if k is Archimedean.

Below we set the value of d to be the $\max d_\alpha$ where $d_\alpha = \dim E_\alpha$ for $\alpha \in \Pi \cup \{0\}$ (recall that we chose to denote by E_0 the adjoint representation, so $d_0 = \dim \mathbb{G}$).

$D = (rk(\mathbb{G}) + 1) \max_{\alpha \in \Pi \cup \{0\}} 24 \cdot 4^{d_\alpha} d_\alpha^2$.

L is defined to be the maximum coefficient in the expression of the heighest weight χ_α (for each $\alpha \in \Pi \cup \{0\}$) as a sum of simple roots.

M is the smallest positive integer k such that $k\chi_{\rho_\alpha} - \beta$ is positive for any choice of simple roots $\alpha, \beta \in \Pi$.

$c_0(v)$ is the maximum of the constants denoted c_0 in Lemma 6.4 for each ρ_α, $\alpha \in \Pi \cup \{0\}$ for a given place v ($c_0 = 1$ when v is finite and $c_0(v)$ is a fixed constant $c_0(\infty)$ if v is infinite).

$c(d_0)_v$ is the constant appearing in the Comparison Lemma, Lemma 2.1. It is 1 if v is finite, a fixed constant $c(d_0)_\infty$ if v is infinite.

g is the Height Gap from Theorem 3.3 in $SL_{d_0}(\overline{\mathbb{Q}})$.

If $char(\Omega) > 1$, then we set $n_1 = 1$, otherwise we set n_1 to be the first integer such that $\exp(\frac{g}{4}\sqrt{\frac{n_1}{8d}}) \geq \max\{c(d_0)^{-2}, d^{4LMd}, c_0(\infty)^{2LM}\}$.

q_0 is the integer obtained by escape in Lemma 6.5.

q_1 is $n_1 d^2$.

ε_0 is $\frac{1}{L}$.

ε is $\varepsilon_0/12d^2$.

T_1 is the maximum of the integers $\tau_1(d_\alpha, \varepsilon)$ obtained in Lemma 5.1 for each representation ρ_α, $\alpha \in \Pi \cup \{0\}$.

C is the constant from Theorem 3.4, applied to $\mathbb{G}$ inside SL_{d_0}.

C_0 is the constant from Lemma 6.3.

Let $m = 48T_1 n_1 CL^2$.

Let $D_1 = D5^{q_1+2q_0}(q_0 + q_1)C_0(1 + g^{-1})$

Let $T_0 = 24CD_1LM$

Let $k_1 = d^2 m$, $k_2 = k_3 = k_5 = q_0$.

Let T_3 be the maximum of the integers τ_3 obtained in Lemma 5.1 for the above values of $d_\alpha, k_1, k_2, k_3, \varepsilon_0, \varepsilon, T_0$ and T_1 for each representation ρ_α, $\alpha \in \Pi \cup \{0\}$.

Let l be the maximum of the integers l obtained in Lemma 5.1 for the above values of $d_\alpha, k_1, k_2, k_3, \varepsilon_0, \varepsilon, T_0, T_1$ and T_3 for each representation ρ_α, $\alpha \in \Pi \cup \{0\}$.

Let $k_4 = 2k_1 l + k_2 + k_3$.

Let $q_4 = n_1 k_4$.

Let $D_2 = D5^{q_4+q_0}(q_4 + q_0)C_0(1 + g^{-1})$.

Let $T_2 = 24CD_2LM$.

Let l_2 be the maximum of each value $l_2(d_\alpha, (k_i)_{1\leq i\leq 5}, \varepsilon, \varepsilon_0, (T_i)_{0\leq i\leq 3})$ obtained in Lemma 5.3 for each representation ρ_α, $\alpha \in \Pi \cup \{0\}$.

Choice of a place v.

Applying Theorem 3.4, we may change F into a conjugate of it by some element in $\mathbb{G}(\Omega)$ and hence get, summing (3), (38) and (39),

$$h(Ad(F)) + \frac{1}{D_1}\delta^1(F) + \frac{1}{D_2}\delta^2(F) \leq 3C \cdot e(Ad(F)) \tag{40}$$

Let K be the (global) field generated by the coefficients of F.

