Model theory of difference fields

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Traditionally, a difference equation is a functional equation of the form

$$P(y(t), y(t+1), \ldots, y(t+n)) \equiv 0$$

where P is a function of $n+1$ variables (usually taken from some restricted class of functions) and the solution functions $y(t)$ are also taken from some restricted class.

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We shall algebraize this situation by

- taking P to be a polynomial and
- replacing the operator on a ring of functions $f(t) \mapsto f(t+1)$ with a general ring endomorphism.

Difference fields

Definition

A difference ring (R, σ) is a pair consisting of a commutative ring R together with a distinguished ring endomorphism $\sigma : R \to R$. If R is a field, we call (R, σ) a difference field. If σ is an automorphism, then we say that (R, σ) is inversive.

Difference rings in practice

- (R, id_R) for R an arbitrary commutative ring.
- If R is any commutative ring, and $S = R^{\mathbb{N}}$ is the set of infinite sequences in R given with coordinatewise addition and multiplication and $\sigma : S \to S$ is defined by $(a_n)_{n\in\mathbb{N}} \mapsto (a_{n+1})_{n\in\mathbb{N}}$, then (S, σ) is a difference ring.
- $(R[[t]], \sigma_q)$ where $\sigma_q(\sum a_i t^i) := \sum a_i q^i t^i$ and $q \in R$ is fixed.
- $(k(V), f^*)$ where V an algebraic variety over some field k, $f : V \to V$ a regular morphism, and $f^* : k(V) \to k(V)$ on the function field of V is defined by $g \mapsto g \circ f$
- $\mathscr{K}_q = (\mathbb{F}_p{}^{\mathrm{alg}}, \tau_q)$ where q is a power of the prime number p and τ_q is the q-power Frobenius defined by $x \mapsto x^q$

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Definable sets

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I have consciously restricted to difference equations rather than the class of general definable sets in difference fields or even difference rings though the more general theory of definability will be implicated.

Goals of this project (internal)

- Determination of the theory of the Frobenius automorphism
- Development of geometric stability theory in unstable theories

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- Algebraic dynamics, especially around issues of descent and classification
- Galois theory for difference equations

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Difference closed fields

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Definition

We say that a difference field $(\mathbb{U}, \sigma)$ is difference closed if it is existentially closed in the class of all difference fields. This is, if some finite system of difference equations and inequations with coefficients over $\mathbb{U}$ has a solution in some extension difference field, then it already has a solution in $\mathbb{U}$.

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Theorem

The class of difference closed fields is axiomatized by a first-order theory ACFA, *the model companion of the theory of difference fields.*

Definition

A field K is pseudofinite if it is elementarily equivalent to an ultraproduct of the form $\prod_{/\mathscr{U}} \mathbb{F}_q$ where $\mathscr{U}$ is a nonprincipal ultrafilter on the set of prime powers. Equivalently, K is infinite but every first-order sentence true in K is true in some finite field.

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Theorem

A field K is pseudofinite if and only if

- *K is perfect,*
- $\mathrm{Gal}(K^{\mathrm{alg}}/K) \cong \widehat{\mathbb{Z}}$, *and*
- *K is pseudoalgebraically closed.*

Fix(σ) is pseudofinite

Proposition

If $(\mathbb{U}, \sigma)$ *is difference closed, then the fixed field*

$$\mathrm{Fix}(\sigma)(\mathbb{U}) := \{a \in \mathbb{U} \ : \ \sigma(a) = a\}$$

is pseudofinite.

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Hence, possibly replacing $\mathbb{U}$ with an elementarily equivalent difference field, we may realized $\mathrm{Fix}(\sigma)(\mathbb{U})$ as an ultraproduct $\prod_{/\mathscr{U}} \mathbb{F}_q$.

Difference closed fields as ultraproducts

Recall that $\mathscr{K}_q := (\mathbb{F}_p{}^{\mathrm{alg}}, \tau_q)$ and that $\mathrm{Fix}(\tau_q)(\mathscr{K}_q) = \mathbb{F}_q$. Thus, if $(\mathscr{K}, \sigma) = \prod_{/\mathscr{U}} \mathscr{K}_q$, we have $\mathrm{Fix}(\sigma)(\mathscr{K}) = \prod_{/\mathscr{U}} \mathbb{F}_q$, a pseudofinite field.

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Theorem

ACFA *is the limit theory of the Frobenius. That is, if* $\mathscr{U}$ *is a nonprincipal ultrafilter on the prime powers, then* $\prod_{/\mathscr{U}} \mathscr{K}_q$ *is difference closed and if* $(\mathbb{U}, \sigma)$ *is any difference closed field, then it is possible to find some ultrafilter* $\mathscr{U}$ *so that* $\prod_{/\mathscr{U}} \mathscr{K}_q \equiv \mathbb{U}$.

