

A NILPOTENT FREIMAN DIMENSION LEMMA

EMMANUEL BREUILLARD, BEN GREEN, AND TERENCE TAO

ABSTRACT. We prove that a K -approximate subgroup of an arbitrary torsion free nilpotent group can be covered by a bounded number of cosets of a nilpotent subgroup of bounded rank, where the bounds are explicit and depend only on K . The result can be seen as a nilpotent analogue to Freiman's dimension lemma.

1. INTRODUCTION

Roughly speaking the Freiman-Ruzsa theorem (see e.g. [\[10, Theorem 5.33\]](#)) asserts that any finite subset $A \subset \mathbb{Z}$ with doubling at most K (i.e. such that $|A + A| \leq K|A|$ for some parameter $K \geq 1$) is contained in a generalized arithmetic progression P , whose rank and size are controlled. A number of papers have appeared in the past few years attempting to prove analogues of the Freiman-Ruzsa theorem in other groups than the additive group of integers, and in particular in non-commutative groups (see [\[9, \[14\], \[10, 11\], \[2\], \[6\], \[3\], \[1\], \[15\], \[4\], \[12\], \[7\], etc.\]](#)).

In [\[5\]](#), we established a structure theorem for sets of small doubling and for *approximate subgroups* of arbitrary groups. Approximate subgroups are symmetric finite subsets A of an ambient group whose product set AA can be covered by a bounded number of translates of the set. In particular they are sets of small doubling. They were introduced in [\[14\]](#), because not only are they easier to handle than arbitrary sets of small doubling, but the study of arbitrary sets of small doubling reduces to a large extent to that of approximate subgroups. We refer the reader to Section [2](#) for a precise definition and a reminder of their basic properties.

One of the main theorems proved in [\[5\]](#) asserts, roughly speaking, that approximate subgroups can be covered by a bounded number of cosets of a certain finite-by-nilpotent subgroup with bounded complexity. The bounds on the number of cosets and on the complexity (rank and step) of the nilpotent subgroup depend only on the doubling parameter K .

It turns out that there is an intimate analogy between approximate groups and neighborhoods of the identity in locally compact groups. In [\[5\]](#), this analogy was worked out making use of ultrafilters in order to build a certain limit approximate group to which we then applied an adequately modified version of the tools developed in the fifties for the solution of Hilbert's fifth problem on the structure of locally compact groups.

The use of ultrafilters makes the results of [\[5\]](#) ineffective in a key way. The aim of the present note is to give, in a fairly special case (that of residually-(torsion free nilpotent) groups), a simple argument independent of [\[5\]](#), which achieves the same goal, but with explicit bounds. Our main results are the following.

freiman

Theorem 1.1. *Let G be a simply connected nilpotent Lie group and let A be a K -approximate subgroup of G . Then A can be covered by at most $\exp(K^{O(1)})$ cosets of a closed connected Lie subgroup of dimension at most $K^{O(1)}$.*

We can be completely explicit about the bounds in this theorem as well as in the corollary below, and take K^{2+29K^9} for the first bound on the number of cosets and K^9 for the bound on the dimension.

This result can be compared with Freiman's dimension lemma (see [16, Theorem 5.20]), according to which a finite set with doubling at most K in an arbitrary vector space can be covered by a bounded number of translates of a proper vector subspace with dimension bounded in terms of K only.

We note that the main result of [5] already proves a result similar to Theorem 1.1, and even with a better bound (of the order of $\log K$) for the dimension of the nilpotent subgroup, but that, unlike Theorem 1.1, it provides no explicit bound on the number of translates whatsoever.

cor

Corollary 1.2. *Let G be a torsion free nilpotent group (or only a residually-(torsion free nilpotent) group) and let A be a finite K -approximate subgroup of G . Then A can be covered by at most $\exp(K^{O(1)})$ cosets of a nilpotent subgroup of nilpotency class at most $K^{O(1)}$.*

Recall that a group G is called *residually nilpotent* if given any $g \in G \setminus \{1\}$, there is a $Q = \pi(G)$ a nilpotent quotient of G such that $\pi(g) \neq 1$. This class of groups includes many non-nilpotent groups, such as all finitely generated free groups.

The main idea behind the proof of Theorem 1.1 and Corollary 1.2 is encapsulated in a simple lemma, Lemma 3.1 below. We came up with this lemma after reading an early article of Gleason [8], which uses an analogous idea in order to establish that the class of compact connected subgroups of a given locally compact group admits a maximal element.