Claim: There is a place v of K such that the following holds: $e_v > 0$ if v is a finite place, $e_v \geq \frac{g}{4}$ if v is infinite and in both cases

(41)
$$\begin{aligned} \log ||Ad(F)||_v &\leq 12C \cdot e_v \\ \delta^1(F)_v &\leq 12CD_1 \cdot e_v \\ \delta^2(F)_v &\leq 12CD_2 \cdot e_v, \end{aligned}$$

where $e_v = \log E_v(Ad(F))$ and $\delta^i(F)_v$ is the part of $\delta^i(F)$ associated to v, i.e.

$$\delta^i(F) = \frac{1}{[K : K_0]} \sum_{v \in V_K} n_v \cdot \delta^i(F)_v$$

Proof of claim: This is an easy verification. Indeed, splitting the infinite part and the finite part write $e = e(Ad(F)) = e_\infty + e_f$. If $e_\infty < \frac{e}{2}$, then $e \leq 2e_f$ and (40) implies

$$h_f(Ad(F)) + \frac{1}{D_1}\delta^1_f(F) + \frac{1}{D_2}\delta^2_f(F) \leq 6C \cdot e_f(Ad(F))$$

where the subscript f means that we have restricted the sum to the finite places. Then the existence of a finite place v such that $e_v > 0$ and (41) holds is guaranteed. On the other hand, if $e_\infty \geq \frac{e}{2}$, then we have

$$h_\infty(Ad(F)) + \frac{1}{D_1}\delta^1_\infty(F) + \frac{1}{D_2}\delta^2_\infty(F) \leq 6C \cdot e_\infty(Ad(F))$$

Let V^+ be the set of places $v \in V_\infty$ for which $e_v \geq \frac{e_\infty}{2}$. We have

$$\frac{e_\infty}{2} \leq \frac{1}{[K : K_0]} \sum_{v \in V^+} n_v e_v$$

And

$$h_\infty(Ad(F)) + \frac{1}{D_1}\delta^1_\infty(F) + \frac{1}{D_2}\delta^2_\infty(F) \leq 12C \cdot \frac{1}{[K : K_0]} \sum_{v \in V^+} n_v e_v$$

which surely guarantees the existence of a place $v \in V^+$ such that (41) holds, and as $v \in V^+$, $e_v \geq \frac{e}{4} \geq \frac{g}{4}$. qed.

Verification of the Ping-Pong conditions (i) **to** (vi) **from Section 5.** We are going to build an element $a \in F^{q_1}$ and choose an $\alpha \in \Pi$ for which all the six conditions of Section 5 are going to be satisfied with a^m in place of a and $\rho_\alpha(F^{n_1})$ in place of F.

According to Lemma 2.3,

$$\begin{aligned} E_v(Ad(F^{n_1})) &\geq E_v(Ad(F))^{\sqrt{\frac{n_1}{8d}}} \\ &\geq \max\{c(d_0)_v^{-2}, C_{K_v,1}^{4LMd}, c_0(v)^{2LM}\}. \end{aligned}$$

From Lemma 2.1,

$$\begin{aligned}\Lambda_v(AdF^{q_1}) &\geq \Lambda_v(AdF^{n_1 d_0^2}) \geq c(d_0)E_v(AdF^{n_1})\\ &\geq E_v(AdF^{n_1})^{\frac{1}{2}} \geq ||AdF||_v^{\frac{1}{12C}} > 1\end{aligned}$$

Let us choose $a \in F^{q_1}$ such that $\Lambda_v(Ad(a)) = \Lambda_v(AdF^{q_1})$. Let also $\alpha \in \Pi$ be such that $|\alpha(a)|_v = \max\{|\beta(a)|_v, \beta \in \Pi\}$. Then the representation ρ_α (defined in Subsection 6.3) and Ad satisfy $\Lambda_v(Ad(a)) \leq |\alpha(a)|_v^L$ and $\Lambda_v(\rho_\alpha(a)) \leq |\alpha(a)|_v^L$ by definition of L. Hence $|\alpha(a)|_v > 1$. Moreover, by definition of ρ_α and $\varepsilon_0 = \frac{1}{L}$ we have

$$\frac{\Lambda_v(\rho_\alpha(a))}{\lambda_v(\rho_\alpha(a))} = |\alpha(a)|_v > 1 \tag{42}$$

It follows that $\rho_\alpha(a)$ is proximal. Moreover

$$\left(\frac{\Lambda_v(\rho_\alpha(a))}{\lambda_v(\rho_\alpha(a))}\right)^{\frac{1}{\varepsilon_0}} = |\alpha(a)|_v^{\frac{1}{\varepsilon_0}} \geq \Lambda_v(\rho_\alpha(a)) \tag{43}$$

and

$$\Lambda_v(\rho_\alpha(a)) \geq |\alpha(a)|_v \geq \Lambda_v(Ad(a))^{\frac{1}{L}} \geq ||AdF||_v^{\frac{1}{12CL}}$$

