A TOPOLOGICAL TITS ALTERNATIVE

E. BREUILLARD AND T. GELANDER

Abstract. Let k be a local field, and $\Gamma \leq \mathrm{GL}_n(k)$ a linear group over k. We prove that either Γ contains a relatively open solvable subgroup, or it contains a relatively dense free subgroup. This result has applications in dynamics, Riemannian foliations and profinite groups.

1. Introduction

In his celebrated 1972 paper [33] J. Tits proved the following fundamental dichotomy for linear groups: *Any finitely generated*[1] *linear group contains either a solvable subgroup of finite index or a non-commutative free subgroup.* This result, known today as "the Tits alternative", answered a conjecture of Bass and Serre and was an important step toward the understanding of linear groups. The purpose of the present paper is to give a topological analog of this dichotomy and to provide various applications of it. Before stating our main result, let us reformulate Tits' alternative in a slightly stronger manner. Note that any linear group $\Gamma \leq \mathrm{GL}_n(K)$ has a Zariski topology, which is, by definition, the topology induced on Γ from the Zariski topology on $\mathrm{GL}_n(K)$.

Theorem 1.1 (Tits' alternative). *Let K be a field and Γ a finitely generated subgroup of $GL_n(K)$. Then either Γ contains a Zariski open solvable subgroup or Γ contains a Zariski dense free subgroup of finite rank.*

Remark 1.2. Theorem 1.1 seems quite close to the original theorem of Tits, stated above. And indeed, it is stated explicitly in [33] in the particular case when the Zariski closure of Γ is assumed to be a semisimple Zariski connected algebraic group. However, the proof of Theorem 1.1 relies on the methods developed in the present paper which allow one to deal with non Zariski connected groups. We will show below how Theorem 1.1 can be easily deduced from Theorem 1.3.

The main purpose of our work is to prove the analog of Theorem 1.1, when the ground field, and hence any linear group over it, carries a more interesting topology than the Zariski topology, namely for local fields.

Date: November 20, 2004.

[1] *In characteristic zero, one may drop the assumption that the group is finitely generated*

Assume that k is a local field, i.e. $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or a field of formal power series in one variable over a finite field. The full linear group $\mathrm{GL}_n(k)$ and hence any subgroup of it, is endowed with the standard topology, that is the topology induced from the local field k. We then prove the following:

Theorem 1.3 (Topological Tits alternative)**.** *Let k be a local field and Γ a subgroup of $GL_n(k)$. Then either Γ contains an open solvable subgroup or Γ contains a dense free subgroup.*

Note that Γ may contain both a dense free subgroup and an open solvable subgroup: in this case Γ has to be discrete and free. For non discrete groups however, the two cases are mutually exclusive.

In general, the dense free subgroup from Theorem 1.3 may have an infinite (but countable) number of free generators. However, in many cases we can find a dense free subgroup on finitely many free generators (see below Theorems 5.1 and 5.8). This is the case, for example, when Γ itself is finitely generated. For another example consider the group $\mathrm{SL}_n(\mathbb{Q})$, $n \geq 2$. It is not finitely generated, yet, we show that it contains a free subgroup of rank 2 which is dense with respect to the topology induced from $\mathrm{SL}_n(\mathbb{R})$. Similarly, for any prime $p \in \mathbb{N}$, we show that $\mathrm{SL}_n(\mathbb{Q})$ contains a free subgroup of finite rank $r = r(n,p) \geq 2$ which is dense with respect to the topology induced from $\mathrm{SL}_n(\mathbb{Q}_p)$.

When $char(k) = 0$, the linearity assumption can be replaced by the weaker assumption that Γ is contained in some second-countable k-analytic Lie group G. In particular, Theorem 1.3 applies to subgroups of any real Lie group with countably many connected components, and to subgroups of any group containing a p-adic analytic pro-p group as an open subgroup of countable index. In particular it has the following consequence:

Corollary 1.1. *Let k be a local field of characteristic 0 and let G be a k-analytic Lie group with no open solvable subgroup. Then G contains a dense free subgroup F. If additionnally G contains a dense subgroup generated by k elements, then F can be taken to be a free group of rank r for any $r \geq k$.*

Let us indicate how Theorem 1.3 implies Theorem 1.1. Let K be a field, $\Gamma \leq \mathrm{GL}_n(K)$ a finitely generated group, and let R be the ring generated by the entries of Γ. By the Noether normalization theorem R can be embedded in the valuation ring $\mathcal{O}$ of some local field k. Such an embedding induces an embedding i of Γ into the linear pro-finite group $\mathrm{GL}_n(\mathcal{O})$. Note also that the topology induced on Γ from the Zariski topology of $\mathrm{GL}_n(K)$ coincides with the one induced from the Zariski topology of $\mathrm{GL}_n(k)$ and this topology is weaker than the topology induced by the local field k. If $i(\Gamma)$ contains a relatively open solvable subgroup then so does its closure, and by compactness, it follows that Γ is virtually solvable, and hence its Zariski connected component is solvable and Zariski open. If $i(\Gamma)$ does not contain an open solvable subgroup then, by Theorem 1.3, it contains a dense free subgroup which, as stated in a paragraph above, we may assume has finite rank. This free subgroup is indeed Zariski dense, which yields Theorem 1.1.

The dichotomy established in Theorem 1.3 strongly depends on the choice of the topology (real, p-adic, or $\mathbb{F}_q((t))$-analytic) assigned to Γ and on the embedding of Γ in $\mathrm{GL}_n(k)$. It can be interesting to consider other topologies as well. However, the existence of a *dense* free subgroup, under the condition that Γ has no open solvable subgroup, is a rather strong property that cannot be generalized to arbitrary topologies on Γ (for example the profinite topology on a surface group, see §1.1 below). Nevertheless, making use of Montgomery-Zippin theory, we show that the following weaker dichotomy holds:

Theorem 1.1. *Let G be a locally compact group and Γ a finitely generated dense subgroup of G. Then one of the following holds:*

(i) Γ *contains a free group F_2 on two generators which is non-discrete in G.*

(ii) *G contains an open amenable subgroup.*

Moreover, if Γ is assumed to be linear and finitely generated, then (ii) can be replaced by "G contains an open solvable subgroup".

For the sake of simplicity, we restrict ourselves throughout this paper to a fixed local field. However, the proof of Theorem 1.3 applies also in the following more general setup:

Theorem 1.4. *Let $k_1, k_2, \ldots, k_r$ be local fields and let Γ be a (finitely generated) subgroup of $\prod_{i=1}^r GL_n(k_i)$. Assume that Γ does not contain an open solvable subgroup, then Γ contains a dense free subgroup (of finite rank).*

We also note that the argument of Section 6, where we build a dense free group on infinitely many generators, is applicable in a much greater generality. For example, we can prove the following adelic version:

Proposition 1.5. *Let K be an algebraic number field and $\mathbb{G}$ a simply connected semisimple algebraic group defined over K. Let V_K be the set of all valuations of K. Then for any $v_0 \in V_K$ such that $\mathbb{G}$ in not K_{v_0} anisotropic, $\mathbb{G}(K)$ contains a free subgroup of infinite rank whose image under the diagonal embedding is dense in the restricted topological product corresponding to $V_K \setminus \{v_0\}$.*

The first step toward Theorem 1.3 was carried out in our previous work [4]. In [4] we made the assumption that $k = \mathbb{R}$ and the closure of Γ is connected.

This considerably simplifies the situation, mainly because it implies that Γ is automatically Zariski connected. One achievement of the present work is the understanding of some dynamical properties of projective representations of non Zariski connected algebraic groups (see Section 4). Another new aspect is the study of representations of finitely generated integral domains into local fields (see Section 2) which allows us to avoid the rationality of the deformation space of Γ in $\mathrm{GL}_n(k)$, and hence to drop the assumption that Γ is finitely generated.

Theorem 1.3 has various applications. We shall now indicate some of them.

1.1. **Applications to the theory of pro-finite groups.** When k is non-Archimedean, Theorem 1.3 provides some new results about pro-finite groups (see Section 8). In particular, we answer a conjecture of Dixon, Pyber, Seress and Shalev (cf. [12] and [25]), by proving:

Theorem 1.6. *Let Γ be a finitely generated linear group over an arbitrary field. Suppose that Γ is not virtually solvable, then its pro-finite completion $\hat{\Gamma}$ contains a dense free subgroup of finite rank.*

In [12], using the classification of finite simple groups, the weaker statement, that $\hat{\Gamma}$ contains a free subgroup whose closure is of finite index, was established. Let us remark that the passage from a subgroup whose closure is of finite index, to a dense subgroup is also a crucial step in the proof of Theorem 1.3. It is exactly this problem that forces us to deal with representations of non Zariski connected algebraic groups. Additionally, our proof of 1.6 does not rely on [12], neither on the classification of finite simple groups.

We also note that Γ itself may not contain a pro-finitely dense free subgroup of finite rank. It was shown in [30] that surface groups have the property that any proper finitely generated subgroup is contained in a proper subgroup of finite index (see also [32]).

In Section 8 we also answer a conjecture of Shalev about coset identities in pro-p groups in the analytic case:

Proposition 1.7. *Let G be an analytic pro-p group. If G satisfies a coset identity with respect to some open subgroup, then G is solvable, and in particular, satisfies an identity.*

1.2. **Applications in dynamics.** The question of the existence of a free subgroup is closely related to questions concerning amenability. It follows from Tits' alternative that for a finitely generated linear group Γ, the following are equivalent:

- Γ is amenable,
- Γ is virtually solvable,
- Γ does not contain a non-abelian free subgroup.

The topology enters the game when considering actions of subgroups on the full group. Let k be a local field, $G \leq \mathrm{GL}_n(k)$ a closed subgroup and $\Gamma \leq G$ a countable subgroup. Let $P \leq G$ be any closed amenable subgroup, and consider the action of Γ on the homogeneous space G/P by left multiplications. Theorem 1.3 implies:

Theorem 1.8. *The following are equivalent:*
(I) The action of Γ on G/P is amenable,
(II) Γ contains an open solvable subgroup,
(III) Γ does not contain a non-discrete free subgroup.

The equivalence between (I) and (II) for the Archimedean case (i.e. $k = \mathbb{R}$) was conjectured by Connes and Sullivan and subsequently proved by Zimmer [35] by means of super-rigidity methods. The equivalence between (III) and (II) was asked by Carrière and Ghys [10] who showed that (I) implies (III) (see also Section 9). For the case $G = \mathrm{SL}_2(\mathbb{R})$ they actually proved that (III) implies (II) and hence concluded the validity of the Connes-Sullivan conjecture for this case (before Zimmer). We remark that the short argument given by Carrière and Ghys relies on the existence of an open subset of elliptic elements in $\mathrm{SL}_2(\mathbb{R})$ and hence does not apply to other real or p-adic Lie groups.

Remark 1.9. 1. When Γ is not both discrete and free, the conditions are also equivalent to: (III') Γ does not contain a dense free subgroup.
2. For k Archimedean, (II) is equivalent to: (II') The connected component of the closure $\overline{\Gamma}^{\circ}$ is solvable.
3. The implication $(II) \to (III)$ is trivial and $(II) \to (I)$ follows easily from the basic properties of amenable actions.

Using Montgomery-Zippin theory (see [22]), we also generalize the Connes-Sullivan conjecture (Zimmer's theorem) for arbitrary locally compact groups as follows (see Section 9):

Theorem 1.10. *(Generalized Connes-Sullivan conjecture) Let Γ be a countable subgroup of a locally compact topological group G. Then the action of Γ on G (as well as on G/P for $P \leq G$ closed amenable) by left multiplication is amenable, if and only if Γ contains a relatively open subgroup which is amenable as an abstract group.*

As a consequence of Theorem 1.10 we obtain the following generalization of Auslander's theorem (see [27] Theorem 8.24):

Theorem 1.11. *Let G be a locally compact topological group, let $P \leq G$ be a closed normal amenable subgroup, and let $\pi : G \to G/P$ be the canonical* *projection. Suppose that $H \leq G$ is a subgroup which contains a relatively open amenable subgroup. Then $\pi(H)$ also contains a relatively open amenable subgroup.*

Theorem 1.11 has many interesting conclusions. For example, it is well known that the original theorem of Auslander (Theorem 1.11 for real Lie groups) directly implies Bieberbach's classical theorem that any compact Euclidean manifold is finitely covered by a torus (part of Hilbert's 18th problem). As a consequence of the general case in Theorem 1.11 we obtain some information on the structure of lattices in general locally compact groups. If $G = G_c \times G_d$ is a direct product of a connected semisimple Lie group and a locally compact totally disconnected group. Then, it is easy to see that, the projection of any lattice in G to the connected factor lies between a lattice to its commensurator. Such information is useful since it says (as follows from Margulis' commensurator criterion for arithmeticity) that if this projection is not a lattice itself then it is a subgroup of the commensurator of some arithmetic lattice (which is, up to finite index, $G_c(\mathbb{Q})$). Theorem 1.11 implies that similar statement holds for general G (see Proposition 9.7).

1.3. **The growth of leaves in Riemannian foliations.** Y. Carrière's interest in the Connes-Sullivan conjecture stemmed from his study of the growth of leaves in Riemannian foliations. In [9] Carrière asked whether there is a dichotomy between polynomial and exponential growth. In order to study this problem, Carrière defined the notion of *local growth* for a subgroup of a Lie group (see Definition 10.3) and showed the equivalence of the growth type of a generic leaf and the local growth of the holonomy group of the foliation viewed as a subgroup of the corresponding structural Lie group associated to the Riemannian foliation (see [21]).

Tits' alternative implies, with some additional argument for solvable non-nilpotent groups, the dichotomy between polynomial and exponential growth for finitely generated linear groups. Similarly, Theorem 1.3, with some additional argument based on its proof for solvable non-nilpotent groups, implies the analogous dichotomy for the local growth:

Theorem 1.12. *Let Γ be a finitely generated dense subgroup of a connected real Lie group G. If G is nilpotent then Γ has polynomial local growth. If G is not nilpotent, then Γ has exponential local growth.*

As a consequence of Theorem 1.12 we obtain:

Theorem 1.13. *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. The leaves of $\mathcal{F}$ have polynomial growth if and only if the structural Lie algebra of $\mathcal{F}$ is nilpotent. Otherwise, generic leaves have exponential growth.*

The first half of Theorem 1.13 was actually proved by Carrière in [9]. Using Zimmer's proof of the Connes-Sullivan conjecture, he first reduced to the solvable case, then he proved the nilpotency of the structural Lie algebra of $\mathcal{F}$ by a delicate direct argument (see also [15]). He then asked whether the second half of this theorem holds. Both parts of Theorem 1.13 follow from Theorem 1.3 and the methods developed in its proof. We remark that although the content of Theorem 1.13 is about dense subgroups of connected Lie groups, its proof relies on methods developed in Section 2 of the current paper, and does not follow from our previous work [4].

If we consider instead the growth of the holonomy cover of each leaf, then the dichotomy shown in Theorem 1.13 holds for every leaf. On the other hand, it is easy to give an example of a Riemannian foliation on a compact manifold in which the growth of a generic leaf is exponential while some of the leaves are compact (see below Example 10.2).

1.4. **Outline of the paper.** The strategy used in this article to prove Theorem 1.3 consists in perturbing the generators γ_i of Γ within Γ and in the topology of $\mathrm{GL}_n(k)$, in order to obtain (under the assumption that Γ has no solvable open subgroup) free generators of a free subgroup which is still dense in Γ. As it turns out, there exists an identity neighborhood U of some non virtually solvable subgroup $\Delta \leq \Gamma$, such that any selection of points x_i in $U\gamma_iU$ generate a dense subgroup in Γ. The argument used here to prove this claim depends on whether k is Archimedean, p-adic or of positive characteristic.

In order to find a free group, we use a variation of the ping-pong method used by Tits, applied to a suitable action of Γ on some projective space over some local field f (which may or may not be isomorphic to k). As in [33] the ping-pong players are the so-called proximal elements (a proximal transformation is a transformation of $\mathbb{P}(f^n)$ which contracts almost all $\mathbb{P}(f^n)$ into a small ball). However, the original method of Tits (via the use of high powers of semisimple elements to produce ping-pong players) is not applicable to our situation and a more careful study of the contraction properties of projective transformations is necessary.

An important step lies in finding a projective representation ρ of Γ into $\mathrm{PGL}_n(f)$ such that the Zariski closure of $\rho(\Delta)$ acts strongly irreducibly (i.e. fixes no finite union of proper projective subspaces) and such that $\rho(U)$ contains very proximal elements. What makes this step much harder is the fact that Γ may not be Zariski connected. We handle this problem in Section 4. We would like to note that we gained motivation and inspiration from the beautiful work of Margulis and Soifer [20] where a similar difficulty arose.

We then make use of the ideas developed in [4] and inspired from [1], where it is shown how the dynamical properties of a projective transformation can be read off on its Cartan decomposition. This allows to produce a set of elements in U which "play ping-pong" on the projective space $\mathbb{P}(f^n)$, and hence generate a free group (see Theorem 4.3). Theorem 4.3 provides a very handy way to generate free subgroups, as soon as some infinite subset of matrices with entries in a given finitely generated ring (e.g. an infinite subset of a finitely generated linear group) is given.

The method used in [33] and in [4] to produce the representation ρ is based on finding a representation of a finitely generated subgroup of Γ into $\mathrm{GL}_n(K)$ for some algebraic number field, and then to replace the number field by a suitable completion of it. However, in [4] and [33], a lot of freedom was possible in the choice of K and in the choice of the representation into $\mathrm{GL}_n(K)$. What played the main role there was the appropriate choice of a completion. This approach is no longer applicable to the situation considered in this paper, and we are forced to choose both K and the representation of Γ in $\mathrm{GL}_n(K)$ in a more careful way. For this purpose, we prove a result (generalizing a lemma of Tits) asserting that in an arbitrary finitely generated integral domain, any infinite set can be sent to an unbounded set under an appropriate embedding of the ring into some local field (see Section 2). This result proves useful in many situations when one needs to find unbounded representations like in the Tits alternative, or in the Margulis super-rigidity theorem, or, as is illustrated below, for subgroups of SL_2 with property (T). It is crucial in particular when dealing with non finitely generated subgroups in Section 6. And it is also used in the proof of the growth of leaves dichotomy, in Section 10. Our proof makes use of a striking simple fact, originally due to Pólya in the case $k = \mathbb{C}$, about the inverse image of the unit disc under polynomial transformations (see Lemma 2.3).

Let us end this introduction by setting a few notations that will be used throughout the paper. The notation $H \leq G$ means that H is a subgroup of the group G. By $[G, G]$ we denote the derived group of G, i.e. the group generated by commutators. Given a group Γ, we denote by $d(\Gamma) \in \mathbb{N}$ the minimal size of a generating set of Γ. If $\Omega \subset G$ is a subset of G, then $\langle \Omega \rangle$ denotes the subgroup of G generated by Ω. If Γ is a subgroup of an algebraic group, we denote by $\overline{\Gamma}^z$ its Zariski closure. Note that the Zariski topology on rational points does not depend on the field of definition, that is if V is an algebraic variety defined over a field K and if L is any extension of K, then the K-Zariski topology on $V(K)$ coincides with the trace of the L-Zariski topology on it. To avoid confusion, we shall always add the prefix "Zariski" to any topological notion regarding the Zariski topology (e.g. "Zariski dense", "Zariski open"). For the topology inherited from the local field k, however, we shall plainly say "dense" or "open" without further notice (e.g. $\mathrm{SL}_n(\mathbb{Z})$ is open and Zariski dense in $\mathrm{SL}_n(\mathbb{Z}[1/p])$, where $k = \mathbb{Q}_p$).

