Algebraic dynamics, function fields and descent

Antoine Chambert-Loir

Algebraic dynamics is the study of dynamical systems defined by rational maps on algebraic varieties. One of its themes consists in interpreting questions and (sometimes) theorems from classical Diophantine geometry as instances of questions for general algebraic dynamical systems. In the first part of these notes, we give a rapid introduction to this circle of ideas. We then focus on the determination of "limited orbits" for algebraic dynamical systems over function fields. In the last part we explain the statement of the proof of a general theorem due to Chatzidakis & Hrushovski (2008*a*,*b*).

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PART I. ALGEBRAIC DYNAMICS

1. Definition and examples

Definition 1.1. — *Let k be a field. An algebraic dynamical system over k consists in a pair (V, f), where V is an algebraic variety over k and $f\colon V \to V$ is an endomorphism.*

For us, *variety over k* really means separated k-scheme of finite type. The reader may think of the locus in affine (*resp.* projective) space defined by a family of polynomials (*resp.* homogeneous polynomials) with coefficients in k. In all interesting cases, V will be irreducible, and even geometrically irreducible. Topological notions will be relative to the Zariski topology.

The *endomorphism f* is assumed to be defined and regular everywhere. From a point $x \in V$, one may compute its image $f(x)$. Dynamics appear when one iterates this process and computes the image $f(f(x)) = f^{(2)}(x)$ of its image, etc. One then gets the *orbit* of x, defined as the set $\{x, f(x), f^{(2)}(x), \dots\}$.

However, there are some natural generalizations to consider.

1.2. Rational maps. — First, it is totally legitimate to study *rational maps $f\colon V \dashrightarrow V$*. This means that there exists a dense open subset U of V and f is a morphism from U to V. Since V is separated, there is a largest open subset of definition; its complementary subset E is called the indetermination locus.

If $x \in V \setminus E$, one can compute $f(x)$; if $f(x) \notin E$, one can iterate the construction, unless at some point, an iterate belongs to E, in which case the process stops.

Already in the definition when f is regular everywhere, an implicit assumption is that its image $f(U)$ be dense in V — one then says that f is *dominant*. Otherwise, the study of the dynamics of f reduces to the dynamics of its restriction to a smaller variety.

1.3. Correspondences. — Another possible generalization considers "probabilistic dynamics" or "multivalued rational maps" — the technical term is *correspondences*: that is, a subvariety $T \subset V \times V$. Then, a point $x \in V$ has a *set* of images, consisting of all points y such that $(x, y) \in T$. A (finite, or infinite) orbit is a sequence $(x_0, x_1, \dots, x_n, \dots)$ such that $(x_i, x_{i+1}) \in T$ for all i.

One assumes that T maps dominantly to V under the two natural projections; the first one being dominant, almost every point has at least an image, using the second one we see that their images are not contained in a strict subvariety. Moreover, it is reasonable to assume that these maps are generically quasi-finite, so that almost all points have only finitely many images. If V is irreducible, this means that $\dim(T) = \dim(V)$.

If f is a rational map from V to itself, defined on the dense open set U, one may consider the closure T_f in $V \times V$ of its graph $\{(x, f(x)) ; x \in U\}$. Except for considerations related to the exceptional locus, the dynamics of f and of T_f are basically the same things.

1.4. Periodic, preperiodic points. — Let (V, f) be an algebraic dynamical system. One says that a point $x \in V$ is *periodic* if it is equal to one of its iterates, that is, if there exists $p > 0$ such that $f^{(p)}(x) = x$; the *period* of x is the smallest such integer p. The dynamics of x under f is then extremely easy to understand: it goes from x, to $f(x)$, then $f(f(x)), \dots, f^{(p-1)}(x)$, and then cycles back to $x = f^{(p)}(x)$, etc.

One says that a point x is *preperiodic* if its orbit is finite. Then, there exist integers $n \geq 0$ and $p > 0$ such that $f^{(n+p)}(x) = f^{(n)}(x)$. If p and n are minimal, the dynamics of x will run through $x, \dots, f^{(n)}(x), \dots, f^{(n+p-1)}(x)$, and then go back to $f^{(n)}(x)$ and cycle indefinitely. Hence, preperiodic are points some iterate of which is periodic.

1.5. Curves. — Let us assume that V is a (geometrically integral) curve. Then, an open subset U of V is smooth over k; let X be the smooth compactification of U. Any rational map $f: V \dashrightarrow V$ comes from an endomorphism $F: X \to X$ and both dynamics basically correspond.

If X has genus $g \geq 2$, it follows from the Riemann-Hurwitz formula that F is an automorphism; moreover, it has finite order, so that the full dynamics is periodic. Consequently, the only interesting examples are in genus 0 and 1, corresponding to the projective line and to elliptic curves.

The projective line $\mathbf{P}^1_k$ carries a lot of dynamical systems. Any endomorphism of $\mathbf{P}^1_k$ (even any rational map) is given by a rational function $f \in k(t)$, and conversely. The poles of f are mapped to the point at infinity.

1.6. Elliptic curves. — Let us detail the case of genus 1. Since one is interested in the iteration of rational points, one may assume that $X(k)$ is non-empty: fixing any of them, say o, as an origin, X becomes an *elliptic curve*: it automatically inherits a commutative group structure for which the origin o is the neutral element.

Then, any endomorphism of X is of the form $x \mapsto \varphi(x) + a$, where φ is an endomorphism of X mapping o to itself, and $a \in X(k)$ is a point. The endomorphism φ is compatible with the group law: it is an endomorphism of the elliptic curve.

If $\varphi = \mathrm{id}_X$, then the dynamics is a translation: one has $f^{(n)}(x) = x + na$ for any $x \in X(k)$ and any $n \in \mathbf{N}$. All orbits have the same shape, finite or periodic, depending on whether there exists an integer $n > 0$ such that $na = o$, or not.

Otherwise, if $\varphi \neq \mathrm{id}_X$, then $\mathrm{id}_X - \varphi$ is surjective and there exists a point o', possibly defined in an extension of k, such that $o' = \varphi(o') + a$. The point o' is fixed under f and up to a conjugation, one assumes that $o' = o$, that is, f is an endomorphism of the elliptic curve.

For most of the elliptic curves, endomorphisms are just self-multiplication a certain number of times. The case $f = [1] = \mathrm{id}_X$ is trivial; the case $f = [-1]$ isn't much more interesting since then $f^{(2)} = \mathrm{id}_X$. Let us assume that $f = [m]$ for some integer $m \in \mathbf{Z}$ such that $|m| \geq 2$; since all automorphisms of an elliptic curve have finite order, the discussion also applies to the complex-multiplication case provided one assumes that $\deg(f) > 1$.

Then, *preperiodic points* precisely coincide with the *torsion points* of X. Indeed, if $f^{(n+p)}(x) = f^{(n)}(x)$, then x is mapped to 0 by the endomorphism $(f^{n+p} - f^n)$ of X. For $f = [m]$, one gets $m^n(m^p - 1)x = 0$, so x is a torsion point; this holds in general when $\deg(f) > 1$. Conversely, if x is a torsion point and, say, $[N]x = 0$, for some integer $N > 1$, then all iterates $f^{(n)}x$ are torsion points and they all satisfy $[N]f^{(n)}(x) = 0$. Since they are finite in number, there must exist integers $n \geq 0$ and $p > 0$ such that $f^{(n+p)}(x) = f^{(n)}(x)$, hence the claim.

For any mathematician versed in XX$^{\text{th}}$ century number theory, this prompts at once a number of questions: what are the analogues for algebraic dynamical systems of all those theorems in Diophantine geometry concerning torsion points of elliptic curves?

1.7. Multiplicative group. — Actually, the case of the multiplicative group already gives rise to interesting analogues. Those are the maps $f_N(x) = x^N$, for some non-zero integer $N \in \mathbf{Z}$. The cases $N = \pm 1$ are rather uninteresting, but for $|N| \geq 2$, one gets a dynamical system which is both easy to describe ($f^{(n)}(x) = x^{N^n}$ can be computed explicitly) but which features the rich arithmetic properties of cyclotomic fields, and of Kummer theory. Indeed, the preperiodic points are 0, ∞, and the roots of unity.

1.8. Lattès maps. — We can view an elliptic curve X as a double covering of the projective line ramified in 4 points. This is explicit when the elliptic curve is written in Weierstrass form, with (affine) equation $y^2 = x^3 + ax + b$, say. In fact, a point $P = (x, y)$ and its opposite $-P$ have the same x-coordinate, namely x; if f is an endomorphism of X, $f(-P) = -f(P)$ also share the same x-coordinate. This implies that f takes the form $f(x, y) = (\varphi(x), \psi(x, y))$, where φ and ψ are rational function, the important point being that φ depends only on x. One has $\deg(\varphi) = \deg(f)$.

Such a dynamical system is called a *Lattès map*. From the point of view of complex dynamics, they are very specific (for example, their measure of minimal entropy is absolutely continuous with respect to the Lebesgue measure.)

Similarly, if one divides the multiplicative group by the action of $x \mapsto 1/x$, one gets the family of *Chebyshev maps* on $\mathbf{P}^1_k$. They are associated to the Chebyshev polynomials T_N such that $T_N(2\cos(x)) = 2\cos(Nx)$, or $T_N(x + 1/x) = x^N + 1/x^N$. The degree of T_N is N. Their dynamical properties are also very particular.

1.9. Higher dimensional varieties. — In higher dimensions, it is much more difficult to classify algebraic dynamical systems, although the picture for surfaces is almost complete.

