Some model theoretic ingredients of the proof

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This talk is completed devoted to a presentation of the use of valued difference fields (and their model theory) in Hrushovski's paper on the non-standard Frobenius automorphisms [2].

1 Valued difference fields

Motivation for the study of valued difference fields:

- They provide a framework for *transformal specialisations* in difference algebraic geometry.
- When studying the non-standard Frobenius automorphism, it is useful to develop it in parallel in the valued category, too. (After the fact, at least in characteristic 0, one may infer a complete description of the valued non-standard Frobenius from the one without valuation, by work of Azgin [1] and Hrushovski.)
- The model theory of valued fields is a very well studied subject, with spectacular applications to standard mathematics, both in the past and the present. Nevertheless, its use in the non-standard Frobenius paper is very mild, and restricted to the quantifier free level.

Notation. If K is a valued field, let $v : K \to \Gamma\dot{\cup}\{\infty\}$ be the valuation map, with $\Gamma = (\Gamma, 0, +, <)$ an ordered abelian group.

We write $\Gamma_K = \Gamma$ for the *value group*, $\mathcal{O}_K = \{a \in K \mid v(a) \geq 0\}$ for the *valuation ring* and $\mathfrak{m}_K = \{a \in K \mid v(a) > 0\}$ for its maximal ideal. Finally, $k_K = \mathcal{O}_K/\mathfrak{m}_K$ denotes the *residue field*, and $res : \mathcal{O}_K \to k_K$ the residue map.

Fact. *If $K \subseteq L$ is an extension of valued fields, one has $\Gamma_K \subseteq \Gamma_L$ and $k_K \subseteq k_L$ (naturally), and the following inequality holds:*

$$\mathrm{td}(L/K) \geq \mathrm{td}(k_L/k_K) + \mathrm{rk}_{\mathbb{Q}}(\Gamma_L/\Gamma_K).$$

Recall that if (K, v) is a valued field, one may consider its *completion* $(\hat{K}, \hat{v})$. The extension $\hat{K}/K$ is *immediate*, i.e. $k_K = k_{\hat{K}}$ and $\Gamma_K = \Gamma_{\hat{K}}$.

Definition.

- A *valued difference field* is a structure (K, v, σ), where (K, v) is a valued field and σ is a field endomorphism satisfying $\sigma(x) \in \mathcal{O}_K$ iff $x \in \mathcal{O}_K$. It induces a field endomorphism σ_k on k_K and an endomorphism (of ordered abelian groups) σ_Γ on Γ_K.
- (K, v, σ) is called *m-increasing* (for $m \geq 1$) if for all $\gamma > 0$, $\gamma \in \Gamma_K$, one has $\sigma_\Gamma(\gamma) \geq m\gamma$. It is called *$\omega$-increasing* if it is m-increasing for all $m \geq 1$.

Note that in the ω-increasing case, Γ gets an ordered module over the ordered ring $\mathbb{Z}[\sigma]$ (ordered such that $1 << \sigma << \sigma^2 \ldots$). In particular, Γ is a torsion free $\mathbb{Z}[\sigma]$-module in this case.

Examples.

1. If (K, v) is a valued field of characteristic $p > 0$ and $q = p^n$, then (K, v, φ_q) is a q-increasing valued difference field.

2. Non-prinicpal ultraproducts of structures from (1), for varying q, give rise to ω-increasing valued difference fields.

3. Let (F,σ) be an arbitrary difference field. On $K = F(t)_{\sigma+}$ one defines a valuation with value group $\Gamma = \mathbb{Z}[\sigma]$ and which is trivial on F, putting $v(t^{\sigma^m}) := \sigma^m$. It is ω-increasing.

Remark. *The last example may be seen as a non-standard version of valued difference fields of the form $M_q := (K_q(t)^{alg}, v, \varphi_q)$, where K_q is trivially valued and $v(t) > 0$, $K_q \models ACF_p$.) Note that this is different from another way of taking a 'non-standard' version of this, namely a non-principal ultraproduct $M = \prod_{\mathcal{U}} M_q$. (The theory of M is understood in characterstic 0.)*

The following generalises the example (3). (It will be enough to check the intution against this example.)

Definition. A valued difference field (K,v,σ) is called a *transformal DVR* over $(F,\sigma) \subseteq (K,\sigma)$ if F is trivially valued and inversive, the valuation on K is non-trivial and ω-increasing, K is finitely σ-generated over F and of transformal dimension 1 over F.

Note in particular that a transformal DVR is never inversive.

The following fact is easily established.