The paper is organised as follows. In Section 2, we recall the definition and some basic properties of approximate groups. In Section 3, we state and prove our key lemma. In Section 4, we establish that approximate subgroups of residually nilpotent groups have elements with a large centralizer. Finally in Section 5, we complete the proof of the main results.

ACKNOWLEDGMENTS. EB is supported in part by the ERC starting grant 208091-GADA. He also acknowledges support from MSRI where part of this work was finalized. TT is supported by a grant from the MacArthur Foundation, by NSF grant DMS-0649473, and by the NSF Waterman award. We would like to thank T. Sanders for a number of valuable discussions.

2. BASIC FACTS ON APPROXIMATE GROUPS

facts

In this section we recall some basic properties of approximate groups. Background on approximate groups and their basic properties can be found in the third author's paper [14].

approx-group-def

Definition 2.1 (Approximate groups). Let $K \geq 1$. A finite subset A of an ambient group G is called *K -approximate subgroup* of G if the following properties hold:

- (i) the set A is symmetric in the sense that $\text{id} \in A$ and $a^{-1} \in A$ if $a \in A$;
- (ii) there is a symmetric subset $X \subset A^3$ with $|X| \leq K$ such that $A \cdot A \subseteq X \cdot A$.

We record the following important, yet easy, fact.

basic

Lemma 2.2. *Let A be a K -approximate subgroup in an ambient group G and H a subgroup of G ,*

(i) $|A| \leq |A^2 \cap H| |AH/H| \leq |A^3|.$

(ii) $A^2 \cap H$ is a K^3 -approximate subgroup of G . Moreover

$$\forall k \geq 1, |A^k \cap H| \leq K^{k-1} |A^2 \cap H|.$$

(iii) if π is a group homomorphism $G \rightarrow Q$, then $\pi(A)$ is a K -approximate subgroup of Q .

Proof. We only prove (ii), and leave the other items to the reader. We have $A^k \subset X^{k-1}A$. But every set of the form $(xA) \cap H$ is contained in $y(A^2 \cap H)$ for some $y \in xA \cap H$. So $A^k \cap H \subset Y(A^2 \cap H)$ for some set Y of size at most K^{k-1} . In particular $A^2 \cap H$ is a K^3 -approximate group. \square

gleaglea

3. A COMBINATORIAL ANALOGUE OF A LEMMA OF GLEASON

In [8] Gleason established that every locally compact group admits a maximal compact connected subgroup. In order to prove that every increasing sequence of such subgroups must stabilize, he used a key lemma, [8, Lemma 1], which once translated in the setting of approximate groups can be formulated as follows.

Let G be an arbitrary group and let $\{1\} = H_0 \subset H_1 \subset \dots \subset H_k$ be a nested sequence of subgroups of G . We denote by $\langle B \rangle$ the subgroup generated by the subset B .

gleason

Lemma 3.1. *Let A be a finite symmetric subset of G . Let $A_i := A^2 \cap H_i$. Assume that $\langle A_{i+1} \rangle \not\subseteq A_{i+1}H_i$ for every $i = 0, \dots, k-1$. Then $|A^5| \geq k|A|$*

Proof. Clearly $A_{i+1}^2 \not\subseteq A_{i+1}H_i$, otherwise this would contradict the standing assumption. Pick $h_{i+1} \in A_{i+1}^2 \setminus A_{i+1}H_i$ for each $i = 0, \dots, k-1$. We claim that all Ah_i 's are disjoint. Indeed, if say $Ah_{i+1} \cap Ah_j \neq \emptyset$, say for $j \leq i$, then $h_{i+1} \in A^2h_j \cap H_{i+1} \subset A_{i+1}H_j \subset A_{i+1}H_i$. This contradicts our assumption. \square

If G is assumed to be a simply connected nilpotent Lie group, then the above lemma can be refined as follows:

nilpotent

Proposition 3.2. *Let G be a simply connected nilpotent Lie group, and $\{1\} = H_0 \subset H_1 \subset \dots \subset H_k$ be a nested sequence of closed connected subgroups of G . Assume that for each $i = 0, \dots, k-1$, we have $A^2 \cap H_i \subsetneq A^2 \cap H_{i+1}$. Then $|A^5| \geq k|A|$.*

In order to prove this last proposition we first recall the following classical fact (see [13, chap. 2]).

connected-closure

Lemma 3.3. *For every subgroup Γ in G , there exists a unique minimal closed connected subgroup containing it. We denote it by $\widetilde{\Gamma}$. Moreover, if Γ_0 has finite index in Γ , then $\widetilde{\Gamma_0} = \widetilde{\Gamma}$.*

