C*-SIMPLICITY AND THE UNIQUE TRACE PROPERTY FOR DISCRETE GROUPS

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ABSTRACT. We prove the simplicity of the reduced C*-algebra of a large class of discrete groups using a new characterization of this property in terms of boundary actions. This class includes most groups for which this result was already known, plus many more. We also completely settle the problem of when the reduced C*-algebra has a unique tracial state.

1. INTRODUCTION

A discrete group is said to be *C*-simple* if the reduced C*-algebra of the group is simple, and is said to have the *unique trace property* if the reduced C*-algebra has a unique trace. The problem of which groups have these properties captured the interest of mathematicians in 1975 with Powers' proof [33] that the free group on two generators is both C*-simple and has the unique trace property. In the ensuing 39 years, the problem has received a great deal of attention, and many more examples of groups with these properties have been found. We direct the reader to [10] for a recent survey.

Recently, the second and third named authors established [22, Theorem 6.2] the following necessary and sufficient condition for the C*-simplicity of a group (see Section 2 for the relevant definitions).

Theorem 1.1. *A discrete group is C*-simple if and only if it has a topologically free boundary action.*

It turns out that it is often possible to prove the existence of a topologically free boundary action for a given group without actually having to construct one. This makes Theorem 1.1 useful in practice for

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establishing C*-simplicity. For example, in [22, Corollary 6.6], by means of an ad hoc argument, this idea was applied to prove the C*-simplicity of Tarski monster groups.

Day [9, Lemma 4.1] showed that every discrete group G has a largest amenable normal subgroup, called the *amenable radical* of G, that contains every amenable normal subgroup of G.

It is well known that if G is C*-simple, then the amenable radical of G is necessarily trivial. However, it is a major open problem to determine if the converse is true (see e.g. [10, Question 4]).

In this paper, we take a more systematic approach to the application of Theorem 1.1, and prove the C*-simplicity of a large class of groups.

Theorem 1.2. *A discrete group with trivial amenable radical having either non-trivial bounded cohomology or non-vanishing ℓ^2-Betti numbers is C*-simple.*

The next result implies the C*-simplicity of (torsion-free) Tarski monster groups and free Burnside groups $B(m, n)$ for $m \geq 2$ and n odd and sufficiently large. The latter result was recently proved in [29].

Theorem 1.3. *A discrete group with only countably many amenable subgroups is C*-simple if and only if its amenable radical is trivial.*

The next result provides a negative answer to [11, Question (Q)].

Theorem 1.4. *Let G be a discrete group and let $N < G$ be a normal subgroup. Then G is C*-simple if and only if both N and $C_G(N)$ are C*-simple, where $C_G(N)$ denotes the centralizer of N in G. In particular, C*-simplicity is closed under extension.*

We also present a new proof of the following result for linear groups from [34].

Theorem 1.5. *A discrete linear group is C*-simple if and only if its amenable radical is trivial.*

The methods typically used to establish the C*-simplicity of a group often also imply that the group has the unique trace property. However, it has been an open problem for some time to determine if this is true in general, i.e. whether the property of C*-simplicity also implies the unique trace property. We prove that this question has an affirmative answer and, more generally, completely settle the problem of which groups have the unique trace property.

Theorem 1.6. *A discrete group has the unique trace property if and only if its amenable radical is trivial. In particular, every C*-simple group has the unique trace property.*

Using this and the argument from [36, Theorem 5.14] we obtain another proof of the fact, recently proved in [4], that amenable invariant random subgroups of a discrete group concentrate of the amenable radical.

In addition to the introduction, this paper has three other sections. In Section 2, we collect some results on boundary actions of discrete groups. In Section 3 we establish our results on C*-simplicity. Finally, in Section 4 we establish our results on the unique trace property.

2. Boundary actions and normalish subgroups

2.1. **Boundary actions.** Let G be a discrete group. A compact Hausdorff space X is said to be a *G-space* if there is a homomorphism from G into the group of homeomorphisms on X. We refer to such a homomorphism as a *G-action* on X.

The notion of a boundary action was introduced in [15] (see also [18]).

Definition 2.1. Let G be a discrete group and let X be a compact G-space.

(1) The G-action on X is *minimal* if Gx is dense in X for every $x \in X$.
(2) The G-action on X is *proximal* if, for ever pair $x, y \in X$, there is a net $t_i \in G$ such that $\lim t_i x = \lim t_i y$.
(3) The G-action on X is *strongly proximal* if the induced G-action on the G-space $\mathcal{P}(X)$, consisting of probability measures on X, is proximal.
(4) The G-action on X is a *G-boundary action* if it is both minimal and strongly proximal. In this case, the G-space X is called a *G-boundary.*

Remark 2.2. It is clear that if the G-action on X is strongly proximal then, in particular, it is proximal. Furthermore, it is easy to see that in this case, for every $x_1, \ldots, x_n \in X$ and every non-empty open subset $U \subset X$, there is $t \in G$ such that $tx_1, \ldots, tx_n \in U$.

Furstenberg showed [15] that every group G has a unique G-boundary $\partial_F G$ which is *universal* in the sense that every G-boundary is a continuous G-equivariant image of $\partial_F G$.

We will require the well-known characterization of the amenable radical of G as the set of elements in G which act trivially on $\partial_F G$ (see e.g. [14, Proposition 7]). A related property, established in the proof of [22, Theorem 6.2], is the following:

Lemma 2.3. *Let G be a discrete group with universal G-boundary $\partial_F G$. For every $x \in \partial_F G$, the stabilizer subgroup G_x is amenable.*

In [22, Theorem 6.2] the following criterion for C*-simplicity was established.

Proposition 2.4. *Let G be a discrete group and X a G-boundary. The reduced crossed product $C(X) \rtimes_{\mathrm{r}} G$ is simple if and only if G is C*-simple.*

For completeness, we briefly recall why. First, suppose that G is C*-simple. Since X is a boundary, we have a G-map from $\partial_F G$ onto X, and this induces an inclusion $\mathrm{C}^*_r(G) = \mathbb{C} \rtimes_{\mathrm{r}} G \subset C(X) \rtimes_{\mathrm{r}} G \subset C(\partial_F G) \rtimes_{\mathrm{r}} G$. By [22, Theorem 3.11] $C(\partial_F G)$ coincides with $I_G(\mathbb{C})$, the G-injective envelope of $\mathbb{C}$. Hence by [20, Theorem 3.4], $C(\partial_F G) \rtimes_{\mathrm{r}} G \subset I(\mathbb{C} \rtimes_{\mathrm{r}} G) = I(\mathrm{C}^*_r(G))$, where $I(\mathrm{C}^*_r(G))$ denotes the injective envelope of $\mathrm{C}^*_r(G)$. However, any C*-algebra sitting between a simple C*-algebra and its injective envelope must be simple. Since $I(\mathbb{C} \rtimes_{\mathrm{r}} G) \subset I(C(X) \rtimes_{\mathrm{r}} G)$, this also shows that the converse reduces to the case when $X = \partial_F G$, which is treated in the implication (2) $\Rightarrow$ (1) of [22, Theorem 6.2].

