Lecture I: The Poisson Boundary.

① Harmonic ~~maps~~ functions. G lcsc. $\mu \in P(G)$. ~~$h \in L^\infty(G)$~~

μ assumed admissible: $\mu <$ haar. $\overline{SG(\mu)} = G$. ~~$\ell(G)$~~ ~~ise~~

$h \in L^\infty(G)$ is μ-harmonic if $\underline{h(g) = \int h(gg')\,d\mu(g')} = \mu * h$

example: harmonic on $\mathbb{Z}$ $\Leftrightarrow$ Arithemetics Progression.

bounded $\Leftrightarrow$ constant.

exercise: find a non-constant bounded harmonic function on F_2.

② Harmonic spaces (Stationary spaces).

X a (measurable) G-space. $G \times X \to X \rightsquigarrow P(G) \times P(X) \xrightarrow{a} P(X)$

$\mu \times \xi \mapsto \mu * \xi$

defined by: $\mu * \xi(f) = \mu \times \xi(f \circ a)$, $f \in B_b(X)$

ξ is μ-harmonic if $\mu * \xi = \xi$. $\left(\text{or if } G \times X \overset{s}{\underset{t}{\rightrightarrows}} X \quad s(\mu \times \xi) = t(\mu \times \xi).\right)$

(X, ξ) - μ-harmonic space.

example: ξ is G-inv. $\Rightarrow$ ξ is harmonic.

example: $G = PSL_2(\mathbb{R})$, $X = \mathbb{P}^1(\mathbb{R})$.

exercise: there is no inv. measure on X.

set: $K = SO(2)$, ξ, μ K-inv. $\Longrightarrow$ ξ is μ-stationary.

Proposition: X compact. $\exists \xi \in P(X)^{\mu}$.

Pf: $\xi' \in P(X)$. any limit point of $\frac{1}{N+1}\sum_{k=0}^{N} \mu^k \xi'$.

remark: Kakutani fixed point, or the amenability of $\mathbb{N}$:

for every compact convex set, $\mathbb{N}$ action. $Q^{\mathbb{N}} \neq \emptyset$.

③ <u>the Poisson transform:</u> (X,ζ) - harmonic. $P: L^\infty(X) \to H^\infty(G)$

$$Pf(g) = \int f \, dg\zeta = \int f(g^{-1}x)\, d\zeta(x).$$

$$\mu * Pf(g) = \int Pf(gg')\, d\mu(g') = \iint f(gg'x)\, d\mu(g')\, d\zeta(x) = \langle \mu * \zeta, f(g\cdot) \rangle = \langle \zeta, f(g\cdot) \rangle$$
$$= \langle g\zeta, f \rangle = Pf(g).$$

④ <u>the Poisson boundary</u>

<u>Thm</u> (Furstenberg): $\exists!$ harmonic space s.t P is an isometric isomorphism.

<u>Construction</u> (Kaimanovich-Vershik, Zimmer)

<u>Digression:</u> Given H, G semi-groups, G-space X, H-space Y and a cocycle: $c: H \times Y \to G$, $c(hh', y) = c(h, h'y)c(h', y)$

denote $Y \ltimes X$ the following -structure on $Y \times X$,

$h(y, x) = (c(h,x)y, hx)$

cocycle condition $\Rightarrow$ action

<u>special case:</u> $X = \Omega = G^{\mathbb{N}}$, $H = \mathbb{N} = \langle S \rangle$, $S(h_1, h_2, \ldots) = (h_2, h_3, \ldots)$

$Y = G$, $c(n, (h_i)) = h_1 \cdots h_n$

$Y \ltimes X = G \times (G \times G \times \ldots) = G \ltimes \Omega$

H-action, $S(g_0, g_1, \ldots g_n, \ldots) = (g_0 g_1, g_2, \ldots)$

left G-action.

measure: $\text{Haar} \times \mu^{\mathbb{N}}$.

<u>(Mackey) Point Realization Thm:</u> For every G-space X and a G-inv alg. of $L^\infty(X)$, $A < L^\infty(X)$
$\exists X \to Y$ s.t $L^\infty(Y) \simeq A < L^\infty(X)$.
G factor map.

Denote $B = \operatorname{spec}(L^\infty(G \ltimes \Omega)^{\mathbb{N}})$. this is a space with a measure class

observe that B is a G-space.

$$\pi : G \ltimes \Omega \to B$$

measure on B first aprox. define $\nu = \pi_*(\delta_e \times \mu^{\mathbb{N}})$.