Fact. *Let K be a transformal DVR over F.*

1. *(K^{alg}, v, σ) is ω-increasing with value group isomorphic to $\mathbb{Q}[\sigma]$.*
2. *σ extends (uniquely) to an endomorphism of $\hat{K}$. [Indeed, $\sigma(\Gamma_K) \subseteq \Gamma_K$ is cofinal.]*
3. *If $F \subseteq F' \subseteq K$ is such that F' is transformally algebraic over F, then F' is trivially valued. [Indeed, if $v(t) > 0$, then t is transformally independent over F, since $t, \sigma(t), \ldots$ are necessarily algebraically independent, as they have $\mathbb{Q}$-independent values.]*

Definition. For L/K an extension of valued difference fields, define

$$\mathrm{rk}_{val}(L/K) := \mathrm{td}(k_L/k_K) + \mathrm{rk}_{\mathbb{Q}}(\Gamma_L/\Gamma_K).$$

For a a tuple from L, we put $\mathrm{rk}_{val}(a/K) := \mathrm{rk}_{val}(K(a)_{\sigma+}/K)$.

Theorem (Key inequality for rk_{val}).
Let L be a transformal DVR over F, $F \subseteq K \subseteq K' \subseteq L$, and a a tuple from L. Then

$$\mathrm{rk}_{val}(a/K') \leq \mathrm{rk}_{val}(a/K).$$

More generally, this inequality holds for $\widehat{L^{alg}}$ in the place of L (if K'/K is transformally algebraic).

This result is a by-product of a structure theory for transformal DVR's (and extensions between these), or rather completions of algebraic closures of such fields. Hrushovski shows for example that under the assumptions of the theorem, one has $\widehat{L^{alg}} \cong \widehat{F'(t)^{alg}_{\sigma+}}$.

The condition on the value group Γ is crucial, as is shown by the following example.

Example.

- Let $F = F^{alg} = F^{inv}$ be a trivially valued difference field, and consider $K = F(t)_\sigma$.
- Then $K = K^{inv}$ and $\Gamma_K = \mathbb{Z}[\sigma, \sigma^{-1}] \subseteq \mathbb{Q}(\sigma)$, the ordered field of rational functions in σ.
- Consider the *Hahn field* (generalised power series field) $H = F((\mathbb{Q}(\sigma))$.
- Let $c = t^{\frac{1}{\sigma-1}} \in H$. Then $\sigma(c) = t^{\frac{\sigma}{\sigma-1}} = t^{\frac{\sigma-1}{\sigma-1}} t^{\frac{1}{\sigma-1}} = tc \Rightarrow K(c) = K(c)_\sigma$ and $rk_{val}(K(c)/K) = 1$.
- For $a = 1 + t^{\sigma^{-1}} + t^{\sigma^{-1}+\sigma^{-2}} + \ldots \in H$, one computes $\sigma(a) = 1 + t + tt^{\sigma^{-1}} + tt^{\sigma^{-1}+\sigma^{-2}} + \ldots = 1 + ta$.
- Easy to see: $K(a) = K(a)_\sigma$ is an immediate extension of K, and $K(a)_\sigma \cong_K K(a+c)_\sigma =: K'$ via $a \mapsto a + c$.
- We get $0 = \mathrm{rk}_{val}(a/K) < \mathrm{rk}_{val}(a/K') = 1$, so a 'counter-example' for $L = K(a,c) = K(a,c)_\sigma$.

2 (Co-)analysability in the residue field and inertial dimension

We consider valued fields in a **3-sorted language**, with sorts (K, Γ, V). On K we put $\mathcal{L}_{rings}$, similarly on the the residue field V, taking $\mathcal{L}^V_{rings}$, on Γ we take the language of ordered abelian groups $\{0, +, -, \leq\}$. Moreover, the valuation map $v : K^* \to \Gamma$ as well as the map $RES : K^2 \to V$, $RES(a, b) = res(ab^{-1})$ if $b \neq 0$ and $v(a) \geq v(b)$, $RES(a, b) = 0$, else.

If we treat valued difference fields, we add function symbols σ, σ_Γ and σ_k.

On the sorts (K, V), we set Fn = the *set of basic definable functions* which are given (componentwise) by terms in the language $\mathcal{L}_{rings}\mathcal{L}^V_{rings} \cup \{RES, \sigma, \sigma_k\}$.

Work in the theory T_{inc} of ω-increasing valued difference fields.