Definition 2.5. Let G be a discrete group and let X be a G-space. For $s \in G$, let X_s denote the set of s-fixed points $X_s = \{x \in X \mid sx = x\}$. The G-action on X is said to be *free* if $X_s = \emptyset$ for every $s \in G \setminus \{e\}$, and *topologically free* if X_s has empty interior for every $s \in G \setminus \{e\}$.

If X is any compact minimal G-space on which the G-action is topologically free, then $C(X) \rtimes_{\mathrm{r}} G$ is simple by [3, Corollary 1]. If in addition X is a G-boundary, on which G acts topologically freely, then G must be C*-simple by Proposition 2.4. In certain cases, the converse also holds.

Proposition 2.6. *Let G be a C*-simple discrete group and let X be a G-boundary. If G_x is amenable for some $x \in X$, then the G-action on X is topologically free.*

Proof. This follows from the argument in the proof of the implication (2) $\Rightarrow$ (4) in [22, Theorem 6.2], which we briefly recall. If the action is not topologically free, there is $s \in G \setminus \{e\}$ such that x belongs to the interior of the set of s-fixed points. Then arguing as in the proof of [22, Theorem 6.2], we construct a representation of the reduced crossed product $C(X) \rtimes_{\mathrm{r}} G$ with non-trivial kernel. In particular, this implies that $C(X) \rtimes_{\mathrm{r}} G$ is not simple, contradicting Proposition 2.4 above. ∎

By Lemma 2.3, this criterion applies to $X = \partial_F G$, and we thus recover Theorem 1.1.

Remark 2.7. The equivalence in Proposition 2.4 above fails if X is only assumed to be a minimal compact G-space. Indeed, as already mentioned, $C(X) \rtimes_{\mathrm{r}} G$ is simple whenever X is topologically free (see [3, Corollary 1]). In particular, one could choose G amenable, and hence not C*-simple, and let X be a minimal subspace of the Stone-Cech compactification of G. However, we will show below (see Theorem 3.19) that the other direction of this equivalence still holds.

We recall from [22, Remark 3.16] that if G is a discrete group, then the universal G-boundary $\partial_F G$ is extremally disconnected (and hence Stonean). This is because it arises as the spectrum of a commutative injective C*-algebra, namely, Hamana's G-injective envelope of $\mathbb{C}$ [20]. Using this fact, the next lemma follows as a simple consequence of Frolík's Theorem [13] (see also [32, Proposition 2.7]).

Lemma 2.8. *Let G be a discrete group with universal G-boundary $\partial_F G$. For $s \in G$, if the set $(\partial_F G)_s$ of s-fixed points is non-empty, then it is clopen.*

The next result follows immediately from Definition 2.5 and Lemma 2.8.

Proposition 2.9. *Let G be a discrete group with universal G-boundary $\partial_F G$. The G-action on $\partial_F G$ is topologically free if and only if it is free.*

Lemma 2.10. *Let G be a discrete group and let X be a G-boundary. If the action is not topologically free, then for every $x \in X$ and every $t_1, \ldots, t_n \in G$, the intersection $\cap_i t_i G_x t_i^{-1}$ is non-trivial. If moreover G has no non-trivial finite normal subgroup, this intersection is infinite.*

Proof. By minimality, for every x in X one can find $s \neq e$ such that x lies in the interior of X_s. Now pick $r \in G$ such that $rt_i x \in X_s$ for each $i = 1, \ldots, n$. Then $r^{-1}sr \in \cap_{i=1}^k t_i G_x t_i^{-1}$. If this intersection is finite, then $\cap_{t \in G} t G_x t^{-1}$ is in fact an intersection of finitely many terms, hence is a non trivial finite normal subgroup. ∎

2.2. **Normalish subgroups.** We make the following definition.

Definition 2.11. Let G be a group. A subgroup $H < G$ is said to be *normalish* if for every $n \geq 1$ and $t_1, \ldots, t_n \in G$ the intersection $\cap_i t_i H t_i^{-1}$ is infinite.

This notion is slightly stronger than the notion of an n-step s-normal subgroup, which was introduced in [5]. Note that a subgroup that is n-step s-normal for every $n \geq 1$ is normalish. Our main observation is the following.

Theorem 2.12. *A discrete group G with no non-trivial finite normal subgroups and no amenable normalish subgroups is C*-simple.*

Proof. Let G be a discrete group with no non-trivial finite normal subgroup. To show that G is C*-simple, it is enough to prove that G acts topologically freely on $\partial_F G$, according to [22, Theorem 6.2]. If that is not the case, then point stabilizers are normalish by Lemma 2.10 and amenable by Lemma 2.3. This ends the proof. ∎

3. C*-simplicity

3.1. **Bounded cohomology and ℓ^2-Betti numbers.** We recall that a group is said to be C*-simple if its reduced C*-algebra is simple. In this section we will prove the C*-simplicity of discrete groups with trivial amenable radical, supposing that they have either non-trivial bounded cohomology, or non-vanishing ℓ^2-Betti numbers. The key idea is that these conditions preclude the existence of amenable normalish subgroups.

Let G be a discrete group. For $n \geq 0$, let $\beta_n^{(2)}$ denote the n-th ℓ^2-Betti number of G (see e.g. [23]). For the result about non-vanishing ℓ^2-Betti numbers, we require the following special case of [5, Theorem 1.3].

Proposition 3.1. *Let G be a discrete group. If G contains an amenable normalish subgroup, then $\beta_n^{(2)} = 0$ for every $n \geq 0$.*

Next, we consider the case when G has non-trivial bounded cohomology. Recall (see e.g. [25]) that a *coefficient G-module* (π, E) is an isometric linear G-representation π on a dual Banach space E (specifically, the dual of a separable Banach space) such that the operators in the image of π are weak*-continuous.