Projective spaces and, more generally, rational varieties carry many dynamical systems. For example, let $F_0, \dots, F_n$ be homogeneous polynomials in $n + 1$ variables, of the same degree d, and without common zeros over the algebraic closure. Then, there is an endomorphism f of $\mathbf{P}^n_k$ which maps a point x with homogeneous coordinates $[x_0 : \dots : x_n]$ to the point with homogeneous coordinates $[F_0(x) : \dots : F_n(x)]$. Conversely, any (possibly rational) algebraic dynamical system on $\mathbf{P}^n_k$ takes this form. The indetermination locus of f is the common locus of the polynomials $F_0, \dots, F_n$.

For example, taking $F_i(x) = x_i^d$; one gets a dynamical system which is a compactification of the dth-power map on of the torus $(\mathbf{G}_{\mathrm{m}})^n$.

Endomorphisms of Abelian varieties (connected projective algebraic groups) give rise to dynamical systems whose arithmetic is very rich, and now quite well understood. As for elliptic curves, preperiodic points are torsion points.

Lattès maps have analogues in higher dimensions. Indeed, if one quotients an Abelian variety by the inverse map $[-1]$, identifying a point x and its opposite $-x$, one gets a Kummer variety. Such a variety inherits dynamical systems whose behavior is close to the ones of the original Abelian variety.

1.10. K3-surfaces. — The study of those examples from a Diophantine point of view has been initiated by SILVERMAN (1991); we describe here later examples due to WANG (1995). Let us consider a smooth irreducible hypersurface X in $\mathbf{P}^1_k \times \mathbf{P}^1_k \times \mathbf{P}^1_k$. If, for $i \in \{1,2,3\}$, one denotes by (T_i, U_i) the homogeneous coordinates on $\mathbf{P}^1_k$, the surface X can be defined by a polynomial $F \in k[T_1, U_1, T_2, U_2, T_3, U_3]$ which is homogeneous of some degree d_i in each of the pairs of variables (T_i, U_i). Let us assume that $d_1 = d_2 = d_3$; then, the canonical class of X is trivial, as well as its first cohomology group $\mathrm{H}^1(X, \mathscr{O}_X)$. Such a surface is called a K3-surface.

Let P_1 and $P_2 \in \mathbf{P}^1_k$. Since F has degree 2 with respect to (T_3, U_3), there are exactly two points (possibly equal) P_3 and P_3' on $\mathbf{P}^1_k$ such that (P_1, P_2, P_3) and (P_1, P_2, P_3') both belong to X. This induces an involution σ_3 of X exchanging (P_1, P_2, P_3) and (P_1, P_2, P_3'). One similarly defines σ_1 and σ_2. These three involutions do not satisfy any non trivial relations and the subgroup of $\mathrm{Aut}(S)$ they generate is the free product $(\mathbf{Z}/2) * (\mathbf{Z}/2) * (\mathbf{Z}/2)$, see (CANTAT, 2011, §2.4.6).

For $i \in \{1,2,3\}$, let $\mathscr{L}_i$ be the inverse image of $\mathscr{O}(1)$ by the projection of index i to $\mathbf{P}^1_k$. By Lefschetz's Theorem, they form a basis of the Picard group $\mathrm{Pic}(X)$. The action of σ_j^* on the line bundles $\mathscr{L}_i$ is given by:

$$\sigma_i^* \mathscr{L}_i \simeq \mathscr{L}_i^{-1} \otimes \mathscr{L}_j^2 \otimes \mathscr{L}_k^2$$

when $\{i,j,k\} = \{1,2,3\}$, and

$$\sigma_i^* \mathscr{L}_j \simeq \mathscr{L}_j$$

when $i \neq j$, see WANG (1995) for the proof. [1]

1.11. Polarized dynamical systems. — Let (V, f) be an algebraic dynamical system, where V is assumed to be projective and (for simplicity) integral. A *polarization* of (V, f) is an ample line bundle $\mathscr{L}$ on V such that $f^*\mathscr{L}$ is isomorphic to some power $\mathscr{L}^q$ of $\mathscr{L}$, with $q > 1$.

This condition implies that f is a finite morphism of degree $q^{\dim(V)}$. If V is smooth, then its Kodaira dimension is ≤ 0.

The simplest example is given by endomorphisms of the projective space $\mathbf{P}^n_k$; then, one may take $\mathscr{L} = \mathscr{O}(1)$, and q is the common degree of the polynomials which define f. (Unless f is an automorphism, one has $q \geq 2$.)

1. One has $\sigma_i^* \mathscr{L}_j = \sigma_i^* p_j^* \mathscr{O}(1) = (p_j \circ \sigma_i)^*(1)$. For $i \neq j$, $p_j \circ \sigma_i = p_j$, hence $\sigma_i^* \mathscr{L}_j \simeq \mathscr{L}_j$. Let us observe the following relations in intersection theory: one has $c_1(\mathscr{L}_i) \cdot c_1(\mathscr{L}_j) = 2$ for $i \neq j$, while $c_1(\mathscr{L}_i)^2 = 0$. Then, for $i \neq j$, $c_1(\sigma_i^* \mathscr{L}_i) \cdot c_1(\mathscr{L}_j) = c_1(\mathscr{L}_i) \cdot \sigma_i^* c_1(\mathscr{L}_j) = 2$. Moreover, let $q_1 = (p_2, p_3)$ be the projection from X to $\mathbf{P}^1_k \times \mathbf{P}^1_k$ and π_1, π_2 the two projections from $\mathbf{P}^1_k \times \mathbf{P}^1_k$ to $\mathbf{P}^1_k$.

Then, $q_{1,*} c_1(\mathscr{L}_1) \cdot \pi_1^* \mathscr{O}(1) = c_1(\mathscr{L}_1) \cdot c_1(\mathscr{L}_2) = 2$ and $q_{1,*} c_1(\mathscr{L}_1) \cdot \pi_2^* \mathscr{O}(1) = c_1(\mathscr{L}_1) \cdot c_1(\mathscr{L}_2) = 2$, so that $q_1^* q_{1,*} c_1(\mathscr{L}_1) = 2c_1(\mathscr{L}_1) + 2c_1(\mathscr{L}_3)$. For any point P, one has $q_1^* q_{1,*} P = P + \sigma_1(P)$, as a zero-cycle hence (understand/explain) $q_1^* q_{1,*} c_1(\mathscr{L}_1) = c_1(\mathscr{L}_1) + c_1(\sigma_1^* \mathscr{L}_1)$. Finally, $\sigma_1^* c_1(\mathscr{L}_1) = -c_1(\mathscr{L}_1) + 2c_1(\mathscr{L}_2) + 2c_1(\mathscr{L}_3)$.

In fact, the following proposition shows that at least when k is infinite, any polarized dynamical system is induced by some endomorphism of a projective space.

***Proposition* (Fakhruddin, 2003). —** *Assume that k is infinite. Let $(V,f,\mathscr{L})$ be a polarized dynamical system, with V proper. Then, there exist positive integers n and p, a closed embedding $i\colon V \hookrightarrow \mathbf{P}^n_k$ such that $i^*\mathscr{O}(1) = \mathscr{L}^p$, and an endomorphism g of $\mathbf{P}^n_k$ such that $g\circ i = i\circ f$.*

Say that (V,f) is *expanding* if there exists an ample line bundle $\mathscr{L}$ on V such that $f^*\mathscr{L}\otimes\mathscr{L}^{-1}$ is ample. Of course, because of the condition $q>1$ on the weight, a polarizable dynamics is expanding, but to be expanding is a weaker notion and there are more examples.

2. Questions

In this section, we recall results for Abelian varieties and discuss their possible extensions to an algebraic dynamical system (V,f) where $f\colon V \dashrightarrow V$ is a dominant rational map.

2.1. Existence of non-preperiodic points. — The first Diophantine question is whether the dynamics of (V,f) is already reflected at the level of rational points, maybe after some finite extension of the ground field.

If A is an Abelian variety over an infinite field k, it is known that there exists a finite extension $k'\supset k$ such that $A(k')$ is Zariski dense in A. More precisely, at least in characteristic zero, there exists a point $a\in A(k')$ whose multiples are already dense in A, see Hassett & Tschinkel (2000); Hassett (2003). This last result is not less general under the assumption that k is algebraically closed.

This suggests the following question of Medvedev & Scanlon (2009): *If k is algebraically closed, does there exist a point $x\in V(k)$ whose orbit is dense for the Zariski topology?* An obvious necessary condition is that f do not preserve any rational fibration, namely there does not exist a rational map $g\colon V\dashrightarrow W$, with $0<\dim(W)<\dim(W)0$, such that $g\circ f = g$ generically. According to Conjecture 5.3 of that paper, this condition should be sufficient. Amerik & Campana (2008) proved that it is indeed the case when k is an uncountable algebraically closed field of characteristic zero (*e.g.*, if $k=\mathbf{C}$).

In general, Amerik (2011) shows that *provided f is not of finite order, there exists a point $x\in V(k)$ whose orbit is infinite.* The proof is based on reduction to finite fields where it relies crucially on the results by Hrushovski (2004) on acfa and Frobenius. In the case of polarized dynamical systems (they cannot preserve any rational fibration), the result can be proved easily using height functions. Indeed, any point of non-zero canonical height satisfies the required property (see below, §3.4).

2.2. Density of periodic points. — In some sense, the theory of dynamical systems aims at classifying invariant (possibly reducible) subvarieties and periodic points give rise to the simplest ones.