2. A generalization of a lemma of Tits

In the original proof of the Tits alternative, Tits used an easy but crucial lemma saying that given a finitely generated field K and an element $\alpha \in K$ which is not a root of unity, there always is a local field k and an embedding $f : K \to k$ such that $|f(\alpha)| > 1$. A natural and useful generalization of this statement is the following lemma:

Lemma 2.1. *Let R be a finitely generated integral domain, and let $I \subset R$ be an infinite subset. Then there exists a local field k and an embedding $i : R \hookrightarrow k$ such that $i(I)$ is unbounded.*

As explained below, this lemma provides a straightforward way to build the proximal elements needed in the construction of dense free subgroups.

Before giving the proof of Lemma 2.1 let us point out a straightforward consequence:

Corollary 2.2 (Zimmer QCITEciteZim3, Theorems 6 and 7, and QCITE-citeHaV 6.26)**.** *There is no faithful conformal action of an infinite Kazhdan group on the Euclidean 2-sphere S^2.*

Proof. Suppose there is an infinite Kazhdan subgroup Γ in $\mathrm{PSL}_2(\mathbb{C})$, the group of conformal transformations of S^2. Since Γ has property (T), it is finitely generated, and hence, Lemma 2.1 could be applied to yield a faithful representation of Γ into $\mathrm{PSL}_2(k)$ for some local field k, with unbounded image. However $\mathrm{PSL}_2(k)$ acts faithfully with compact isotropy groups by isometries on the hyperbolic space $\mathbb{H}^3$ if k is Archimedean, and on a tree if it is not. As Γ has property-(T), it must fix a point (c.f. [16] 6.4 and 6.23 or [37] Prop. 18) and hence lie in some compact group. A contradiction. $\square$

When R is integral over $\mathbb{Z}$, the lemma follows easily by considering the diagonal embedding of R into a product of finitely many completions of its field of fractions. The main difficulty comes from the possible presence of transcendental elements. Our proof of Lemma 2.1 relies on the following interesting fact. Let k be a local field, and let $\mu = \mu_k$ denote the standard Haar measure on k, i.e. the Lebesgue measure if k is Archimedean, and the Haar measure giving measure 1 to the ring of integers $\mathcal{O}_k$ of k when k is non-Archimedean. Given a polynomial P in $k[X]$, let

$$A_P = \{x \in k, |P(x)| \leq 1\}.$$

Lemma 2.3. *For any local field k, there is a constant $c = c(k)$ such that $\mu(A_P) \leq c$ for any monic polynomial $P \in k[X]$.*

Proof. Let $\overline{k}$ be an algebraic closure of k, and P a monic polynomial in $k[X]$. We can write $P(X) = \prod (X - x_i)$ for some $x_i \in \overline{k}$. The absolute value of k

extends uniquely to an absolute value in $\overline{k}$ (see [18] XII, 4, Theorem 4.1 p. 482). Now if $x \in A_P$ then $|P(x)| \leq 1$, and hence

$$\sum \log |x - x_i| = \log |P(x)| \leq 0.$$

But A_P is measurable and bounded, therefore, integrating with respect to μ, we obtain

$$\sum \int_{A_P} \log |x - x_i| \, d\mu(x) = \int_{A_P} \sum \log |x - x_i| \, d\mu(x) \leq 0.$$

The lemma will now follow from the following **claim**: *for any measurable set $B \subset k$ and any point $z \in \overline{k}$,*

$$\int_B \log |x - z| \, d\mu(x) \geq \mu(B) - c, \tag{1}$$

where $c = c(k) > 0$ is some constant independent of z and B.

Indeed, let $\widetilde{z} \in k$ be such that $|\widetilde{z} - z| = \min_{x \in k} |x - z|$, then $|x - z| \geq |x - \widetilde{z}|$ for all $x \in k$, so

$$\int_B \log |x - z| \, d\mu(x) \geq \int_B \log |x - \widetilde{z}| \, d\mu(x) = \int_{B - \widetilde{z}} \log |x| \, d\mu(x).$$

Therefore, it suffices to show (1) when $z = 0$. But a direct computation for each possible field k shows that $-\int_{|x| \leq 1} \log |x| d\mu(x) < \infty$. Therefore taking $c = \mu \{x \in k, |x| \leq e\} + |\int_{|x| \leq 1} \log |x| d\mu(x)|$ we obtain (1). This concludes the proof of the lemma. $\square$

Lemma 2.3 was proved by Pólya in [24] for the case $k = \mathbb{C}$ by means of potential theory. Pólya's proof gives the best constant $c(\mathbb{C}) = \pi$. For $k = \mathbb{R}$ one can show that the best constant is $c(\mathbb{R}) = 4$ and that it can be realized as the limit of the sequence of lengths of the pre-image of $[-1, 1]$ by the Chebyshev polynomials (under an appropriate normalization of these polynomials). In the real case, this result admits generalizations to arbitrary smooth functions such as the Van der Corput lemma (see [8] for a multi-dimensional analog). For k non-Archimedean, the constant is always < 2 and it tends to 1 as the residue field $\mathcal{O}_k / \pi \mathcal{O}_k$ gets larger.

Let us just explain how, with a little more consideration, one can improve the constant c in the above proof[2]. We wish to find the minimal $c > 0$ such

[2]Let us also remark that there is a natural generalization of Lemma 2.3 to higher dimension which follows by an analogous argument: For any local field k and $n \in \mathbb{N}$, there is a constant $c(k, n)$, such that for any finite set $\{x_1, \ldots, x_m\} \in k^n$, we have $\mu\big(\{y \in k^n : \prod_1^m \|y - x_i\| \leq 1\}\big) \leq c(k, n)$.

that for every compact subset B of k whose measure is $\mu(B) \geq c$ we have

$$\int_B \log|x| d\mu(x) \geq 0.$$

Suppose $k = \mathbb{C}$. Since $\log|x|$ is increasing with $|x|$, for any B

$$\int_B \log|x| d\mu(x) \geq \int_C \log|x| d\mu(x)$$

where C is a ball around 0 ($C = \{x \in k : |x| \leq t\}$) with the same area as B. Therefore $c = \pi t^2$ where t is such that $2\pi \int_0^t r \log(r) dr = 0$. The unique positive root of this equation is $t = \sqrt{e}$. Thus we can take

$$c = \pi e.$$

For $k = \mathbb{R}$ the same argument gives a possible constant $c = 2e$, while for k non-Archimedean it gives $c = 1 + \frac{1}{f(q-1)}$ where $q = p^f$ is the size of the residue class field and f is its dimension over its prime field $\mathbb{F}_p$.

As in the proof of Lemma 2.3, there is a positive constant c_1 such that the integral of $\log|x|$ over a ball of measure c_1 centered at 0 is at least 1. This implies:

Corollary 2.4. *For any monic polynomial $P \in k[X]$, the integral of $\log|P(x)|$ over any set of measure grater then c_1 is at least the degree $d^\circ P$.*

We shall also need the following two propositions:

Proposition 2.5. *Let k be a local field and k_0 its prime field. If $(P_n)_n$ is a sequence of monic polynomials in $k[X]$ such that the degrees $d^\circ P_n \to +\infty$ as $n \to \infty$, and $\xi_1, \ldots, \xi_m$ are given numbers in k, then there exists a number $\xi \in k$, transcendental over $k_0(\xi_1, \ldots, \xi_m)$, such that $(|P_n(\xi)|)_n$ is unbounded in k.*

Proof. Let T be the set of numbers in k which are transcendental over $k_0(\xi_1, \ldots, \xi_m)$. Then T has full measure. For every $r > 0$ we consider the compact set

$$K_r = \{x \in k : \forall n \ |P_n(x)| \leq r\}.$$

We now proceed by contradiction. Suppose $T \subset \bigcup_{r>0} K_r$. Then for some large r, we have $\mu(K_r) \geq c_1$, where $c_1 > 0$ is the constant from Corollary 2.4. This implies

$$d^\circ P_n \leq \int_{K_r} \log\big(|P_n(x)|\big) d\mu(x) \leq \mu(K_r) \log r,$$

contradicting the assumption of the proposition. ∎

Proposition 2.6. *If $(P_n)_n$ is a sequence of distinct polynomials in $\mathbb{Z}[X_1, \ldots, X_m]$ such that $\sup_n d^\circ P_n < \infty$, then there exist algebraically independent numbers $\xi_1, \ldots, \xi_m$ in $\mathbb{C}$ such that $(|P_n(\xi_1, \ldots, \xi_m)|)_n$ is unbounded in $\mathbb{C}$.*

Proof. Let $d = \max_n d^\circ P_n$ and let T be the set of all m-tuples of complex numbers algebraically independent over $\mathbb{Z}$. The P_n's lie in

$$\{P \in \mathbb{C}[X_1, \ldots, X_m] : d^\circ P \leq d\}$$

which can be identified, since T is dense and polynomials are continuous, as a finite dimensional vector subspace V of the $\mathbb{C}$-vector space of all functions from T to $\mathbb{C}$. Let $l = \dim_{\mathbb{C}} V$. Then, as it is easy to see, there exist $(\overline{x}_1, \ldots, \overline{x}_l) \in T^l$, such that the evaluation map $P \mapsto \big(P(\overline{x}_1), \ldots, P(\overline{x}_l)\big)$ from V to $\mathbb{C}^l$ is a linear isomorphism from V to $\mathbb{C}^{\dim V}$. Since the P_n's belong to a $\mathbb{Z}$-lattice in V, so does their image under the evaluation map. Since the P_n's are all distinct, $\{P_n(\overline{x}_i)\}$ is unbounded for an appropriate $i \leq l$. $\square$

Proof of Lemma 2.1. Let us first assume that the characteristic of the field of fractions of R is 0. By Noether's normalization theorem, $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is integral over $\mathbb{Q}[\xi_1, \ldots, \xi_m]$ for some algebraically independent elements $\xi_1, \ldots, \xi_m$ in R. Since R is finitely generated, there exists an integer $l \in \mathbb{N}$ such that the generators of R, hence all elements of R, are roots of monic polynomials with coefficients in $S = \mathbb{Z}[\frac{1}{l}, \xi_1, \ldots, \xi_m]$. Hence $R_0 := R[\frac{1}{l}]$ is integral over S. Let F be the field of fractions of R_0 and K that of S. Then F is a finite extension of K and there are finitely many embeddings $\sigma_1, \ldots, \sigma_r$ of F into some (fixed) algebraic closure $\overline{K}$ of K. Note that S is integrally closed. Therefore if $x \in R$, the characteristic polynomial of x over F belongs to $S[X]$ and equals

$$\prod_{1 \leq i \leq r} (X - \sigma_i(x)) = X^r + \alpha_r(x) X^{r-1} + \ldots + \alpha_1(x)$$

where each $\alpha_i(x) \in S$. Since I is infinite, we can find i_0 such that $\{\alpha_{i_0}(x)\}_{x \in I}$ is infinite. This reduces the problem to the case $R = S$, for if S can be embedded in a local field k such that $\{|\alpha_{i_0}(x)|\}_{x \in I}$ is unbounded, then for at least one i, the $|\sigma_i(x)|$'s will be unbounded in some finite extension of k in which F embeds.

So assume $I \subset S = \mathbb{Z}[\frac{1}{l}, \xi_1, \ldots, \xi_m]$ and proceed by induction on the transcendence degree m.

The case $m = 0$ is easy since $S = \mathbb{Z}[\frac{1}{l}]$ embeds discretely (by the diagonal embedding) in the finite product $\mathbb{R} \prod_{p \backslash l} \mathbb{Q}_p$.

Now assume $m \geq 1$. Suppose first that the total degrees of the x's in I are unbounded. Then, for say ξ_m, $\sup_{x \in I} d^\circ_{\xi_m} x = +\infty$. Let $a(x)$ be the dominant coefficient of x in its expansion as a polynomial in ξ_m. Then $a(x) \in \mathbb{Z}[\frac{1}{l}, \xi_1, \ldots, \xi_{m-1}]$ and is non zero.

If $\{a(x)\}_{x\in I}$ is infinite, then we can apply the induction hypothesis and find an embedding of $\mathbb{Z}[\frac{1}{l}, \xi_1, \ldots, \xi_{m-1}]$ into some local field k for which $\{|a(x)|\}_{x\in I}$ is unbounded. Hence $I' := \{x \in I, |a(x)| \geq 1\}$ is infinite. Now $\frac{x}{a(x)}$ is a monic polynomial in $k[\xi_m]$, so we can then apply Proposition 2.5 and extend the embedding to $\mathbb{Z}[\frac{1}{l}, \xi_1, \ldots, \xi_{m-1}][\xi_m] = S$ in k, such that $\{\frac{x}{a(x)}\}_{x\in I'}$ is unbounded in k. The image of I, under this embedding, is unbounded in k.

Suppose now that $\{a(x)\}_{x\in I}$ is finite. Then either $a(x) \in \mathbb{Z}[\frac{1}{l}]$ for all but finitely many x's or not. In the first case we can embed $\mathbb{Z}[\frac{1}{l}, \xi_1, \ldots, \xi_{m-1}]$ into either $\mathbb{R}$ or $\mathbb{Q}_p$ (for some prime p dividing l) so that $|a(x)| \geq 1$ for infinitely many x's, while in the second case we can find $\xi_1, \ldots, \xi_{m-1}$ algebraically independent in $\mathbb{C}$, such that $|a(x)| \geq 1$ for infinitely many x in I, Then, the same argument as above, using Proposition 2.5 applies.

Now suppose that the total degrees of the x's in I are bounded. If for some infinite subset of I, the powers of $\frac{1}{l}$ in the coefficients of x (lying in $\mathbb{Z}[\frac{1}{l}]$) are bounded from above, then we can apply Proposition 2.6 to conclude. If not, then for some prime factor p of l, we can write $x = \frac{1}{p^{n(x)}}\tilde{x}$ where $\tilde{x} \in \mathbb{Z}_p[\xi_1, \ldots, \xi_m]$ with at least one coefficient of p-adic absolute value 1, and the $n(x) \in \mathbb{Z}$ are not bounded from above. By compactness, we can pick a subsequence $(\tilde{x})_{x\in I'}$ which converges in $\mathbb{Z}_p[\xi_1, \ldots, \xi_m]$, and we may assume that $n(x) \to \infty$ on this subsequence. The limit will be a non-zero polynomial $\tilde{x}_0$. Pick arbitrary algebraically independent numbers $z_1, \ldots, z_m \in \mathbb{Q}_p$. The limit polynomial $\tilde{x}_0$ evaluated at the point $(z_1, \ldots, z_m) \in \mathbb{Q}_p^m$ is not 0, and the sequence of polynomial $(\tilde{x})_{x\in I'}$ evaluated at $(z_1, \ldots, z_m)$ tends to $\tilde{x}_0(z_1, \ldots, z_m) \neq 0$. Hence $\big(x(z_1, \ldots, z_m)\big)_{x\in I'}$ tends to ∞ in $\mathbb{Q}_p$. Sending the ξ_i's to the z_i's we obtain the desired embedding. (Note that in this case, after p is selected, the specific values of the z_i's are not important.)

Finally, let us turn to the case when $char(k) = p > 0$. The first part of the argument remains valid : R is integral over $S = \mathbb{F}_q[\xi_1, \ldots, \xi_m]$ where $\xi_1, \ldots, \xi_m$ are algebraically independent over $\mathbb{F}_q$ and this enables to reduce to the case $R = S$. Then we proceed by induction on the transcendence degree m. If $m = 1$, then the assignment $\xi_1 \mapsto \frac{1}{t}$ gives the desired embedding of S into $\mathbb{F}_q((t))$. Let $m \geq 2$ and note that the total degrees of elements of I are necessarily unbounded. From this point the proof works verbatim as in the corresponding paragraphs above. $\square$

3. Contracting projective transformations

In this section and the next, unless otherwise stated, k is assumed to be a local field, with no assumption on the characteristic.

3.1. **Proximality and ping-pong.** Let us first recall some basic facts about projective transformations on $\mathbb{P}(k^n)$, where k is a local field. For proofs and a detailed (and self-contained) exposition, see [4], Section 3. We let $\|\cdot\|$ be the standard norm on k^n, i.e. the standard Euclidean norm if k is Archimedean and $\|x\| = \max_{1\leq i\leq n} |x_i|$ where $x = \sum x_i e_i$ when k is non-Archimedean and $(e_1, \ldots, e_n)$ is the canonical basis of k^n. This norm extends in the usual way to $\Lambda^2 k^n$. Then we define the *standard metric* on $\mathbb{P}(k^n)$ by

$$d([v], [w]) = \frac{\|v \wedge w\|}{\|v\| \, \|w\|}.$$

With respect to this metric, every projective transformation is bi-Lipschitz on $\mathbb{P}(k^n)$. For $\epsilon \in (0, 1)$, we call a projective transformation $[g] \in \mathrm{PGL}_n(k)$ **ϵ-contracting** if there exist a point $v_g \in \mathbb{P}^{n-1}(k)$, called an attracting point of $[g]$, and a projective hyperplane H_g, called a repelling hyperplane of $[g]$, such that $[g]$ maps the complement of the ϵ-neighborhood of $H_g \subset \mathbb{P}(k^n)$ (the repelling neighborhood of $[g]$) into the ϵ-ball around v_g (the attracting neighborhood of $[g]$). We say that $[g]$ is **ϵ-very contracting** if both $[g]$ and $[g^{-1}]$ are ϵ-contracting. A projective transformation $[g] \in \mathrm{PGL}_n(k)$ is called **(r, ϵ)-proximal** ($r > 2\epsilon > 0$) if it is ϵ-contracting with respect to some attracting point $v_g \in \mathbb{P}(k^n)$ and some repelling hyperplane H_g, such that $d(v_g, H_g) \geq r$. The transformation $[g]$ is called **(r, ϵ)-very proximal** if both $[g]$ and $[g]^{-1}$ are (r, ϵ)-proximal. Finally $[g]$ is simply called **proximal** (resp. **very proximal**) if it is (r, ϵ)-proximal (resp. (r, ϵ)-very proximal) for some $r > 2\epsilon > 0$.

The attracting point v_g and repelling hyperplane H_g of an ϵ-contracting transformation are not uniquely defined. Yet, if $[g]$ is proximal we have the following nice choice of v_g and H_g.

Lemma 3.1. *Let $\epsilon \in (0, \frac{1}{4})$. There exist two constants $c_1, c_2 \geq 1$ (depending only on the local field k) such that if $[g]$ is an (r, ϵ)-proximal transformation with $r \geq c_1\epsilon$ then it must fix a unique point $\overline{v}_g$ inside its attracting neighborhood and a unique projective hyperplane $\overline{H}_g$ lying inside its repelling neighborhood. Moreover, if $r \geq c_1\epsilon^{2/3}$, then all positive powers $[g^n]$, $n \geq 1$, are $(r - 2\epsilon, (c_2\epsilon)^{\frac{n}{3}})$-proximal transformations with respect to these same $\overline{v}_g$ and $\overline{H}_g$.*

Let us postpone the proof of this lemma till the next paragraph.

An m-tuple of projective transformations $a_1, \ldots, a_m$ is called a **ping-pong m-tuple** if all the a_i's are (r, ϵ)-very proximal (for some $r > 2\epsilon > 0$) and the attracting points of a_i and a_i^{-1} are at least r-apart from the repelling hyperplanes of a_j and a_j^{-1}, for any $i \neq j$. Ping-pong m-tuples give rise to free groups by the following variant of the *ping-pong lemma* (see [33] 1.1) :

Lemma 3.2. *If $a_1, \ldots, a_m \in PGL_n(k)$ form a ping-pong m-tuple, then $\langle a_1, \ldots, a_m \rangle$ is a free group of rank m.*

A *finite* subset $F \subset \mathrm{PGL}_n(k)$ is called (m,r)**-separating** ($r > 0$, $m \in \mathbb{N}$) if for every choice of $2m$ points $v_1, \ldots, v_{2m}$ in $\mathbb{P}(k^n)$ and $2m$ projective hyperplanes $H_1, \ldots, H_{2m}$ there exists $\gamma \in F$ such that

$$\min_{1 \leq i,j \leq 2m} \{d(\gamma v_i, H_j), d(\gamma^{-1} v_i, H_j)\} > r.$$

A separating set and an ϵ-contracting element for small ϵ are precisely the two ingredients needed to generate a ping-pong m-tuple. This is summarized by the following proposition (see [4] Propositions 3.8 and 3.11).