For $n \geq 0$, the *bounded cohomology* group $H_b^n(G, E)$ of G with coefficient module (π, E) is the n-th cohomology group of the complex

$$\cdots \longrightarrow \ell^\infty(G^n, E)^G \longrightarrow \ell^\infty(G^{n+1}, E)^G \longrightarrow \cdots$$

consisting of bounded G-invariant functions.

The dual Banach G-module E is said to be *mixing* if the stabilizer subgroup $G_x = \{s \in G \mid sx = x\}$ is finite for every $x \in E \setminus \{0\}$. Examples of such G-modules include $\ell^p(G)$ for $1 \leq p < \infty$.

The proof of the next result is similar to the proofs of [25, Corollary 7.5.9, Corollary 7.5.10].

Proposition 3.2. *Let G be a discrete group, and let (π, E) be a coefficient G-module with E mixing. If G contains an amenable normalish subgroup, then $H_b^n(G, E)$ is trivial for every $n \geq 0$ and every mixing dual Banach G-module.*

Proof. Let $H < G$ be an amenable normalish subgroup. We can compute the bounded cohomology of G using the complex

$$\cdots \longrightarrow \ell^\infty((G/H)^n, E)^G \longrightarrow \ell^\infty((G/H)^{n+1}, E)^G \longrightarrow \cdots$$

Here, we have written $\ell^\infty((G/H)^n, E)^G$ for the Banach space of bounded G-equivariant maps from $(G/H)^n$ into E. We claim that $\ell^\infty((G/H)^n, E)^G$ vanishes, and hence that $H_b^n(G, E)$ also vanishes. To see this, fix $f \in \ell^\infty((G/H)^n, E)^G$. Then for every $t = (t_1, \ldots, t_n) \in (G/H)^n$, the element $f(t) \in E$ is left invariant by every element in $\cap_i t_i H t_i^{-1}$. Since E is mixing, it follows that $f(t) = 0$. ∎

We will say that a discrete group G has *non-trivial bounded cohomology* if there is a coefficient G-module (π, E) with E mixing such that $H_b^n(G, E)$ is non-trivial for some $n \geq 0$.

Theorem 3.3. *Let G be a discrete group with trivial amenable radical such that either*

(1) *G has non-trivial bounded cohomology, or*
(2) *G has non-vanishing ℓ^2-Betti numbers.*

Then G is C-simple.*

Proof. Proposition 3.1 and Proposition 3.2 imply that G has no amenable normalish subgroup. The conclusion follows from Theorem 2.12. ∎

The class of groups $\mathcal{C}_{\mathrm{reg}}$ was introduced in [26, Notation 1.2]. It consists of those countable discrete groups G satisfying $H_b^2(G, \ell^2(G)) \neq 0$, which can be seen as a cohomological analogue of the property of having negative curvature. This includes groups admitting a non-elementary proper isometric action on some Gromov-hyperbolic graph of bounded valency, groups admitting a non-elementary proper isometric action on some proper CAT(-1) space, and groups admitting a non-elementary simplicial action on some simplicial tree.

The closely related class of groups $\mathcal{D}_{\mathrm{reg}}$ was introduced in [35, Definition 2.6] as a variation on the class $\mathcal{C}_{\mathrm{reg}}$. It consists of those countable discrete groups with the property that there exists an unbounded quasi-cocycle from G to $\ell^2(G)$.

The classes $\mathcal{C}_{\mathrm{reg}}$ and $\mathcal{D}_{\mathrm{reg}}$ both properly contain the class of acylindrically hyperbolic groups introduced in [30]. This latter class includes all non-elementary hyperbolic and relatively hyperbolic groups, outer automorphism groups of free groups on two or more generators, all but finitely many mapping class groups of punctured closed surfaces and most 3-manifold groups. It was proved in [8, Theorem 2.32] that an acylindrically hyperbolic group is C*-simple if and only if its amenable radical is trivial.

By [35, Lemma 2.8], every group in $\mathcal{D}_{\mathrm{reg}}$ has either non-vanishing first ℓ^2-Betti number or non-trivial second bounded cohomology with coefficients in $\ell^2(G)$. Therefore, Theorem 3.3 implies the following generalization of [8, Theorem 2.32].

Corollary 3.4. *A group in $\mathcal{C}_{reg}$ or $\mathcal{D}_{reg}$ is C*-simple if and only if its amenable radical is trivial.*

Recall that a discrete group is said to be *strongly non-amenable* if it has positive first ℓ^2-Betti number.

Corollary 3.5. *A strongly non-amenable group is C*-simple if and only if its amenable radical is trivial.*

We note that by [16, Lemme V.3], if a group G contains an amenable 1-step s-normal subgroup, then the cost is zero for every probability measure preserving action of G.

3.2. **Linear groups.** A linear group is a subgroup of $GL_n(K)$ for some field K. A discrete linear group is a linear group endowed with the discrete topology.

Theorem 3.6. *Let G be a discrete linear group with trivial amenable radical. Then every normalish subgroup of G is non-amenable.*

Proof. Let G be a subgroup of $GL_n(K)$ with trivial amenable radical, for some algebraically closed field K. For the sake of contradiction, suppose that $H < G$ is an amenable normalish subgroup.

For a finite subset $F \subset G$, let $H_F = \cap_{t \in F} tHt^{-1}$. Since H is normalish, H_F is non-trivial. Let L_F denote the Zariski closure of H_F, which is an algebraic subgroup of $GL_n(K)$.

Applying the descending chain condition for varieties, the intersection $L = \cap_F L_F$ over finite subsets $F \subset G$ is actually a finite intersection, and hence $L = L_{F_0}$ for some finite subset $F_0 \subset G$.

Observe that for $t \in G$, $tL_{F_0}t^{-1} = L_{tF_0}$. But by construction, $L_{F_0} \subset L_{tF_0}$. Thus, $L \subset tLt^{-1}$, and it follows that L is normalized by G.

Note that H_{F_0} is an amenable normalish subgroup of G. By replacing H with H_{F_0}, we can assume that H_F is Zariski dense in L for every finite subset $F \subset G$. Moreover, since $L \cap G$ is a normal subgroup of G, it must have trivial amenable radical. It follows that we can replace G with $L \cap G$. We can therefore assume that G is a subgroup of the algebraic group L and contains an amenable normalish subgroup H such that H_F is Zariski-dense in L for every finite subset $F \subset G$.