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The heart of this theorem is a refinement of Deligne's proof of the Weil conjectures, which may be understood as counting the number of solutions to the difference equations $a \in X(\mathbb{F}_p{}^{\mathrm{alg}})$ and $\tau_q(a) = a$, to more general zero dimensional systems of difference equations.

False conjecture

If X is a strongly minimal set, then exactly one of the following is true of X.

- X is bi-interpretable with an algebraically closed field,
- X is inter-algebraic with a group whose induced structue is essentially just that of a vector space, or
- X is trivial in the sense that all dependencies on X are binary.

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The theory ACFA is not stable. However, it is supersimple and quantifier-free stable. The initial analysis of ACFA required the development of a general theory of stability theoretic constructions under these weaker hypotheses.

Manin-Mumford conjecture

Theorem

If G is an abelian variety over $\mathbb{C}$ and $X \subseteq G$ is a closed subvariety, then $X(\mathbb{C}) \cap G(\mathbb{C})_{\text{tor}}$ is a finite union of coset of subgroups of the torsion group $G(\mathbb{C})_{\text{tor}} := \{a \in G(\mathbb{C}) \ : \ (\exists n \in \mathbb{Z}_+)[n]_G(a) = 0_G\}$.

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- The Manin-Mumford conjecture in this form was proven by Raynaud using p-adic methods. Many other proofs have been offered in subsequent years.
- The conjecture holds for G an arbitrary commutative algebraic group.
- Hrushovski's proof, which yields some explicit bounds on the complexity of the defining equations for $X(\mathbb{C}) \cap G(\mathbb{C})_{\mathrm{tor}}$, is based on an analysis of definable sets in difference closed fields.

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Mann's theorem and ACFA

Let us consider the case $G(\mathbb{C}) = {\mathbb{G}_m}^n(\mathbb{C}) = (\mathbb{C}^\times)^n$ (which was solved by Mann years before the Manin-Mumford conjecture itself was formulated).

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- From the theory of theory of cyclotomic fields, one finds an automorphism σ of the algebraic numbers for which $\sigma(\zeta) = \zeta^2$ for every root of unity ζ of odd order and $\sigma(\xi) = \xi^3$ for every root of unity ξ of order a power of two.
- Using the fact that there is only one algebraically closed field of characteristic zero of cardinality continuum up to isomorphism, we may extend σ to an automorphism of $\mathbb{C}$ for which $(\mathbb{C}, \sigma)$ is difference closed.
- The multiplicative group $\Gamma := \{a \in \mathbb{C}^\times \ : \ \sigma^2(x)x^6 = \sigma(x)^5\}$ is then definable and contains all of the roots of unity.
- It follows from the trichotomy theorem and a classification theorem for definable groups, that each definable subset of Γ^n (in particular $X(\mathbb{C}) \cap \Gamma^n$) is a finite union of cosets of groups. The Manin-Mumford statement now follows.

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The techniques behind the proof of the Manin-Mumford conjecture have been applied to related Diophantine problems for which there is no obvious algebraic group action.

- Some instances of the André-Oort conjecture in which algebraic relations on special points in (mixed) Shimura varieties (for example CM-moduli points on moduli spaces of abelian varieties) are conjecturally reducible to geometric relations coming from special varieties
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Noncommutative Diophantine problems

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We have two natural (and not unrelated) ways to study algebraic dynamical systems with difference fields.

- Consider the definable set $(X, f)^\sharp$ defined by $x \in X$ and $\sigma(x) = f(x)$. If $(\mathbb{U}, \sigma)$ is a difference closed field extending k for which $k \subseteq \text{Fix}(\sigma)(\mathbb{U})$, then $(X, f)^\sharp(\mathbb{U})$ is Zariski dense in X.
- If $K := k(X)$ is the field of rational functions on X, then $f^* : k(X) \to k(X)$ defined by $g \mapsto g \circ f$ is a difference field.
- We may embed $(K, f^*) \hookrightarrow (\mathbb{U}, \sigma)$ over (k, id_k). In so doing, the generic point of X is taken to a point of $(X, f)^\sharp(\mathbb{U})$.

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If (X, f) is an algebraic dynamical system, then by an invariant subvariety we mean a subvariety $Y \subseteq X$ for which f maps Y back to itself. If $Y \subseteq X$ is an invariant variety, then $Y(\mathbb{U}) \cap (X, f)^\sharp(\mathbb{U})$ is a definable subset of $(X, f)^\sharp$.