Proposition 3.3. *Let F be an (m,r)-separating set ($r < 1$, $m \in \mathbb{N}$) in $PGL_n(k)$. Then there is $C \geq 1$ such that for every ϵ, $0 < \epsilon < 1/C$, we have*

(i) If $[g] \in PGL_n(k)$ is an ϵ-contracting transformation, one can find an element $[f] \in F$, such that $[gfg^{-1}]$ is $C\epsilon$-very contracting.

(ii) If $a_1, \ldots, a_m \in PGL_n(k)$, and γ is an ϵ-very contracting transformation, then there are $h_1, \ldots, h_m \in F$ and $g_1, \ldots, g_m \in F$ such that

$$(g_1 \gamma a_1 h_1, g_2 \gamma a_2 h_2, \ldots, g_m \gamma a_m h_m)$$

forms a ping-pong m-tuple and hence are free generators of a free group.

3.2. **The Cartan decomposition.** Now let $\mathbb{H}$ be a Zariski connected reductive k-split algebraic k-group and $H = \mathbb{H}(k)$. Let $\mathbb{T}$ be a maximal k-split torus and $T = \mathbb{T}(k)$. Fix a system Φ of k-roots of $\mathbb{H}$ relative to $\mathbb{T}$ and a basis Δ of simple roots. Let $\mathbb{X}(\mathbb{T})$ be the group of k-rational multiplicative characters of $\mathbb{T}$ and $V' = \mathbb{X}(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ and V the dual vector space of V'. We denote by C^+ the positive Weyl chamber:

$$C^+ = \{v \in V : \forall \alpha \in \Delta,\ \alpha(v) > 0\}.$$

The Weyl group will be denoted by W and is identified with the quotient $N_H(T)/Z_H(T)$ of the normalizer by the centralizer of T in H. Let K be a maximal compact subgroup of H such that $N_K(T)$ contains representatives of every element of W. If k is Archimedean, let A be the subset of T consisting of elements t such that $|\alpha(t)| \geq 1$ for every simple root $\alpha \in \Delta$. And if k is non-Archimedean, let A be the subset of T consisting of elements such that $\alpha(t) = \pi^{-n_\alpha}$ for some $n_\alpha \in \mathbb{N} \cup \{0\}$ for any simple root $\alpha \in \Delta$, where π is a given uniformizer for k (i.e. the valuation of π is 1). Then we have the following *Cartan decomposition* (see Bruhat-Tits [5])

$$H = KAK. \tag{2}$$

In this decomposition, the A component is uniquely defined. We can therefore associate to every element $g \in H$ a uniquely defined $a_g \in A$.

Then, in what follows, *we define* $\chi(g)$ to be equal to $\chi(a_g)$ for any character $\chi \in \mathbb{X}(\mathbb{T})$ and element $g \in H$. Although this conflicts with the original meaning of $\chi(g)$ when g belongs to the torus $\mathbb{T}(k)$, we will keep this notation throughout the paper. Thus we always have $|\alpha(g)| \geq 1$ for any simple root α and $g \in H$.

Let us note that the above decomposition (2) is no longer true when $\mathbb{H}$ is not assumed to be k-split (see Bruhat-Tits [5] or [26] for the Cartan decomposition in the general case).

If $\mathbb{H} = \mathbb{GL}_n$ and α is the simple root corresponding to the difference of the first two eigenvalues $\lambda_1 - \lambda_2$, then a_g is a diagonal matrix $diag(a_1(g), \ldots, a_n(g))$ where $|\alpha(g)| = |\frac{a_1(g)}{a_2(g)}|$. Then we have the following nice criterion for ϵ-contraction, which justifies the introduction of this notion (see [4] Proposition 3.3).

Lemma 3.4. *Let $\epsilon < \frac{1}{4}$. If $|\frac{a_1(g)}{a_2(g)}| \geq 1/\epsilon^2$, then $[g] \in PGL_n(k)$ is ϵ-contracting on $\mathbb{P}(k^n)$. Conversely, suppose $[g]$ is ϵ-contracting on $\mathbb{P}(k^n)$ and k is non-Archimedean with uniformizer π (resp. Archimedean), then $|\frac{a_1(g)}{a_2(g)}| \geq \frac{|\pi|}{\epsilon^2}$ (resp. $|\frac{a_1(g)}{a_2(g)}| \geq \frac{1}{4\epsilon^2}$).*

The proof of Lemma 3.1, as well as of Proposition 3.3, is based on the latter characterization of ϵ-contraction and on the following crucial lemma (see [4] Lemmas 3.4 and 3.5) :

Lemma 3.5. *Let $r, \epsilon \in (0,1]$. If $|\frac{a_1(g)}{a_2(g)}| \geq \frac{1}{\epsilon^2}$, then $[g]$ is ϵ-contracting with respect to the repelling hyperplane*

$$H_g = [span\{k'^{-1}(e_i)\}_{i=2}^n]$$

and the attracting point $v_g = [ke_1]$, where $g = ka_gk'$ is a Cartan decomposition of g. Moreover, $[g]$ is $\frac{\epsilon^2}{r^2}$-Lipschitz outside the r-neighborhood of H_g. Conversely assume that the restriction of $[g]$ to some open set $O \subset \mathbb{P}(k^n)$ is ϵ-Lipschitz, then $|\frac{a_1(g)}{a_2(g)}| \geq \frac{1}{2\epsilon}$.

3.3. **The proof of Lemma 3.1.** Given a projective transformation $[h]$ and $\delta > 0$, we say that (H, v) is a *δ-related pair* of a repelling hyperplane and attracting point for $[h]$, if $[h]$ maps the complementary of the δ-neighborhood of H inside the δ-ball around v.

The attracting point and repelling hyperplane of an δ-contracting transformation $[h]$ are not uniquely defined. However, note that if $\delta < \frac{1}{4}$ then for any two δ-related pairs of $[h]$ (H_h^i, v_h^i), $i = 1, 2$, we have $d(v_h^1, v_h^2) < 2\delta$. Indeed, since $\delta < \frac{1}{4}$, the union of the δ -neighborhoods of the H_h^i's does not cover $\mathbb{P}(k^n)$. Let $p \in \mathbb{P}(k^n)$ be a point lying outside this union, then $d([h]p, v_h^i) < \delta$ for $i = 1, 2$.

Now consider two δ-related pairs (H_h^i, v_h^i), $i = 1, 2$ of some projective transformation $[h]$, satisfying $d(v_h^1, H_h^1) \geq r$ and no further assumption on the pair (H_h^2, v_h^2). Suppose that $1 \geq r > 4\delta$. Then we claim that $\mathrm{Hd}(H_h^1, H_h^2) \leq 2\delta$, where Hd denotes the standard distance between hyperplanes, i.e. the Hausdorff distance. (Note that $\mathrm{Hd}(H^1, H^2) = max_{x \in H^1}\{\frac{|f_2(x)|}{\|x\|}\}$ where f_2 is the unique (up to sign) norm one functional whose kernel is the hyperplane H_2 (for details see [4] section 3).) To see this, notice that if $\mathrm{Hd}(H_h^1, H_h^2)$ were greater than 2δ then any projective hyperplane H would contain a point outside the δ-neighborhood of either H_h^1 or H_h^2. Such a point is mapped under $[h]$ to the δ-ball around either v_h^1 or v_h^2, hence to the 3δ-ball around v_h^1. This in particular applies to the hyperplane $[h^{-1}]H_h^1$. A contradiction to the assumption $d(H_h^1, v_h^1) > 4\delta$. We also conclude that when $r > 8\delta$, then for any two δ-related pairs (H^i, v^i) $i = 1, 2$ of $[h]$, we have $d(v^i, H^j) > \frac{r}{2}$ for all $i, j \in \{1, 2\}$.

Let us now fix an arbitrary ϵ-related pair (H, v) of the (r, ϵ)-proximal transformation $[g]$ from the statement of Lemma 3.1. Let also (H_g, v_g) be the hyperplane and point introduced in Lemma 3.5. From Lemmas 3.4 and 3.5, we see that the pair (H_g, v_g) is a $C\epsilon$-related pair for $[g]$ for some constant $C \geq 1$ depending only on k. Assume $d(v, H) \geq r > 8C\epsilon$. Then it follows from the above that the ϵ-ball around v is mapped into itself under $[g]$, and that $d(v, H_g) > \frac{r}{2}$. From Lemma 3.5, we obtain that $[g]$ is $(\frac{4C\epsilon}{r})^2$-Lipschitz in this ball, and hence $[g^n]$ is $(\frac{4C\epsilon}{r})^{2n}$-Lipschitz there. Hence $[g]$ has a unique fixed point $\overline{v}_g$ in this ball which is the desired attracting point for all the powers of $[g]$. Note that $d(v, \overline{v}_g) \leq \epsilon$.

Since $[g^n]$ is $(\frac{4C\epsilon}{r})^{2n}$-Lipschitz on some open set, it follows from Lemma 3.5 that $|\frac{a_2(g^n)}{a_1(g^n)}| \leq 2(\frac{4C\epsilon}{r})^{2n}$, and from Lemma 3.4 that $[g^n]$ is $2(\frac{4C\epsilon}{r})^n$-contracting. Moreover, it is now easy to see that if $r > (4C)^2\epsilon$, then for every $2(\frac{4C\epsilon}{r})^n$-related pair (H_n, v_n) for $[g^n]$ $n \geq 2$, we have $d(\overline{v}_g, v_n) \leq 4(\frac{4C\epsilon}{r})^n$. (To see this apply $[g^n]$ to some point of the ϵ-ball around v which lies outside the $2(\frac{4C\epsilon}{r})^n$-neighborhood of H_n). Therefore $(H_n, \overline{v}_g)$ is a $6(\frac{4C\epsilon}{r})^n$-related pair for $[g^n]$, $n \geq 2$.

We shall now show that the ϵ-neighborhood of H contains a unique $[g]$-invariant hyperplane which can be used as a common repelling hyperplane for all the powers of $[g]$. The set $\mathcal{F}$ of all projective points at distance at most ϵ from H is mapped into itself under $[g^{-1}]$. Similarly the set $\mathfrak{H}$ of all projective hyperplanes which are contained in $\mathcal{F}$ is mapped into itself under $[g^{-1}]$. Both sets $\mathcal{F}$, and $\mathfrak{H}$ are compact with respect to the corresponding Grassmann topologies. The intersection $\mathcal{F}_\infty = \cap[g^{-n}]\mathcal{F}$ is therefore non empty and contains some hyperplane $\overline{H}_g$ which corresponds to any point of the intersection $\cap[g^{-n}]\mathfrak{H}$. We claim that $\mathcal{F}_\infty = \overline{H}_g$. Indeed, the set $\mathcal{F}_\infty$ is invariant under $[g^{-1}]$ and hence under $[g]$ and $[g^n]$. Since $(H_n, \overline{v}_g)$ is a $6(\frac{4C\epsilon}{r})^n$-related pair for $[g^n]$, $n \geq 2$, and since $\overline{v}_g$ is "far" (at least $r-2\epsilon$ away) from the invariant set $\mathcal{F}_\infty$, it follows that for large n, $\mathcal{F}_\infty$ must lie inside the $6(\frac{4C\epsilon}{r})^n$-neighborhood of H_n. Since $\mathcal{F}_\infty$ contains a hyperplane, and since it is arbitrarily close to a hyperplane, it must coincide with a hyperplane. Hence $\mathcal{F}_\infty = \overline{H}_g$. It follows that $(\overline{H}_g, \overline{v}_g)$ is a $12(\frac{4C\epsilon}{r})^n$-related pair for $[g^n]$ for any large enough n. Note that then $d(\overline{v}_g, \overline{H}_g) > r - 2\epsilon$, since $d(\overline{v}_g, v) \leq \epsilon$ and $\mathrm{Hd}(\overline{H}_g, H) \leq \epsilon$. This proves existence and uniqueness of $(\overline{H}_g, \overline{v}_g)$ as soon as $r > c_1\epsilon$ where $c_1 \geq (4C)^2 + 8C$.

If we assume further that $r^3 \geq 12(4C\epsilon)^2$, then $\mathcal{F}_\infty$ lies inside the $6(\frac{4C\epsilon}{r})^n$-neighborhood of H_n as soon as $n \geq 2$. Then $(\overline{H}_g, \overline{v}_g)$ is a $12(\frac{4C\epsilon}{r})^n$-related pair for $[g^n]$, hence a $(c_2\epsilon)^{n/3}$-related pair for $[g^n]$ whenever $n \geq 1$, where $c_2 \geq 1$ is a constant easily computable in terms of C. This finishes the proof of the lemma.

In what follows, whenever we add the article *the* to an attracting point and repelling hyperplane of a proximal transformation $[g]$, we shall mean these fixed point $\overline{v}_g$ and fixed hyperplane $\overline{H}_g$ obtained in Lemma 3.1.

3.4. **The case of general semisimple group.** Now let us assume that $\mathbb{H}$ is a Zariski connected semisimple k-algebraic group, and let (ρ, V_ρ) be a finite dimensional k-rational representation of $\mathbb{H}$ with highest weight χ_ρ. Let Θ_ρ be the set of simple roots α such that χ_ρ/α is again a non-trivial weight of ρ

$$\Theta_\rho = \{\alpha \in \Delta : \chi_\rho/\alpha \text{ is a weight of } \rho\}.$$

It turns out that Θ_ρ is precisely the set of simple roots α such that the associated fundamental weight π_α appears in the decomposition of χ_ρ as a sum of fundamental weights. Suppose that the weight space V_{χ_ρ} corresponding to χ_ρ has dimension 1, then we have the following lemma.

Lemma 3.6. *There are positive constants $C_1 \leq 1 \leq C_2$, such that for any $\epsilon \in (0,1)$ and any $g \in \mathbb{H}(k)$, if $|\alpha(g)| > \frac{C_2}{\epsilon^2}$ for all $\alpha \in \Theta_\rho$ then the projective transformation $[\rho(g)] \in PGL(V_\rho)$ is ϵ-contracting, and conversely, if $[\rho(g)]$ is ϵ-contracting, then $|\alpha(g)| > \frac{C_1}{\epsilon^2}$ for all $\alpha \in \Theta_\rho$.*

Proof. Let $V_\rho = \bigoplus V_\chi$ be the decomposition of V_ρ into a direct sum of weight spaces. Let us fix a basis $(e_1, \ldots, e_n)$ of V_ρ compatible with this decomposition and such that $V_{\chi_\rho} = ke_1$. We then identify V_ρ with k^n via this choice of basis. Let $g = k_1 a_g k_2$ be a Cartan decomposition of g in H. We have $\rho(g) = \rho(k_1)\rho(a_g)\rho(k_2) \in \rho(K)D\rho(K)$ where $D \subset SL_n(k)$ is the set of diagonal matrices. Since $\rho(K)$ is compact, there exists a positive constant C such that if $[\rho(g)]$ is ϵ-contracting then $[\rho(a_g)]$ is $C\epsilon$-contracting, and conversely if $[\rho(a_g)]$ is ϵ-contracting then $[\rho(g)]$ is $C\epsilon$-contracting. Therefore, it is equivalent to prove the lemma for $\rho(a_g)$ instead of $\rho(g)$. Now the coefficient $|a_1(\rho(a_g))|$ in the Cartan decomposition on $SL_n(k)$ equals $\max_\chi |\chi(a_g)| = |\chi_\rho(g)|$, and the coefficient $|a_2(\rho(a_g))|$ is the second highest diagonal coefficient and hence of the form $|\chi_\rho(a_g)/\alpha(a_g)|$ where α is some simple root. Now the conclusion follows from Lemma 3.4. $\square$

4. Irreducible representations of non-Zariski connected algebraic groups

In the process of constructing dense free groups, we need to find some suitable linear representation of the group Γ we started with. In general, the Zariski closure of Γ may not be Zariski connected, and yet we cannot pass to a subgroup of finite index in Γ in Theorem 1.3. Therefore we will need to consider representations of non Zariski connected groups.

Let $\mathbb{H}^\circ$ be a connected semisimple k-split algebraic k-group. The group $Aut_k(\mathbb{H}^\circ)$ of k-automorphisms of $\mathbb{H}^\circ$ acts naturally on the characters $\mathbb{X}(\mathbb{T})$ of a maximal split torus $\mathbb{T}$. Indeed, for every $\sigma \in Aut_k(\mathbb{H}^\circ)$, the torus $\sigma(T)$ is conjugate to $T = \mathbb{T}(k)$ by some element $g \in H = \mathbb{H}(k)$ and we can define the character $\sigma(\chi)$ by $\sigma(\chi)(t) = \chi(g^{-1}\sigma(t)g)$. This is not well defined, since the choice of g is not unique (it is up to multiplication by an element of the normalizer $N_H(T)$). But if we require $\sigma(\chi)$ to lie in the same Weyl chamber as χ, then this determines g up to multiplication by an element from the centralizer $Z_H(T)$, hence it determines $\sigma(\chi)$ uniquely. Note also that every σ sends roots to roots and simple roots to simple roots.

In fact, what we need are representations of algebraic groups whose restriction to the connected component is irreducible. As explained below, it turns out that an irreducible representation ρ of a connected semisimple algebraic group $\mathbb{H}^\circ$ extends to the full group $\mathbb{H}$ if and only if its highest weight is invariant under the action of $\mathbb{H}$ by conjugation.

We thus have to face the problem of finding elements in $\mathbb{H}^\circ(k) \cap \Gamma$ which are ε-contracting under such a representation ρ. By Lemma 3.6 this amounts to finding elements h such that $\alpha(h)$ is large for all simple roots α in the set Θ_ρ defined in Paragraph 3.4. As will be explained below, we can find such a representation ρ such that all simple roots belonging to Θ_ρ are images by some outer automorphisms σ's of $\mathbb{H}^\circ$ (coming from conjugation by an element of $\mathbb{H}$) of a single simple root α. But $\sigma(\alpha)(h)$ and $\alpha(\sigma(h))$ are comparable. The idea of the proof below is then to find elements h in $\mathbb{H}^\circ(k)$ such that all relevant $\alpha(\sigma(h))$'s are large. But, according to the converse statement in Lemma 3.6, this amounts to finding elements h such that all relevant $\sigma(h)$'s are ε-contracting under a representation ρ_α such that $\Theta_{\rho_\alpha} = \{\alpha\}$. This is the content of the forthcoming proposition.

Before stating the proposition, let us note that, $\mathbb{H}^\circ$ being k-split, to every simple root $\alpha \in \Delta$ corresponds an irreducible k-rational representation of $\mathbb{H}^\circ(k)$ whose highest weight χ_{ρ_α} is the fundamental weight π_α associated to α and has multiplicity one. In this case the set Θ_{ρ_α} defined in Paragraph 3.4 is reduced to the singleton $\{\alpha\}$.