Changing G into $G \cap L^0$, where L^0 is the connected component of the identity in L, we may further assume that L is connected as an algebraic group. Finally L is not solvable, for otherwise G would be amenable. Hence L admits a center-free simple quotient. Projecting G to that quotient, we see that we may assume additionally that L is a connected center-free simple algebraic K-group.

Recall that according to the Tits alternative every amenable finitely generated linear group is virtually solvable. In characteristic zero this remains true without the finite generation assumption. So first suppose that K has characteristic zero. Then we may conclude that H is virtually solvable. Hence L is also virtually solvable, contradicting our assumption that L is simple.

Now suppose that K has characteristic $p > 0$. We first claim that H is locally finite. To see this, we require the following definition. For an arbitrary subgroup Γ of $GL_n(K)$, let $\mathrm{core}(\Gamma)$ denote the inductive limit

$$\mathrm{core}(\Gamma) = \varinjlim \mathrm{cls}(\Lambda)^0,$$

as Λ ranges over all finitely generated subgroups of Γ, where $\mathrm{cls}(\Lambda)^0$ denotes the connected component of the identify of the Zariski closure of Λ. Then $\mathrm{core}(\Gamma)$ is a connected algebraic subgroup of $GL_n(K)$ that is normalized by Γ and $\mathrm{core}(\Gamma) \cap \Gamma$ is Zariski-dense in $\mathrm{core}(\Gamma)$. Moreover, Γ is locally finite if and only if $\mathrm{core}(\Gamma)$ is trivial.

Now $\mathrm{core}(H)$ is a connected algebraic subgroup that is normalized by H, hence by L. According to the Tits alternative every finitely generated subgroup of H is virtually solvable. Hence its Zariski closure has a solvable connected component of the identity. It follows that $\mathrm{core}(H)$ is solvable. Since L is simple, we conclude that $\mathrm{core}(H)$ is trivial, and hence that H is locally finite.

Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of the prime field $\mathbb{F}_p$ in K. Let $GL_n(K)$ act on $M_n(K)$ by left multiplication. We claim that there is a G-invariant K-vector subspace W in $M_n(K)$, which admits a $\overline{\mathbb{F}_p}$-structure, i.e. an $\overline{\mathbb{F}_p}$-vector subspace W_0 such that $W = W_0 \otimes_{\overline{\mathbb{F}_p}} K$, such that H preserves W_0. Since W is G-invariant, it is also L-invariant, and making L act on W, we obtain an embedding of L as an algebraic subgroup of $GL_m(K)$, $m = \dim W$. Under this embedding $G \leq GL_m(K)$ and $H \leq GL_m(\overline{\mathbb{F}_p})$ are Zariski-dense in L. Hence, up to changing n into m, and modulo the claim, we may assume that $H \leq GL_n(\overline{\mathbb{F}_p})$.

Now it is easy to reach a contradiction: for every $t \in G$, $H \cap tHt^{-1}$ is Zariski-dense in L. In particular, since L is center-free, the intersection of the centralizers $C_L(h) := \{x \in L; xh = hx\}$ for h varying in $H \cap tHt^{-1}$ is trivial. And by the descending chain condition, there is a finite set of h's, say $h_1, \ldots, h_k$ in $H \cap tHt^{-1}$ such that $\cap_1^k C_L(h_i)$ is trivial. By assumption $t^{-1}h_i t \in GL_n(\overline{\mathbb{F}_p})$ for each $i = 1, \ldots, k$. Thus there is a Galois automorphism σ of K, a power of the Frobenius map $x \mapsto x^p$, such that $\sigma(t^{-1}h_i t) = t^{-1}h_i t$ for each $i = 1, \ldots, k$. This implies that $\sigma(t)t^{-1}$ commutes with each h_i, and hence is trivial. This means that $\sigma(t) = t$, i.e. that $t \in GL_n(\overline{\mathbb{F}_p})$. Hence $G \leq GL_n(\overline{\mathbb{F}_p})$, and G is locally finite, hence amenable, which is a contradiction.

It remains to verify the claim. Note that since every element h of H has finite order, its eigenvalues belong to $\overline{\mathbb{F}_p}$. Now consider the trace $tr(xy)$ on $M_n(K)$ as a non-degenerate bilinear form. Let W be the K-vector subspace of $M_n(K)$ generated by all matrices $h \in H$. Pick a basis $h_1, \ldots, h_m \in H$ of W. Then the linear map sending $w \in W$ to the m-tuple $(tr(wh_1), \ldots, tr(wh_m))$ is a K-linear isomorphism, which sends the $\overline{\mathbb{F}_p}$-linear span of the h_i's to $(\overline{\mathbb{F}_p})^m$. This gives the desired $\overline{\mathbb{F}_p}$-structure on W. Since H is Zariski-dense in G and L, it follows that W is fixed by L, proving the claim. This ends the proof. ∎

In an important paper [6], Bekka, Cowling and de la Harpe proved that lattices in semi-simple real Lie groups with trivial center are C*-simple. This was vastly extended by Poznansky [34] to all linear groups in the form below. Combining Theorem 3.6 with Theorem 2.12, we thus obtain a new proof of this result.

Corollary 3.7. *A discrete linear group is C*-simple if and only if its amenable radical is trivial.*

3.3. **Groups with few amenable subgroups.**

Theorem 3.8. *A discrete group with only countably many amenable subgroups is C*-simple if and only if its amenable radical is trivial.*

Proof. Let G be a discrete group with only countably many amenable subgroups. If G is C*-simple, then the amenable radical of G is necessarily trivial (see e.g. [10]).

Conversely, suppose the amenable radical of G is trivial. Consider the universal G-boundary $\partial_F G$. Suppose for the sake of contradiction that G is not C*-simple. Then by [22, Theorem 6.2], the G-action on $\partial_F G$ is not topologically free. Let $s \in G$ be an element such that the set of s-fixed points $(\partial_F G)_s$ is non-empty. By Lemma 2.8, $(\partial_F G)_s$ is clopen.

For $x \in (\partial_F G)_s$, let $F_x = \cap_{t \in G_x} (\partial_F G)_t$ denote the set of points fixed by every element in the stabilizer G_x. Note that F_x is closed and contains x. By Lemma 2.3, the stabilizer G_x is amenable. Therefore, by the assumption that G contains only countably many amenable subgroups, there is a countable sequence $(x_k) \in (\partial_F G)_s$ such that $\cup_{x \in (\partial_F G)_s} F_x = \cup_k F_{x_k}$. By the Baire category theorem, it follows that there is $y \in (\partial_F G)_s$ such that F_y has non-empty interior, say U.