On the other hand, by Lemma 6.4, we have $||\rho_\alpha(F^{n_1})||_v \leq c_0(v)||AdF^{n_1}||_v^L \leq ||AdF^{n_1}||_v^{L+1} \leq ||AdF||_v^{n_1 2L}$ so

$$\Lambda_v(\rho_\alpha(a)) \geq ||\rho_\alpha(F^{n_1})||_v^{\frac{1}{24CL^2 n_1}}$$

And

$$\Lambda_v(\rho_\alpha(a^m)) \geq ||\rho_\alpha(F^{n_1})||_v^{T_1} \tag{44}$$

Raising a to the power m, (42) gives condition (i), while (43) and (44) give condition (iii). On the other hand Lemma 6.4 gives

$$||\rho_\alpha(F^{n_1})||_v \geq c_0(v)^{-1}||AdF^{n_1}||_v^{\frac{1}{LM}} \geq ||AdF^{n_1}||_v^{\frac{1}{2LM}} \geq C_{K_v,1}^{2d}. \tag{45}$$

Hence condition (ii) is fulfilled.

We now check (iv) and (v). By (41) we have $\delta^1(F)_v \leq 12CD_1 \cdot e_v$. Since $\delta^1(F)_v$ is a sum of positive terms, we get in particular for any (b,t) such that $(a,b,t) \in \mathcal{Q}_\alpha$ (just pick one!)

$$\sum_W \delta({}^t\rho_\alpha(b)H_a^\perp; W^\perp)_v + \delta({}^t\rho_\alpha(b^{-1})H_a^\perp; W^\perp)_v \leq 12CD_1 \cdot e_v$$

where H_a the generalized eigenspace of $\rho_\alpha(a)$ corresponding to eigenvalues that are $< \Lambda_v(\rho_\alpha(a))$ (it is a hyperplane since $\rho_\alpha(a)$ is proximal), and the sum is made over all non trivial $\rho_\alpha(a)$-admissible subspaces W. This gives

$$\begin{aligned} d_v({}^t\rho_\alpha(b)H_a^\perp; W^\perp) &\geq E_v(Ad(F^{n_1}))^{-12CD_1} \geq ||Ad(F^{n_1})||_v^{-12CD_1} \\ &\geq ||\rho_\alpha(F^{n_1})||_v^{-24CD_1LM} \\ &\geq ||\rho_\alpha(F^{n_1})||_v^{-T_0} \end{aligned}$$

Similarly

$$d_v({}^t\rho_\alpha(b^{-1})H_a^\perp; W^\perp) \geq ||\rho_\alpha(F^{n_1})||_v^{-T_0}$$

This proves (iv). Condition (v) is derived in exactly the same way.

Therefore we are in the situation where we may apply Lemma 5.1. It yields an element $x \in F^{n_1 k_4} = F^{q_4}$ such that $\rho_\alpha(x)$ is very proximal and satisfies the conclusions of Lemma 5.1. Pick $c \in F^{q_0}$ such that $(x, c) \in \mathcal{R}_\alpha$ (there are such c by Lemma 6.5). The third inequality in (41) gives for every $\rho_\alpha(x)$-admissible subspaces V and W with $\dim V = 1$,

$$\delta(\rho_\alpha(c)V; W)_v + \delta(\rho_\alpha(c^{-1})V; W)_v \leq 12CD_2 \cdot e_v$$

We may take $V = V_x$ or $V_{x^{-1}}$ and $W = H_x$ or $H_{x^{-1}}$ and this indeed gives condition (v) with $T_2 = 24CD_2LM$.

Finally Lemma 5.3 yields that x^n and cx^nc^{-1} generate a free subgroup as soon as n is larger than the constant l_2 (l_2 is expressible explicitly in terms of all the other constants introduced so far).

This ends the proof of the main theorem. Q.E.D.

7. Applications

In this section we briefly discuss the corollaries. We shall be brief as each of them is derived in exactly the same way as in the GL_2 case, so we will refer the reader to the paper [12] for details. The proofs of Corollaries 1.7, 1.9 and 1.10 rely only on the characteristic 0 part of Theorem 1.1 and on a reformulation of that theorem in terms of algebraic varieties. So we will content ourselves to give this reformulation and briefly explain below what makes this translation possible. The following fact is standard,

Proposition 7.1. (see e.g. [13] Proposition 7.4.) Let $G = GL_d(\mathbb{C})$. For every integer k, let $\mathcal{V}$ be the set of k-tuples $(a_1, ..., a_k) \in G^k$ which generate a virtually solvable subgroup. Then $\mathcal{V}$ is a closed algebraic subvariety of G^k.