Of course, a translation dynamics, as $f(z)=z+a$ on $\mathbf{C}$, does not admit a lot of periodic points, so that some hypothesis is necessary if one wants to show that periodic points are dense.

Over the complex numbers, using tools from complex analysis among which plurisubharmonic functions and currents one may show that this holds in the case of polarized dynamics. In that case, the "repelling" periodic points of order n are finite in number, and equidistribute, when n goes to infinity, to some canonical probability measure on $V(\mathbf{C})$ whose support is Zariski dense. In particular, periodic points are Zariski dense. Such a property holds, more generally, for dominant rational maps f of a projective variety V whose "last dynamical degree" is the largest — one could say that f is "cohomologically expanding".

The story began with LJUBICH (1983) who treated the case of a rational function of degree > 2 acting on the Riemann sphere; extensions to higher dimensions have been given by BRIEND & DUVAL (2001); GUEDJ (2005); DINH & SIBONY (2010), the last reference providing an extensive and useful survey of the field.

In the case of polarized dynamics over an algebraically closed field, an algebraic proof that periodic points are Zariski dense has been given by FAKHRUDDIN (2003), using reduction to finite fields and HRUSHOVSKI (2004). Moreover, the same proof establishes the weaker result that preperiodic points are Zariski dense, without needing to appeal to the results of HRUSHOVSKI (2004). This proof is also valid for expanding algebraic dynamical systems.

2.3. Uniform boundedness. — We now turn to the question of understanding preperiodic points over an arithmetic ground field, a number field, say.

For an elliptic curve, we have seen that preperiodic points correspond to torsion points. Moreover, a celebrated theorem of MEREL (1996) claims that for any number field k, the order of the torsion subgroup of any elliptic curve over k is bounded uniquely in terms of $[k:\mathbf{Q}]$.

The analogous result for Abelian varieties is still wide open.

MORTON & SILVERMAN (1994) conjecture a vast generalization: there should exist a constant $\kappa(D,n,d)$ bounding the number of preperiodic points of any endomorphism of degree $D\geq 2$ of $\mathbf{P}^n_k$ defined over a number field k such that $[k:\mathbf{Q}]\leq d$.

Indeed, using Prop. 1.11 and basic properties of ample line bundles on Abelian varieties, one can prove that this conjecture implies that the order of the torsion subgroup of Abelian varieties over a number field is bounded in terms of the dimension and the degree of the field, see FAKHRUDDIN (2003) for details.

2.4. Manin–Mumford. — Assume here that (V,f) is a polarized dynamical system over an algebraically closed field k, V being projective. We have seen that the periodic points of $V(k)$ are dense. Consequently, any closed integral subvariety $W\subset V$ which is invariant under f will also contain a dense set of periodic points; it suffices to apply the above result to the induced dynamical system $(W, f|_W)$.

More genarally, any subvariety W which is preperiodic, meaning that there exist integers $m>n\geq 0$ such that $f^{(m)}(W)=f^{(n)}(W)$, will contain a dense subset of preperiodic points.

In the case of Abelian varieties over an algebraically closed field of characteristic zero, Manin and Mumford asked, and RAYNAUD (1983) proved that conversely, if an integral subvariety W of an Abelian variety V contains a dense set of torsion points, there exists an

Abelian subvariety W_0 of V and a torsion point $a \in W(k)$ such that $W = a + W_0$. There is a similar theorem over fields of positive characteristic, but it is more subtle, see Hrushovski (2001).

However, as shown by Ghioca *et al* (2011), this picture does not generalize easily to general polarized algebraic dynamics. Indeed, let E be an elliptic curve with complex multiplication by $\mathbf{Z}[i]$, let $V = E \times E$ and let $f = ([3+4i],[5])$. Since $|3+4i|^2 = 3^2 + 4^2 = 5^2$, this is a polarized endomorphism; preperiodic points are torsion points, and the diagonal Δ contains a dense set of preperiodic points. However, since $(3+4i)/5$ is not a root of unity (it is even not an algebraic integer), Δ is not preperiodic.

2.5. Mordell–Lang. — The last question I would like to mention here is related to the Mordell–Lang conjecture. Indeed, combining the Mordell–Weil theorem and Mordell conjecture, Lang suggested the following conjecture, now a theorem.

***Theorem* (Faltings, 1994). —** *Let A be an Abelian variety over an algebraically closed field k of characteristic zero, let $\Gamma \subset A(k)$ be a finitely generated subgroup and let V be an integral subvariety of A. If $V \cap \Gamma$ is dense in V, then there exists an Abelian subvariety V_0 and a point $a \in A(k)$ such that $V = V_0 + a$.*

Let now (V, f) be an algebraic dynamical system over an algebraically closed field k. Let $W \subset V$ be an integral subscheme and let $x \in V(k)$. In that context, the Mordell–Lang problem aims at understanding the intersection of the orbit of x with W, equivalently, the set $\mathscr{N}_W$ of integers n such that $f^{(n)}(x) \in W$.

Let us observe what happens when V is an Abelian variety and f the translation by a point $a \in V(k)$. If a has finite order, then the sequence $(f^{(n)}(x)) = (x + na)$ is periodic and the result holds trivially, so assume that a has infinite order. Let us consider an irreducible component W' of the Zariski closure of $W \cap (x + \mathbf{N}a)$. By assumption, $W' \cap (x + \mathbf{N}a)$ is dense in W'; by Faltings's Theorem, $W' = x' + W'_0$ is a translate of an Abelian subvariety W'_0 by a point $x' \in W'(k)$. If $W'_0 \neq 0$, there are infinitely many integers n such that $x + na \in W'$, hence infinitely many integers n such that $na \in W'_0$. Let p be their least common divisor; one has $pa \in W'_0$ and the set of integers n such that $x + na \in W'$ is a union of arithmetic progressions modulo p. Therefore, the set $\mathscr{N}_W$ is a finite union of arithmetic progressions (possibly of step 0).

One expects, see for example Ghioca & Tucker (2009), that this property holds in general. There are indeed a lot of results in that direction, due to Benedetto, Ghioca, Kurlberg, Scanlon, Tucker, Zieve, and probably others. In most of them, p-adic techniques play an important rôle, close in spirit to the approach of Chabauty (1941) to the Mordell conjecture, or to the proof of the Skolem-Mahler-Lech theorem.

3. Heights

We shall sometimes be brief; more details can be found in many places, notably the excellent textbooks by Hindry & Silverman (2000) or Bombieri & Gubler (2006).

3.1. Fields and the product formula. — We now assume that k is a field of one of the following types:

- a number field;
- a function field in one variable over a finite field;
- a function field over an algebraically closed field.

In these three cases, one can define a set $M(k)$ of pairwise inequivalent absolute values of k such that the product formula holds: for any $a \in k^\times$, only finitely many $|a|_v$ are distinct from 1, and $\prod_{v \in M(k)} |a|_v = 1$

For $k = \mathbf{Q}$, one takes for $M(k)$ the set of all p-adic absolute values $|\cdot|_p$ (normalized for any prime number p by $|p|_p = 1/p$), as well as the usual archimedean value $|a|_\infty$. The product formula follows from factorization into prime powers: for $a \in \mathbf{Q}^\times$, one can write $a = \pm \prod_{p \text{ prime}} p^{a_p}$, then $|a|_p = p^{-a_p}$ for each prime number p, while $|a|_\infty = \prod_p p^{a_p}$.

A function field in one variable over a finite field can be viewed as the field of functions of a smooth, projective, geometrically irreducible curve C over a finite field κ. The set $M(k)$ is indexed by closed points of the curve C. The local ring at for any closed point $x \in C$ is a discrete valuation ring; let v_x be the corresponding normalized valuation (with image $\mathbf{Z}$): for $a \in k^\times$, $v_x(a)$ is the order of the zero of a, or minus the order of the pole. The absolute value $|\cdot|_x$ is then defined by the formula $|a|_x = |\kappa(x)|^{-v_x(a)}$, where $\kappa(x)$ is the residue field of C at x. The product formula corresponds to the fact that a nonzero rational function on a curve possesses as many zeros as poles, counted with appropriate multiplicities.

General function fields over an algebraically closed field κ can be treated similarly, but there is no canonical choice in general. Let k be such a field, and view it as the field of functions of a projective integral scheme S over κ. The set $M(k)$ is now indexed by the set of prime divisors of S (meaning, integral subschemes of codimension 1 in S); any prime divisor D gives rise to a normalized valuation $v_D \colon \kappa(S)^\times \to \mathbf{Z}$. Let us also fix an ample line bundle $\mathscr{L}$ on S; it allows to define the degree of any integral subscheme of S, more generally, of any linear combination of them. Then, for any prime divisor D, one defines $|a|_D = e^{-\deg_{\mathscr{L}}(D) v_D(a)}$. The product formula is equivalent to the relation $\sum_D \deg_{\mathscr{L}}(D) v_D(a) = 0$, which follows from the fact that the divisor of a, defined as the cycle $\sum_D v_D(a)[D]$, has degree 0.

We shall say that k is a M-field. In all of these three cases, it is possible to define naturally a M-field structure on any finite extension k' of k, which is compatible with that of $M(k)$: there is a surjective map with finite fibers, $\pi \colon M(k') \to M(k)$, such that $|a|_v = \prod_{\pi(v')=v} |a|_{v'}$ for any $a \in k^\times$ and any $v \in M(k)$.