The elements in G_y fix every point in U. By compactness and minimality, there are $s_1, \ldots, s_n \in G$ such that $s_1 U, \ldots, s_n U$ cover $\partial_F G$. Notice that the elements in $s_k G_y s_k^{-1}$ fix every point in $s_k U$. Therefore, the elements in the intersection $\cap_k s_k G_y s_k^{-1}$ fix every point in $\partial_F G$. Hence by [14, Proposition 7], every element in this intersection belongs to the amenable radical of G. By Lemma 2.10, this intersection is non-trivial, which contradicts the fact that the amenable radical of G is trivial. It follows that G is C*-simple. ∎

For a prime number p, a Tarski monster group of order p is an infinite group with the property that every non-trivial subgroup is cyclic of order p. Tarski monster groups of order p were first constructed in [27] for $p > 10^{75}$ as a counterexample to von Neumann's conjecture about the amenability of groups which do not contain free subgroups. In [28], torsion-free Tarski monster groups were constructed. These are infinite groups with the property that every non-trivial subgroup is cyclic of infinite order.

It was shown in [22, Corollary 6.6], using an ad hoc argument, that Tarski monster groups are C*-simple. However, since both Tarski monster groups and torsion-free Tarski monster groups are finitely generated and have trivial amenable radicals, they satisfy the hypotheses of Theorem 3.8. Thus we obtain the following generalization of [22, Corollary 6.6].

Corollary 3.9. *Tarski monster groups and torsion-free Tarski monster groups are C*-simple.*

The free Burnside group of rank m and exponent n, written $B(m,n)$ is an infinite group that is, in a certain specific sense, the "largest" group with m generators such that every element in the group has order n. It was recently shown in [29] that $B(m,n)$ is C*-simple for $m \geq 2$ and n odd and sufficiently large.

It was shown in [21] that for n sufficiently large, every non-cyclic subgroup of the free Burnside group $B(m,n)$ contains a subgroup isomorphic to $B(2,n)$. By [2], if n is odd, then $B(2,n)$ is non-amenable. Thus for $m \geq 2$ and n odd and sufficiently large, $B(m,n)$ satisfies the hypotheses of Theorem 3.8, and we recover the following result from [29].

Corollary 3.10. *For $m \geq 2$ and n odd and sufficiently large, the free Burnside group $B(m,n)$ is C*-simple.*

3.4. **Normal subgroups.** Let G be a group. Recall that for a subgroup $H < G$, the centralizer of H is $C_G(H) = \{s \in G \mid st = ts \ \forall t \in H\}$.

Lemma 3.11. *Let N be a discrete group, let X be an N-boundary, and let $U \subset X$ be a non-empty open subset. Then the set $\{t \in N \mid tU \cap U \neq \emptyset\}$ generates N.*

Proof. Let $H < N$ denote the subgroup generated by $\{t \in N \mid tU \cap U \neq \emptyset\}$. Then HU is a non-empty open subset of X such that $tHU \cap HU = \emptyset$ for all $t \in N \setminus H$. By minimality and compactness, $(tHU)_{tH \in N/H}$ is necessarily a finite partition of X. Note in particular that N/H is finite. The corresponding equivalence relation on X induces a continuous G-map from X to N/H. Since X is proximal, N/H is proximal. Being finite, it follows that N/H is a singleton, and hence that $H = N$. ∎

Lemma 3.12. *Let G be a discrete group and let $N < G$ be a normal subgroup with universal N-boundary $\partial_F N$. The N-action on $\partial_F N$ extends to a G-boundary action on $\partial_F N$.*

Proof. For $s \in G$, define $\sigma_s \in \mathrm{Aut}(N)$ by $\sigma_s(t) = sts^{-1}$ for $t \in N$. Mapping N into $Aut(N)$ via σ, it follows from [18, Proposition II.4.3] that the N-action on $\partial_F N$ has an extension to an action of $\mathrm{Aut}(N)$ on $\partial_F N$. Composing this action with the map from G into $\mathrm{Aut}(N)$ gives a G-action on $\partial_F N$ that extends the N-action. Since $\partial_F N$ is an N-boundary, it is clear that this is a G-boundary action ∎

Lemma 3.13. *Let G be a discrete group and let $N < G$ be a C*-simple normal subgroup with universal N-boundary $\partial_F N$. Then the action of $s \in G$ on $\partial_F N$ is either trivial or free, where the G-action on $\partial_F N$ is defined as in Lemma 3.12. The former possibility occurs if and only if $s \in C_G(N)$.*

Proof. If $s \in G$ belongs to $C_G(N)$, then the automorphism $\sigma_s \in \mathrm{Aut}(N)$ defined as in the proof of Lemma 3.12 is trivial, and it follows from the construction of the G-action on $\partial_F N$ that s acts trivially on $\partial_F N$.

For the converse, fix $s \in G$ such that the set of s-fixed points $(\partial_F N)_s$ is non-empty. For every $t \in N$ such that $t(\partial_F N)_s \cap (\partial_F N)_s \neq \emptyset$, the actions of the elements sts^{-1} and t coincide on $(\partial_F N)_s \cap t^{-1}(\partial_F N)_s$. However, since N is C*-simple, [22, Theorem 6.2] and Proposition 2.9 imply that N acts freely on $\partial_F N$. Thus $sts^{-1} = t$. By Lemma 3.11, the elements with this property generate N. Hence $s \in C_G(N)$. ∎

The next result provides a negative answer to [11, Question (Q)].

Theorem 3.14. *Let G be a discrete group and let $N < G$ be a normal subgroup. Then G is C*-simple if and only if both N and $C_G(N)$ are C*-simple. In particular, C*-simplicity is closed under extension.*

Proof. For brevity, let $K = C_G(N)$ and let $L = NK$. Note that K and L are normal in G.

Consider the following three G-boundary actions. First, by Lemma 3.12, the N-action on $\partial_F N$ extends to a G-boundary action on $\partial_F N$. Similarly, the K-action on $\partial_F K$ extends to a G-action on $\partial_F K$. Finally, composing the quotient map from G to G/L with the G/L-action on $\partial_F(G/L)$ gives a G-boundary action on $\partial_F(G/L)$.