It is proved via the following proposition:

Proposition 7.2. There exists $N = N(d)$ such that $(a_1, ..., a_k) \in G^k$ generates a virtually solvable subgroup if and only if they leave invariant a common finite subset of at most N points on the flag variety G/B, where B is the subgroup of upper triangular matrices.

Let $N = N(d)$ be the integer obtained in the statement of Theorem 1.1 and let $B(n)$ be the ball of radius n in the free group F_2 on two generators. For $n \geq 1$ let $\mathcal{W}_n$ be the set of couples $(A, B) \in GL_d(\mathbb{C})^2$ such that for any words w_1 and w_2 in $B(N)$ there exists a word $w \in B(n)\backslash\{1\}$ such that $w(w_1(A, B), w_2(A, B)) = 1$. Clearly $\mathcal{W}_n$ is a closed subvariety of $GL_d(\mathbb{C})^2$. We obtain:

Proposition 7.3. Theorem 1.1 for $K = \mathbb{C}$ is equivalent to the statement: $\mathcal{W}_n \subset \mathcal{V}$ for every $n \geq 1$.

This allows to use the following effective version of Hilbert's Nullstellensatz:

Theorem 7.4. ([29]) Let $r, d \in \mathbb{N}$, $h > 0$ and $f, q_1, ..., q_k$ be polynomials in $\mathbb{Z}[X_1, ..., X_r]$ with logarithmic height at most h and degree at most d. Assume that f vanishes at all common zeros (if any) of $q_1, ..., q_k$ in $\mathbb{C}[X_1, ..., X_r]$. Then there exist $a, e \in \mathbb{N}$ and polynomials $b_1, ..., b_k \in \mathbb{Z}[X_1, ..., X_r]$ such that

$$af^e = b_1q_1 + ... + b_kq_k$$

with $e \leq (8d)^{2^r}$, the total degree of each b_i at most $(8d)^{2^r+1}$ and the logarithmic height of each b_i as well as a is at most $(8d)^{2^{r+1}+1}(h + 8d\log(8d))$.

Since the polynomial equations defining $\mathcal{W}_n$ have degree linear in n and height exponential in n, one can get from Theorem 7.4 the desired bound on the degree and height of the b_i's and on a and e. This readily allows to deduce Corollaries 1.7, 1.9 and 1.10 from Theorem 1.1 and Corollary 1.5. Corollary 1.8 is derived in a similar fashion. See [12] for more details.

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References

[1] H. Abels, G. Margulis, G. Soifer, *Semigroups containing proximal linear maps,* Israel J. Math. **91** (1995), no. 1-3, 1–30.
[2] Barnea Y., Larsen M., *Random Generation for semisimple algebraic groups over local fields*, J. Algebra 271 (2004), no. 1, 1–10.
[3] Bartholdi L., de Cornulier Y., *Infinite groups with large balls of torsion elements and small entropy*, to appear in Archiv der Mathematik.
[4] Y. Benoist, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. 7 (1997), no. 1, 1–47.
[5] Bilu, Y, *Limit distribution of small points on algebraic tori*, Duke Math. J. **89** (1997), no. 3, 465–476.