Proposition 4.1. *Let α be a simple root. Let I be a subset of $\mathbb{H}^\circ(k)$ such that $\{|\alpha(g)|\}_{g\in I}$ is unbounded in $\mathbb{R}$. Let $\Omega \subset \mathbb{H}^\circ(k)$ be a Zariski dense subset. Let $\sigma_1, \ldots, \sigma_m$ be algebraic k-automorphisms of $\mathbb{H}^\circ$. Then for any arbitrary large $M > 0$, there exists an element $h \in \mathbb{H}^\circ(k)$ of the form $h = f_1\sigma_1^{-1}(g)\ldots f_m\sigma_m^{-1}(g)$ where $g \in I$ and the f_i's belong to Ω, such that $|\sigma_i(\alpha)(h)| > M$ for all $1 \le i \le m$.*

Proof. Let $\epsilon \in (0,1)$ and $g \in I$ such that $|\alpha(g)| \ge \frac{1}{\epsilon^2}$. Let (ρ_α, V) be the irreducible representation of $\mathbb{H}^\circ(k)$ corresponding to α as described above. Consider the weight space decomposition $V_{\rho_\alpha} = \bigoplus V_\chi$ and fix a basis $(e_1, \ldots, e_n)$ of $V = V_{\rho_\alpha}$ compatible with this decomposition and such that $V_{\chi_{\rho_\alpha}} = ke_1$. We then identify V with k^n via this choice of basis, and in particular, endow $\mathbb{P}(V)$ with the standard metric defined in the previous section. It follows from Lemma 3.6 above that $[\rho_\alpha(g)]$ is ϵC-contracting on $\mathbb{P}(V)$ for some constant $C \ge 1$ depending only on ρ_α. Now from Lemma 3.5, there exists for any $x \in \mathbb{H}^\circ(k)$ a point $u_x \in \mathbb{P}(V)$ such that $[\rho_\alpha(x)]$ is 2-Lipschitz over some open neighborhood of u_x. Similarly there exists a projective hyperplane H_x such that $[\rho_\alpha(x)]$ is $\frac{1}{r^2}$-Lipschitz outside the r-neighborhood of H_x. Moreover, combining Lemmas 3.4 and 3.5 (and up to changing C if necessary to a larger constant depending this time only on k), we see that $[\rho_\alpha(g)]$ is $\frac{\epsilon^2C^2}{r^2}$-Lipschitz outside the r-neighborhood of the repelling hyperplane H_g defined in Lemma 3.5. We pick u_g outside this r-neighborhood.

By modifying slightly the definition of a finite (m,r)-separating set (see above Paragraph 3.1), we can say that a finite subset F of $\mathbb{H}^\circ(k)$ is an (m,r)-separating set *with respect to* ρ_α and $\sigma_1, \ldots, \sigma_m$ if for every choice of m points $v_1, \ldots, v_m$ in $\mathbb{P}(V)$ and m projective hyperplanes $H_1, \ldots, H_m$ there exists $\gamma \in F$ such that

$$\min_{1\le i,j,k\le m,} d(\rho_\alpha(\sigma_k(\gamma))v_i, H_j) > r > 0.$$

Claim : The Zariski dense subset Ω contains a finite (m,r)-separating set with respect to ρ_α and $\sigma_1, \ldots, \sigma_m$, for some positive number r.

Proof of claim : For $\gamma \in \Omega$, we let M_γ be the set of all tuples $(v_i, H_i)_{1\le i\le m}$ such that there exists some i, j and l for which $\rho_\alpha(\sigma_l(\gamma))v_i \in H_j$. Now $\bigcap_{\gamma\in\Omega} M_\gamma$ is empty, for otherwise there would be points $v_1, \ldots, v_m$ in $P(V)$ and projective hyperplanes $H_1, \ldots, H_m$ such that Ω is included in the union of the closed algebraic k-subvarieties $\{x \in \mathbb{H}^\circ(k),\ \rho_\alpha(\sigma_l(x))v_i \in H_j\}$ where

i, j and l range between 1 and m. But, by irreducibility of ρ_α each of these subvarieties is proper, and this would contradict the Zariski density of Ω or the Zariski connectedness of $\mathbb{H}^\circ$. Now, since each M_γ is compact in the appropriate product of Grassmannians, it follows that for some finite subset $F \subset \Omega$, $\bigcap_{\gamma \in F} M_\gamma = \emptyset$. Finally, since $\max_{\gamma \in F} \min_{1 \le i,j,l \le m} d(\rho_\alpha(\sigma_l(\gamma)v_i, H_j)$ depends continuously on $(v_i, H_i)_{i=1}^m$ and never vanishes, it must attain a positive minimum r, by compactness of the set of all tuples $(v_i, H_i)_{i=1}^m$ in

$$\big(\mathbb{P}(V) \times \mathbb{G}r_{\dim(V)-1}(V)\big)^{2m}.$$

Therefore F is the desired (m, r)-separating set.

Up to taking a bigger constant C, we can assume that C is larger than the bi-Lipschitz constant of every $\rho_\alpha(x)$ on $\mathbb{P}(k^n)$ when x ranges over the finite set $\{\sigma_k(f), f \in F, 1 \le k \le m\}$.

Now let us explain how to find the element $h = f_m \sigma_m^{-1}(g) \dots f_1 \sigma_1^{-1}(g)$ we are looking for. We shall choose the f_j's recursively, starting from $j = 1$, in such a way that all the elements $\sigma_i(h)$, $1 \le i \le m$, will be contracting. Write

$$\begin{array}{ccc} \sigma_i(h) & = & \sigma_i(f_m \sigma_m^{-1}(g) \dots f_1 \sigma_1^{-1}(g)) = \\ \big(\sigma_i(f_m)\sigma_i\sigma_m^{-1}(g) \cdot \ldots \cdot \sigma_i(f_i)\big) & g & \big(\sigma_i(f_{i-1})\sigma_i\sigma_{i-1}(g) \cdot \ldots \cdot \sigma_i(f_1)\sigma_i\sigma_1^{-1}(g)\big). \end{array}$$

In order to make $\sigma_i(h)$ contracting, we shall require that:

- For $m \ge i \ge 2$, $\sigma_i(f_{i-1})$ takes the image under $\sigma_i\sigma_{i-1}(g) \cdot \ldots \cdot \sigma_i(f_1)\sigma_i\sigma_1^{-1}(g)$ of some open set on which $\sigma_i\sigma_{i-1}(g) \cdot \ldots \cdot \sigma_i(f_1)\sigma_i\sigma_1^{-1}(g)$ is 2-Lipschitz, e.g. a small neighborhood of the point
  $$u_i := \big(\sigma_i\sigma_{i-1}(g) \cdot \ldots \cdot \sigma_i(f_1)\sigma_i\sigma_1^{-1}(g)\big)(u_{\sigma_i\sigma_{i-1}(g) \cdot \ldots \cdot \sigma_i(f_1)\sigma_i\sigma_1^{-1}(g)})$$
  at least r apart from the hyperplane H_g, and
- For $m > j \ge i$, $\sigma_i(f_j)$ takes the image of u_i under $\big(\sigma_i\sigma_j^{-1}(g) \cdot \ldots \cdot \sigma_i(f_i)\big)g$ at least r apart from the hyperplane $H_{\sigma_i\sigma_{j+1}^{-1}(g)}$ of $\sigma_i\sigma_{j+1}^{-1}(g)$ (i.e. of the next element on the left in the expression of $\sigma_i(h)$).

Assembling the conditions on each f_i we see that there are $\le m$ points that the $\sigma_j(f_i)$'s, $1 \le j \le m$ should send r apart from $\le m$ projective hyperplanes.

This appropriate choice of $f_1, \dots, f_m$ in F forces each of $\sigma_1(h), \dots, \sigma_m(h)$ to be $\frac{2C^{m+2}\epsilon^2}{r^{2m}}$-Lipschitz in some open subset of $\mathbb{P}(V)$. Lemma 3.5 now implies that $\sigma_1(h), \dots, \sigma_m(h)$ are $C_0\epsilon$-contracting on $\mathbb{P}(V)$ for some constant C_0 depending only on (ρ, V).

Moreover $h \in Ka_hK$ and each of the $\sigma_i(K)$ is compact, we conclude that $\sigma_1(a_h), \dots, \sigma_m(a_h)$ are also $C_1\epsilon$-contracting on $\mathbb{P}(V)$ for some constant C_1. But for every σ_i there exists an element $b_i \in \mathbb{H}^\circ(k)$ such that $\sigma_i(T) = b_i T b_i^{-1}$ and $\sigma_i(\alpha)(t) = \alpha(b_i^{-1}\sigma_i(t)b_i)$ for every element t in the positive Weyl chamber of the maximal k-split torus $T = \mathbb{T}(k)$. Up to taking a larger constant C_1 (depending on the b_i's) we therefore obtain that $b_1^{-1}\sigma_1(a_h)b_1, \ldots, b_m^{-1}\sigma_m(a_h)b_m$ are also $C_1\epsilon$-contracting on $\mathbb{P}(V)$ via the representation ρ_α. Finally Lemma 3.6 yields the conclusion that $|\sigma_i(\alpha)(h)| = |\alpha(b_i^{-1}\sigma_i(a_h)b_i)| \geq \frac{1}{C_2\epsilon^2}$ for some other positive constant C_2. Since ϵ can be chosen arbitrarily small, we are done. $\square$

Now let $\mathbb{H}$ be an arbitrary algebraic k-group, whose identity connected component $\mathbb{H}^\circ$ is semisimple. Let us fix a system Σ of k-roots for $\mathbb{H}^\circ$ and a simple root α. For every element g in $\mathbb{H}(k)$ let σ_g be the automorphism of $\mathbb{H}^\circ(k)$ which is induced by g under conjugation, and let $\mathcal{S}$ be the group of all such automorphisms. As was described above, $\mathcal{S}$ acts naturally on the set Δ of simple roots. Let $\mathcal{S}\cdot\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be the orbit of α under this action. Suppose $I \subset \mathbb{H}^\circ(k)$ satisfies the conclusion of the last proposition for $\mathcal{S}\cdot\alpha$, that is for any $\epsilon > 0$, there exists $g \in I$ such that $|\alpha_i(g)| > 1/\epsilon^2$ for all $i = 1, \ldots, p$. Then the following proposition shows that under some suitable irreducible projective representation of the full group $\mathbb{H}(k)$, for arbitrary small ϵ, some elements of I act as ϵ-contracting transformations.

Proposition 4.2. *Let $I \subset \mathbb{H}^\circ(k)$ be as above. Then there exists a finite extension K of k, $[K : k] < \infty$, and a non-trivial finite dimensional irreducible K-rational representation of $\mathbb{H}^\circ$ into a K-vector space V which extends to an irreducible projective representation $\rho : \mathbb{H}(K) \to PGL(V)$, satisfying the following property : for every positive $\epsilon > 0$ there exists $\gamma_\epsilon \in I$ such that $\rho(\gamma_\epsilon)$ is an ϵ-contracting projective transformation of $\mathbb{P}(V)$.*

Proof. Up to taking a finite extension of k, we can assume that $\mathbb{H}^\circ$ is k-split. Let (ρ, V) be an irreducible k-rational representation of $\mathbb{H}^\circ$ whose highest weight χ_ρ is a multiple of $\alpha_1 + \ldots + \alpha_p$ and such that the highest weight space V_{χ_ρ} has dimension 1 over k. Burnside's theorem implies that, up to passing to a finite extension of k, we can also assume that the group algebra $k[\mathbb{H}^\circ(k)]$ is mapped under ρ to the full algebra of endomorphisms of V, i.e. $End_k(V)$. For a k-automorphism σ of $\mathbb{H}^\circ$ let $\sigma(\rho)$ be the representation of $\mathbb{H}^\circ$ given on V by $\sigma(\rho)(g) = \rho(\sigma(g))$. It is a k-rational irreducible representation of $\mathbb{H}^\circ$ whose highest weight is precisely $\sigma(\chi_\rho)$. But $\chi_\rho = d(\alpha_1 + \ldots + \alpha_p)$ for some $d \in \mathbb{N}$, and is invariant under the action of $\mathcal{S}$. Hence for any $\sigma \in \mathcal{S}$, $\sigma(\rho)$ is equivalent to ρ. So there must exists a linear automorphism $J_\sigma \in \mathrm{GL}(V)$ such that $\sigma(\rho)(h) = J_\sigma \rho(h) J_\sigma^{-1}$ for all $h \in \mathbb{H}^\circ(k)$. Now set $\widetilde{\rho}(g) = [\rho(g)] \in PGL(V)$ if $g \in \mathbb{H}^\circ(k)$ and $\widetilde{\rho}(g) = [J_{\sigma_g}] \in \mathrm{PGL}(V)$ otherwise. Since the $\rho(g)$'s when g ranges over $\mathbb{H}^\circ(k)$ generate the whole of $End_k(V)$, it follows from Schur's lemma that $\widetilde{\rho}$ is a well defined projective representation of the whole of $\mathbb{H}(k)$. Now the set Θ_ρ of simple roots α such that χ_ρ/α is a non-trivial weight of ρ is precisely $\{\alpha_1, \ldots, \alpha_p\}$. Hence if $\gamma_\epsilon \in I$ satisfies

$|\alpha_i(\gamma_\epsilon)| > \frac{1}{\epsilon^2}$ for all $i = 1, \ldots, p$, then we have by Lemma 3.6 that $\widetilde{\rho}(\gamma_\epsilon)$ is $C_2\epsilon$-contracting on $\mathbb{P}(V)$ for some constant C_2 independent of ϵ. □

We can now state and prove the main result of this paragraph, and the only one which will be used in the sequel. Let here K be an arbitrary field which is finitely generated over its prime field and $\mathbb{H}$ an algebraic K-group such that its Zariski connected component $\mathbb{H}^\circ$ is semisimple and non-trivial. Fix some faithful K-rational representation $\mathbb{H} \hookrightarrow \mathbb{GL}_d$. Let R be a finitely generated subring of K. We shall denote by $\mathbb{H}(R)$ (resp. $\mathbb{H}^\circ(R)$) the subset of points of $\mathbb{H}(K)$ (resp. $\mathbb{H}^\circ(K)$) which are mapped into $\mathbb{GL}_d(R)$ under the latter embedding.

Theorem 4.3. *Let $\Omega_0 \subset \mathbb{H}^\circ(R)$ be a Zariski dense subset of $\mathbb{H}^\circ$ with $\Omega_0 = \Omega_0^{-1}$. Suppose $\{g_1, \ldots, g_m\}$ is a finite subset of $\mathbb{H}(K)$ exhausting all cosets of $\mathbb{H}^\circ$ in $\mathbb{H}$, and let*

$$\Omega = g_1\Omega_0 g_1^{-1} \cup \ldots \cup g_m\Omega_0 g_m^{-1}.$$

Then we can find a number $r > 0$, a local field k, an embedding $K \hookrightarrow k$, and a strongly irreducible projective representation $\rho : \mathbb{H}(k) \to PGL_d(k)$ defined over k with the following property. If $\epsilon \in (0, \frac{r}{2})$ and $a_1, \ldots, a_n \in \mathbb{H}(K)$ are n arbitrary points ($n \in \mathbb{N}$), then there exist n elements $x_1, \ldots, x_n$ with

$$x_i \in \Omega^{4m+2} a_i \Omega$$

such that the $\rho(x_i)$'s form a ping-pong n-tuple of (r, ϵ)-very proximal transformations on $\mathbb{P}(k^d)$, and in particular are generators of a free group F_n.

Proof. Up to enlarging the subring R if necessary, we can assume that K is the field of fractions of R. We shall make use of Lemma 2.1. Since Ω_0 is infinite, we can apply this lemma and obtain an embedding of K into a local field k such that Ω_0 becomes an unbounded set in $\mathbb{H}(k)$. Up to enlarging k if necessary we can assume that $\mathbb{H}^\circ(k)$ is k-split. We fix a maximal k-split torus and a system of k-roots with a base Δ of simple roots. Then, in the corresponding Cartan decomposition of $\mathbb{H}(k)$ the elements of Ω_0 have unbounded A component (see Paragraph 3.2). Therefore, there exists a simple root α such that the set $\{|\alpha(g)|\}_{g \in \Omega_0}$ is unbounded in $\mathbb{R}$. Let σ_{g_i} be the automorphism of $\mathbb{H}^\circ(k)$ given by the conjugation by g_i. The orbit of α under the group generated by the σ_{g_i}'s is denoted by $\{\alpha_1, \ldots, \alpha_p\}$. Now it follows from Proposition 4.1 that for every $\epsilon > 0$ there exists an element $h \in \Omega^{2p}$ such that $|\alpha_i(h)| > 1/\epsilon^2$ for every $i = 1, \ldots, p$. We are now in a position to apply the last Proposition 4.2 and obtain (up to taking a finite extension of k if necessary) an irreducible projective representation $\rho : \mathbb{H}(k) \to$PGL(V), such that the restriction of ρ to $\mathbb{H}^\circ(k)$ is also irreducible and with the following property: for every positive $\epsilon > 0$ there exists $h_\epsilon \in \Omega^{2p}$ such that $\rho(h_\epsilon)$ is an ϵ-contracting projective transformation of $\mathbb{P}(V)$.

Moreover, since $\rho_{|\mathbb{H}^\circ}$ is also irreducible and Ω_0 is Zariski dense in $\mathbb{H}^\circ$ we can find an (n,r)-separating set with respect to $\rho_{|\mathbb{H}^\circ}$ for some $r>0$ (for this terminology, see definitions in Paragraph 3.1). This follows from the proof of the claim in Proposition 4.1 above (see also Lemma 4.3. in [4]). By Proposition 3.3 (i) above, we obtain for every small $\epsilon>0$ an ϵ-very contracting element γ_ϵ in $h_\epsilon\Omega_0 h_\epsilon^{-1}\subset\Omega^{4p+1}$. Similarly, statement (ii) of the same Proposition gives elements $f_1,\ldots,f_n\in\Omega_0$ and $f_1',\ldots,f_n'\in\Omega_0$ such that, for ϵ small enough, $(x_1,\ldots,x_n)=(f_1'\gamma_\epsilon a_1 f_1,\ldots,f_n'\gamma_\epsilon a_n f_n)$ form under ρ a ping-pong n-tuple of proximal transformations on $\mathbb{P}(V)$. Then each x_i lies in $\Omega^{4p+2}a_i\Omega$ and together the x_i's form generators of a free group F_n of rank n. □

4.1. **Further remarks.** For further use in later sections we shall state two more facts. Let $\Gamma\subset\mathbb{G}(K)$ be a Zariski dense subgroup of some algebraic group $\mathbb{G}$. Suppose Γ is not virtually solvable and finitely generated and let $\Delta\leq\Gamma$ be a subgroup of finite index. Taking the quotient by the solvable radical of $\mathbb{G}^\circ$, we obtain a homomorphism π of Γ into an algebraic group $\mathbb{H}$ whose connected component is semisimple. Let $g_1,\ldots,g_m\in\Gamma$ be arbitrary elements in Γ. Then $\Omega_0=\pi(\cap_{i=1}^m g_i\Delta g_i^{-1})\cap\mathbb{H}^0$ is clearly Zariski dense in $\mathbb{H}^0$ and satisfies the conditions of Theorem 4.3. Hence taking $a_i=\pi(g_i)$ in the theorem, we obtain:

Corollary 4.4. *Let Γ be a finitely generated linear group which is not virtually solvable, and let $\Delta\subset\Gamma$ be a subgroup of finite index. Then for any choice of elements $g_1,\ldots,g_m\in\Gamma$ one can find free generators of a free group $a_1,\ldots,a_m$ lying in the same cosets, i.e. $a_i\in g_i\Delta$.*

The following lemma will be useful when dealing with the non-Archimedean case.

Lemma 4.5. *Let k be a non-Archimedean local field. Let $\Gamma\leq GL_n(k)$ be a linear group over k which contains no open solvable subgroup. Then there exists a homomorphism ρ from Γ into a k-algebraic group $\mathbb{H}$ such that the Zariski closure of the image of any open subgroup of Γ contains the connected component of the identity $\mathbb{H}^\circ$. Moreover, we can take $\rho:\Gamma\to\mathbb{H}(k)$ to be continuous in the topology induced by k, and we can find $\mathbb{H}$ such that $\mathbb{H}^\circ$ is semisimple and* $\dim(\mathbb{H}^\circ)\leq\dim\overline{\Gamma}^z$.