3.2. Height functions and their functoriality properties. — Let k be a M-field. The (exponential) Height function on the rational points of the projective space $\mathbf{P}^n$ is defined by the formula:

$$H([x_0 : \ldots : x_n]) = \prod_{v \in M(k)} \max(|x_0|_v, \ldots, |x_n|_v).$$

The product formula assures that this is well defined, independently of the choice of the homogeneous coordinates. Let us give an example, assuming that $k = \mathbf{Q}$: any point x of $\mathbf{P}^n(\mathbf{Q})$ has a system of homogeneous coordinates $[x_0 : \ldots : x_n]$ consisting of coprime integers, unique up to multiplication by ±1; then, $H(x) = \max(|x_0|, \ldots, |x_n|)$. Indeed, for any prime number p, one of the x_j is not divisible by p, hence $\max(|x_0|_p, \ldots, |x_n|_v) = 1$; there only remains the factor corresponding to the archimedean absolute value, whence the result.

In fact, the Height function extends naturally to a function, still denoted H, on $\mathbf{P}^n(k')$, for any finite extension k' of k, and then to $\mathbf{P}^n(\bar{k})$, where $\bar{k}$ is an algebraic closure of k.

In some applications, and in the following propositions, it is convenient to introduce the height function h on $\mathbf{P}^n(\bar{k})$ defined as the logarithm of the Height function.

Proposition. — *Let $f\colon \mathbf{P}^n_k \dashrightarrow \mathbf{P}^m_k$ be a rational map defined by homogeneous polynomials $f_0, \ldots, f_m$ of degree d, without common factors. Let $E = V(f_0, \ldots, f_m)$ be the locus they define in $\mathbf{P}^n_k$; this is the locus of indetermination of f.*

(1) *There exists a positive real number c such that $H(f(x)) \leq cH(x)^d$ for any point $x \in \mathbf{P}^n(\bar{k})$ such that $x \notin E$.*

(2) *For any closed subscheme V of $\mathbf{P}^n_k$ such that $V \cap E = \varnothing$, there exists a positive real number c_V such that $H(f(x)) \geq c_V H(x)^d$ for any $x \in V(\bar{k})$.*

3.3. The Height machine. — From this Proposition and basic properties of line bundles on projective varieties, one constructs the *height machine*.

Proposition. — *Let V be a projective variety over k. Let $\mathscr{F}(V)$ be the real vector space of real-valued functions on $V(\bar{k})$ and let $\mathscr{F}_{\mathrm{b}}(V)$ be its subspace of bounded functions. There is a unique linear map*

$$h\colon \operatorname{Pic}(V) \otimes \mathbf{R} \to \mathscr{F}(V)/\mathscr{F}_{\mathrm{b}}(V), \qquad \mathscr{L} \mapsto h_{\mathscr{L}}$$

such that $h_{\mathscr{L}} = h \circ \varphi \pmod{\mathscr{F}_b(V)}$ for any closed embedding $\varphi\colon V \hookrightarrow \mathbf{P}^n_k$ and $\mathscr{L} = \varphi^*\mathscr{O}(1)$.

Moreover, this formula holds for any morphism $\varphi\colon V \to \mathbf{P}^n_k$, without assuming that it is an embedding.

Corollary. — *Let V and W be projective varieties over k, let $f\colon V \to W$ be a morphism. For any $\mathscr{L} \in \operatorname{Pic}(W) \otimes \mathbf{R}$, one has $h_{f^*\mathscr{L}} = h_{\mathscr{L}} \circ f$.*

In practice, one chooses representatives of the height function, still denoted $h_{\mathscr{L}}$. The preceding equalities then become formulae up to a bounded function.

Remark. — By a Theorem of Néron (see Serre (1997), §2.9 and 3.11 [2]), the morphism $\operatorname{Pic}(V) \otimes \mathbf{R} \to \mathscr{F}(V)/\mathscr{F}_{\mathrm{b}}(V)$ is injective.

2. This reference only treats the case when k is a number field, but the proof seems to establish the result in general, up to some minor modifications.

3.4. Polarized dynamics, canonical heights. — Let $(V, f, \mathscr{L})$ be a polarized dynamical system, let q be its weight, and let $h_{\mathscr{L}}$ be a height function for $\mathscr{L}$. One has $h_{\mathscr{L}}(f(x)) = qh_{\mathscr{L}}(x) + \mathscr{O}(1)$.

***Proposition* (Tate, Call & Silverman (1993)).** — *There exists a unique height function $\hat{h}_{\mathscr{L}}$ for $\mathscr{L}$ such that $\hat{h}_{\mathscr{L}}(f(x)) = q\hat{h}_{\mathscr{L}}(x)$ for any $x \in V(\bar{k})$.*

Proof. — The space of height functions for $\mathscr{L}$ is a real affine space directed by the space $\mathscr{F}_{\mathrm{b}}(V)$. Let us endow that space with the supremum norm. For any height function h for $\mathscr{L}$, one has $h(f(x)) = qh(x) + \mathrm{O}(1)$, so that $\frac{1}{q}h \circ f$ is still a height function for $\mathscr{L}$. Moreover, the map $h \mapsto \frac{1}{q}h \circ f$ is contracting, so it has a unique fixed point $\hat{h}$.

In fact, the proof of the fixed point theorem shows that $\hat{h}$ can be defined by the explicit limit formula:

$$\hat{h}(x) = \lim_{n\to\infty} q^{-n} h(f^{(n)}(x)), \qquad x \in V(\bar{k}).$$ □

This last formula shows that $\hat{h}$ is nonnegative. The existence of the canonical height has beautiful consequences. For example, the functional equation implies at once that preperiodic points have canonical height zero. Actually, it was already the main point of Northcott (1950) to prove that the set of preperiodic points in $V(\bar{k})$ has bounded height, a property which does not depend on the actual choice of a representative.

3.5. Northcott finiteness and height zero. — From the explicit computation of the height of a point in $\mathbf{P}^n(\mathbf{Q})$, it is obvious that given any bound B, there are only finitely many points $x \in \mathbf{P}^n(\mathbf{Q})$ such that $H(x) \leq B$. This observation generalizes as follows.

***Proposition* (Northcott, 1950).** — *Assume that k is either a number field, or a function field in one variable over a finite field. Then, for any real number B and any positive integer d, there are only finitely many points $x \in \mathbf{P}^n(\bar{k})$ such that $[k(x) : k] \leq d$ and $H(x) \leq B$.*

Proof. — For $d = 1$ and k a number field, the proof is the above observation; still when $d = 1$ and $k = \kappa(T)$, for κ a finite field, a similar computation can be done. From the definition of the Height, we see that we may assume that $n = 1$. Finally, one passes from $d = 1$ to arbitrary d by consideration of symmetric products and the fact that $\mathrm{Sym}^d(\mathbf{P}^1_k)$ is isomorphic to $\mathbf{P}^d_k$ — this is basically nothing more than the theory of elementary symmetric functions. □

We will call N-field a M-field in which the preceding finiteness result holds. There are interesting ongoing investigations in number theory to enlarge the class of known N-fields. Observe however that the field of functions $k = \kappa(S)$ of a projective integral variety S over an algebraically closed field κ is not a N-field: the points of $\mathbf{P}^n(\kappa) \subset \mathbf{P}^n(k)$ all have height 0.

Corollary. — *Let $(V, f, \mathscr{L})$ be a polarized dynamical system over a N-field. Any point $x \in V(\bar{k})$ such that $\hat{h}(x) = 0$ is preperiodic.*

Proof. — Let x be such a point; any point y in its orbit satisfies $\hat{h}(y) = 0$ and $[k(y) : k] \leq [k(x) : k]$. Since k is a N-field, the orbit of x is finite, which means that x is preperiodic. □

Describing what happens when k is a function field over an algebraically closed field will be the subject of the second part of this paper.

PART II. ALGEBRAIC DYNAMICS OVER FUNCTION FIELDS

The second part of this text is about algebraic dynamics over a function field. Specifically, in presence of a polarized dynamical system, we want to characterize the points of height zero. But first, we need to return once again to Abelian varieties.

4. Abelian varieties over function fields

4.1. The Néron–Tate height. — Let A be an Abelian variety over a M-field k and let $\mathscr{L}$ be an ample symmetric line bundle on A (symmetric means that $[-1]^*\mathscr{L} \simeq \mathscr{L}$). Basic properties of line bundles on Abelian varieties (the Theorem of the cube, see Mumford (1974)) imply that for any integer $n \geq 2$, $[n]^*\mathscr{L} \simeq \mathscr{L}^{n^2}$. In other words, the algebraic dynamical system $(A, [2])$ is polarized by $\mathscr{L}$. Let $\hat{h}_{\mathscr{L}}$ be a canonical height for $\mathscr{L}$. The Theorem of the cube implies that $\hat{h}_{\mathscr{L}}$ is a quadratic form on the Abelian group $A(\bar{k})$; it is nonnegative.

Assume that k is a N-field. Then, we have seen how Northcott's theorem implies that $\hat{h}_{\mathscr{L}}$ is positive on non-torsion points of $A(\bar{k})$, so that $\hat{h}_{\mathscr{L}}$ is a positive definite quadratic form on $A(\bar{k}) \otimes \mathbf{Q}$. In fact, using the full strength of Northcott's theorem, one can show, see (Serre, 1997, Lemma 2, p. 42), that $\hat{h}_{\mathscr{L}}$ is a positive definite quadratic form on $A(\bar{k}) \otimes \mathbf{R}$. This finiteness property is also an important step in the proof of the Mordell-Weil theorem that asserts $A(k)$ is finitely generated.