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On the other hand, if $D \subseteq (X,f)^\sharp$ is a definable subset of $(X,f)^\sharp$ and Y is an irreducible component of the Zariski closure of $D(\mathbb{U})$, then Y is skew-invariant: f maps Y back to Y^σ, the transform of Y under σ. Thus, the description of the (quantifier-free) definable subsets of $(X,f)^\sharp$ is identical to that of the skew-invariant subvarieties of X.

Coarse classification of algebraic dynamics: "dimension" one

If (X, f) is an algebraic dynamical system over some difference closed field $(\mathbb{U}, \sigma)$ (where we assume that $X = X^\sigma$ and $f^\sigma = f$), then the definable set $(X, f)^\sharp$ is "finite dimensional" in several senses. In the special case that $\dim(X) = 1$, then $(X, f)^\sharp$ has dimension one for any of our reasonable notions of dimension. It follows from the trichotomy (together with additional geometric work) that exactly one of the following holds.

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- (X, f) is (skew-)isomorphic to a (twisted-)constant dynamical system on a curve: That is, in characteristic zero there is an automorphism $g : X \to X$ for which $g^\sigma \circ f \circ g^{-1} = \mathrm{id}_X$. In characteristic p, we may need to first map X to an isomorphic curve defined over a finite field and then $g^\sigma \circ f \circ g^{-1}$ might be a power of the Frobenius.
- There is a commutative algebraic group G, a map of group $\phi : G \to G$, and a dominant map $g : G \to X$ so that $g^\sigma \circ \phi = f \circ g$ and every skew-invariant subvariety of $(G, \phi)^n$ is a translate of a group.
- (X, f) is trivial in the sense that every skew-invariant subvariety of $(X, f)^n$ comes from skew-invariant subvarieties of $(X, f)^2$ via pull-back and intersection.

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- There is a commutative algebraic group G, a map of group $\phi : G \to G$, and a dominant map $g : G \to X$ so that $g^\sigma \circ \phi = f \circ g$ and every skew-invariant subvariety of $(G, \phi)^n$ is a translate of a group.
- (X, f) is trivial in the sense that every skew-invariant subvariety of $(X, f)^n$ comes from skew-invariant subvarieties of $(X, f)^2$ via pull-back and intersection.

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Descent problems: heights

Let $K = \mathbb{C}(t)$ be the field of rational functions over $\mathbb{C}$. For $f = \frac{g}{h} \in K$ with g and h having no common divisor, we define $h(f) = \max\{\deg(g), \deg(h)\}$.

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- If $G \in \mathbb{C}(x)$ (that is, if t does not appear in G), then for $a \in \mathbb{C}$, $\widehat{h}_G(c) = 0$.

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Theorem

If $G \in \mathbb{C}(t)(x)$ is a rational function (in the variable x) over the field of rational functions (in the variable t) and there is some non-preperiodic point $a \in \mathbb{C}(t)$ with $\widehat{h}_G(a) = 0$, then there is some fractional linear transformation γ for which $\gamma \circ G \circ \gamma^{-1} \in \mathbb{C}(x)$.

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We shall see a proof this week of a generalization of this theorem based on the theory of canonical bases for types in difference fields.

Topics not directly discussed this week

- Galois theory for difference equations
- Valued difference fields

Difference Galois theory

Traditionally, one starts with some $A \in \mathrm{GL}_n(K)$ (where (K, σ) is a difference field) and studies the equation $\sigma(x) = Ax$. Under suitable hypotheses, there is a Picard-Vessiot field L for this equation which is generated by a basis of solutions and has the same fixed field as K. We then define the Galois group of the equation to be $\mathrm{Aut}(L/K)$ and show that this group may be realized as the constant points of an algebraic group and that the intermediate algebraic groups correspond to intermediate difference fields.

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In the material we discuss this week, these groups will be behind the scenes in the analysis of higher rank definable sets and types in terms of minimal sets.

A valued difference field is a difference field (K, σ) given together with a valuation v for which we usually assume that σ and v interact in a strong way.

- Isometry: $(\forall x) v(\sigma(x)) = v(x)$ This theory (and its model companion) are useful for studying liftings of the Frobenius from characteristic p to characteristic zero. We shall see this implicitly in the application of ACFA to Manin-Mumford.
- q-multiplicative $(\forall x) v(\sigma(x)) = qv(x)$ (for $q \geq 2$) or ω-increasing $(\forall x)(v(x) > 0 \rightarrow v(\sigma(x)) > nv(x)$ for all $n \in \mathbb{Z}_+$ If (K, v) is a nontrivially valued field of characteristic p and q is a power of p, the τ_q gives a q-multiplicative valued difference field. An ultraproduct will yield an ω-increasing valued difference field. These constructions will be used in the analysis of the limit theory of the Frobenius.

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