Proof. Let U_i be a decreasing sequence of open subgroups in $\mathrm{GL}_n(k)$ forming a base of identity neighborhoods. Consider the decreasing sequence of algebraic groups $\overline{\Gamma\cap U_i}^z$. This sequence must stabilize after a finite step s. The limiting group $\mathbb{G}=\overline{\Gamma\cap U_s}^z$ must be Zariski connected. Indeed, the intersection of $\Gamma\cap U_s$ with the Zariski connected component of the identity in $\mathbb{G}$ is a relatively open subgroup and contains $\Gamma\cap U_t$ for some large t. If $\mathbb{G}$

were not Zariski connected, then $\overline{\Gamma \cap U_t}^z$ would be a smaller algebraic group. Moreover, from the assumption on Γ, we get that $\mathbb{G}$ is not solvable.

Note that the conjugation by an element of Γ fixes $\mathbb{G}$, since $\gamma U_i \gamma^{-1} \cap U_i$ is again open if $\gamma \in \Gamma$ and hence contains some U_j. Since the solvable radical $Rad(\mathbb{G})$ of $\mathbb{G}$ is a characteristic subgroup of $\mathbb{G}$, it is also fixed under conjugation by elements of Γ. We thus obtain a homomorphism ρ from Γ to the k-points of the group of k-automorphisms $\mathbb{H} = Aut(\mathbb{S})$ of the Zariski connected semisimple k-group $\mathbb{S} = \mathbb{G}/Rad(\mathbb{G})$. This homomorphism is clearly continuous. Since the image of $\Gamma \cap U$ is Zariski dense in $\mathbb{G}$ for all open $U \subset \mathrm{GL}_n(k)$, $\Gamma \cap U$ is mapped under this homomorphism to a Zariski dense subgroup of the group of inner automorphisms $Int(\mathbb{S})$ of $\mathbb{S}$. But $Int(\mathbb{S})$ is a semisimple algebraic k-group which is precisely the Zariski connected component of the identity in $\mathbb{H} = Aut(\mathbb{S})$ (see for example [3], 14.9). Finally, it is clear from the construction that $\dim \mathbb{H} \leq \dim \overline{\Gamma}^z$. $\square$

5. The proof of Theorem 1.3 in the finitely generated case

In this section we prove our main result, Theorem 1.3, in the case when Γ is finitely generated. We obtain in fact a more precise result which yields some control on the number of generators required for the free group.

Theorem 5.1. *Let $\Gamma \leq GL_n(k)$ be a finitely generated linear group over a local field k. Suppose Γ contains no solvable open subgroup. Then, there is a constant $h(\Gamma) \in \mathbb{N}$ such that for any integer $r \geq h(\Gamma)$, Γ contains a dense free subgroup of rank r. Moreover, if $char(k) = 0$ we can take $h(\Gamma) = d(\Gamma)$ (i.e. the minimal size of a generating set for Γ), while if $char(k) > 0$ we can take $h(\Gamma) = d(\Gamma) + n^2$.*

In the following paragraphs we split the proof to three cases (Archimedean, non-Archimedean of characteristic zero, and positive characteristic) which have to be dealt with independently.

5.1. **The Archimedean case.** Consider first the case $k = \mathbb{R}$ or $\mathbb{C}$. Let G be the linear Lie group $G = \overline{\Gamma}$, and let G° be the connected component of the identity in G. The condition "Γ contains no open solvable subgroup" means simply "G° is not solvable". Note also that $d(G/G^\circ) \leq d(\Gamma) < \infty$.

Define inductively $G^\circ_0 = G^\circ$ and $G^\circ_{n+1} = \overline{[G^\circ_n, G^\circ_n]}$. This sequence stabilizes after some finite step t to a normal topologically perfect subgroup $H := G^\circ_t$ (i.e. the commutator group $[H, H]$ is dense in H). As was shown in [4] Theorem 2.1, any topologically perfect group H contains a finite set of elements $\{h_1, \ldots, h_l\}$, $l \leq \dim(H)$ and a relatively open identity neighborhood $V \subset H$ such that, for any selection of points $x_i \in V h_i V$, the group $\langle x_1, \ldots, x_l \rangle$ is dense in H. Moreover, H is clearly a characteristic subgroup of G°, hence it is normal in G. It is also clear from the definition of H that if Γ is a dense subgroup of G then $\Gamma \cap H$ is dense in H.

Let $r \geq d(\Gamma)$, and let $\{\gamma_1, \ldots, \gamma_r\}$ be a generating set for Γ. Then one can find a smaller identity neighborhood $U \subset V \subset H$ such that for any selection of points $y_j \in U\gamma_j U,\ j = 1, \ldots, r$, the group they generate $\langle y_1, \ldots, y_r \rangle$ is dense in G. Indeed, as $\Gamma \cap H$ is dense in H, there are l words w_i in r letters such that $w_i(\gamma_1, \ldots, \gamma_r) \in Vh_iV$ for $i = 1, \ldots, l$. Hence, for some smaller neighborhood $U \subset V \subset H$ and for any selection of points $y_j \in U\gamma_j U,\ j = 1, \ldots, r$, we will have $w_i(y_1, \ldots, y_r) \in Vh_iV$ for $i = 1, \ldots, l$. But then $\langle y_1, \ldots, y_r \rangle$ is dense in G, since its intersection with the normal subgroup H is dense in H, and its projection to G/H coincides with the projection of Γ to G/H.

Let $R \leq G^\circ$ be the solvable radical of G°. The group G/R is a semisimple Lie group with connected component G°/R and H clearly projects onto G°/R. Composing the projection $G \to G/R$ with the adjoint representation of G/R on its Lie algebra $\mathfrak{v} = Lie(G^\circ/R)$, we get a homomorphism $\pi : G \to \mathrm{GL}(\mathfrak{v})$. The image $\pi(G)$ is open in the group of real points of some real algebraic group $\mathbb{H}$ whose connected identity component $\mathbb{H}^\circ$ is semisimple. Moreover $\pi(\Gamma)$ is dense in $\pi(G)$. Let $m = |\mathbb{H}/\mathbb{H}^\circ|$ and let $g_1, \ldots, g_m \in \Gamma$ be elements which are sent under π to representatives of all cosets of $\mathbb{H}^\circ$ in $\mathbb{H}$. Let U_0 be an even smaller symmetric identity neighborhood $U_0 \subset U \subset H$ such that $U_1^{4m+2} \subset U$ where $U_1 = \cup_{j=1}^m g_j U_0 g_j^{-1}$, and set $\Omega_0 = \pi(U_0 \cap \Gamma)$. Then the conditions of Theorem 4.3 are satisfied, since Ω_0 is Zariski dense in $\mathbb{H}^\circ(\mathbb{R})$ (see [4] Lemma 5.2 applied to H). Thus we can choose $\alpha_i \in \Gamma \cap U_1^{4m+2}\gamma_i U_1$ which generate a free group $\langle \alpha_1, \ldots, \alpha_r \rangle$. It will also be dense by the discussion above.

5.2. **The p-adic case.** Suppose now that k is a non-Archimedean local field of characteristic 0, i.e. it is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ for some prime $p \in \mathbb{N}$. Let $\mathcal{O} = \mathcal{O}_k$ be the valuation ring of k and $\mathfrak{p}$ its maximal ideal. Let $\Gamma \leq \mathrm{GL}_n(k)$ be a finitely generated linear group over k, let G be the closure of Γ in $\mathrm{GL}_n(k)$. Let $G(\mathcal{O}) = G \cap \mathrm{GL}_n(\mathcal{O})$ (and $\Gamma(\mathcal{O}) = \Gamma \cap G(\mathcal{O})$) and denote by $\mathrm{GL}_n^1(\mathcal{O})$ the first congruence subgroup, i.e. the kernel of the homomorphism $\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{p})$. The subgroup $G^1(\mathcal{O}) = G \cap \mathrm{GL}_n^1(\mathcal{O})$ is an open compact subgroup of G and is a p-adic analytic pro-p group. The group $\mathrm{GL}_n(\mathcal{O})$ has finite rank (i.e. there is an upper bound on the minimal number of topological generators for all closed subgroups of $\mathrm{GL}_n(\mathcal{O})$) as it follows for instance from Theorem 5.2 in [13]. Consequently, $G(\mathcal{O})$ itself is finitely generated as a pro-finite group and it contains the finitely generated pro-p group $G^1(\mathcal{O})$ as a subgroup of finite index. This implies that the Frattini subgroup $F \leq G(\mathcal{O})$ (the intersection of all maximal open subgroups of $G(\mathcal{O})$) is open (normal), hence of finite index in $G(\mathcal{O})$ (Proposition 1.14 in [13]). In this situation, generating a dense group in G is an open condition. More precisely:

Lemma 5.2. *Suppose $x_1, \dots, x_r \in G$ generate a dense subgroup of G, then there is a neighborhood of the identity $U \subset G$, such that for any selection of points $y_i \in Ux_iU$, $1 \le i \le r$, the y_i's generate a dense subgroup of G.*

Proof. Note that a subgroup of the pro-finite group $G(\mathcal{O})$ is dense if and only if it intersects every coset of the Frattini subgroup F. Now since $\langle x_1, \dots, x_r\rangle$ is dense, there are $l = [G(\mathcal{O}) : F]$ words $\{w_i\}_{i=1}^{l}$ on r letters, such that the $w_i(x_1, \dots, x_r)$'s are representatives of all cosets of F in $G(\mathcal{O})$. But then, if U is small enough, and $y_i \in Ux_iU$, the elements $w_i(y_1, \dots, y_r)$ form again a full set of representatives for the cosets of F in $G(\mathcal{O})$. This implies that $\langle y_1, \dots, y_r\rangle \cap G(\mathcal{O})$ is dense in $G(\mathcal{O})$. Now if we assume further that U lies inside the open subgroup $G(\mathcal{O})$, then we have $x_i \in \overline{G(\mathcal{O})y_iG(\mathcal{O})}$, hence $x_i \in \overline{\langle y_1, \dots, y_r\rangle}$ for $i = 1, \dots, r$. This implies that $G = \overline{\langle y_1, \dots, y_r\rangle}$. $\square$

The proof of the theorem now follows easily. Let $\{x_1, \dots, x_r\}$ be a generating set for Γ. Choose U as in the lemma, and take it to be an open subgroup. Hence it satisfies $U^l = U$ for all $l \ge 1$. By Lemma 4.5 we have a representation $\rho : \Gamma \to \mathbb{H}$ into some semisimple k-algebraic group $\mathbb{H}$ such that the image of $U \cap \Gamma$ is Zariski dense in $\mathbb{H}^0$. Thus we can use Theorem 4.3 in order to find elements $\alpha_i \in \Gamma \cap Ux_iU$ that generate a free group. It will be dense by Lemma 5.2.

5.3. **The positive characteristic case.** Finally, consider the case where k is a local field of characteristic $p > 0$, i.e. a field of formal power series $\mathbb{F}_q[[t]]$ over some finite field extension $\mathbb{F}_q$ of $\mathbb{F}_p$. First, we do not suppose that Γ is finitely generated (in particular in the lemmas below). We use the same notations as those introduced at the beginning of the last Paragraph 5.2 in the p-adic case. In particular G is the closure of Γ, $G(\mathcal{O})$ is the intersection of G with $\mathrm{GL}_n(\mathcal{O})$ where $\mathcal{O}$ is the valuation ring of k. In positive characteristic, we have to deal with the additional difficulty that, even when Γ is finitely generated, $G(\mathcal{O})$ may not be topologically finitely generated.

However, when $G(\mathcal{O})$ is topologically finitely generated, then the argument used in the p-adic case (via Lemmas 4.5 and 5.2) applies here as well without changes. In particular, if $\overline{\Gamma}$ is compact, then we do not have to take more than $d(\Gamma)$ generators for the dense free subgroup. This fact will be used in Section 8. We thus have:

Proposition 5.3. *Let k be a non-Archimedean local field and let $\mathcal{O}$ be its valuation ring. Let $\Gamma \le GL_n(\mathcal{O})$ be a finitely generated group which is not virtually solvable, then Γ contains a dense free group F_r for any $r \ge d(\Gamma)$.*

Moreover, it is shown in [2] that if k is a local field of positive characteristic, and $G = \mathbb{G}(k)$ for some semisimple simply connected k-algebraic group $\mathbb{G}$, then G and $G(\mathcal{O})$ are finitely generated. Thus, the above proof applies also to this case and we obtain:

Proposition 5.4. *Let k be a local field of positive characteristic, and let G be the group of k points of some semisimple simply connected k-algebraic group. Let Γ be a finitely generated dense subgroup of G, then Γ contains a dense F_r for any $r \geq d(\Gamma)$.*

Let us now turn to the general case, when $G(\mathcal{O})$ is not assumed topologically finitely generated. As above, we denote by $\mathrm{GL}_n^1(\mathcal{O})$ the first congruence subgroup $\mathrm{Ker}\big(\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{p})\big)$. This group is pro-$p$ and, as it is easy to see, the elements of torsion in $\mathrm{GL}_n^1(\mathcal{O})$ are precisely the unipotent matrices. In particular the order of every torsion element is $\leq p^n$. Moreover, every open subgroup of $\mathrm{GL}_n^1(\mathcal{O})$ contains elements of infinite order. Hence the torsion elements are not Zariski dense in $\mathrm{GL}_n^1(\mathcal{O})$. More generally we have:

Lemma 5.5. *Let k be a non-Archimedean local field of arbitrary characteristic and n a positive integer. There is an integer m such that the order of every torsion element in $\mathbb{GL}_n(k)$ divides m. In particular, if $\mathbb{H}$ is a semisimple algebraic k-group, then the set of torsion elements in $\mathbb{H}(k)$ is contained in a proper subvariety.*

Proof. Let $\mathrm{char}(k) = p \geq 0$. Suppose $x \in \mathbb{GL}_n(k)$ is an element of torsion, then x^{p^n} (resp. x if $char(k) = 0$) is semisimple. Since the minimal polynomial of x is of degree at most n, its eigenvalues, which are roots of unity, lie in an extension of degree at most n of k. But, as k is a local field, there are only finitely many such extensions. Moreover, in a given non-Archimedean local field, there are only finitely many roots of unity. Hence there is an integer m such that $x^m = 1$.

The last claim follows from the obvious fact that if $\mathbb{H}$ is semisimple then there are elements if infinite order in $\mathbb{H}(k)$. □

Let $\rho : \Gamma \to \mathbb{H}$ be the representation given by Lemma 4.5. Then for any sufficiently small open subgroup U of $\mathrm{GL}_n(\mathcal{O})$ (for instance some small congruence subgroup), $\rho\big(\Gamma \cap U\big)$ is Zariski dense in $\mathbb{H}^\circ$. We then have (Γ is not assumed finitely generated):

Lemma 5.6. *There are $t := \dim(\mathbb{H})$ elements $x_1, \ldots, x_t \in \Gamma \cap G(\mathcal{O})$ such that $\rho\big(\langle x_1, \ldots, x_t \rangle\big)$ is Zariski dense in $\mathbb{H}^0$.*

Proof. Let U be an open subgroup of $\mathrm{GL}_n(\mathcal{O})$ so that $\rho\big(\Gamma \cap U\big)$ lies in $\mathbb{H}^\circ$ and is Zariski dense in it. It follows from the above lemma that there is

$x_1 \in \Gamma \cap U$ such that $\rho(x_1)$ is of infinite order. Then the algebraic group $A = \overline{\langle x_1 \rangle}^z$ is at least one dimensional.

Let the integer i, $1 \leq i \leq t$, be maximal for the property that there exist i elements $x_1, \ldots, x_i \in \Gamma \cap U$ whose images in $\mathbb{H}$ generate a group whose Zariski closure is of dimension $\geq i$. We have to show that $i = t$. Suppose this is not the case. Fix such $x_1, \ldots, x_i$ and let A be the Zariski connected component of identity of $\overline{\langle \rho(x_1), \ldots, \rho(x_i) \rangle}^z$. Then for any $x \in \Gamma \cap U$, $\overline{\langle \rho(x_1), \ldots, \rho(x_i), \rho(x) \rangle}^z$ is i-dimensional. This implies that $\rho(x)$ normalizes A. Since $\rho\big(\Gamma \cap U\big)$ is Zariski dense in $\mathbb{H}^0$, we see that A is a normal subgroup of $\mathbb{H}^\circ$. Dividing $\mathbb{H}^\circ$ by A we obtain a Zariski connected semisimple k-group of positive dimension, and a map from $\Gamma \cap U$ with Zariski dense image into the k-points of this semisimple group. But then, again by Lemma 5.5 above, there is an element $x_{i+1} \in \Gamma \cap U$ whose image in $\mathbb{H}^\circ / A$ has infinite order — a contradiction to the maximality of i. □

End of the proof in the positive characteristic case. Suppose now that Γ is finitely generated and let Δ be the closure in G of the subgroup generated by $x_1, \ldots, x_t$ given by Lemma 5.6 above. It is a topologically finitely generated pro-finite group containing $\Delta \cap G^1(\mathcal{O})$ as a subgroup of finite index. Hence the Frattini subgroup F of Δ (i.e. the intersection of all maximal open subgroups) is open and of finite index ([13] Proposition 1.14). In particular, $\rho(F \cap \langle x_1, \ldots, x_t \rangle)$ is Zariski dense in $\mathbb{H}^\circ$, and we can use Theorem 4.3 with $\Omega_0 = \rho(F \cap \langle x_1, \ldots, x_t \rangle)$. Note that F, being a group, satisfies $F^m = F$ for $m \in \mathbb{N}$. Also F is normal in Δ. Let $\gamma_1, \ldots, \gamma_r$ be generators for Γ. By Theorem 4.3, we can choose $\alpha_i \in F\gamma_i F,\ i = 1, \ldots, r$, and $\alpha_{i+r} \in x_i F,\ i = 1, \ldots, t$, so that $D = \langle \alpha_1, \ldots, \alpha_{r+t} \rangle$ is isomorphic to the free group F_{r+t} on $r+t$ generators. Clearly $D \cap \Delta$ is dense in Δ and $D \cap F$ is dense in F. This implies that each γ_i lies in $F\alpha_i F \subset \overline{D}$. As the γ_i's generate Γ, we see that D is dense in $\overline{\Gamma}$ and this finishes the proof. □

5.4. **Some stronger statements.** In this section we shall formulate some stronger statements which follow from the proof of Theorem 1.3. In case $\mathrm{char}(k) = 0$, the proof actually gives the following:

Theorem 5.7. *Let k be a local field of characteristic 0 and let $\Gamma \leq GL_n(k)$ be a finitely generated group containing no open solvable subgroup. Suppose that $\Gamma = \langle \gamma_1, \ldots, \gamma_d \rangle$ then for any identity neighborhood $\Omega \subset \Gamma$, there are $\alpha_i \in \Omega\gamma_i\Omega$ for $i = 1, \ldots, d$ such that $\alpha_1, \ldots, \alpha_d$ are free generators of a dense free subgroup of Γ.*

The argument in the previous section combined with the argument of [4], Section 2 provides the following generalization:

Theorem 5.8. *Let k be a local field and let $G \leq GL_n(k)$ be a closed linear group containing no open solvable subgroup. Assume also that $G(\mathcal{O})$ is topologically finitely generated in case k is non-Archimedean of positive characteristic. Then there is an integer $h(G)$ which satisfies*

- *$h(G) \leq 2\dim(G) - 1 + d(G/G^\circ)$ if k is Archimedean, and*
- *$h(G)$ is the minimal cardinality of a set generating a dense subgroup of G if k is non-Archimedean,*

such that any finitely generated dense subgroup $\Gamma \leq G$ contains a dense F_r, for any $r \geq \min\{d(\Gamma), h(G)\}$. Furthermore, if k is non-Archimedean or if G° is topologically perfect (i.e. $\overline{[G^\circ, G^\circ]} = G^\circ$) and $d(G/G^\circ) < \infty$, then we can drop the assumption that Γ is finitely generated. In these cases, any dense subgroup Γ in G contains a dense F_r for any $r \geq h(G)$.