4.2. The Theorem of Lang-Néron. — We now leave the world of N-fields. Let S be a projective integral variety over an algebraically closed field κ and let $k = \kappa(S)$ be its function field. As in §3.1, we endow it with the structure of a M-field.

We want to describe the set of points $x \in A(k)$ such that $\hat{h}_{\mathscr{L}}(x) = 0$.

Let d be any positive integer such that $\mathscr{L}^d$ is very ample and let $\varphi\colon A \hookrightarrow \mathbf{P}^N_k$ be a closed embedding of A into a projective space such that $\varphi^*\mathscr{O}(1)$ is isomorphic to $\mathscr{L}^d$. Let B a real number such that $\left|\hat{h}_{\mathscr{L}}(x) - h(\varphi(x))\right| \leq B$ for all $x \in A(\bar{k})$.

Let $x \in A(k)$ be a point such that $\hat{h}_{\mathscr{L}}(x) = 0$; assume that x is not a torsion point. Any of its multiples $[n]x$ satisfies $\hat{h}_{\mathscr{L}}([n]x) = 0$, so that the points $x_n = \varphi([n]x) \in \mathbf{P}^N(k)$ furnish an infinite sequence of rational points whose heights are uniformly bounded.

4.3. Limited families. — The following lemma shows that points of bounded height in $\mathbf{P}^N(k)$ can be parameterized by a finite dimensional variety over κ.

Lemma. — *Let B be a positive integer. Let η be the generic point of S. There exists a κ-scheme of finite type T and a rational map $\xi\colon S \times T \dashrightarrow \mathbf{P}^N_\kappa$ such that, for any point $x \in \mathbf{P}^N(k)$ such that $h(x) \leq B$, there exists a point $t \in T(\kappa)$ satisfying $\xi(\eta, t) = x$.*

Proof. — We assume that S is a curve [3] Let $x = [x_0 : \ldots : x_N] \in \mathbf{P}^N(k)$ be a point such that $h(x) \leq B$. Without loss of generality, we assume that $x_0 = 1$. Then, for each $j \in \{1, \ldots, N\}$, the degree of the rational functions x_j is at most B. For any j, the function x_j is determined by its polar divisor D_j, itself an element of some symmetric product $\operatorname{Sym}^b S$, with $0 \leq b \leq B$, up to the finite dimensional ambiguity corresponding to the Riemann-Roch space $\mathscr{L}(D_j)$. The lemma follows from that; to be honest, it should follow from that. □

Let us now explain the title of this Section. Let S is a Noetherian scheme and X a S-scheme of finite type, Exposé XIII of Grothendieck *et al* (1971) defines and studies the notion of limited families of coherent sheaves on the fibers of X/S. A family of coherent sheaves is the datum, for any point $s \in S$ and any extension K of $\kappa(s)$, of a set $\mathscr{F}(K)$ of coherent sheaves on the K-variety X_K. One says that such a family is *limited* if there exists a S-scheme of finite type T and a coherent sheaf $\mathscr{E}$ on $X \times_S T$ such that, for any $s \in S$, and any extension K of $\kappa(s)$, and any sheaf $F \in \mathscr{F}_K$, there exists a point $t \in T_{\kappa(s)}$, a common extension K' of $\kappa(s)$ and K such that $F_{K'} \simeq \mathscr{E}_{t,K'}$.

Here is the crucial criterion proved in (Grothendieck *et al*, 1971, Exposé XIII, Théorème 1.13): a family of coherent sheaves is bounded if and only if 1) all of the sheaves in the family can be presented as a quotient of one fixed coherent sheaf on X, and 2) the set formed by their Hilbert polynomials is finite.

The most important case is given by $S = \operatorname{Spec} \kappa$ and $X = \mathbf{P}^N$. Associating to any closed integral subscheme $V \subset X$ its structure sheaf $\mathscr{O}_V$ allows to talk about a families, especially limited families, of closed integral subschemes of $\mathbf{P}^N$. From the preceding criterion, one can deduce the theorem of Chow: *the family of all closed integral subschemes of $\mathbf{P}^N$ with given degree is limited.*

4.4. The Theorem of Lang–Néron (conclusion of the proof). — Let us apply Lemma 4.3 on limited sets to the points $x_n = \varphi([n]x)$. The scheme T has only finitely many irreducible components, so we may choose two distinct integers m and n such that t_m and t_n belong to the same irreducible component of T. Let C be an smooth irreducible curve over κ and $\pi\colon C \to T$ a morphism such that $\pi(C)$ contains t_m and t_n. From the morphism π and the rational map ξ, one constructs a morphism $\alpha\colon C_k \to A$, given by $\alpha(c) = \xi(\eta, \pi(c))$. Fix a point $c_0 \in C$. Necessarily, α factors through the Albanese map $C_k \to \operatorname{Alb}(C)_k$ which is the universal morphism to an Abelian variety mapping c_0 to 0. Therefore, we obtain a morphism of Abelian varieties, $\alpha\colon \operatorname{Alb}(C)_k \to A$, whose image, up to some translation, contains x_m and x_n. In particular, $x_m - x_n$ belongs to $\alpha(\operatorname{Alb}(C)_k)$.

In other words, the difference between the points x_m and x_n, while not zero, is explained by an Abelian subvariety of A the modulus of which belongs to κ. The theory of the trace asserts the existence of a largest Abelian variety A_0 over κ together with an injective

3. I have to think whether the following proof can be adapted to treat the general case or not.

morphism $\tau\colon (A_0)_k \to A$. The variety A_0 is called the k/κ-trace of A. What precedes shows that $[m-n]x \in \tau(A_0(\kappa))$

This concludes the proof of the following Theorem:

Theorem **(Lang & Néron, 1959).** — *Let k be the function field of a projective smooth integral curve over an algebraically closed field κ. let A be an Abelian variety over k, let A_0 be its k/κ-trace and let $\tau\colon (A_0)_k \to A$ be the canonical morphism. Let $\mathscr{L}$ be an ample symmetric line bundle on A and let $\hat{h}_{\mathscr{L}}$ be the associated canonical height.*

For any point $x \in A(k)$ such that $\hat{h}_{\mathscr{L}}(x) = 0$, there exists a positive integer n such that $[n]x \in \tau(A_0(\kappa))$.

Let us consider the particular case where A is a simple k-variety; then, there are only two possibilities concerning the k/κ-trace: either $A_0 = 0$, or $A_0 = A$ and $A = (A_0)_k$ is defined over κ. In the first case, points of canonical height zero are torsion points; in the second one, they are the "constant points", that is, the points of $A_0(\kappa)$ seen within $A(k)$.

5. Points of height zero in polarized algebraic dynamics

Let κ be an algebraically closed field, let S be a projective integral variety over κ and let $k = \kappa(S)$. We endow it with the structure of a M-field.

5.1. Dynamics on the projective line. — Let $f \in k(T)$ be a rational function of degree $d \geq 2$. It defines an endomorphism of $\mathbf{P}^1_k$, hence an algebraic dynamical system. Since $f^*\mathscr{O}(1) \simeq \mathscr{O}(d)$, this system is polarized. Let $\hat{h}$ be the corresponding canonical height.

Theorem **(Baker, 2009).** — *If there exists a non-preperiodic point $x \in \mathbf{P}^1(\bar{k})$ such that $\hat{h}(x) = 0$, then there is an homography $u \in \mathrm{PGL}(2,\bar{k})$ such that $u(x) \in \mathbf{P}^1(\kappa)$ and $u \circ f \circ u^{-1} \in \kappa(T)$.*

Baker's proof belongs to Diophantine geometry and relies on a careful study of "canonical Green functions" on the projective line in the sense of Berkovich (1990), as developed by Baker & Rumely (2010); see also Thuillier (2005). However, and unfortunately, the analysis of Green functions is insensitive to extensions of the ground field. Consequently, the proof does not show that one can one find an adequate homography in $\mathrm{PGL}(2,k)$ when there exists a k-rational which has height 0 but is not preperiodic.

As an example, taking for f a Lattès map, one recovers the theorem of Lang–Néron, up to this minor discrepancy between being constant over k, or becoming constant after base change to $\bar{k}$.

5.2. Height zero, *vs.* limited families. — As we have already seen in the proof of the theorem of Lang–Néron, heights over function fields are but a way of measuring that families of geometric objects are limited.

Any point $x \in \mathbf{P}^N(k)$ defines a rational map $\varepsilon_x\colon S \dashrightarrow \mathbf{P}^N$ (which is in fact regular everywhere if S is a smooth curve). Let S_x be the closure of its image. One says that a subset Σ

of rational points in $\mathbf{P}^N(k)$ is limited if the set of corresponding subvarieties $S_x \subset \mathbf{P}^N$, for $x \in \Sigma$, is limited.

As a consequence of Lemma 4.3, one has the following lemma.