By the construction of these G-actions, N acts trivially on $\partial_F K$ and $\partial_F(G/L)$, but minimally on $\partial_F N$. On the other hand, K acts trivially on $\partial_F N$ and $\partial_F(G/L)$, but minimally on $\partial_F K$. Since G acts minimally on $\partial_F(G/L)$, it is not difficult to see that the diagonal G-action on

$$X := \partial_F N \times \partial_F K \times \partial_F(G/L)$$

is a boundary action.

First suppose that both N and K are C*-simple. Then by [22, Theorem 6.2] and Proposition 2.9, the N-action on $\partial_F N$ and the K-action on $\partial_F K$ are free. If $s \in G$ does not act freely on X, then in particular s does not act freely on either $\partial_F N$ or $\partial_F K$. Hence by Lemma 3.13, $s \in K$. Then since K acts freely on $\partial_F K$, it follows that $s = e$. Therefore, G acts freely on X, and hence G is C*-simple by [22, Theorem 6.2].

Conversely, suppose that G is C*-simple. For $(x, y, z) \in X$, consider the stabilizer $G_{(x,y,z)}$. Lemma 2.3 implies that $G_{(x,y,z)} \cap L = N_x K_y$ and $G_{(x,y,z)}/(G_{(x,y,z)} \cap L) \subset (G/L)_z$ are both amenable. Hence $G_{(x,y,z)}$ is amenable. Therefore, by Proposition 2.6, the G-action on X is topologically free. It follows that the N-action on $\partial_F N$ and the K-action on $\partial_F K$ are also topologically free. Hence by [22, Theorem 6.2], N and K are C*-simple. ∎

Remark 3.15. We note that the analogue of Theorem 3.14 for the property of having trivial amenable radical is proved in [36, Lemma B.6].

3.5. **Stabilizer subgroups.** Let G be a discrete group and let X be a G-boundary. In this section, we relate the C*-simplicity of G to the C*-simplicity of a stabilizer subgroup G_x for $x \in X$.

Let $H < G$ be a subgroup and let E_H denote the canonical conditional expectation from $C^*_r(G)$ onto $C^*_r(H)$ defined by $E_H(\lambda_s) = \lambda_s$ for $s \in H$ and $E_H(\lambda_s) = 0$ otherwise. Thus the canonical tracial state τ_λ on $C^*_r(G)$ corresponds to $E_{\{e\}}$.

For every closed non-trivial ideal I of $\mathrm{C}^*_r(G)$, the subspace $E_H(I)$ is a (possibly non-closed) non-zero ideal of $\mathrm{C}^*_r(H)$. Indeed, $E_H(I)$ is an ideal since E_H is a $\mathrm{C}^*_r(H)$-bimodule map, and it is non-zero since $\tau_\lambda = \tau_\lambda \circ E_H$ is faithful on $\mathrm{C}^*_r(G)$.

For $x \in X$, the conditional expectation E_{G_x} extends to a conditional expectation E_x from $C(X) \rtimes_{\mathrm{r}} G$ onto $\mathrm{C}^*_r(G_x)$ satisfying $E_x(f\lambda_s) = f(x)E_{G_x}(\lambda_s)$ for $f \in C(X)$ and $s \in G$.

Proposition 3.16. *Let G be a discrete group and let X be a G-boundary. Assume that G is not C^*-simple and let I be a non-trivial closed ideal in $\mathrm{C}^*_r(G)$. Then for every $x \in X$, the closure of $E_{G_x}(I)$ is a non-trivial closed ideal of $\mathrm{C}^*_r(G_x)$, where E_{G_x} denotes the conditional expectation from $\mathrm{C}^*_r(G)$ onto $\mathrm{C}^*_r(G_x)$. In particular, G_x is not C^*-simple.*

Proof. Let $x \in X$ be given. It suffices to show $E_{G_x}(I)$ is not dense in $\mathrm{C}^*_r(G_x)$. By the proofs of (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) in [22, Theorem 6.2], the closed ideal J of $C(X) \rtimes_{\mathrm{r}} G$ generated by I is a proper ideal which has zero intersection with $C(X)$. As usual, we consider the state on $C(X) + J$ obtained by composing the map

$$C(X) + J \to (C(X) + J)/J \cong C(X)$$

with point evaluation at x. Let ϕ_x be a state extension on $C(X) \rtimes_{\mathrm{r}} G$.

Since $C(X)$ is contained in the multiplicative domain of ϕ_x, we have $\phi_x(f\lambda_s) = f(x)\phi_x(\lambda_s)$ for every $f \in C(X)$ and $s \in G$. We claim that $\phi_x(\lambda_s) = 0$ for every $s \in G \setminus G_x$. Indeed, there is $h \in C(X)$ such that $h(x) = 1$ and $\mathrm{supp}(h) \cap s\,\mathrm{supp}(h) = \emptyset$, and hence $\phi_x(\lambda_s) = \phi_x(h\lambda_s h) = \phi_x(h(sh)\lambda_s) = 0$. It follows that $\phi_x = \phi_x \circ E_x$. Since $\phi_x(I) = 0$, the ideal $E_{G_x}(I) = E_x(I)$ is not dense. ∎

3.6. **Thompson's groups.** Consider Thompson's groups F and T. Recall that F is a subgroup of T, and that T acts as a group of homeomorphisms on the interval $[0, 1]$ with the endpoints identified. In fact, F is the stabilizer subgroup T_0 of T corresponding to the point 0.

It is not difficult to see that this is a boundary action for T that is not topologically free. Hence Proposition 2.6 implies the following result from [19].

Theorem 3.17. *If Thompson's group T is C*-simple, then Thompson's group F is non-amenable.*

Proposition 3.16 implies the following result, which can be seen as a partial converse to Theorem 3.17.

Proposition 3.18. *If Thompson's group T is not C*-simple, then Thompson's group F is not C*-simple.*

We note that the recent paper [7] also considers the converse of Theorem 3.17.

3.7. **Amalgamated free products, Baumslag-Solitar groups.** In [11, Theorem 2] de la Harpe and Préaux consider the C*-simplicity of certain amalgamated free products and HNN extensions including the Baumslag-Solitar groups $BS(n,m) = \langle a,t \mid t^{-1}a^m t = a^n\rangle$ with $|n| \neq |m|$ and $|n|,|m| \geq 2$. Their analysis implies that the action of these groups on the boundary of their Bass-Serre tree is a topologically free boundary action. Thus C*-simplicity follows from [22, Theorem 6.2]. The topological freeness of these actions can also be shown by use of Lemma 2.10. We note that the cyclic subgroup $\langle a\rangle < BS(n,m)$ is amenable and normalish.