Remark 5.9. The interested reader is referred to [4] for a sharper estimation of $h(G)$ in the Archimedean case. For instance, if G is a connected and semisimple real Lie group, then $h(G) = 2$.

Let us also remark that in the characteristic zero case, we can drop the linearity assumption, and assume only that Γ is a subgroup of some second countable k-analytic Lie group. To see this, simply note that the procedure of generating a dense subgroup does not rely upon the linearity of $G = \overline{\Gamma}$, and for generating a free subgroup, we can look at the image of G under the adjoint representation which is a linear group. The main difference in the positive characteristic case is that we do not know in that case whether or not the image $\mathrm{Ad}(G)$ is solvable. For this reason we make the additional linearity assumption in positive characteristic.

6. Dense free subgroups with infinitely many generators

In this section we shall prove the following:

Theorem 6.1. *Let k be a local field and $\Gamma \leq GL_n(k)$ a linear group over k. Assume that Γ contains no open solvable subgroup, then Γ contains a countable dense free subgroup of infinite rank.*

As above, we denote by G the closure of Γ. Since $\mathrm{GL}_n(k)$ and hence G is second countable, we can assume that Γ is countable. If k is non-Archimedean we let $\mathcal{O}$ denote the valuation ring of k, $G(\mathcal{O}) := G \cap \mathrm{GL}_n(\mathcal{O})$ the corresponding open pro-finite group and $\Gamma(\mathcal{O}) = \Gamma \cap \mathrm{GL}_n(\mathcal{O})$. Set $G_j := G \cap \mathrm{GL}_n^j(\mathcal{O})$ where $\mathrm{GL}_n^j(\mathcal{O}) = \mathrm{Ker}\big(\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{p}^j)\big)$ is the j'th congruence subgroup, and write $\Gamma_j := \Gamma \cap G_j$. In order to treat both the Archimedean and the non-Archimedean cases at the same time, we will say *by convention* that in the Archimedean case, Γ_j denotes always the same group $H \cap \Gamma$ where H is the limit of the sequence of closed commutators introduced in Paragraph 5.1.

We now fix once and for all a sequence (x_j) of elements of Γ which is dense in G. In the non-Archimedean case, we can also require that the $G_{k_j} x_j G_{k_j}$'s form a base for the topology of G for some choice of a sequence of integers (k_j). We are going to perturb the x_i's by choosing elements y_i's inside $\Gamma_{k_j} x_j \Gamma_{k_j}$ which play ping-pong all together on some projective space, and hence generate a dense free group.

From Lemma 4.5 in the non-Archimedean case, and from the discussion in Paragraph 5.1 in the Archimedean case, we have a homomorphism $\pi : \Gamma \to \mathbb{H}(k)$, where $\mathbb{H}$ is an algebraic k-group with $\mathbb{H}^\circ$ semisimple, such that the Zariski closure of $\pi(\Gamma_j)$ contains $\mathbb{H}^\circ$ for all $j \geq 1$. It now follows from Lemma 5.6 when $char(k) > 0$ and from the discussion in Paragraphs 5.1 and 5.2 in the other cases (i.e. from the fact that H and G_j are topologically finitely generated) that Γ_1 contains a finitely generated subgroup Δ_1 such that $\pi(\Delta_1)$ is also Zariski dense in $\mathbb{H}^\circ$ (we also take Δ_1 to be dense in H when k is Archimedean). From Theorem 4.3 we can find a local field k' and an irreducible projective representation ρ of $\mathbb{H}$ on $\mathbb{P}(V_{k'})$ defined over k' such that, under this representation, some elements of Δ_1 play ping-pong in the projective space $\mathbb{P}(V_{k'})$. In particular, for some $r > 0$ and for every positive $\epsilon < \frac{r}{2}$, there is an element in Δ_1 acting on $\mathbb{P}(V_{k'})$ by an (r, ϵ)-very proximal transformation (c.f. Paragraph 3.1). Furthermore, there is a field extension K of k' such that under this representation the full group Γ is map into $\mathrm{PGL}(V_K)$ where $V_K = V_{k'} \otimes K$. This field extension may not be finitely generated. Nevertheless, the absolute value on k' extends to an absolute value on K (see [18] XII, 4, Theorem 4.1 p. 482) and the projective space $\mathbb{P}(V_K)$ is still a metric space (although not compact in general) for the metric introduced in Paragraph 3.1. Moreover, if $[g] \in \mathrm{PGL}(V_{k'})$ is ϵ-contracting on $\mathbb{P}(V_{k'})$, it is $c\epsilon$-contracting on $\mathbb{P}(V_K)$ for some constant $c = c(k', K) \geq 1$. Similarly, if $[g] \in \mathrm{PGL}_n(V_{k'})$ is (r, ϵ)-proximal transformation on $\mathbb{P}(V_{k'})$ then it is $(\frac{r}{c}, c\epsilon)$-proximal on $\mathbb{P}(V_K)$. Let $\rho : \Gamma \to \mathrm{PGL}(V_K)$ be this representation. (The reason why we may not reduce to the case where K is local is that there may not be a finitely generated dense subgroup in Γ.)

In the Archimedean case, the discussion in Paragraph 5.1 shows that we can find inside Δ_1 elements $z_1, \ldots, z_l$, generating a dense subgroup of $\Gamma_1 = H \cap \Gamma$, and such that, under the above representation, they act as a ping-pong l-tuple of projective transformations. We can find another element $g \in \Delta_1$ such that $(z_1, \ldots, z_l, g)$ acts as a ping-pong $(l+1)$-tuple and g acts as an (r, ϵ)-very proximal transformation on $\mathbb{P}(V_{k'})$ where the pair (r, ϵ) satisfies the conditions of Lemma 3.1 with respect to k'. In the non-Archimedean case, let simply g be some element of Δ_1 acting as an (r, ϵ)-very proximal transformation on the projective space $\mathbb{P}(V_{k'})$ with (r,ϵ) as in Lemma 3.1. As follows from Lemma 3.1, g (resp. g^{-1}) fixes an attracting point $\overline{v}_g$ (resp. $\overline{v}_{g^{-1}}$) and a repelling hyperplane $\overline{H}_g$ (resp. $\overline{H}_{g^{-1}}$) and the positive (resp. negative) powers g^n behave as $(\frac{r}{C},(C\epsilon)^{\frac{n}{3}})$-very proximal transformations with respect to these same attracting points and repelling hyperplanes. Note that in the non-Archimedean case, if n_j is the index of the j'th congruence subgroup G_j in $G(\mathcal{O})$, then $g^{n_j}\in G_j$ and in particular $g^{n_j}\to 1$ as j tends to infinity.

We are now going to construct an infinite sequence (g_j) of elements in Δ_1 acting on $\mathbb{P}(V_{k'})$ by very proximal transformations and such that they play ping-pong all together on $\mathbb{P}(V_{k'})$ (and also together with $z_1,\ldots,z_l$ in the Archimedean case). Since $\pi(\Delta_1)$ is Zariski dense in $\mathbb{H}^\circ$ and the representation ρ of $\mathbb{H}^\circ$ is irreducible, we may pick an element $\gamma\in\Delta_1$ such that

$$\{\rho(\gamma)\overline{v}_g,\ \rho(\gamma)\overline{v}_{g^{-1}},\ \rho(\gamma^{-1})\overline{v}_g,\ \rho(\gamma^{-1})\overline{v}_{g^{-1}}\}\cap\big(\overline{H}_g\cup\overline{H}_{g^{-1}}\cup\{\overline{v}_g,\overline{v}_{g^{-1}}\}\big)=\emptyset.$$

Now consider the element $\delta_{m_1}=g^{m_1}\gamma g^{m_1}$. When m_1 is large enough, δ_{m_1} acts on $\mathbb{P}(V_{k'})$ under ρ as a very proximal transformation, whose repelling neighborhoods lie inside the ϵ-repelling neighborhood of g and whose attracting points lies inside the ϵ-attracting neighborhood of g. We can certainly assume that $\rho(\delta_{m_1})$ satisfies the conditions of Lemma 3.1. Hence δ_{m_1} fixes some attracting points $\overline{v}_{\delta_{m_1}},\overline{v}_{\delta_{m_1}^{-1}}$ which are close to, but distinct from $\overline{v}_g,\overline{v}_{g^{-1}}$ respectively. Similarly the repelling neighborhoods of $\delta_{m_1},\delta_{m_1}^{-1}$ lie inside the ϵ-repelling neighborhood of g,g^{-1}, and the repelling hyperplanes $\overline{H}_{\delta_{m_1}},\overline{H}_{\delta_{m_1}^{-1}}$ are close to that of g. We claim that for all large enough m_1

$$(3)\quad \{\overline{v}_g,\overline{v}_{g^{-1}}\}\cap\big(\overline{H}_{\delta_{m_1}}\cup\overline{H}_{\delta_{m_1}^{-1}}\big)=\emptyset,\ \text{ and } \{\overline{v}_{\delta_{m_1}},\overline{v}_{\delta_{m_1}^{-1}}\}\cap\big(\overline{H}_g\cup\overline{H}_{g^{-1}}\big)=\emptyset.$$

Let us explain, for example, why $\overline{v}_{g^{-1}}\notin\overline{H}_{\delta_{m_1}}$ and why $\overline{v}_{\delta_{m_1}}\notin\overline{H}_{g^{-1}}$ (the other six conditions are similarly verified). Apply δ_{m_1} to the point $\overline{v}_{g^{-1}}$. As g stabilizes $\overline{v}_{g^{-1}}$ we see that

$$\delta_{m_1}(\overline{v}_{g^{-1}})=g^{m_1}\gamma g^{m_1}(\overline{v}_{g^{-1}})=g^{m_1}\gamma(\overline{v}_{g^{-1}}).$$

Now, by our assumption, $\gamma(\overline{v}_{g^{-1}})\notin\overline{H}_g$. Moreover when m_1 is large, g^{m_1} is a ϵ_{m_1}-contracting with $\overline{H}_{g^{m_1}}=\overline{H}_g$, $\overline{v}_{g^{m_1}}=\overline{v}_g$ and ϵ_{m_1} arbitrarily small. Hence, we may assume that $\gamma(\overline{v}_{g^{-1}})$ is outside the ϵ_{m_1} repelling neighborhood of g^{m_1}. Hence $\delta_{m_1}(\overline{v}_{g^{-1}})=g^{m_1}\big(\gamma(\overline{v}_{g^{-1}})\big)$ lie near $\overline{v}_g$ which is far from $\overline{H}_{\delta_{m_1}}$. Since $\overline{H}_{\delta_{m_1}}$ is invariant under δ_{m_1}, we conclude that $\delta_{m_1}\overline{v}_{g^{-1}}\notin\overline{H}_{\delta_{m_1}}$.

To show that $\overline{v}_{\delta_{m_1}}\notin\overline{H}_{g^{-1}}$ we shall apply g^{-2m_1} to $\overline{v}_{\delta_{m_1}}$. If m_1 is very large then $\overline{v}_{\delta_{m_1}}$ is very close to $\overline{v}_g$, and hence also $g^{m_1}(\overline{v}_{\delta_{m_1}})$ is very close to $\overline{v}_g$. As we assume that γ takes $\overline{v}_g$ outside $\overline{H}_{g^{-1}}$, we get (by taking m_1 sufficiently large) that γ also takes $g^{m_1}\overline{v}_{\delta_{m_1}}$ outside $\overline{H}_{g^{-1}}$. Taking m_1 even larger if necessary we get that g^{-m_1} takes $\gamma g^{m_1}\overline{v}_{\delta_{m_1}}$ to a small neighborhood of $\overline{v}_{g^{-1}}$. Hence

$$g^{-2m_1}\overline{v}_{\delta_{m_1}} = g^{-2m_1}\delta_{m_1}\overline{v}_{\delta_{m_1}} = g^{-m_1}\gamma g^{m_1}\overline{v}_{\delta_{m_1}}$$

lies near $\overline{v}_{g^{-1}}$. Since $\overline{H}_{g^{-1}}$ is g^{-2m_1} invariant and is far from $\overline{v}_{g^{-1}}$, we conclude that $\overline{v}_{\delta_{m_1}} \notin \overline{H}_{g^{-1}}$.

Now it follows from (3) and Lemma 3.1 that for every $\epsilon_1 > 0$ we can take j_1 sufficiently large so that g^{j_1} and $\delta_{m_1}^{j_1}$ are ϵ_1-very proximal transformations, and the ϵ_1-repelling neighborhoods of each of them are disjoint from the ϵ_1-attracting points of the other, and hence they form a ping-pong pair. Set $g_1 = \delta_{m_1}^{j_1}$.

In a second step, we construct g_2 in an analogous way to the first step, working with g^{j_1} instead of g. In this way we would get g_2 which is ϵ_2-very proximal, and play ping-pong with $g^{j_1 j_2}$. Moreover, by construction, the ϵ_2-repelling neighborhoods of g_2 lie inside the ϵ_1-repelling neighborhoods of g^{j_1}, and the ϵ_2-attracting neighborhoods of g_2 lie inside the ϵ_1-attracting neighborhoods of g^{j_1}. Hence the three elements g_1, g_2 and $g^{j_1 j_2}$ form a ping-pong 3-tuple.

We continue recursively and construct the desired sequence (g_n). Note that in the Archimedean case, we have to make sure that the g_n's form a ping-pong $\aleph_0$-tuple also when we add to them the finitely many z_i's. This can be done by declaring $g_i = z_i$ for $i = 1, \dots l$, and starting the recursive argument by constructing g_{l+1}.

Now since all $\Gamma_{k_j} = G_{k_j} \cap \Gamma$'s are mapped under the homomorphism π to Zariski dense subsets of $\mathbb{H}^\circ$, we can multiply x_j on the left and on the right by some elements of Γ_{k_j} so that, if we call this new element x_j again, $\rho(x_j)\overline{v}_{g_j} \notin \overline{H}_{g_j}$ and $\rho(x_j^{-1})\overline{v}_{g_j^{-1}} \notin \overline{H}_{g_j^{-1}}$. Considering the element $y_j = g_j^{l_j} x_j g_j^{l_j}$ for some positive power l_j, we see that it lies in $\Gamma_{k_j} x_j \Gamma_{k_j}$. Moreover, if we take l_j large enough, it will behave on $\mathbb{P}(V_K)$ like a very proximal transformation whose attracting and repelling neighborhoods are contained in those of g_j. Therefore, the y_j's also form an infinite ping-pong tuple and in the Archimedean case they do so together with $z_1, \dots, z_l$. Hence the family $(y_j)_j$ (resp. $(z_1, \dots, z_l, (y_j)_j)$) generates a free group.

In the non-Archimedean case, the y_j's are already dense in G since we selected them from sets which form a base for the topology. In the Archimedean case, the elements $z_1, \dots, z_k$ already generate a dense subgroup of H, and since the g_j's belong to H and the x_j's are dense. Hence the group generated by the z_i's and y_j's is dense in G. This completes the proof of Theorem 6.1.

7. Multiple fields, adelic versions and other topologies

In this section we prove Statements 1.1, 1.4, 1.5 and 1.1 from the introduction.

Corollary 7.1. *(Corollary 1.1 from the introduction) Let k be a local field of characteristic* 0 *and let G be a k-analytic Lie group with no open solvable subgroup. Then G contains a dense free subgroup F. If additionnally G contains a dense subgroup generated by k elements, then F can be taken to be a free group of rank r for any $r \geq k$.*

This fairly general fact had been known earlier in some cases including the case of connected real Lie groups. For connected Lie groups, a very short argument can be given using analyticity and the Baire category theorem because the existence of a free subgroup implies that the locus of vanishing of a given non trivial word in k letters has empty interior in G^k (see Kuranishi [17] for the original reference and also [14] for a more recent reference).

Theorem 7.1. *(Theorem 1.4 from the introduction) Let $k_1, \ldots, k_r$ be local fields and let Γ be a (finitely generated) subgroup of $\Pi_{i=1}^r GL_n(k_i)$. Assume that Γ does not contain an open solvable subgroup, then it contains a dense free subgroup (of finite rank).*

Proof. We may assume that $k_1 = \mathbb{R}$ and $k_i,\ i > 1$ are non-Archimedean. Denote by G the closure of Γ, by $G_{\mathbb{R}}$ the closure of its projection to $\mathrm{GL}_n(k_1)$ and by G_d the closure of its projection to $\Pi_{i>1}\mathrm{GL}_n(k_i)$. Let $G^1(\mathcal{O})$ be the closure of the set of elements of Γ whose real coordinate lies in $G_{\mathbb{R}}^{\circ}$ while the other coordinates belong to the product of the first congruence subgroups $\Pi_{i>1}\mathrm{GL}_n^1(\mathcal{O}_{k_i})$. This is an open subgroup of G and $\Gamma \cap G^1(\mathcal{O})$ does not contain an open solvable subgroup. We shall distinguish two cases:

(1) $G_{\mathbb{R}}^{\circ}$ is solvable (including the case where $G_{\mathbb{R}}$ is trivial).
(2) $G_{\mathbb{R}}^{\circ}$ is not solvable.

In the first case, for one of the non-Archimedean field say k_i, we can find a homomorphism ρ of Γ into a k_i-algebraic group $\mathbb{H}$, whose Zariski-connected component $\mathbb{H}^{\circ}$ is semi-simple, by first factoring through $\mathrm{GL}_n(k_i)$ and then applying Lemma 4.5. Then the Zariski closure of $\rho(\Gamma \cap G^1(\mathcal{O}))$ will contain $\mathbb{H}^{\circ}(k_i)$, and, as can be seen by taking successive commutators, so will the Zariski closure of $\rho(A)$ where A is the subgroup of $\Gamma \cap G^1(\mathcal{O})$ consisting of elements whose real coordinate is trivial. Now by Lemma 5.6, we can pick finitely many element $x_1, \ldots, x_t \in A$ such that $\rho(\langle x_1, \ldots, x_t \rangle)$ is Zariski dense in $\mathbb{H}^{\circ}(k_i)$. If Γ is finitely generated, the end of the argument follows *verbatim* the end of the proof of Theorem 1.3 in the positive characteristic case (see above §5.3). And if Γ is not finitely generated we continue the argument as in Section 6. Note that $\Delta := \overline{\langle x_1, \ldots, x_t \rangle}$ is a topologically finitely generated profinite group and that its Frattini subgroup is the intersection of Δ with the product of the Frattini subgroups of the projections of Δ to the $\mathrm{GL}_n^1(\mathcal{O}_{k_j})$'s, hence it is also open in Δ.