Lemma. — *Let $(V, f, \mathscr{L})$ be a polarized dynamical system over k. Let $\hat{h}$ be the corresponding canonical height. Let x be a any point in $V(k)$. The following properties are equivalent:*

(1) *the point x has canonical height zero: $\hat{h}(x) = 0$;*

(2) *for any choice of a height function on V, its orbit $\{x, f(x), \dots\}$ has bounded height;*

(3) *its orbit $\{x, f(x), \dots\}$ is limited.*

5.3. Small points *vs.* limited families. — In fact, the proof of Theorem 5.1 yields a finer result: the conclusion holds under the assumption that *there exists a sequence (x_n) of distinct points in $\mathbf{P}^1(k)$ such that $\hat{h}(x_n) \to 0$.* This hypothesis translates as follows into the language of limited families:

Lemma. — *Let $(V, f, \mathscr{L})$ be a polarized dynamical system over k. Let $\hat{h}$ be the corresponding canonical height. Let (x_n) be a sequence of distinct points in $\mathbf{P}^1(k)$. The following properties are equivalent*

(1) *one has $\hat{h}(x_n) \to 0$;*

(2) *for a given choice of a height function on V and any integer n, define N_n as the smallest positive integer m such that $h(f^{(m)}(x_n)) \geq 1$. Then, the sequence (N_n) tends to ∞.*

(3) *There exists a limited set Σ, and a sequence (N_n) of integers converging to ∞ such that $f^{(m)}(x_n) \in \Sigma$ for any n and any $m \leq N_n$.*

Remark. — The last assumption implies that there exists an algebraically closed extension κ^* of κ a point $x \in V(\kappa^* k)$ the orbit of which is limited. If κ is uncountable, one may even take $\kappa^* = \kappa$.

By assumption, there exists a κ-scheme T of finite type and a morphism $\sigma\colon T_k \to V$ such that $\Sigma \subset \sigma(T(\kappa))$. Let Γ be the closure in $T \times T$ of the set of points $(t, t') \in T(\kappa)^2$ such that $f(\sigma(t)) = t'$. For any integer N, the set of points $t \in T(\kappa)$ for which there exists points $t = t_0, t_1, \dots, t_N \in T(\kappa)$ such that $(t_{i-1}, t_i) \in \Gamma$ for $i \in \{1, \dots, N\}$ is the set of κ-points of a constructible algebraic subset O_N of T. By assumption, $O_N(\kappa) \neq \varnothing$. By compactness of the constructible topology, (Grothendieck, 1961, Chap. 0, (9.2.4)), their intersection O_∞ is non-empty and there exists an extension κ^* of κ such that $O_\infty(\kappa^*) \neq \varnothing$. If κ is uncountable, the κ-points of a countable union of strict subvarieties cannot exhaust the whole space, so one may even take $\kappa^* = \kappa$.

When κ is the field of complex numbers, this last property is often proved by invoking the theorem of Baire. But here is an algebraic proof, given to me by J.-L. Colliot-Thélène some years ago: it is sufficient to show that, given a sequence (f_n) of non-zero polynomials in N variables, there exists $x \in \kappa^N$ such that $f_n(\kappa) \neq 0$ for each integer n. Let us prove the result by induction on N. If $N = 1$, then the equation $f_n(T) = 0$ has only finitely many solutions in κ, and the countable union of these finite sets does not exhaust κ. Let $N \geq 2$; for any integer n, let $g_n \in \kappa[T_1, \dots, T_{N-1}]$ be the leading coefficient of f_n, viewed as a polynomial in T_N. By induction, there exists $(t_1, \dots, t_{N-1}) \in \kappa^{N-1}$ such that for all n, $f_n(t_1, \dots, t_{N-1}, T_N)$ is a non-zero polynomial in the variable T_N. By induction again, there exists $t_N \in \kappa$ such that $f_n(t_1, \dots, t_N) \neq 0$ for all n.

In model theoretic language, within the theory ACF of algebraically closed fields, O_∞ is described by the countable list of formulae defining the O_N. It follows from the compactness theorem that it is satisfiable, hence defines a (partial) type p. This shows that it is realized in some algebraically closed extension κ^* of κ. If κ is uncountable, then κ is *saturated*, see Exercise 4.5.17 of Marker (2002), so that p is realized in κ.

PART III. DIFFERENCE FIELD AND DESCENT

From now on, essentially all results are taken from the two papers of Chatzidakis & Hrushovski (2008*a*,*b*). Inaccurracies, misunderstandings and mistakes are mine.

6. Algebraic dynamics and algebraically closed fields with an automorphism

6.1. Difference fields. — We recall that a difference field (K, σ) is a field K equipped with an endomorphism σ. The difference field is called *inversive* when σ is an automorphism.

We shall work in the language of rings, enlarged by a symbol σ for an endomorphism. In that language, the theory of *algebraically closed fields with an automorphism*, in short, ACFA, has for structures pairs (K, σ), where σ is a function from K to K subject to the following families of axioms:

(1) axioms that say that K is an algebraically closed field;

(2) axioms that say that σ is an automorphism of K;

(3) for any irreducible K-variety V and any irreducible subvariety $S \subset V \times V^\sigma$ such that both projections from S to V, and from S to V^σ, are generically surjective, there exists $a \in V$ such that $(a, \sigma(a)) \in V$.

These axioms are expressible by an infinite set of formulae in first order logic.

***Theorem* (Macintyre (1997); Chatzidakis & Hrushovski (1999)).**
The theory ACFA *is model-complete; it is the model companion of the theory of difference fields.*

In the sequel, we shall work within some "large" model **M** of ACFA.

6.2. Difference fields and algebraic dynamical systems. — Let (V, S) be an algebraic dynamical system over some given ground field F, S being a correspondence in V, dominant over each factor. If one prefers, one may assume that S is the graph of an endomorphism f of V; then, (V, f) is an algebraic dynamical system.

View F as a difference field by imposing $\sigma_F = \mathrm{id}_F$. For any extension (K, σ) of F as a difference field, one may consider the set

$$S^\sharp(K) = \{(x, y) \in S(K) \subset V(K) \times V(K)\,;\, y = \sigma(x)\}$$

together with its two projections to $V(K)$. We shall see below that if K is a model of ACFA, then $S^\sharp(K)$ is Zariski dense in S.

If one is ready to adopt the language and techniques from model theory of difference fields, it is more natural not to assume that the dynamics is trivial on the ground field; this is in fact an unnatural hypothesis, for it forbids to consider the "fibers" of a morphism $(V, S) \to (W, T)$ of dynamical systems.

So let (F, σ) be a difference field. A dynamical system is a pair (V, S), where V and $S \subset V \times V^\sigma$ are algebraic varieties over F such that the images of S by the two projections to V and V^σ are dominant. Here, V^σ is the algebraic variety obtained from V by applying σ to all coefficients of the equations of V; in particular, for any extension (K, σ) of (F, σ) as a difference fields, the morphism $\sigma\colon K \to K$ induces a map $\sigma\colon V(K) \to V^\sigma(K)$. As above, one defines

$$S^\sharp(K) = \{(x, y) \in S(K) \subset V(K) \times V^\sigma(K)\,;\, y = \sigma(x)\}.$$

Lemma. — *Assume that K is a model of* ACFA. *Then, $S^\sharp(K)$ is Zariski dense in S.*

Proof. — One may assume that V and S are irreducible. Let T be a non-empty open subset of S; then, T maps dominantly to V and V^σ. The third series of axioms of ACFA then guarantees that $T^\sharp(K) \neq \varnothing$. □

6.3. Some classification. — A dynamical system (V, f) is said to have *constant* dynamics if $f = \mathrm{id}_V$, *periodic* dynamics if there exists a positive integer n such that $f^{(n)} = \mathrm{id}_V$. If V is defined over a finite field, one says that (V, f) has *twisted-periodic* dynamics if there exists a positive integer n such that $f^{(n)}$ is a power of the Frobenius morphism.

Let (U, g) be a dynamical system and $p\colon (V, f) \to (U, g)$ be a morphism.

One says that (V, f) is constant, or periodic, or twisted periodic, over (U, g) if there exists a dynamical system (W, h) with constant (*resp.* periodic, twisted periodic) dynamics such that V is a subsystem of $W \times U$.

One says that (V, f) is *fixed-field internal* (*resp.* twisted-field internal) if there exists a dynamical system (W, h) over (U, g) such that some irreducible component $V \times_U W$ is periodic (*resp.* twisted-periodic).

One says finally that (V, f) is *fixed-field-free* (*resp.* field-free) over U. if for any factorization $V \to W \to U' \to U$ such that W is fixed-field internal (*resp.* field internal) over U', then $W \to U$ has generically finite fibers.

When (U, g) is a point, one simply says *fixed-field-free* (or field-free).

Finally, one says that the dynamical system (V, f) is *primitive* if there is no morphism $(V, f) \to (U, g)$ with $\dim(U) > 0$.

Field-free and field-internal dynamics are orthogonal: no non-trivial dynamics can be defined in the product of two orthogonal dynamics. Precisely, one has the following result.

Proposition **(Chatzidakis & Hrushovski, 2008*a*, Proposition 1.3).**

Let (V,f) and (U,g) be dynamical systems defined over K. Assume that (V,f) is field-free and (U,g) is field-internal.

(1) *Any irreducible difference variety $R \subset U \times V$ which is dominant over U and V is a component of $U \times V$.*

(2) *Let $L = K(U)$, let W be a finite cover of V_L. There exists a finite cover V' of V such that V'_L is a finite cover of W.*

Proof. — (1) Let b be a generic point of V. Since V is field free, $K(b) \cap \mathrm{Fix}(\sigma) \subset K^{\mathrm{alg}}$. (Any element $c \in K(b) \cap \mathrm{Fix}(\sigma)$ which is not algebraic over K would yield a positive-dimensional quotient of (V,f) with constant dynamics.)