3.8. **Simplicity of reduced crossed products.** Let A be a *G-C*-algebra*, i.e. a C*-algebra equipped with a G-action. Then the reduced crossed product $A \rtimes_{\mathrm{r}} G$ is also a G-C*-algebra with respect to the conjugation G-action. There is a canonical conditional expectation E from $A \rtimes_{\mathrm{r}} G$ onto A defined by $E(a\lambda_s) = \delta_{s,1}a$ for $a \in A$ and $s \in G$.

The following theorem generalizes [12, Theorem I].

Theorem 3.19. *Let G be a discrete C*-simple group. Then for any unital G-C*-algebra A having no non-trivial G-invariant closed ideal, the reduced crossed product $A \rtimes_{\mathrm{r}} G$ is simple.*

The proof of Theorem 3.19 is divided into several steps.

Lemma 3.20. *Let A be a unital G-C*-algebra and let X be a G-boundary. Then for any non-trivial closed ideal I of $A \rtimes_{\mathrm{r}} G$, the ideal J of $(A \otimes C(X)) \rtimes_{\mathrm{r}} G$ generated by I is non-trivial.*

Proof. We proceed as in the proof of (2) $\Rightarrow$ (1) in [22, Theorem 6.2]. Let $\pi : A \rtimes_{\mathrm{r}} G \to \mathcal{B}(H)$ be a *-representation such that $\ker\pi = I$. We extend π to a unital completely positive map $\bar\pi$ from $(A \otimes C(X)) \rtimes_{\mathrm{r}} G$ into $\mathcal{B}(H)$.

Note that $A \rtimes_{\mathrm{r}} G$ is contained in the multiplicative domain of $\bar\pi$, that is $\bar\pi(afb) = \pi(a)\bar\pi(f)\pi(b)$ for every $a, b \in A \rtimes_{\mathrm{r}} G$ and $f \in C(X)$. In particular, $\pi(A)$ and $\bar\pi(C(X))$ commute.

Since X is a G-boundary, it is easy to check that $\bar\pi$ is completely isometric on $C(X)$. Hence there is a *-homomorphism Q from $\mathrm{C}^*(\bar\pi(C(X)))$ onto $C(X)$ such that $Q \circ \bar\pi = \mathrm{id}_{C(X)}$. The C*-algebra $\mathrm{C}^*(\bar\pi(C(X)))$ is a G-C*-algebra with the conjugation G-action through π, and the ideal $K = \ker Q$ is G-invariant.

We consider

$$D = \mathrm{C}^*\Big(\bar\pi\big((A \otimes C(X)) \rtimes_{\mathrm{r}} G\big)\Big) = \overline{\mathrm{C}^*(\bar\pi(C(X))) \cdot \pi(A \rtimes_{\mathrm{r}} G)}$$

and the ideal L of D defined by

$$L = \overline{K \cdot \pi(A \rtimes_{\mathrm{r}} G)}.$$

An element $d \in D$ belongs to L if and only if $e_i d \to d$ for an approximate unit (e_i) of K. This implies that $L \cap \mathrm{C}^*(\bar{\pi}(C(X))) = K$. Let Q now denote the quotient map from D onto D/L. Then $Q \circ \bar{\pi}$ is a *-homomorphism, since it a unital completely positive map which is multiplicative on $A \otimes C(X)$ and G-equivariant. The ideal $\ker(Q \circ \bar{\pi})$ is proper and contains I. ∎

Let B be a G-C*-algebra and let K be a G-invariant closed ideal of B. Then, letting $K \bar{\rtimes}_{\mathrm{r}} G$ denote the kernel of the map $B \rtimes_{\mathrm{r}} G \to (B/K) \rtimes_{\mathrm{r}} G$,

$$K \bar{\rtimes}_{\mathrm{r}} G = \{b \in B \rtimes_{\mathrm{r}} G : E(b\lambda_s^*) \in K \,\forall s \in G\},$$

and $K \bar{\rtimes}_{\mathrm{r}} G$ is a closed ideal in $B \rtimes_{\mathrm{r}} G$ which contains $K \rtimes_{\mathrm{r}} G$. (In fact, these two ideals coincide whenever G is exact.) The following lemma is inspired by [3].

Lemma 3.21. *Let G be discrete C*-simple group with universal G-boundary $\partial_F G$. Let A be a unital G-C*-algebra and let J be a closed ideal in $(A \otimes C(\partial_F G)) \rtimes_{\mathrm{r}} G$. Then setting $J_A = J \cap (A \otimes C(\partial_F G))$,*

$$J_A \rtimes_{\mathrm{r}} G \subset J \subset J_A \bar{\rtimes}_{\mathrm{r}} G.$$

Proof. For $x \in \partial_F G$, let $J_A^x = (\mathrm{id}_A \otimes \delta_x)(J_A)$, where $\mathrm{id}_A \otimes \delta_x$ is the homomorphism from $A \otimes C(\partial_F G)$ onto A given by evaluation at x. Then J_A^x is a (potentially non-proper) ideal of A.

Let π_x denote the induced homomorphism from $(A \otimes C(\partial_F G))/J_A$ onto A/J_A^x. Note that any irreducible representation of $(A \otimes C(\partial_F G))/J_A$ factors through some π_x, and hence the set $\{\pi_x \mid x \in \partial_F G\}$ is a faithful family of representations.

Let $x \in \partial_F G$ be such that $J_A^x \neq A$. Consider the composition of the map

$$\begin{aligned} J + A \otimes C(\partial_F G) &\to (J + A \otimes C(\partial_F G))/J \\ &\cong (A \otimes C(\partial_F G))/J_A \stackrel{\pi_x}{\to} A/J_A^x \end{aligned}$$

with any faithful representation of A/J_A^x into $\mathcal{B}(H)$. By Arveson's extension theorem, this extends to a unital completely positive map Φ_x from $(A \otimes C(\partial_F G)) \rtimes_{\mathrm{r}} G$ into $\mathcal{B}(H)$.

We claim that $\Phi_x = \Phi_x \circ E$, where E is the canonical conditional expectation onto $A \otimes C(\partial_F G)$. Since $A \otimes C(\partial_F G)$ is contained in the multiplicative domain of Φ_x and $\Phi_x(f) = f(x)$ for $f \in C(\partial_F G)$, it suffices to show that $\Phi_x(\lambda_s) = 0$ for every $s \in G \setminus \{e\}$. By [22, Theorem 6.2] and Proposition 2.9, G acts freely on $\partial_F G$. Hence there is $h \in$

$C(\partial_F G)$ such that $h(x) = 1$ and $\mathrm{supp}(h) \cap s\,\mathrm{supp}(h) = \emptyset$. Then $\Phi_x(\lambda_s) = \Phi_x(h\lambda_s h) = \Phi_x(h(sh)\lambda_s) = 0$, which proves the claim.