In the second case, let $H \leq G_{\mathbb{R}}^{\circ}$ be the topologically perfect subgroup introduced in the proof of Theorem 1.3 in the Archimedean case (H is the intersection of all closed commutator subgroups of $G_{\mathbb{R}}^{\circ}$). Then $\Gamma \cap G^1(\mathcal{O})$ projects densely to H, and if the Lie algebra $\mathrm{Lie}(H)$ is generated by t elements, we can find $x_1, \ldots, x_t \in \Gamma \cap G^1(\mathcal{O})$ whose projection to H generate H topologically. It is then easy to see, by taking subgroups of arbitrarily large index, that $H \subset \overline{\langle x_1, \ldots, x_t \rangle}$ (where $H = H \times \{1\}$ is viewed as a subgroup of G). We now find a homomorphism ρ of Γ into an $\mathbb{R}$-algebraic group $\mathbb{H}$, whose Zariski-connected component $\mathbb{H}^{\circ}$ is semi-simple, by first factoring through $G_{\mathbb{R}}$. Denote by Δ the closure of the projection to G_d of $\langle x_1, \ldots, x_t \rangle$. It is a topologically finitely generated profinite group and again its Frattini subgroup is open in Δ. Moreover, we have $\overline{\langle x_1, \ldots, x_t \rangle} = H \cdot \Delta$. With these notations, we can proceed, arguing *verbatim* as in the proof of Theorem 1.3 in the positive characteristic case when Γ is assumed finitely generated, and as in Section 6 when Γ is infinitely generated. □

Proposition 7.2. *(Proposition 1.5 from the introduction) Let K be an algebraic number field and $\mathbb{G}$ a simply connected semisimple algebraic group defined over K. Let V_K be the set of all valuations of K. Then for any $v_0 \in V_K$ such that $\mathbb{G}$ in not K_{v_0} anisotropic, $\mathbb{G}(K)$ contains a free subgroup of infinite rank whose image under the diagonal embedding is dense in the restricted topological product corresponding to $V_K \setminus \{v_0\}$.*

Proof. Let $G = G_\infty G_f$ be the restricted topological product corresponding to $V_K \setminus \{v_0\}$ where G_∞ (resp. G_f) corresponds to the product of the Archimedean (resp. non-Archimedean) places. By the strong approximation theorem (see [23] Theorem 7.12) $\mathbb{G}(K)$ is dense in G. We choose one non-Archimedean place v and proceed as in the proof of Theorem 1.3 in the infinite generated case in the last section by playing ping-pong in a representation of $\mathbb{G}(K_v)$. First, we pick two elements in $\mathbb{G}(K)$ generating a free subgroup whose projections to G_∞ generate a dense subgroup (see Remark 5.9), while their projections to G_f lie inside the profinite group (corresponding to the closure of $\mathbb{G}(\mathcal{O}_K)$). Note that the closure of the group generated by these two elements contains G_∞. Then continue as in the proof of Theorem 1.3 in the infinite generated case, to add generators of a free group of rank $\aleph_0$ whose projection to G_f exhaust a countable base for the topology, i.e. there is one generator in each basic set. The closure of the (infinitely generated) free group obtained in this way contains G_∞ and its projection to G_f is dense. Hence it is a dense free subgroup in G. □

Theorem 7.1. *Let G be a locally compact group and Γ a dense subgroup of G. Then one of the following holds:*

(*i*) Γ *contains a free group* F_2 *on two generators which is non-discrete in* G.

(*ii*) G *contains an open amenable subgroup.*

Moreover, if Γ *is assumed to be linear and finitely generated, then* (*ii*) *can be replaced by*

(*ii*)′ G *contains an open solvable subgroup.*

Proof. Let G° denote the connected component of the identity in G. Then G/G° is a totally disconnected locally compact group, and hence contains an open compact subgroup P. We may of course replace G by $P \cdot G^\circ$ and G by $\Gamma \cap P \cdot G^\circ$, and assume that G is a compact extension of a connected group. By Montgomery-Zippin theory (see [22] Theorem 4.6) there is a compact normal subgroup $K \subset G$ such that the quotient $L = G/K$ is a real Lie group. Let $f : \Gamma \to L$ be the restriction of the quotient map. If L does not contain an open solvable subgroup, then by Theorem 1.3 $f(\Gamma)$ contains a free subgroup which is dense in L. We can pick two elements in this free subgroup so that they generate a non-discrete subgroup F_2. The pre-image in G of this F_2 is a non-discrete (since K is compact) free subgroup of Γ so we are in case (*i*). On the other hand, if L contains an open solvable subgroup, then G is amenable and we have (*ii*).

To prove the last part of the Theorem, we now assume that Γ is linear and finitely generated. Up to changing G into an open subgroup of G, we can assume that L itself is solvable. Now if Γ is virtually solvable, then G contains an open solvable subgroup of finite index, so we are in case (*ii*)′. If, on the other hand, Γ is not virtually solvable, then by Tits' alternative (applicable because Γ is linear and finitely generated) it contains a non-commutative free subgroup F. However, since L is solvable, $F \cap \ker(f)$ is also a non-commutative free subgroup. Hence it contains a free subgroup of rank 2, which cannot be discrete because it lies in the compact subgroup K. So we are in case (*i*). □

Remark 7.1. *If* Γ *is not assumed to be linear, assertion* (*ii*) *just cannot be replaced by* (*ii*)′*. As A. Mann* [19] *pointed out to us, there are examples of finitely generated, residually free groups* Γ *which are not virtually solvable, although they satisfy a law (i.e. some non-trivial word in* F_k *kills all of* Γ^k*). Residually freeness ensures that* Γ *embeds densely inside its profinite completion* $G = \widehat{\Gamma}$*. However, the existence of a law shows that* G *contains no non-commutative free subgroups.*

8. Applications to pro-finite groups

We derive two conclusions in the theory of pro-finite groups. The following was conjectured by Dixon, Pyber, Seress and Shalev (see [12]):

Theorem 8.1. *Let Γ be a finitely generated linear group over some field. Assume that Γ is not virtually solvable. Then, for any integer $r \geq d(\Gamma)$, its pro-finite completion $\hat{\Gamma}$ contains a dense free subgroup of rank r.*

Proof. Let R be the ring generated by the matrix entries of the elements of Γ. It follows from the Noether normalization theorem that R can be embedded in the valuation ring $\mathcal{O}$ of some local field k. Such an embedding induces an embedding of Γ in the pro-finite group $\mathrm{GL}_n(\mathcal{O})$. By the universal property of $\hat{\Gamma}$ this embedding induces a surjective map $\hat{\Gamma} \to \overline{\Gamma} \leq \mathrm{GL}_n(\mathcal{O})$ onto the closure of the image of Γ in $\mathrm{GL}_n(k)$. Since Γ is not virtually solvable, $\overline{\Gamma}$ contains no open solvable subgroup, and hence by Theorem 1.3 (see also Proposition 5.3), $\overline{\Gamma}$ contains a dense F_r (in fact we can find such an F_r inside Γ). By Gaschütz's lemma (see [28], Proposition 2.5.4) it is possible to lift the r generators of this F_r to r elements in $\hat{\Gamma}$ generating a dense subgroup in $\hat{\Gamma}$. These lifts, thus, generate a dense F_r in $\hat{\Gamma}$. □

Let now H be a subgroup of a group G. Following [31] we define the notion of coset identity as follows:

Definition 8.2. A group G satisfies a coset identity with respect to H if there exist

- a non-trivial reduced word W on l letters,
- l fixed elements $g_1, \ldots, g_l$,

such that the identity

$$W(g_1h_1, \ldots, g_lh_l) = 1$$

holds for any $h_1, \ldots, h_l \in H$.

It was conjectured by Shalev [31] that if there is a coset identity with respect to some open subgroup in a pro-p group G, then there is also an identity in G. The following immediate consequence of Corollary 4.4 settles this conjecture in the case where G is a p-adic analytic pro-p group, and in fact, shows that a stronger statement is true in this case:

Theorem 8.3. *Let G be a p-adic analytic pro-p group. If G satisfies a coset identity with respect to some open subgroup H, then G is solvable.*

Proof. If G is not solvable then it is not virtually solvable, since any finite p-group is nilpotent. In that case Corollary 4.4 allows us to choose free generators of a free group in each coset g_iH. □

In fact the analogous statement holds also for finitely generated linear groups:

Theorem 8.4. *Let Γ be a finitely generated linear group over any field. If Γ satisfies a coset identity with respect to some finite index subgroup Δ, then Γ is virtually solvable.*

9. Applications to amenable actions

For convenience, we introduce the following definition:

Definition 9.1. We shall say that a topological group G has property (OS) if it contains an open solvable subgroup.

Our main result, Theorem 1.3, states that if Γ is a (finitely generated) linear topological group over a local field, then either Γ has property (OS) or Γ contains a dense (finitely generated) free subgroup. In the previous section we proved the analogous statement for pro-finite completions of linear groups over an arbitrary field. For real Lie groups, property (OS) is equivalent to "the identity component is solvable".

It was conjectured by Connes and Sullivan and proved subsequently by Zimmer [35] that if Γ is a countable subgroup of a real Lie group G, then the action of Γ on G by left multiplications is amenable if and only if Γ has property (OS). Note also that if Γ acts amenably on G, then it also acts amenably on G/P whenever $P \leq G$ is closed amenable subgroup. We refer the reader to [36] Chapter 4 for an introduction and background on amenable actions. The harder part of the equivalence is to show that if Γ acts amenably then it has (OS). As noted by Carrière and Ghys [10], the Connes-Sullivan conjecture is a straightforward consequence of Theorem 1.3. Let us reexplain this claim: by Theorem 1.3, it is enough to show that if Γ contains a non-discrete free subgroup, then it cannot act amenably.

Proof. (*non-discrete free subgroup$\Rightarrow$action is non-amenable*). By contradiction, if Γ were acting amenably, then any subgroup would do so too, hence we can assume that Γ itself is a non-discrete free group $\langle x, y\rangle$. By Proposition 4.3.9 in [36], it follows that there exists a Γ-equivariant Borel map $g \mapsto m_g$ from G to the space of probability measures on the boundary $\partial\Gamma$. Let X (resp. Y) be the set of infinite words starting with a non trivial power of x (resp. y). Let (ξ_n) (resp. θ_n) be a sequence of elements of Γ tending to the identity element in G and consisting of reduced words starting with y (resp. y^{-1}). The function $g \mapsto m_g(X)$ is measurable and up to passing to a subsequence if necessary, we obtain that for almost all $g \in G$, $m_{g\xi_n}(X)$ and $m_{g\theta_n}(X)$ converge to $m_g(X)$. However, for almost every $g \in G$, $m_{g\xi_n}(X) = m_g(\xi_n X)$ and $m_{g\theta_n}(X) = m_g(\theta_n X)$. Moreover $\xi_n X$ and $\theta_n X$ are disjoint subsets of Y. Hence, for almost every $g \in G$, $2m_g(X) \leq m_g(Y)$. Reversing the roles of X and Y we get a contradiction. $\square$

Clearly, this proof is valid whenever G is a locally compact group. In particular, Theorem 1.3 implies the following general result:

Theorem 9.2. *Let k be a local field, G a closed subgroup of $GL_n(k)$, P a closed amenable subgroup of G and $\Gamma \leq G$ a countable subgroup. Then the following are equivalent:*

(1) Γ *has property (OS).*
(2) *The action of Γ by left multiplications on the homogeneous space G/P is amenable.*
(3) Γ *contains no non-discrete free subgroup.*

A theorem of Auslander (see [27] 8.24) states that if G is a real Lie group, R a closed normal solvable subgroup, and Γ a subgroup with property (OS), then the image of Γ in G/R also has property (OS). Taking G to be the group of Euclidean motions, R the subgroup of translations, and $\Gamma \leq G$ a torsion free lattice, one obtains as a corollary the classical theorem of Bieberbach that any compact Euclidean manifold is finitely covered by a torus.

Following Zimmer ([35]), we remark that Auslander's theorem follows from Zimmer's theorem. To see this, note that G (hence also Γ) being second countable, we can always replace Γ by a countable dense subgroup of it. Then, if Γ has property (OS) it must act amenably on G. As R is closed and amenable, this implies that $\Gamma R/R$ acts amenably on G/R (see [36] chapter 4), which in turn implies, by Zimmer's theorem, that $\Gamma R/R$ has property (OS).

A discrete linear group is amenable if and only if it is virtually solvable. It follows that for a countable linear group over some topological field, being (OS) is the same as containing an open amenable subgroup.

Definition 9.3. We shall say that a countable topological group Γ has property (OA) if it contains an open subgroup, which is amenable in the abstract sense (i.e. amenable with respect to the discrete topology).

The following is a generalization of Zimmer's theorem:

Theorem 9.4. *Let G be a locally compact group, and let $\Gamma \leq G$ be a countable subgroup. Then the action of Γ on G by left multiplications is amenable if and only if Γ has property (OA).*

Proof. The proof makes use of the structure theory for locally compact groups (see [22]). We shall reduce the general case to the already known case of real Lie groups.

The "if" side is clear.

Assume that Γ acts amenably. Let G^0 be the identity connected component of G. Then G^0 is normal in G and $F = G/G^0$ is a totally disconnected locally compact group, and as such, has an open profinite subgroup. Since

Γ acts amenably, its intersection with an open subgroup acts amenably on the open subgroup. Therefore we can assume that G/G^0 itself is profinite. By [22] Theorem 4.6, there is a compact normal subgroup K in G such that the quotient G/K is a Lie group. Up to passing to an open subgroup of G again, we can assume that G/K is connected. Since $\Gamma \cap K$ acts amenably on K and K is amenable, $\Gamma \cap K$ is amenable (see [36], Chapter 4). Moreover, as K is amenable, Γ, and hence also $\Gamma K/K$, acts amenably on the connected Lie group G/K. We conclude that $\Gamma K/K$ has property (OS) and hence Γ has property (OA). $\square$

As an immediate corollary we obtain the following generalization of Auslander's theorem:

Theorem 9.5. *Let G be locally compact group, $R \leq G$ a closed normal amenable subgroup, and $\Gamma \leq G$ a subgroup with property (OA). Then the image of Γ in G/R has also (OA).*

Remark 9.6. The original statement of Auslander follows easily from 9.5.

As a consequence of Theorem 9.5 we derive a structural result for lattices in general locally compact groups. Let G be a locally compact group. Then G admits a unique maximal closed normal amenable subgroup P, and G/P is isomorphic (up to finite index) to a direct product

$$G/P \cong G_d \times G_c,$$

of a totally disconnected group G_d with a connected center-free semisimple Lie group without compact factors G_c (see [6] Theorem 3.3.3). Let $\Gamma \leq G$ be a lattice, then we have:

Proposition 9.7 (This result was proved in a conversation with Marc Burger)**.** *The projection of Γ to the connected factor G_c lies between a lattice in G_c to its commensurator.*

Proof. Let π, π_d, π_c denote the quotient maps from G to $G/P, G_d, G_c$ respectively. Since Γ is discrete, and hence has property (OA) it follows from Theorem 9.5 that $\Delta = \pi(\Gamma)$ also has (OA). Let $A \leq G_d$ be an open compact subgroup, and let $\Delta^0 = \Delta \cap (A \times G_c)$. Then Δ^0 has (OA) being open in Δ. Clearly, Δ commensurates Δ^0. Let Σ be the projection of Δ^0 to G_c. By Theorem 9.5 Σ has (OA), and hence also $\overline{\Sigma}$ has (OA), i.e. the identity connected component $\overline{\Sigma}^0$ is solvable. However, since the homogeneous space $G_c/\overline{\Sigma}$ clearly carries a finite G_c-invariant Borel regular measure, it follows from Borel's density theorem that $\overline{\Sigma}^0$ is normal in G_c. Since $\overline{\Sigma}^0$ is also solvable and G_c is semisimple, it follows that $\overline{\Sigma}$ is discrete. Therefore $\overline{\Sigma} = \Sigma$ and Σ is a lattice in G_c, and $\Sigma \leq \pi_c(\Gamma) \leq Comm_{G_c}(\Sigma)$. $\square$

10. The growth of leaves in Riemannian foliations

Let M be a compact manifold. A foliation $\mathcal{F}$ on M is said to be *Riemannian* if one can find a Riemannian metric on M for which the leaves of $\mathcal{F}$ are (locally) equidistant. An arbitrary Riemannian metric on M induces a corresponding Riemannian metric on each leaf of $\mathcal{F}$. Then one can ask about the volume growth of large balls lying inside a given leaf. Note that, since M is compact, two different choices of a Riemannian metric on M will lead to coarsely equivalent asymptotic behaviors for the volume of large metric balls in a given leaf. Hence one can speak about the type of growth of a leaf. The main result of this section is the following theorem which answers a question of Carrière [9] (see also [15]):

Theorem 10.1. *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. Either there is an integer $d \geq 1$ such that all leaves of $\mathcal{F}$ have polynomial growth of degree less or equal to d, or there is a G_δ-dense subset of leaves with exponential growth.*

Molino's theory assigns to every Riemannian foliation $\mathcal{F}$ a finite dimensional Lie algebra $\mathfrak{g}$, called the structural Lie algebra of $\mathcal{F}$. In Theorem 10.1, the case of polynomial growth holds when $\mathfrak{g}$ is nilpotent while the other case holds whenever $\mathfrak{g}$ is not nilpotent. In fact, in the second case, we show that the holonomy cover of an arbitrary leaf has exponential volume growth. The result then follows from the fact that a generic leaf has no holonomy (see [7]). For background and definitions about Riemannian foliations, see [7] [9], [15] and [21]. The example below show that it is possible that some leaves are compact while generic leaves have exponential growth.

Example 10.2. We give here an example of a Riemannian foliation on a compact manifold which has two compact leaves although every other leaf has exponential volume growth. This example was shown to us by E. Ghys. Let Γ be the fundamental group of a surface of genus $g \geq 2$ and $\pi : \Gamma \to SO(3)$ be a faithful representation. Such a map can be obtained, for instance, by realizing Γ as a torsion free co-compact arithmetic lattice in $SO(2,1)$ coming from a quadratic form over a number field which has signature $(2,1)$ over $\mathbb{R}$ and all of its Galois conjugate are $\mathbb{R}$-anisotropic. The group Γ acts freely and co-compactly on the hyperbolic plane $\mathbb{H}^2$ by deck transformations and it acts via the homomorphism π by rotations on the 3-sphere S^3 viewed as $SO(4)/SO(3)$. We now *suspend* π to the quotient manifold $M = (\mathbb{H}^2 \times S^3)/\Gamma$ and obtain a Riemannian foliation on M whose leaves are projections to M of each $(\mathbb{H}^2, y)$, $y \in S^3$. There are three types of leaves. The group $SO(3)$ has two fixed points on S^3, the north and south poles, each giving rise to a compact leaf $\mathbb{H}^2/\Gamma$ in M. Each $\gamma \in \Gamma \setminus \{1\}$ stabilizes a circle in S^3 and the leaf through each point of this circle other than the poles will be a hyperbolic cylinder $\mathbb{H}^2/\langle\gamma\rangle$. Any other point in S^3 has a trivial stabilizer in Γ and gives rise to a leaf isometric to $\mathbb{H}^2$. These are the generic holonomy-free leaves. Apart from the two compact leaves all others have exponential volume growth.

Following Carrière [9], [11] we define the **local growth** of a finitely generated subgroup Γ in a given connected real Lie group G in the following way. Fix a left-invariant Riemannian metric on G and consider the open ball B_R of radius $R > 0$ around the identity. Suppose that S is a finite symmetric set of generators of Γ. Let $B(n)$ be the ball of radius n in Γ for the word metric determined by S, and let $B_R(n)$ be the subset of $B(n)$ consisting of those elements $\gamma \in B(n)$ which can be written as a product $\gamma = \gamma_1 \cdot \ldots \cdot \gamma_k$, $k \leq n$, of generators $\gamma_i \in S$ in such a way that whenever $1 \leq i \leq k$ the element $\gamma_1 \cdot \ldots \cdot \gamma_i$ belongs to B_R. In this situation, we say that γ can be written as a word with letters in S which *stays all its life in* B_R. Let $f_{R,S}(n) = card(B_R(n))$. As it is easy to check, if S_1 and S_2 are two symmetric sets of generators of Γ, then there exist integers $N_0, N_1 > 0$ such that $f_{R,S_1}(n) \leq f_{R+N_0,S_2}(N_1 n)$.

Definition 10.3. The local growth of Γ in G with respect to a set S of generators and a ball B_R of radius R is the growth type of $f_{R,S}(n)$.

The growth type of $f_{R,S}(n)$ is *polynomial* if there are positive constants A and B such that $f_{R,S}(n) \leq An^B$ and is *exponential* if there are constants $C > 0$ and $\rho > 1$ such that $f_{R,S}(n) \geq C\rho^n$. It can be seen that Γ is discrete in G if and only if the local growth is bounded for any S and R.