(2) □

6.4. Modularity. — In Chatzidakis & Hrushovski (2008*b*), *modularity* is defined under the name of *one-basedness*: a dynamical system (V,f) over a difference field K is said to be *one-based* if for any extension L of K as a difference field, any finite family a in $V(L)$ and any finite family b in L, the fields $K(a)^{\mathrm{alg}}$ and $K(b)^{\mathrm{alg}}$ are linearly independent over their intersections. According to the fundamental dichotomy, see Chatzidakis & Hrushovski (1999); Chatzidakis *et al* (2002), *field-free dynamics are modular.* As a consequence, one has the following inequality.

***Proposition* (Chatzidakis & Hrushovski, 2008*a*, Theorem 1.4).**
Let (V,f) be a field-free dynamical system, let (U,g) be a dynamical system and let $W \subset U \times V$ be a m-dimensional irreducible family of n-dimensional irreducible difference subvarieties of V. Then, $m + n \leq \dim(V)$.

***Proposition* 6.5 (Chatzidakis & Hrushovski, 2008*a*, Proposition 1.5).**
Let (V,f) be a primitive dynamical system. If f has separable degree > 1, then (V,f) is modular.

For the proof, we need to recall the following algebraic notions from difference algebra, see §5.16 of Cohn (1965), Let a be a finite family of elements in $\mathbf{M}$ such that $\sigma(a) \in K(a)^{\mathrm{alg}}$. When $n \to \infty$, the degrees of the extensions

$$[K(a,\sigma(a),\dots,\sigma^{n+1}(a)) : K(a,\dots,\sigma^n(a))]$$

are non-increasing, since

$$\begin{aligned}[K(a,\sigma(a),\dots,\sigma^{(n+1)}(a)) : K(a,\dots,\sigma^n(a))]&\\ &= [K^\sigma(\sigma(a),\dots,\sigma^{(n+2)}(a)) : K^\sigma(\sigma(a),\dots,\sigma^{n+1}(a))]\\ &\geq [K(a,\sigma(a),\dots,\sigma^{(n+2)}(a)) : K(a,\dots,\sigma^{n+1}(a))].\end{aligned}$$

Consequently, these degrees have a limit, when $n \to \infty$, denoted $\mathrm{ldeg}(a/K)$ and called the *limit degree* of a over K. Similarly, the degrees

$$[K(a,\sigma^{-1}(a),\dots,\sigma^{(-n-1)}(a)) : K(a,\dots,\sigma^{-n}(a))]$$

have a limit when $n \to \infty$, denoted $\mathrm{ildeg}(a/K)$ and called the *inverse limit degree*of a over K. Both degrees depend only on the extension $K(a)_{\sigma,\sigma^{-1}}/K$. We define accordingly the limit degree $\operatorname{ldeg}(L/K)$ and inverse limit degree $\operatorname{ildeg}(L/K)$ of an extension L/K of difference fields. They multiply in towers: if M/L and L/K are extensions of difference fields, then $\operatorname{ldeg}(M/K) = \operatorname{ldeg}(M/L)\operatorname{ldeg}(L/K)$ and $\operatorname{ildeg}(M/K) = \operatorname{ildeg}(M/L)\operatorname{ildeg}(L/K)$, see (Cohn, 1965, §5.19).

***Lemma* (Chatzidakis & Hrushovski, 2008*b*, Lemma 1.11).**

Assume that K is an inversive difference field. Let a and b be finite families of elements in **M** *such that $\sigma(a) \in K(a)^{\mathrm{alg}}$ and $b \in K(a)^{\mathrm{alg}}$.* [4]

(1) *If $b \in K(a)_{\sigma,\sigma^{-1}}$, then* $\operatorname{ldeg}(b/K) \le \operatorname{ldeg}(a/K)$ *and* $\operatorname{ildeg}(b/K) \le \operatorname{ildeg}(a/K)$.

(2) *If $a \in K^{\mathrm{alg}}$, then* $\operatorname{ldeg}(a/K) = \operatorname{ildeg}(a/K)$.

(3) *If* $\operatorname{ldeg}(b/K) = 1$, *then $K(a,b)_{\sigma,\sigma^{-1}}$ is a finite extension of $K(a)_{\sigma,\sigma^{-1}}$.*

(4) *Assume that some analysis of the type* $\operatorname{tp}(a/K)$ *only involves types non-orthogonal to* $\operatorname{Fix}(\sigma)$, *then* $\operatorname{ldeg}(a/K) = \operatorname{ildeg}(a/K)$.

Proof. — (1) By the above properties of limit degrees, one has

$$\begin{aligned}\operatorname{ldeg}(a/K) &= \operatorname{ldeg}(K(a)_\sigma/K) = \operatorname{ldeg}(K(a,b)_\sigma/K)\\ &= \operatorname{ldeg}(K(a,b)_\sigma/K(b)_\sigma)\operatorname{ldeg}(K(b)_\sigma/K)\\ &\ge \operatorname{ldeg}(b/K).\end{aligned}$$

The other formula is proved similarly.

(2) Replacing a by $(a,\sigma(a),\dots,\sigma^n(a))$ for some large enough integer n, we may assume that $\operatorname{ldeg}(a/K) = [K(a,\sigma(a)) : K(a)]$. Therefore,

$$\operatorname{ldeg}(a/K)[K(a):K] = [K(a,\sigma(a):K] = [K(a,\sigma(a)):K(\sigma(a)][K(\sigma(a):K].$$

The field K being inversive, one has

$$\begin{aligned}[K(a,\sigma(a)):K(\sigma(a)) &= [K^\sigma(a,\sigma(a):K^\sigma(\sigma(a))] = [K(\sigma^{-1}(a),a):K(a)]\\ &= \operatorname{ildeg}(a/K).\end{aligned}$$

This shows that ,

$$\operatorname{ldeg}(a/K)[K(a):K] = [K(a,\sigma(a):K] = \operatorname{ildeg}(a/K)[K(\sigma(a):K].$$

The result follows since $[K(a):K] = [K(\sigma(a)):K]$.

(3)

(4) The hypothesis means that there exists a finite sequence $(a_1,\dots,a_n)$ such that for all i, $\operatorname{tp}(a_i/\operatorname{acl}(K,a_1,\dots,a_{i-1}))$ is almost-internal to a type of SU-rank 1 which is non-orthogonal to $\operatorname{Fix}(\sigma)$. By the multiplicative properties of limit and inverse limit degrees, we reduce to the case where $n=1$.

Therefore, there exists a closed difference field L, independent from a over K, and a finite family $b \in \operatorname{Fix}(\sigma)$ such that $\operatorname{acl}(La) = \operatorname{acl}(Lb)$. This implies that $a \in L(b)^{\mathrm{alg}}$ and $b \in L(a)_\sigma^{\mathrm{alg}}$.

4. Hence $\sigma(b) \in K(b)^{\mathrm{alg}}$...

Moreover, just by definition, one has $\operatorname{ldeg}(L(b)/L) = \operatorname{ildeg}(L(b)/L) = 1$. Since $b \in L(a)^{\mathrm{alg}}_{\sigma}$, $\operatorname{ldeg}(b/L(a)^{\mathrm{alg}}_{\sigma}) = \operatorname{ildeg}(b/L(a)^{\mathrm{alg}}_{\sigma}$. Consequently, the relations

$$\operatorname{ldeg}(a,b/L) = \operatorname{ldeg}(b/L(a))\operatorname{ldeg}(L(a)/L)$$

and

$$\operatorname{ildeg}(a,b/L) = \operatorname{ildeg}(b/L(a))\operatorname{ildeg}(L(a)/L)$$

imply that

$$\frac{\operatorname{ldeg}(a,b/L)}{\operatorname{ildeg}(a,b/L)} = \frac{\operatorname{ldeg}(a/L)}{\operatorname{ildeg}(a/L)} = \frac{\operatorname{ldeg}(a/K)}{\operatorname{ildeg}(a/K)}$$

where we used that a and L were independent. Moreover,

$$\frac{\operatorname{ldeg}(a,b/L)}{\operatorname{ildeg}(a,b/L)} = \frac{\operatorname{ldeg}(a,b/Lb)\operatorname{ldeg}(b/L)}{\operatorname{ildeg}(a,b/Lb)\operatorname{ildeg}(Lb/L)} = \frac{\operatorname{ldeg}(a,b/Lb)}{\operatorname{ildeg}(a,b/Lb)} = 1$$

since $a \in L(b)^{\mathrm{alg}}$. □

We now prove Proposition 6.5. So let (V,f) be a primitive dynamical system which is not field free. Let a be a generic point of V. Since V is primitive, it satisfies the assumption of the Lemma, (4). Consequently, $\operatorname{ldeg}(a/K) = \operatorname{ildeg}(a/K)$. Since f is a morphism, $\operatorname{ldeg}(a/K) = 1$. Moreover, $\operatorname{ildeg}(a/K) = \deg(f)$, so that $\deg(f) = 1$.

This implies modularity in characteristic zero, but in characteristic p, one needs to exclude non-field-free dynamics which however would be fixed-field free

In fact, simply replacing degrees by *separable* degrees, one may define the reduced limit degree and reduced inverse limit degree of an extension of difference fields, in a similar manner to the definition of the limit and inverse limit degrees. Entirely analogous to the previous lemma, on can then state an prove a lemma for reduced degrees; in (4), non-orthogonality to $\operatorname{Fix}(\sigma)$ is replaced by non-orthogonality to a field $\operatorname{Fix}(\tau)$, with $\tau = \sigma^m \operatorname{Frob}^n$. For a generic point a of a dynamical system (V,f), one has $\operatorname{rldeg}(a/K) = 1$ and $\operatorname{rildeg}(a/K) = \deg_{\mathrm{s}}(f)$, hence the proposition.