Claim: ν is μ-harmonic.

$$\mu * \nu = \mu * \pi(\delta_e \times \mu^{\mathbb{N}}) = \pi(\mu * (\delta_e \times \mu^{\mathbb{N}})) = \pi(\mu \times \mu^{\mathbb{N}}) = \pi(\cancel{T_*}(\delta_e \times \mu^{\mathbb{N}}))$$
$$= \pi(\delta_e \times \mu^{\mathbb{N}}) = \nu.$$

problem: $\delta_e \times \mu^{\mathbb{N}} \not< \mathrm{Haar} \times \mu^{\mathbb{N}}$. take instead $\mu \times \mu^{\mathbb{N}} = T_*(\delta_e \times \mu^{\mathbb{N}})$

(5) Universality: We have $P : L^\infty(B) \to H^\infty(G)$, $Pf(g) = \langle f, g\nu \rangle$.

claim: P is invertible.

Pf: we construct $H^\infty(G) \to L^\infty(G \ltimes \Omega)^T$

$h \mapsto Qh(g_0, \dots g_n, \dots) = \lim h(g_0 g_1 \dots g_n)$

exist (a.e) by the MCT, $h(g_0 \dots g_n) = \int h(g_0 \dots g_{n+1}) d\mu(g_{n+1})$.

claim: $PQ = \cancel{Id}$, $QP = Id$.

$h \in H^\infty$, $\quad h(g_0) = \int h(g_0 \dots g_n) d\mu^n(g_1, \dots g_n) \to \lim \int = \int \lim = P\cancel{Q h(g_0)}\, h(g_0 \dots g_n)$

$\cancel{h} = \int Qh(g_0(e, g_1, g_2, \dots g_n \dots)) = PQh(g_0)$

$f \in L^\infty(G \ltimes \Omega)^S$, $\quad QPf(g_0, \dots g_n, \dots) = \lim Pf(g_0 \dots g_n) = \lim \int f(g_0 g_1 \cdots g_2 \cdots (g_k')) d(g')$

$= \lim \int \cancel{T}f(g_0 g_1, g_2, \dots g_n, g_0', g_1', g_2' \dots) d(g')$

$= \lim \int f(g_0, \dots g_n, g_0', \dots) d(g') = f(g_0, g_1, \dots)$

$\xrightarrow{L^2}$ hence a.e (cause a.e limit $\exists$).

exercises: P, Q are isometries

(hint: both $\|P\|, \|Q\| \le 1$).

(6) ~~Ergodicity~~ Second def of B (Furstenberg):

pull back the L^∞ product from B to H^∞:

$$\varphi \times \psi(g) = P(Q\varphi \cdot Q\psi) = \int \lim \varphi(g_0 \dots g_n) \lim \psi(g_0 \dots g_n) dg\nu = \int \lim \varphi \cdot \psi(g g_0 \dots g_n) d(g_0 \dots g_n)$$
$$= \lim \int \varphi(g g_0 \dots g_n) \psi(g g_0 \dots g_n) d\mu^{n+1}(g_0, \dots g_n)$$

$$B = \mathrm{Spec}(H^\infty(G), \times).$$

⑦ Ergodicity: B is ergodic.

pf 1: $L^\infty(B)^G \simeq H^\infty(G)^G = \mathbb{C}$

pf 2: $L^\infty(B)^G \simeq L^\infty(\Omega)^{T \times G} \simeq L^\infty((g_1, g_2, \dots))^S = \mathbb{C}$

⑧ Double ergodicity $\quad g \xrightarrow{-1} G,\ \mu \mapsto \check{\mu} \rightsquigarrow \check{B}$

$(G^{\mathbb{N}}, \check{\mu}^{\mathbb{N}}, s) \longleftrightarrow (G^{-\mathbb{N}}, \mu^{-\mathbb{N}}, \check{S}) \quad (\dots g_2, g_1, g_0) \xrightarrow{s} (\dots g_2, g_1 g_0)$

$(\dots g_2, g_1, g_0) \xmapsto{g} (\dots g_2, g_1, g_0 g^{-1})$

$G^{\mathbb{Z}}: (\dots \underline{g_{-2}, g_{-1}, g_0}, \underline{g_1, g_2, \dots}) \xrightarrow{\phi} \check{B} \times B$

$g_0 S((g)) = g_0(\dots \underline{g_{-2}, g_{-1}, g_0}, \underline{g_1, g_2} \dots) = (\dots \underline{g_{-2}, g_{-1}, 1}, \underline{g_0, g_1, g_2} \dots)$

$\Rightarrow \quad \phi(g_0 S((g))) = \phi((g))$

$\Rightarrow L^\infty(\check{B} \times B)^G \hookrightarrow L^\infty(G^{\mathbb{Z}})^{G,S} \subset L^\infty(G^{\mathbb{Z}})^S = \mathbb{C}$ Remark: Ergodicity w/ coef.

corrections: Bernoulli, $Aff(\mathbb{R}) = (\mathbb{C} \ltimes \mathbb{C}) \curvearrowright \mathbb{R}$ non ergodic on $\mathbb{R}^2$.