Proposition. *Let $K \models T_{inc}$ and X be an (affine) difference variety of finite total dimension defined over $K_0 \subseteq K$. Then X is* **analysable** *in the residue field, i.e. there are quantifier free K_0-definable equivalence relations $\Delta_X = E_0 \subseteq \ldots \subseteq E_n$ such that for every i and every E_{i+1}-equivalence class Y there is a basic definable function (using extra parameters from Y) $f_i^Y : Y \to V$ inducing an embedding of Y/E_i into V.*

Proof. **Step 1: X is scattered**, i.e. if $X \subseteq K^n$, for every projection $pr_i : K^n \to K$, the set $\{v(pr_i(x) - pr_i(y)) \mid x, y \in X\} \subseteq \Gamma_K$ is finite.

To prove this, since $Y_i = \{pr_i(x) - pr_i(y) \mid x, y \in X\}$ is of finite total dimension, and thus contained in $\{x \mid H(x) = 0\}$ for some $H \in \mathcal{O}_K[X]_\sigma$, it is enough to show that $\{v(a) \mid H(a) = 0\}$ is finite.

Let $H(X) = \sum_{\mu \in \mathbb{N}[\sigma]} d^\mu X^\mu$. If $H(a) = 0$, there are $\mu \neq \nu$ such that $v(d_\mu a^\mu) = v(d_\nu a^\nu) < \infty$. But this means that $(\mu - \nu)v(a) = v(d_\nu) - v(d_\mu)$. So there are only finitely many choices for $v(a)$, since Γ_K is a torsion free $\mathbb{Z}[\sigma]$-module.

Step 2: Every scattered set is analysable in the residue field. Using projections and appropriate equivalence relations (equality of coordinates), it is enough to treat the case where $X \subseteq K$. Let $\infty = \gamma_0 > \ldots > \gamma_n$ be the set of values of differences of elements from X. Choose $c_i \in K$ with $v(c_i) = \gamma_i$ $(i > 0)$.

It is enough to define $E_i(x, y) :\Leftrightarrow v(x - y) \geq \gamma_i$, and if Y is an E_{i+1}-equivalence class, to put $f_i^Y(y) := res(c^{-1}(y - y_0))$, where $y_0 \in Y$ is some fixed element. □

Context:

- We work over a ground model $M \models T_{inc}$, $M = M_0^{alg}$ for some transformal DVR M_0. Let $T = T_{inc}(M)$.
- Let x, u, y be tuples of variables from K or k, and let v be variables from V.
- Let Φ be the set of quantifier free formulas $\varphi(x, v)$ such that if $N \models \varphi(a, r)$ for some $N \models T$, then a is transformally algebraic over M, and r is transformally algebraic over r. [By compactness, we thus have $T \models (\varphi(x, v) \to Z(x) \wedge W(v))$, where Z (resp. W) is a difference variety of finite total dimension defined over M (resp. over k_M)].
- We write $\varphi(x; y)$ to indicate that y is a variable for a parameter (so we think about definable families). Nevertheless, both the parameter variable and the variable variable admit only solutions which are transformally algebraic over M.
- Let $\varphi = \varphi(v; y) \in \Phi$. We let $d_V(\varphi) := \max\{\dim_{\text{tot}}(\varphi(v, b)) \mid b \in N \models T\}$

The following technical notion is more flexible than the notion of analysability, and it leads to a good dimension notion (inertial dimension) useful in Frobenius specialisation arguments.

Definition. By induction on $\frac{1}{2}\mathbb{N}$, we define the notion of co-analysability (and the inertial dimension) for $P \in \Phi$ as follows:

1. (Initial step) $P(x; u)$ is **co-analysable in 0 steps (of inertial dimension** ≤ 0**)** if for every $b \in N \models T$, $|P(N, b)| \leq 1$.

2. (Fibration) $P(x;u)$ is **co-analysable in $h+1/2$ steps (of inertial dimension $\leq n$)** if there exists $Q(x;y,v) \in \Phi$ co-analysable in h steps of inertial dimension $\leq n_1$, $R(v;,u) \in \Phi$ with $d_V(R) \leq n_2$ ($n_1+n_2 \leq n$), and $g(x,u)=v$ a basic definable function such that for every $b \in N \models T$, one has $P(N,b) \subseteq \{a \in N \mid N \models Q(a,b,g(a,b)) \wedge R(g(a,b),b))\}$.

3. (Extra parameters) $P(x;u)$ is **co-analysable in $h+1$ steps (of inertial dimension $\leq n$)** if there exist $Q_1(y;u),\ldots,Q_l(y;u) \in \Phi$ such that for $j=1,\ldots,l$, $\varphi_j(x;y,u) := P(x,u) \wedge Q_j(y,u)$ is co-analysable in $h+1/2$ steps (of inertial dimension $\leq n$) and for every $b \in N \models T$ such that $\emptyset \neq P(N,b)$ there exists $c \in N$ and j such that $N \models Q_j(c,b)$.