Proof of Proposition 3.2. Let $A_i := A^2 \cap H_i$. In view of Lemma 3.1, it is enough to prove that $\langle A_{i+1} \rangle \not\subseteq A_{i+1}H_i$ for each $i = 0, \dots, k-1$. If that were not the case, then $\langle A_{i+1} \rangle \cap H_i$ would have finite index in $\langle A_{i+1} \rangle$. Therefore, by Lemma 3.3, $\widetilde{\langle A_{i+1} \rangle} = \widetilde{\langle A_{i+1} \rangle} \cap H_i \subset H_i$ and thus $A_{i+1} = A_i$ contrary to our standing assumption. \square

commutator

4. FINDING AN ELEMENT WITH LARGE CENTRALIZER IN RESIDUALLY NILPOTENT GROUPS

Another key part of our proof is to establish that in any approximate subgroups there are elements which commute with a large proportion of it. This is of course a necessary step towards exhibiting the sought after nilpotent structure given by our theorem.

This important idea has intervened several times before under slightly different guises in the classification of approximate groups, be it in Helfgott's original paper [\[10\]](#), in the first two authors classification of approximate subgroups of compact Lie groups [\[1\]](#), or in our recent paper on the structure of approximate groups in general [\[5\]](#). It is also closely related to the key idea in the proof of the Margulis lemma in Riemannian geometry, or in the well-known geometric proof by Frobenius and Bieberbach of Jordan's theorem on finite linear groups.

Proposition 4.1. *Let G be a residually nilpotent group and let A be a K -approximate subgroup of G . Then, there is an element $\gamma \in A^2$ such that*

$$|A^2 \cap C_G(\gamma)| \geq \frac{|A|}{K^6}.$$

Proof. Let $C_1 = G$ and $C_{i+1} = [G, C_i]$ be the central descending series of G . Let k be the largest integer such that $A^2 \cap C_k \neq \{1\}$. By the residually nilpotent assumption, and the fact that A is a finite set, k is finite. It follows that the map $A \times (A^6 \cap C_{k+1}), (a, x) \mapsto ax$ is injective: indeed if $ax = a'y$, then $a^{-1}a' \in A^2 \cap C_{k+1} = \{1\}$, so $a = a'$ and $x = y$. As a consequence we get: $|A||A^6 \cap C_{k+1}| \leq |A^7| \leq K^6|A|$, and thus

$$|A^6 \cap C_{k+1}| \leq K^6.$$

Now let $\gamma \in A^2 \cap C_k \neq \{1\}$ and consider the map $A \rightarrow A^6, a \mapsto [\gamma, a]$. Clearly $[\gamma, a] \in C_{k+1} \cap A^6$ and we conclude that $[\gamma, a]$ can take at most K^6 possible values as a varies in A . Let $[\gamma, x]$ be the most popular such value. We have $[\gamma, x] = [\gamma, y]$ for at least $\frac{|A|}{K^6}$ elements $x, y \in A$. For any two such x, y , $x^{-1}y \in C_G(\gamma)$. We conclude $|A^2 \cap C_G(\gamma)| \geq \frac{|A|}{K^6}$, which is what we wished to prove. \square

end

5. PROOF OF STATEMENTS OF THE INTRODUCTION

We first prove Theorem [1.1](#). We will build inductively two nested sequences of connected closed subgroups of G ,

$$\begin{aligned} G &= G_0 \supseteq G_1 \supseteq \dots \supseteq G_k, \\ \{1\} &= H_0 \subsetneq H_1 \subsetneq \dots \subsetneq H_k, \end{aligned}$$

and elements $\gamma_i \in A^4$ such that for each $i = 1, \dots, k$,

- (i) each H_i is a normal subgroup of G_i .
- (ii) γ_i normalizes H_{i-1} and $H_i = \langle H_{i-1}, \gamma_i \rangle$ is the connected subgroup generated by H_{i-1} and γ_i .
- (iii) $\gamma_i \in H_i \setminus H_{i-1}$.
- (iv) $|A^2 \cap G_i| \geq |A|/K^{n_i}$, where n_i is an increasing sequence of integers to be determined later.

Moreover, assuming that such data has been built up to level k , we will be able to build a new level $k+1$ as soon as $A^2 \cap G_k \not\subseteq H_k$.