① ⑨ Amenability: Let Q be compact and convex. G-space.

fix $q \in Q$, set $\phi: \Omega \to Q$, $\phi(g_0, g_1, \dots) = g_0 q$.

$\Rightarrow \mathrm{Map}_G(\Omega, Q) \neq \phi$ compact convex

$\Rightarrow \mathrm{Map}_G(B, Q) = \mathrm{Map}_G(\Omega /\!/ S, Q) = \mathrm{Map}_G(\Omega, Q)^S \neq \phi.$

Def: G act on C (Lebesgue space) Amenably if $\forall Q$, $\mathrm{Map}_G(C, Q) \neq \phi$.

Cor: The action on B is amenable.

Cor: G not amenable $\Rightarrow$ B not trivial

Fact: converse hold for some measure (Rosenblat, Kai-Ver)

(2) (10) <u>The Boundary Map</u>.

<u>barycenter</u>: every measure on Q gives a barycenter $\mathrm{bar}(\nu) \in Q$, defined by $\phi \in \mathrm{Aff}(Q)$,

$\phi(\mathrm{bar}\,\nu) = \int d\nu\, \phi(q)$.

<u>exercise</u>: ν ~~Harmonic~~ harmonic $\Rightarrow$ bar ν harmonic

(~~barycent~~ (bar is e.v, bar $\mu * \nu = \mu * \mathrm{bar}\,\nu$)).

<u>Cor</u>: for every map $B \xrightarrow{\pi} Q$ we get bar $\pi \in Q$ harmonic.

<u>Converse</u>: $q \in Q$ harmonic. $\Rightarrow \forall \phi \in \mathrm{Aff}(Q)$, $\phi(gq)$ harmonic

$\Rightarrow \forall$ a.e $(g_n) \in \Omega$, $\phi(g_n q)$ converges

$\rightsquigarrow$ $B \to Q$ with bar $\nu = q$.

<u>Cheating</u>: take a dense set of ϕ in order to set $B' < B$.

2 (11) <u>Equivalence</u>: $\mathrm{Map}_G(B, Q) \simeq Q^{\mu}$

In particular, X compact: harmonic measure $\Leftrightarrow$ boundary maps.

existence of harmonic measure $\Rightarrow$ B is amenable.

(3) (12) <u>Uniqueness Properties - Sharpness</u>

$\xi \sim \xi'$ ergodic harmonic $\Rightarrow \xi = \xi'$

<u>Remark</u>: Must assume $\xi \sim \xi'$, otherwise may take $X = X_1 \cup X_2$

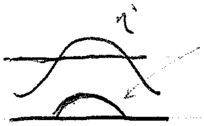
" " ergodic, " " $\xi' = \xi_2$.

Lemma: $\eta' < \eta \Rightarrow f = \frac{d\eta'}{d\eta}$ is invariant.

Lemma': $\mathrm{Meas}(X)^{\mu}$ is a sublattice of $\mathrm{Meas}(X)$

Remark: $\alpha, \beta \in \mathrm{Meas}(X) \Rightarrow \exists\, \alpha \vee \beta, \alpha \wedge \beta$, $\alpha \vee \beta = \max\left(\frac{d\alpha}{d\gamma}, \frac{d\beta}{d\gamma}\right) \cdot \gamma$, $\gamma = \alpha + \beta$.

idea of pf of Lemma: μ harmonic $\Rightarrow$ $\mathrm{supp}(\mu)$ invariant.
Show that level sets of f are inv. by constructing new harmonic measures with that supp:

$a\eta$  η' $\eta' - \eta' \wedge a\eta$. $\{f > a\} = \mathrm{supp}(\eta' - \eta' \wedge a\eta)$

pf of lemma': $\eta_1 > \eta_1 \wedge \eta_2$, $\eta_2 > \eta_1 \wedge \eta_2 \Rightarrow \eta_i = \mu\eta_i \geq \mu(\eta_1 \wedge \eta_2) \Rightarrow \eta_1 \wedge \eta_2 \geq \mu(\eta_1 \wedge \eta_2)$

but $\mu \in P(G) \Rightarrow \eta_1 \wedge \eta_2(x) = \mu(\eta_1 \wedge \eta_2)(x) \Rightarrow \eta_1 \wedge \eta_2 = \mu(\eta_1 \wedge \eta_2)$.

Harvest

(4) Cor: $\mathrm{Aut}_G(B) = 1$, B has no automorphism as a Lebesgue space

pf: auto as a Lebegue space $\overset{\text{sharp}}{=}$ auto as harmonic space. $B \xrightarrow{\delta} P(B)$
correspond to $\nu \Rightarrow$ equal.