7. Descent

7.1. Notions of isotriviality. — Let $k \hookrightarrow K$ be an extension of difference fields, and let (V,S) be an algebraic dynamical system over K. There are various notions for (V,S) to "come from k". In all cases, this demands that there exists a dynamical system (W,T) over k whose base-change to K "recovers" our original (V,f).

The difference lies in the strength of the identification between $(W,T)_K$ and (V,f).

The strongest one demands that (V,S) be isomorphic to $(W,T)_K$: here we require an isomorphism, $W_K \simeq V$, of algebraic varieties over K, that maps $T_K \subset (W \times W^{\sigma})_K$ to $S \subset V \times V^{\sigma}$. Although the more desirable to obtain, it seems to lie (slightly) beyond the scope of the model theoretic techniques which are based on *fields*.

Therefore, we shall say that (V,S) *descends* to k if there is a *birational isomorphism* of W_K to V which maps T_K to S. Anyway, if V is a smooth projective curve, *e.g.*, the projective line, then such a birational isomorphism extends to a true isomorphism and this notion would imply the previous one.

A weaker notion consists in requiring that there is a *constructible isomorphism* of W_K to V which, again, maps T_K to S. (In characteristic zero, this is of course equivalent to birational isomorphy.)

A still weaker notion asks for the existence of of an *isogeny* between (V,S) and $(W,T)_K$, namely, a dynamical system (V',S') over K which is a finite cover of both (V,S) and $(W,T)_K$. We then say that (V,S) is isogeny-isotrivial.

Finally, the weakest notion which we shall use here is that of *domination*: (V,S) is dominated by (W,T) if there exists a morphism $(W,T)_K \to (V,S)$ consisting of a dominant rational map from W_K to V that maps T_K to S.

The following propositions show that these notions are closely related.

***Proposition 7.2* (Chatzidakis & Hrushovski, 2008*a*, Proposition 1.9).**
Let $k \hookrightarrow K$ be an extension of difference fields, k being algebraically closed. Let (U,f) be a dynamical system over k, and let (W,h) be dynamical system over U. Assume that W is field-free and is dominated by a dynamical system over k. Then, W is isogeny-isotrivial.

Proof. — The assumptions give a dynamical system (V,g) over k as well as a dominant morphism $p\colon U \times V \to W$ over U.

Firstly, using elimination of imaginaries in ACF, we quotient V by the equivalence relation according to which $v \sim v'$ if $p(u,v) = p(u,v')$ for any generic point u of U. Observe that then, $h(p(u,v)) = p(f(u),g(v)) = p(f(u),g(v')) = h(p(u,v'))$, so that this equivalence relation is compatible with the dynamics. This allows to assume that assume that for $v \neq v'$, and for generic $u \in U$, $p(u,v) \neq p(u,v')$.

Let us then define $p_m\colon U^m \times V \to W^m$ by $p_m(u_1,\dots,u_m,v) = (p(u_1,v),\dots,p(u_m,v))$. By compactness, p_m is generically injective for m large enough. Let $q\colon U \times V \to W^m$ be given by $q(u,v) = p_m(u,\dots,u,v) = (p(u,v),\dots,p(u,v))$. This embeds $U \times V$ as a dynamical subsystem of $W \times_U \dots \times_U W$. Since W is field-free, so is V.

Proposition 6.5 then implies that V is modular.

Let us view the variety V as parameterizing functions $U \to W$. Their graphs are irreducible subvarieties of $U \times W$ of dimension $\dim(U)$, distinct for distinct v. Consequently, Proposition 6.4 implies that $\dim(V) + \dim(U) \leq \dim(U \times W)$, *i.e.*, $\dim(V) \leq \dim(W)$. Since $p\colon U \times V \to W$ is dominant, it is a finite cover of W, so that W is isogeny-isotrivial, as claimed. ☐

Corollary. — *With the notation and hypothesis of Proposition 7.2, let us moreover assume that U has trivial dynamics. Then, W descends constructibly to k.*

Proof. — We begin with a dominant morphism $p\colon U \times V \to W$ as at the end of the proof of Proposition 7.2. Moreover, V is field-free. For any $u \in U$, let $\mathscr{R}_u$ be the equivalence relation on V defined by $x \mathrel{\mathscr{R}_u} y$ if $p(u,x) = p(u,y)$. Observe that this equality implies that $p(u,g(x)) = h(p(u,x)) = h(p(u,y)) = p(u,g(y))$ so that $g(x) \mathrel{\mathscr{R}_u} g(y)$; in other words, the morphism g is compatible with the relation $\mathscr{R}_u$. When u varies, the graph of the relation $\mathscr{R}$ gives a definable subset $\mathscr{R}$ of $U \times V \times V$. Since U is fixed-field internal and V is fixed field-free, this subset $\mathscr{R}$ must be essentially trivial. This implies that, for generic $u \in U$, the relation $\mathscr{R}_u$ does not depend on u. By elimination of imaginaries in ACF, we may thus assume that the relation $\mathscr{R}$ is trivial. Then, p is a constructible isomorphism onto its image, hence the corollary. □

8. Limited orbits and descent

***Theorem 8.1.** — Let $k \hookrightarrow K$ be a regular extension of fields, let (X, φ) be a dynamical system over K which is field-free. Assume that there exists a limited set $\Sigma \subset X(K)$ which contains a dense orbit $\{x, \varphi(x), \varphi^{(2)}(x), \dots\}$. Then, (X, φ) descends constructibly to k.*

Observe that the theorem applies when (X, φ) is primitive and $\deg_{\mathrm{s}}(\varphi) > 1$.

Proof. — We may assume that K is finitely generated. Let U be an irreducible k-variety whose function field is isomorphic to K; let also $(V, f) \to U$ be a model of (X, φ). By definition, a limited set Σ is given by a (possibly reducible) k-variety V and a morphism $p\colon U \times V \to W$. For each integer n, let $v_n \in V(k)$ be a point such that $p(\cdot, v_n)$ corresponds to $\varphi^{(n)}(x)$. We may replace V by an irreducible component of the closure of the set $\{v_0, v_1, \dots\}$, and φ by some iterate $\varphi^{(d)}$, so that $v_{dn} \in V$ for all integers $n \geq 1$. We also replace $f^{(d)}$ by the morphism whose graph is the closure of the set of pairs $(v_{dn}, v_{d(n+1)})$ in $V \times V$. All these reductions may require to replace U by a dense open subset of it.

Since the orbit $\{x, \varphi^{(d)}(x), \dots\}$ is dense in X, the morphism $p\colon U \times V \to W$ is dominant. This implies that $(W, f^{(d)})$ is dominated by a dynamical system over k. By Proposition 7.2 and its corollary, $(W, f^{(d)})$ is isogeny isotrivial, and descends constructibly to k. □

***Corollary.** — Let (X, φ) be a primitive dynamical system over K which does not descend constructibly to k. For any limited subset Σ of $X(K)$, there exist an integer n, a dense open subset U of X defined over K, such that there is no point $x \in U(K)$ such that $x, \varphi(x), \dots, \varphi^{(n)}(x)$ be contained in Σ.*

Proof. — The finite bounds follow from the compactness theorem. □

8.2. A counterexample. — Let A be an Abelian variety over a field k and let H be an extension of A by a vector group V; in other words, there is an exact sequence of algebraic groups

$$0 \to V \to H \to A \to 0$$

and V is a power of the additive group $\mathbf{G}_{\mathrm{a}}$. We assume that $\dim(V) = 2$ and $\mathrm{Hom}(H, \mathbf{G}_{\mathrm{a}}) = 0$. This can be achieved, *e.g.*, by choosing A such that $\dim(A) = 2$ and considering its universal vector extension or, more generally. (Any Abelian variety A possesses a universal vector extension, and the corresponding vector group, isomorphic to $\omega_{A^\vee}$ has dimension $\dim(A)$. If one pushes it out by a surjective morphism of $\omega_{A^\vee}$ onto a 2-dimensional vector group, the resulting extension satisfies the required assumptions.)

Let us now make a base-change from k to the projective space $\mathbf{P}(V)$. We get an extension of group-schemes

$$0 \to V_{\mathbf{P}(V)} \to H_{\mathbf{P}(V)} \to A_{\mathbf{P}(V)} \to 0.$$

Now, the group-scheme $V_{\mathbf{P}(V)}$ possesses a tautological subscheme W: any $t \in \mathbf{P}(V)$ corresponds to a one-dimensional quotient of V, and W_t is its kernel. Let E be the generic fiber of the $\mathbf{P}(V)$-group scheme $H_{\mathbf{P}(V)}/W$. Let $K \simeq k(t)$ be the function field of $\mathbf{P}(V)$. Then, E

is an algebraic group over K, extension of A_K by $\mathbf{G}_a$. By construction, this extension is not defined over k.

Let $x \in E$ be any point which generates E: for that, it suffices that its image a in A generates A. If A is geometrically simple, it suffices to ensure that a is not a torsion point, and this can be achieved after some finite extension K' of K

Finally, let τ_x be the translation by x in E. Let $y \in H$ be any point mapping to E by the natural projection $H \to E$. The diagram $H \times \mathbf{P}(V) \to E$ shows that the dynamical system (E, τ_x) is dominated by the constant dynamical system (H, τ_y), where y is the translation by y in H.

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