Before establishing the existence of the above data, let us see how this concludes the proof of Lemma [1.1](#). From item (ii) and (iii) we see that $\dim H_i = \dim H_{i-1} + 1$, and hence $\dim H_i = i$ for

all $i \leq k$. Since G is finite dimensional, this implies that the process must stop at some finite time k , for which we thus have $A^2 \cap G_k \subseteq H_k$. However, item (iii) implies that $\gamma_i \in A^4 \cap (H_i \setminus H_{i-1})$ and thus $A^4 \cap H_{i-1} \subsetneq A^4 \cap H_i$ for all $i = 1, \dots, k$. From Proposition [3.2](#), we conclude that $|A^{10}| \geq k|A^2|$. Since A is a K -approximate group, $|A^{10}| \leq K^9|A|$ and it follows that $k \leq K^9$. But by item (iv), $|A^2 \cap G_k| \geq |A|/K^{n_k}$. Therefore $|A^2 \cap H_k| \geq |A|/K^{n_k}$ and we conclude by Lemma [2.2\(i\)](#) that A can be covered by at most K^{2+n_k} translates of the closed subgroup H_k , which has dimension $k \leq K^9$. An explicit value for n_i is computed below; it gives the desired bound in Lemma [1.1](#) and we are done.

It remains to establish that the data above can indeed be constructed. This will be done by induction on k . So we suppose that $k \geq 0$ and that the data has been built up to level k . We also make the assumption that $A^2 \cap G_k \not\subseteq H_k$ as required above and proceed to build the $k+1$ -th level.

Let $A'_k := (A^2 \cap G_k)/H_k$. By our assumption $A'_k \neq \{1\}$ and from Lemma [2.2](#) we see that A'_k is a K^3 -approximate subgroup of G_k/H_k . Since G_k/H_k is a simply connected nilpotent Lie group, we may apply Proposition [4.1](#) to it and conclude that there exists some element $\gamma'_{k+1} \in A'^2_k \setminus \{1\}$ such that

$$\boxed{\text{commbound}} \quad (5.1) \quad |A'^2_k \cap C_{G_k/H_k}(\gamma'_{k+1})| \geq |A'_k|/K^{18}.$$

Note that $C_{G_k/H_k}(\gamma'_{k+1})$ is a closed connected subgroup of G_k/H_k , because in a simply connected nilpotent Lie group the commutator subgroup of an element coincides with the analytic subgroup whose Lie algebra is the commutator subalgebra of the logarithm of that element. We can then define G_{k+1} to be the closed connected subgroup of G_k such that $C_{G_k/H_k}(\gamma'_{k+1}) = G_{k+1}/H_k$.

We can also lift γ'_{k+1} to an element $\gamma_{k+1} \in (A^2 \cap G_k)^2$ so that $\gamma_{k+1}H_k = \gamma'_{k+1}$. Note that $\gamma_{k+1} \notin H_k$ and that $\gamma_{k+1} \in G_{k+1}$.

We then define H_{k+1} to be the closed connected subgroup of G_{k+1} generated by H_k and γ_{k+1} . Since G_{k+1} commutes with γ_{k+1} modulo H_k , we conclude that H_{k+1} is a normal subgroup of G_{k+1} . This shows the first three items (i), (ii) and (iii).

In order to establish item (iv), the last remaining item, we first get from [\(5.1\)](#), [commbound](#)

$$\boxed{\text{newcommbound}} \quad (5.2) \quad |(A^4 \cap G_{k+1})/H_k| \geq |A'_k|/K^{18},$$

and from Lemma [2.2\(i\)](#) we have

$$K^{11}|A^2 \cap G_{k+1}| \geq |A^{12} \cap G_{k+1}| \geq |(A^4 \cap G_{k+1})^3| \geq |(A^4 \cap G_{k+1})/H_k| |A^4 \cap H_k|.$$

Now combining this last line with [\(5.2\)](#) we get [newcommbound](#)

$$K^{11}|A^2 \cap G_{k+1}| \geq |(A^2 \cap G_k)/H_k| |A^2 \cap H_k|/K^{18} \geq |A^2 \cap G_k|/K^{18},$$

where the last inequality follows from Lemma [2.2\(i\)](#). Therefore [basic](#)

$$|A^2 \cap G_{k+1}| \geq |A^2 \cap G_k|/K^{29} \geq |A|/K^{29+n_k}.$$

We thus see that we can set $n_{k+1} = n_k + 29$, or alternatively $n_i = 29i$ for all i . This ends the proof of Theorem [1.1](#). [freiman](#)

Proof of Corollary [1.2](#). [cor](#) Suppose first that G is torsion free nilpotent. Without loss of generality, we may assume that G is generated by the finite set A . We may then consider the Malcev closure \tilde{G} of G

over \mathbb{R} (see [13, Chap. 2]) and apply Theorem 1.1 and conclude that A is covered by $\exp(K^{O(1)})$ cosets of a closed connected subgroup L of \tilde{G} with dimension at most $K^{O(1)}$. One of these cosets intersects A in more than $|A|/\exp(K^{O(1)})$ elements and it follows that $|A^2 \cap L| \geq |A|/\exp(K^{O(1)})$. We let H be the subgroup generated by $A^2 \cap L$. It is a finitely generated torsion free nilpotent group with nilpotency class (and even length of any composition series) at most the dimension of L , that is at most K^9 . Finally, by Lemma 2.2(1), we see that A can be covered by at most $\exp(K^{O(1)})$ cosets of H , as desired.