Thus, we see that $\Phi_x(E(J)) = \Phi_x(J) = 0$ for all $x \in \partial_F G$. Since $E(J) \subset A \otimes C(\partial_F G)$, one obtains $E(J) \subset J_A$, or equivalently that $J \subset J_A \bar{\rtimes}_{\mathrm{r}} G$. The other inclusion, $J_A \rtimes_{\mathrm{r}} G \subset J$ is obvious. ∎

Proof of Theorem 3.19. Let A be a unital G-C*-algebra and let I be a non-trivial closed ideal in $A \rtimes_{\mathrm{r}} G$. By Lemma 3.20, the ideal J of $(A \otimes C(\partial_F G)) \rtimes_{\mathrm{r}} G$ generated by I is non-trivial, and by Lemma 3.21, for $J_A = J \cap (A \otimes C(\partial_F G))$ we have $J \subset J_A \bar{\rtimes}_r G$. It follows that $I_A = J \cap A$ is a proper ideal such that $I \subset I_A \bar{\rtimes}_r G$. By the assumption that A has no non-trivial G-invariant closed ideal, it follows that $I_A = \{0\}$, and hence that $I = \{0\}$. ∎

Remark 3.22. Theorem 3.19 applies to the case when $A = C(X)$ is a minimal compact G-space. In particular this implies that Proposition 2.6 holds for minimal compact G-spaces that are not necessarily G-boundaries.

4. Uniqueness of the trace

We recall that a discrete group is said to have the *unique trace property* if its reduced C*-algebra has a unique tracial state.

Theorem 4.1. *Let G be a discrete group, and let $R_a(G)$ denote the amenable radical of G. For every tracial state τ on the reduced C*-algebra $\mathrm{C}^*_r(G)$, $\tau = \tau \circ E_{R_a(G)}$, where $E_{R_a(G)}$ denotes the canonical conditional expectation from $\mathrm{C}^*_r(G)$ onto $\mathrm{C}^*_r(R_a(G))$.*

Proof. Let $\tau : \mathrm{C}^*_r(G) \to \mathbb{C}$ be a tracial state. Note that τ is G-equivariant with respect to the conjugation G-action on $\mathrm{C}^*_r(G)$ and the trivial G-action on $\mathbb{C}$. Identify $\mathbb{C}$ with the scalar subalgebra of $C(\partial_F G)$, where $\partial_F G$ denotes the universal G-boundary. Then by the G-injectivity of $C(\partial_F G)$ (see [20]), we can extend τ to a G-equivariant completely positive map $\Phi : C(\partial_F G) \rtimes_{\mathrm{r}} G \to C(\partial_F G)$. By [22, Theorem 3.12], the restriction of Φ to $C(\partial_F G)$ is the identity map. In particular, this implies that Φ is a $C(\partial_F G)$-bimodule map.

Let $s \in G$ be an element that does not belong to the amenable radical of G. Then by [14, Proposition 7], s acts non-trivially on $\partial_F G$. Hence there is a clopen subset $U \subset \partial_F G$ such that $U \cap sU = \emptyset$. Let $1_U \in C(\partial_F G)$ denote the indicator function for U. Then

$$1_U \tau(\lambda_s) = 1_U \Phi(\lambda_s) 1_U = \Phi(1_U \lambda_s 1_U) = \Phi(1_U 1_{sU} \lambda_s) = 0,$$

since 1_U and 1_{sU} have disjoint support. Therefore, $\tau(\lambda_s) = 0$, and the result follows. ∎

Remark 4.2. Let G be a discrete group. A state on a G-C*-algebra A is said to be *G-central* if it is invariant under the G-action. It follows as in the proof of Theorem 4.1 that every G-central state ϕ on the reduced crossed product $A \rtimes_{\mathrm{r}} G$ satisfies $\phi = (\phi|_A) \circ E_{R_a(G)}$ for the canonical conditional expectation $E_{R_a(G)}$ from $A \rtimes_{\mathrm{r}} G$ onto $A \rtimes_{\mathrm{r}} R_a(G)$.

Theorem 4.1 implies the following result.

Corollary 4.3. *A discrete group has the unique trace property if and only if its amenable radical is trivial. In particular, every C*-simple group has the unique trace property.*

Proof. Let G be a discrete group, and let $R_a(G)$ denote the amenable radical of G. If $R_a(G)$ is trivial, then the result follows immediately from Theorem 4.1.

Conversely, any trace τ_0 on the reduced C*-algebra $\mathrm{C}^*_r(R_a(G))$ gives rise to a trace τ on the reduced C*-algebra $\mathrm{C}^*_r(G)$ via $\tau = \tau_0 \circ E_{R_a(G)}$, where $E_{R_a(G)}$ denotes the conditional expectation from $\mathrm{C}^*_r(G)$ onto $\mathrm{C}^*_r(R_a(G))$. If $R_a(G)$ is non-trivial, then because it is amenable, the unit character extends to a trace on $\mathrm{C}^*_r(R_a(G))$ which is non-canonical. ∎

Let G be a discrete group, and let $\mathcal{S}(G)$ denote the set of subgroups of G. The set $\mathcal{S}(G)$ is compact with respect to the Chabauty topology, which corresponds to the product topology on $\{0,1\}^G$, and $\mathcal{S}(G)$ forms a G-space with respect to the conjugation action of G.

An *invariant random subgroup* of G is a probability measure μ on $\mathcal{S}(G)$ which is invariant with respect to the adjoint of the conjugation action of G on $\mathcal{S}(G)$. Let $\mathcal{S}_a(G)$ denote the set of amenable subgroups of G. Then μ is said to be *amenable* if $\mathcal{S}_a(G)$ is μ-measurable and $\mu(\mathcal{S}_a(G)) = 1$.

The notion of an invariant random subgroup was introduced in [1], and the problem was raised whether every amenable invariant random subgroup is concentrated on the amenable radical. This problem was recently solved affirmatively in [4]. Combining Theorem 4.1 and [36, Corollary 5.15], we obtain a different proof.

Corollary 4.4. *Every amenable invariant random subgroup on a discrete group is concentrated on the amenable radical.*

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