[6] Bombieri, E., Gubler, W., *Heights in Diophantine geometry*, New Mathematical Monographs, 4. Cambridge University Press, Cambridge, (2006).
[7] Borel, A., *Linear algebraic groups*, Notes taken by Hyman Bass W. A. Benjamin, Inc., New York-Amsterdam 1969
[8] Bourbaki, N. *Groupes et Algèbres de Lie*, Chapitres 4-5-6 and 7-8, Hermann ed.
[9] Bourgain, J. Gamburd, A., *On the spectral gap for finitely generated subgroups of $SU(2)$*, to appear in Invent. Math.
[10] Bourgain, J. Gamburd, A., *Uniform expansion bounds for Cayley graphs of $SL_2(\mathbb{F}_p)$*, to appear in Annals of Math.
[11] Breuillard E., *On uniform exponential growth for solvable groups,* to appear in the Margulis Volume, Pure and Applied Math. Quart.
[12] Breuillard E., *Heights on GL_2 and free subgroups*, preprint December 2007.
[13] Breuillard E., *A height gap theorem for finite subsets of $SL_n(\overline{\mathbb{Q}})$ and non amenable subgroups*, preprint April 2008.
[14] Breuillard E., Gelander, T., *Uniform independence in linear groups*, to appear in Invent. Math.
[15] Bridson M., Haefliger A., *Metric spaces of non-positive curvature*, Springer-Verlag, (1999), vii, 643 p.
[16] C.W. Curtis, I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, (Interscience, New York) (1962).
[17] P. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Math. (1996).
[18] Eskin, Alex; Mozes, Shahar; Oh, Hee, *On uniform exponential growth for linear groups*, Invent. Math. **160** (2005), no. 1, 1–30
[19] Gamburd, A., Jakobson, D., Sarnak, P., *Spectra of elements in the group ring of $SU(2)$*, J. Eur. Math. Soc. (JEMS) 1 (1999), no. 1, 51–85.
[20] Gamburd, A., Hoory S., Shahshahani M., Shalev A., Virag, B., *On the girth of random Cayley graphs,* arXiv preprint (2005).
[21] R. Grigorchuk, P. de la Harpe, *Limit behaviour of exponential growth rates for finitely generated group*s, in Essays on geometry and related topics, Vol. 1, 2, 351–370, Monogr. Enseign. Math., **38**, (2001).
[22] Helfgott H., *Growth and Generation in $SL_2(\mathbb{F}_p)$*, to appear in Annals of Math.
[23] Kazhdan, D., Margulis, G., *A proof of Selberg's hypothesis*, Mat. Sb. (N.S.) **75** (117) 1968 163–168
[24] Kesten, H., *Symmetric walks on groups*, Trans. Amer. Math. Soc. **92** (1959) 336–354.
[25] Iwahori, N., Matsumoto, H., *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. No. **25** (1965) 5–48.
[26] Kaloshin, V. Rodnianski, I, *Diophantine properties of elements of $SO(3)$*, Geom. Funct. Anal. 11 (2001), no. 5, 953–970.
[27] Landvogt, E., *Some functorial properties of the Bruhat-Tits building*, J. Reine Angew. Math. **518** (2000), 213–241.
[28] Lang, S., *Fundamentals of Diophantine geometry*, Springer-Verlag, New York, (1983).
[29] Masser, Wustholz, *Fields of large transcendence degree generated by values of elliptic functions*, Invent. Math. (1983).
[30] Mostow, G. D., *Self-adjoint groups*, Ann. of Math. (2) **62**, (1955). 44–55.

[31] Onishchik, A. L.; Vinberg, È. B. *Lie groups and algebraic groups*, Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, (1990).

[32] Quint, J.-F., *Cônes limites des sous-groupes discrets des groupes réductifs sur un corps local,* Transform. Groups **7** (2002), no. 3, 247–266.

[33] M.S. Raghunathan, *Discrete Subgroups of Lie Groups,* Ergebnisse der Mathematik und Ihrer Grenzgebiete. Band **68** (1972).

[34] Sarnak, P. *Applications of modular forms*, Cambridge University Press.

[35] Schinzel, A. *Polynomials with special regard to reducibility*, With an appendix by Umberto Zannier. Encyclopedia of Mathematics and its Applications, 77. Cambridge University Press, Cambridge, (2000).

[36] Y. Shalom, *Explicit Kazhdan constants for representations of semisimple and arithmetic groups*, Ann. Inst. Fourier, **50** (2000), no. 3, 833–863.

[37] L. Szpiro, E. Ullmo, S. Zhang, *Equirépartition des petits points*, Invent. Math. **127** (1997), 337–347.

[38] Steinberg, R., *Lectures on Chevalley groups*, Notes prepared by John Faulkner and Robert Wilson. Yale University, New Haven, Conn., (1968).

[39] J. Tits, *Free subgroups of Linear groups*, Journal of Algebra **20** (1972), 250-270.

[40] Thurston, W, *Three-dimensional geometry and topology*, Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, **35**. Princeton University Press, (1997).

[41] Ullmo, E. *Positivité et discrétion des points algébriques des courbes*, Ann. of Math. (2) **147** (1998), no. 1, 167–179.

[42] Wang, H. C., *Topics on totally discontinuous groups*, in Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970), pp. 459–487. Pure and Appl. Math., Vol. 8, Dekker, (1972).

[43] A. Weil, *Basic Number Theory*, Springer-Verlag, (1967).

[44] Wehrfritz, B., *Infinite linear groups. An account of the group-theoretic properties of infinite groups of matrices*, Ergeb. Mat. Grenz., **76**, Springer-Verlag, (1973).

[45] Zhang, S., *Small points and adelic metrics*, J. Algebraic Geom. 4 (1995), no. **2**, 281–300

[46] Zhang, S-W., *Equidistribution of small points on abelian varieties*, Ann. of Math. (2) **147** (1998), no. 1, 159–165.

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