Now if G is only assumed to be residually torsion free nilpotent, then for every constant $C \geq 1$, there exists a torsion free nilpotent quotient $G/\ker \pi$ of G such that A^C intersects the kernel $\ker \pi$ trivially. If C is taken larger than the word length of every commutator of length K^9 , then all commutators of length K^9 in the elements of $\pi^{-1}(\pi(A)^2 \cap L)$ are trivial, and thus the subgroup H of G generated by $\pi^{-1}(\pi(A)^2 \cap L)$ is nilpotent with nilpotency class at most K^9 . By Lemma 2.2(1), we see that A intersects only at most $\exp(K^{O(1)})$ cosets of H and we are done. \square

REFERENCES

- [1] E. Breuillard and B. Green. Approximate groups. III: the unitary case. *arXiv:1006.5160*, 2010. Preprint.
- [2] E. Breuillard and B. Green. Approximate groups. I: the torsion-free nilpotent case. *J. Inst. Math. Jussieu*, 10(1):37–57, 2011.
- [3] E. Breuillard and B. Green. Approximate groups. II: the solvable linear case. *Quart. J. of Math., Oxford*, 62(3):513–521, 2011.
- [4] E. Breuillard, B. Green, and T. Tao. Approximate subgroups of linear groups. *Geom. Funct. Anal.*, 21(4):774–819, 2011.
- [5] E. Breuillard, B. Green, and T. Tao. The structure of approximate groups. *preprint arXiv*, page 91, 2011.
- [6] D. Fisher, N. H. Katz, and I. Peng. Approximate multiplicative groups in nilpotent Lie groups. *Proc. Amer. Math. Soc.*, 138(5):1575–1580, 2010.
- [7] N. Gill and H. Helfgott. Growth in solvable subgroups of $GL_r(\mathbb{Z}/p\mathbb{Z})$. *arXiv:1008.5264*, 2010. Preprint.
- [8] A. M. Gleason. Compact subgroups. *P. Nat. Acad. Sc.*, pages 622–623, 1951.
- [9] B. Green and I. Z. Ruzsa. Freiman’s theorem in an arbitrary abelian group. *J. Lond. Math. Soc. (2)*, 75(1):163–175, 2007.
- [10] H. A. Helfgott. Growth and generation in $SL_2(\mathbb{Z}/p\mathbb{Z})$. *Ann. of Math. (2)*, 167(2):601–623, 2008.
- [11] H. A. Helfgott. Growth in $SL_3(\mathbb{Z}/p\mathbb{Z})$. *J. Eur. Math. Soc. (JEMS)*, 13(3):761–851, 2011.
- [12] L. Pyber and E. Szabó. Growth in finite simple groups of lie type of bounded rank. *arXiv:1005.1858*, 2010. Preprint.
- [13] M. Raghunathan. *Discrete Subgroups of Lie Groups*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin.
- [14] T. Tao. Product set estimates for non-commutative groups. *Combinatorica*, 28(5):547–594, 2008.
- [15] T. Tao. Freiman’s theorem for solvable groups. *Contrib. Discrete Math.*, 5(2):137–184, 2010.
- [16] T. Tao and V. Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

LABORATOIRE DE MATHÉMATIQUES, BÂTIMENT 425, UNIVERSITÉ PARIS SUD 11, 91405 ORSAY, FRANCE

E-mail address: `emmanuel.breuillard@math.u-psud.fr`

CENTRE FOR MATHEMATICAL SCIENCES, WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, ENGLAND

E-mail address: `b.j.green@dpms.cam.ac.uk`

DEPARTMENT OF MATHEMATICS, UCLA, 405 HILGARD AVE, LOS ANGELES CA 90095, USA

E-mail address: `tao@math.ucla.edu`

rd-green-unitary
breuillard-green

bg

bgt

bgt-structure
fisher-katz-peng

gill-helfgott

gleason3

green-ruzsa

helfgott-sl2

helfgott-sl3

pyber-szabo

raghunathan

o-noncommutative

tao-solvable

tv-book