Thm: B is sharply amenable (has no autos) and doubly ergodi with coef (if $\mu = \check{\mu}$).

Cor: $\exists$ such a Space!

(5) Functoriality: $(G_1, \mu_1) \longrightarrow (G_2, \mu_2) \Rightarrow$
$$\begin{tikzcd} \Omega_1 \arrow[r] \arrow[d] & \Omega_2 \arrow[d] \\ \Omega_1/S_1 \arrow[r, dashed] & \Omega_2/S_2 \end{tikzcd} \Rightarrow B \to B_2$$

in fact $H^\infty(G_2) = H^\infty(G_1)^N \Rightarrow B_2 = B_1/N$.

Cor: The center of G acts trivially on $\mathcal{B}$.

$\Rightarrow$ $\mathcal{B} = \mathcal{B}/\!/Z = \mathcal{B}(G/Z)$ if $Z < G$ central.

(6) examples: $- \; G = \mathbb{Z} \Rightarrow \mathcal{B} = *$

$- \; G$ nilpotent $\Rightarrow \mathcal{B} = *$.

(7) $G = SL_2(\mathbb{R})$: $K = SO(2)$. $m_K =$ haar. $m_K^2 = m_K$ $\mu \in P(G)$, $m_K \mu = \mu$. $\Rightarrow \mu$ K-inv.

Claim: $h \in H^\infty(G) \Rightarrow h(gk) = h(g) \Rightarrow h \in L^\infty(G/K) = L^\infty(\mathbb{H}^2)$.

$h(gk) = \int h(gkg')\,d\mu = \int h(gg')\,d\mu = h(g)$.

Claim: K acts ergodically on $\mathcal{B}$.

$H^\infty(G)^K =$ constant $\Leftrightarrow$ K left inv. harmonic functions are constant.
$\Leftrightarrow$ functions on $A = \ominus$.

Cor: $\mathcal{B} = \mathbb{P}^1(\mathbb{R}) = G/P$.
K ergodic $\Rightarrow$ K transitive. $\Rightarrow$ G/P or $*$
$\mathcal{B}$ amenable + G not $\Rightarrow$ G/P.

Thm: G semisimple $\Rightarrow$ $\mathcal{B} = G/P$.

(8) $G = G_1 \times G_2$, $\mu = \mu_1 \times \mu_2$ $\Rightarrow$ $\Omega = \Omega_1 \times \Omega_2$, $S = S_1 \times S_2$
$\Rightarrow \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$.

(9) The Weyl group: $\mu = \check{\mu}$. $W := \mathrm{Aut}_G(B \times B)$.

example: G amenable $\Rightarrow W = 1$

G not amenable $\Rightarrow \mathbb{Z}_2 < W$. w_0 = the long element

the flip.

$G = G_1 \times G_2 \Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 < W$

G semi-simple $\Rightarrow$ classical Weyl grp.

(10) The Weyl map. Fix Γ, μ, B, $U = W_\Gamma$.

G - alg. grp. $\rho : \Gamma \to G$ Zariski dense.

Claim: $\exists$ natural $W_\rho \longrightarrow W_G$.

(with further properties ...)

idea: X Γ ergodic, $\Gamma \to G$. $\Rightarrow \exists!\ \underbrace{H < G}_{\text{up to conj}}$, $\underbrace{X \to G/H}_{\text{up to } \mathrm{Aut}(G/H)}$ s.t $\forall V$,

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & V \\ & G/H \arrow[ur] & \end{tikzcd}$$

$$\Rightarrow \mathrm{Aut}^{Leb}_\Gamma(X) \hookrightarrow \mathrm{Aut}^{Alg}_G(G/H) = N(H)/H$$

(11) How to prove things? $\Gamma < SL_3(\mathbb{R})$, $G = SL_2(\mathbb{R})$

$\rightsquigarrow W_\rho \to W_G$ not apriori impossible, but $w_0 \overset{(321)}{\longmapsto} 1$

$W_\rho = S_3$, $W_G = S_2$

$B \times B \xrightarrow[\phi w_0]{\phi} P(SL_3(\mathbb{R})/Q) \Rightarrow$ constant $\Rightarrow \Gamma$ fixed point measure

What about $SL_4(\mathbb{R})$?

(12) One More idea: Galois Relation

$$\begin{tikzcd} & \mathrm{Flag} \arrow[dl] \arrow[dr] & \\ Gr(1,3) \arrow[dr] & & Gr(2,3) \arrow[dl] \\ & * & \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 1 \times S_2 \arrow[dr] & & S_2 \times 1 \arrow[dl] \\ & W & \end{tikzcd}$$

factors of $B \Leftrightarrow$ sub-grp of W.

$\boxed{\Gamma \text{ a lattice} \Rightarrow \text{the Weyl map is injective}}$