According to Molino's theory (see [15], [21]), every Riemannian foliation $\mathcal{F}$ on a compact manifold M lifts to a foliation $\mathcal{F}'$ of the same dimension on the bundle M' of transverse orthonormal frames over M. In general, every leaf of $\mathcal{F}'$ is a Riemannian cover over its projection in $\mathcal{F}$, but when the leaf below in $\mathcal{F}$ has no holonomy, then it is actually isometric to its lift. Considering such a lifted foliation has a remarkable advantage: the new foliation $\mathcal{F}'$ is transversely parallelizable. This means that one can find $q = \mathrm{codim}(\mathcal{F}')$ vector fields on M' whose transverse parts are linearly independent at every point of M. It is easy to see that, for such a foliation, the group of automorphisms of $\mathcal{F}'$ (i.e. those diffeomorphisms of M' which send leaves to leaves) acts transitively on M' and in particular all leaves are diffeomorphic to one another. Moreover, one can show that the leaves closures are compact submanifolds that form a simple foliation defined by a locally trivial fibration $M' \to W$ for some manifold W. Hence one can reduce the situation to one leaf closure, say N'.

When restricting the foliation $\mathcal{F}'$ to N' we have a foliation with dense leaves which remains transversely parallelizable. In this situation, we can consider the Lie subalgebra $\mathfrak{t}$ of vector fields on N' that are tangent to the leaves and we define the Lie algebra $\mathfrak{g}$ to be the quotient $Norm(\mathfrak{t})/\mathfrak{t}$, where $Norm(\mathfrak{t})$ is the normalizer of $\mathfrak{t}$ within the Lie algebra of all vector fields on N'. Changing N' into another leaf closure would lead to an isomorphic Lie algebra, because leaves are conjugate by automorphisms of the foliation. The Lie algebra $\mathfrak{g}$ is called *the structural Lie algebra* of the Riemannian foliation $\mathcal{F}$.

The one-parameter groups of diffeomorphisms associated to elements of $Norm(\mathfrak{t})$ are automorphisms of the foliation and the elements of $\mathfrak{g}$ give rise to local diffeomorphisms of the local transverse manifolds. The fact that leaves are dense in N' shows that $\mathfrak{g}$ is finite dimensional (the evaluation map $X \to X_x$, from vector fields in $\mathfrak{g}$ to vectors in the quotient tangent space T_xN'/T_xL where L is the leaf at $x \in N'$, is a linear isomorphism). If G denotes the simply connected Lie group corresponding to $\mathfrak{g}$, we see that one can find an open cover $(U_i)_{i \in I}$ of N' such that the foliation $\mathcal{F}'$ is given locally on each U_i by the fibers of local submersions $f_i : U_i \to G$ in such a way that each transition map $h_{i,j} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ coincides locally with a translation by an element of G. In this situation, we say that the foliation is a *G-Lie foliation.* Thus the above discussion allows to reduce Theorem 10.1 to Lie foliations with dense leaves.

Given a base point x_0 in M we obtain a natural map $D : (\widetilde{M}, \widetilde{x_0}) \to G$ from the universal cover of M into G (the *developing map*), together with a natural homomorphism ρ of the fundamental group of M into G such that D is ρ-equivariant. The image of ρ is called the holonomy group of the Lie foliation and is a dense subgroup Γ in G when the leaves are dense. As Carrière pointed out in [9], the volume growth of any leaf of a G-Lie foliation is coarsely equivalent to the local growth of Γ in G. Hence Theorem 10.1 is a consequence of the following:

Theorem 10.4. *Let Γ be a finitely generated dense subgroup of a connected real Lie group G. If G is nilpotent then Γ has polynomial local growth (for any choice of S and R). If G is not nilpotent, then Γ has exponential local growth (for any choice of S and any R big enough).*

The rest of this section is therefore devoted to the proof of Theorem 10.4. As it turns out, Theorem 10.4 is an easy corollary of Theorem 1.3 in the case when G is not solvable. When G is solvable we can adapt the argument as shown below. The main proposition is the following:

Proposition 10.5. *Let G be a non-nilpotent connected real Lie group and Γ a finitely generated dense subgroup. For any finite set $S = \{s_1, \dots, s_k\}$ of generators of Γ, and any $\varepsilon > 0$, one can find perturbations $t_i \in \Gamma$ of the s_i, $i = 1, \dots, k$ such that $t_i \in s_iB_\varepsilon$ and the t_i's are free generators of a free semi-group on k generators.*

Before going through the proof of this proposition, let us explain how we deduce from it a proof of Theorem 10.4.

Proof of Theorem 10.4. Suppose that $\Sigma := \{g_1, \dots, g_N, h_1, \dots, h_N\}$ is a subset of B_R consisting of pairwise distinct elements such that both $\{g_1, \dots, g_N\}$ and $\{h_1, \dots, h_N\}$ are maximal $R/2$-discrete subsets of $\overline{B_R}$ (that is $d(g_i, g_j), d(h_i, h_j) \geq R/2$ if $i \neq j$). Then

$$\overline{B_R} \subset \bigcup_{1 \leq i,j \leq N} (g_i B_{R/2} \cap h_j B_{R/2}). \tag{4}$$

Lemma 10.6. *Let G be a connected real Lie group endowed with a left-invariant Riemannian metric. Let B_R be the open ball of radius R centered at the identity. Let $\Sigma = \{s_1, \dots, s_k\}$ be a finite subset of pairwise distinct elements of B_R such that*

$$\overline{B_R} \subset \bigcup_{i<j} (s_i^{-1} B_{R/2} \cap s_j^{-1} B_{R/2}). \tag{5}$$

Assume also that the elements of Σ are free generators of a free semi-group. Then any finitely generated subgroup of G containing Σ has exponential local growth.

Proof. Let $S(n)$ be the sphere of radius n in the free semi-group for the word metric determined by the generating set Σ. Let $w \in S(n) \cap B_R$. By (5) there are indices $i \neq j$ such that $w \in s_i^{-1} B_{R/2}$ and $w \in s_j^{-1} B_{R/2}$. This implies that $s_i w$ and $s_j w$ belong to $S(n+1) \cap B_R(n+1)$. All elements obtained in this way are pairwise distinct, hence $card(S(n+1) \cap B_R(n+1)) \geq 2 \cdot card(S(n) \cap B_R(n))$. This yields $card(S(n) \cap B_R(n)) \geq 2^n$ for all $n \geq 0$. □

Now observe that any small enough perturbation of the g_i's and h_i's in Σ in B_R satisfying (4) still satisfies (4). Hence we can apply Lemma 10.6 and exponential local growth for dense subgroups in non-nilpotent connected real Lie groups follows from Proposition 10.5. □

Proof of Proposition 10.5. When G is not solvable, we already know this fact from the proof of Theorem 1.3 for connected Lie groups (see Paragraph 5.1). In that case, we showed that we could even take the t_i's to generate a free subgroup. Thus we may assume that G is solvable. By Ado's theorem it is locally isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{C})$, and it is easy to check that the property to be shown in Proposition 10.5 does not change by local isomorphisms. Thus, we may also assume that $G \leq \mathrm{GL}_n(\mathbb{C})$. Let $\mathbb{G}$ be the Zariski closure of G in $GL_n(\mathbb{C})$. It is a Zariski connected solvable algebraic group over $\mathbb{C}$ which is not nilpotent. We need the following elementary lemma for $k = \mathbb{C}$.

Lemma 10.7. *Let $\mathbb{G}$ be a solvable connected algebraic k-group which is not nilpotent. Suppose it is k-split, then there is an algebraic k-morphism from $\mathbb{G}(k)$ to $\mathbb{GL}_2(k)$ whose image is the full affine group*

$$\mathbb{A}(k) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} a, b \in k \right\}. \tag{6}$$

Proof. We proceed by induction on $\dim \mathbb{G}$. We can write $G := \mathbb{G}(k) = T \cdot N$ where $T = \mathbb{T}(k)$ is a split torus and $N = \mathbb{N}(k)$ is the unipotent radical of G (see [3], Chapter III). Let Z be the center of N. It is a non trivial normal algebraic subgroup of G. If T acts trivially on Z by conjugation then G/Z is again non-nilpotent k-split solvable k-group and we can use induction. thus we may assume that T acts non-trivially on Z by conjugation. As T is split over k, its action on Z also splits, and there is a non-trivial algebraic multiplicative character $\chi : T \to \mathbb{G}_m(k)$ defined over k and a 1-dimensional subgroup Z_χ of Z such that, identifying Z_χ with the additive group $\mathbb{G}_a(k)$, we have $tzt^{-1} = \chi(t)z$ for all $t \in T$ and $z \in Z_\chi$. It follows that Z_χ is a normal subgroup in G, and we can assume that T acts trivially on N/Z_χ, for otherwise we could apply the induction assumption on G/Z_χ. For all $\gamma \in T$, this yields a homomorphism $\pi_\gamma : N \to Z_\chi$ given by the formula $\pi_\gamma(n) = \gamma n \gamma^{-1} n^{-1}$. Since T and N do not commute, π_γ is non trivial for at least one $\gamma \in T$. Fix such a γ and let N act on Z_χ by left multiplication by $\pi_\gamma(n)$. Let T act on Z_χ by conjugation. One can verify that this yields an algebraic action of the whole of G on Z_χ. Identifying Z_χ with the additive group $\mathbb{G}_a(k)$, we have that N acts unipotently and non-trivially and T acts via the non-trivial character χ. We have found a k-algebraic affine action of G on the line, and hence a k-map $\mathbb{G} \to \mathbb{A}$. Clearly this map is onto. $\square$

By Lemma 10.7, $\mathbb{G}$ surjects onto the affine group of the complex line, which we denote by $A = \mathbb{A}(\mathbb{C})$. The image of G is a real connected subgroup of A which is Zariski dense. Hence it is enough to prove Proposition 10.5 for Zariski dense connected subgroups of A. We need the following technical lemma:

Lemma 10.8. *Let Γ be a non-discrete finitely generated Zariski dense subgroup of $\mathbb{A}(\mathbb{C})$ with connected closure. Let $R \subset \mathbb{C}$ be the subring generated by the matrix entries of elements in Γ. Then there exists a sequence $(\gamma_n)_n$ of points of Γ, together with a ring embedding $\sigma : R \hookrightarrow k$ into another local field k, such that $\gamma_n = (a_n, b_n) \to (1, 0)$ in $\mathbb{A}(\mathbb{C})$ and $\sigma(\gamma_n) = \big(\sigma(a_n), \sigma(b_n)\big) \to (0, \sigma(\beta))$ in the topology of k for some number β in the field of fractions of R.*

Proof. Let $g_n = (a_n, b_n)$ be a sequence of distinct elements of Γ converging to identity in $\mathbb{A}(\mathbb{C})$ and such that $|a_n|_\mathbb{C} \leq 1$ and $a_n \neq 1$ for all integers n. From Lemma 2.1 one can find a ring embedding $\sigma : R \hookrightarrow k$ for some local field k such that, up to passing to a subsequence of g_n's, we have $\sigma(a_n) \to 0$ in k. We can assume $|\sigma(a_n)|_k < 1$ for all n. Now let $\xi = (a, b) := g_0$ and consider the element

$$\xi^m g_n \xi^{-m} = (a_n, \frac{1-a^m}{1-a} b(1-a_n) + a^m b_n).$$

Since $|a|_{\mathbb{C}} \leq 1$, the second component remains $\leq \frac{2}{|1-a|_{\mathbb{C}}} |b|_{\mathbb{C}} |1-a_n|_{\mathbb{C}} + |b_n|_{\mathbb{C}}$ for all m, and tends to 0 in $\mathbb{C}$ when $n \to \infty$ uniformly in m. Applying the isomorphism σ, we have:

$$\sigma(\xi^m g_n \xi^{-m}) = \big(\sigma(a_n), \frac{1-\sigma(a)^m}{1-\sigma(a)} \sigma(b)(1-\sigma(a_n)) + \sigma(a)^m \sigma(b_n)\big). \tag{7}$$

Since $|\sigma(a)|_k < 1$, for any given n, choosing m large, we can make $|\sigma(a)^m \sigma(b_n)|_k$ arbitrarily small. Hence for some sequence $m_n \to +\infty$ the second component in (7) tends to $\sigma(\beta)$ where $\beta := \frac{b}{1-a}$ as n tends to $+\infty$. $\square$

We shall now complete the proof of Proposition 10.5. Note that if k is some local field and $\gamma = (a_0, b_0) \in \mathbb{A}(k)$ with $|a_0|_k < 1$, then γ acts on the affine line k with a fixed point $x_0 = b_0/(1-a_0)$ and it contracts the disc of radius R around x_0 to the disc of radius $|a_0|_k \cdot R$. Therefore, if we are given t distinct points $b_1, \ldots, b_t$ in k, there exists $\varepsilon > 0$ such that for all $a_1, \ldots, a_t \in k$ with $|a_i|_k \leq \varepsilon$, $i = 1, \ldots, t$, the elements (a_i, b_i)'s play ping-pong on the affine line, hence are free generators of a free semi-group. The group $G = \overline{\Gamma}$ is a connected and Zariski dense subgroup of $\mathbb{A}(\mathbb{C})$: it follows that we can find arbitrary small perturbations $\widetilde{s_i}$ of the s_i's within Γ such that the $a(\tilde{s}_i)\beta + b(\tilde{s}_i)$'s are pairwise distinct complex numbers. If $(\gamma_n)_n$ is the sequence obtained in the last Lemma, then for some n large enough the points $t_i := \widetilde{s_i}\gamma_n$ will be small perturbations of the s_i's (i.e. belong to $s_i B_\varepsilon$) and the $\sigma(t_i) = \big(\sigma(a(\tilde{s}_i)a_n), \sigma(a(\tilde{s}_i)b_n) + \sigma(b(\tilde{s}_i))\big)$ will play ping-pong on k for the reason we just explained (the $\sigma(a(\tilde{s}_i))\sigma(\beta) + \sigma(b(\tilde{s}_i))$'s are all distinct). $\square$

Acknowledgments 10.9. We thank the following persons for their help and the interesting conversations we had about various points in this article: Y. Barnea, M. Burger, D. Getz, E. Ghys, M. Larsen, A. Lubotzky, A. Mann, G.A. Margulis, S. Mozes, N. Nikolov, J.F. Quint, A. Salehi-Golsefidi, G.A. Soifer. We also thank Olivia Barbarroux for her hospitality during our stay at Luminy in July 2002 and H. Abels for inviting us to Bielefeld in July 2003, where part of this work was conducted.

References

[1] H. Abels, G.A. Margulis, G.A. Soifer, *Semigroups containing proximal linear maps*, Israel J. Math. **91** (1995), no. 1-3, p. 1–30.

[2] Y. Barnea, M. Larsen, *Random generation in semisimple algebraic groups over local fields*, (preprint).

[3] A. Borel, *Linear Algebraic Groups*, Springer Verlag.

[4] E. Breuillard, T. Gelander, *On dense free subgroups of Lie groups*, J. Algebra **261** (2003), no. 2, p. 448–467.

[5] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local, I. Données radicielles valuées*, Publ. math. IHES **41** (1972).

[6] M. Burger, N. Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. 12 (2002), **2**, p. 219–280.

[7] A. Candel, L. Conlon, *Foliations I*, Graduate Studies in Mathematics, **23**, AMS Providence, RI, 402 pp, (2000).

[8] A. Carbery, M. Christ, J. Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. **12** (1999), no. 4, p. 981–1015.

[9] Y. Carrière *Feuilletages riemanniens à croissance polynômiale*, Comment. Math. Helv. **63** (1988), no. 1, p. 1-20.

[10] Y. Carrière, E. Ghys, *Relations d'équivalence moyennables sur les groupes de Lie*, C. R. Acad. Sci. Paris Sér. I Math. **300** (1985), no. 19, p. 677-680.

[11] Y. Carrière, F. Dal'bo, *Généralisations du premier théorème de Bieberbach sur les groupes cristallographiques*, Enseign. Math. (2) **35** (1989), no. 3-4, p. 245-262.

[12] J.D. Dixon, L. Pyber, A. Seress, A. Shalev, *Residual properties of free groups and probabilistic methods*, J. Reine Angew. Math. **556** (2003), p.159-172.

[13] J. Dixon, M. du Sautoy, A. Mann, D. Segal, *Analytic pro-p Groups*, London Math. Soc. Lect. Note. Ser. **157**, (1991).

[14] D. B. Epstein, *Almost all subgroups of a Lie group are free*, J. of Algebra. **19** (1971) 261-262.

[15] A. Haefliger, *Feuilletages Riemanniens*, Séminaire Bourbaki, Vol. 1988/89. Astérisque No. **177-178** (1989), Exp. No. 707, p. 183–197.

[16] P. de la Harpe, A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)*, Astérisque No. **175** (1989), 158 pp.

[17] M. Kuranishi, *On everywhere dense embedding of free groups in Lie groups,* Nagoya Math. **J 2** (1951), 63-71.

[18] S. Lang, *Algebra,* Third Edition, Addison-Wesley (1994).

[19] A. Mann, personal communication.

[20] G. A. Margulis, G. A. Soifer, *Maximal subgroups of infinite index in finitely generated linear groups*, J. Algebra **69**, (1981), no. 1, p. 1-23.

[21] P. Molino, *Riemannian foliations*, Prog. in Math. **73**, Birkhäuser, (1988), 339 pp.

[22] D. Montgomery, L. Zippin, *Topological transformation groups*, Reprint of the 1955 original, R.E. Krieger Publ. Co., Huntington, N.Y., (1974).

[23] V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, 1994.

[24] G. Pólya*, Beitrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach Zusammenhangende Gebiete*, S.B. Preuss. Akad. Wiss., Berlin, K.L. Math. Phys. Tech. (1928), p. 228-232 and p. 280-282.

[25] L. Pyber, *Group enumeration and where it leads us*, Birkhäuser, Prog. Math.**169**.

[26] J.F. Quint, thèse de l'université Paris VII, (2001).

[27] M. S. Raghunathan, *Discrete Subgroups of Discrete Groups*, Springer Verlag (1972).

[28] Ribes, L. Zalevskii, P., *Profinite groups*, Ergeb. Math. vol. **40**, Springer Verlag (2000)

[29] J.P. Serre, *Lie algebras and Lie groups*, Springer Verlag LNM 1500.

[30] P. Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) 17 (1978), no. 3, p. 555–565 and erratum J. London Math. Soc. (2) 32 (1985), no. 2, p. 217–220.
[31] A. Shalev, *Lie methods in the theory of pro-p groups*, in *New horizons in pro-p groups*, p. 1-54, Progr. Math., **184**, Birkhäuser (2000).
[32] G. Soifer, T. Venkataramana, *Finitely generated pro-finitely dense free groups in higher rank semisimple groups*, Transform. Groups 5 (2000), no. 1, 93–100.
[33] J. Tits, *Free subgroups in linear groups*, J. of Algebra, **20**, (1972).
[34] J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971) p. 196–220.
[35] R. Zimmer, *Amenable actions and dense subgroups of Lie groups*, Journal of Functional analysis, **72** (1987), no. 1, p. 58–64.
[36] R. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Math., **81**, Birkhäuser Verlag, (1984), 209 pp.
[37] R. Zimmer, *Kazhdan groups acting on compact manifolds*, Invent. Math. **75** (1984), no. 3, 425–436.

Emmanuel Breuillard, Dept. of Math, Yale University, 10 Hillhouse Ave, New Haven 06520 CT, USA

E-mail address: emmanuel.breuillard@yale.edu

Tsachik Gelander, Dept. of Math, Yale University, 10 Hillhouse Ave, New Haven 06520 CT, USA

E-mail address: tsachik.gelander@yale.edu