If the inertial dimension of P is $\leq n$ and $\not\leq n-1$, we put $\dim_{\text{in}}(P)=n$.

Lemma. *1. If $E_0 \subseteq \ldots \subseteq E_n$ is an analysis of X, and if $f_i^Y(Y)$ is of total dimension $\leq n_i$ for all Y, then X is co-analysable in n steps and $\dim_{\text{in}}(X) \leq \sum_{i=0}^{n-1} n_i$.*

2. Every formula from Φ is co-anaysable, of finite inertial dimension.

Proof. The proof uses induction on n and is easy. □

3 Bounds for the inertial dimension

Proposition (Upper bound for the inertial dimension)**.**
Let $\varphi(x,v) \in \Phi(x,v)$. Assume that whenever $N \models \varphi(c,r)$ for $N \models T$ and $(c,r) \in N^m \times k_N^l$, one has $\mathrm{rk}_{val}(c,r/M) \leq n$. Then $\dim_{\text{in}}(\varphi) \leq n$.

Idea of the proof. Here, we put $\mathrm{rk}_{val}(c,r/M) := \mathrm{rk}_{\mathbb{Q}}(\Gamma_{M(c)_{\sigma+}}/\Gamma_M) + td(k_{M(c)_{\sigma+}}(r)_{\sigma+}/k_M)$.

One argues by induction on the number of steps in a co-analysis of $\varphi(x,v)$, the rank inequality for rk_{val} being the crucial ingredient.

One also uses that rk_{val} is additive in towers, and the following easy fact:

If $r \in k_N$ is such that $\mathrm{rk}_{val}(r/M) = \mathrm{td}(k_M(r)_\sigma/k_M) \leq n$, there exists $\psi(v;u) \in \Phi(v;u)$ and $b \in M$ such that $d_V(\psi) \leq n$ and $N \models \psi(r,b)$. □

Lemma. *Let $P(x;y) \in \Phi(x;y)$. There exists finite $T_0 \subseteq T$, $\Phi_0 \subseteq \Phi$ and $\mathrm{Fn}_0 \subseteq \mathrm{Fn}$ with $P \in \Phi_0$ and such that for any $Q \in \Phi_0$, if $\dim_{\text{in}}(Q)=n$, this may be witnessed by a co-analysis using elements from Φ_0 and Fn_0 only, provably in the theory T_0.*

Proof. By compactness and induction on the number of steps in a co-analysis of P. □

Lemma ('Premier lemme de majoration' — talks by Giabicani-Laszlo)**.**
Let $X(x;y)$ be a family of difference varieties (or more generally quantifier free definable sets in $\mathcal{L}_{an} \cup \{\sigma\}$). Assume that $X(x;b)$ is of total dimension $\leq n$ for every parameter b from some difference field.

Then there is $\beta \in \mathbb{N}$ such that, for $q >> 0$, for all $b \in K_q$ one has

$$\mathrm{card}(X(K_q,b)) \leq \beta q^n.$$

Recall: $M_q = (K_q(t)^{alg}, v, \varphi_q)$.

Proposition (Upper bound for the Frobenius specialisation in terms of the inertial dimension)**.** *Let $P(x;y) \in \Phi(x;y)$ with $\dim_{\text{in}}(P) \leq n$, and let $T_0, \Phi_0, \mathrm{Fn}_0$ be as in the first lemma.*

1. For $q >> 0$, $M_q \models T_0$ (w.r.t. to some interpretation of the constants from $M = F(t)_\sigma^{alg}$ by elements of M_q).

2. There is $\beta \in \mathbb{N}$ such that for $q >> 0$, for all $b \in M_q \models T_0$, one has

$$\mathrm{card}(P(M_q,b)) \leq \beta q^n.$$

Proof. The first part is clear, since T_0 is finite.

The second part is proved by induction on the number of steps in a co-analysis of X:

For the step $h+\frac{1}{2} \mapsto h+1$, let $Q_1(y,u),\ldots,Q_l(y,u)$ be as in the definition. By induction, there are β_i such that

$$|P(M_q,b)\wedge Q_i(b,c)|\leq \beta_i q^n \text{ for } q>>0.$$

Clearly, $\beta := \max\{\beta_i\}$ works for P.

For the step $h \mapsto h+\frac{1}{2}$, suppose $Q(x;u,v)$ and $R(v;u)$ are as in the definition, with $\dim_{\text{in}}(Q)\leq n_1$ and total dimension of $R\leq n_2$, $(n_1+n_2\leq n)$. By induction hypothesis, there is β_1 for Q, and by the previous lemma there is β_2 such that, for $q>>0$, $|Q(M_q,b,r)|\leq \beta_1 q^{n_1}$ and $|R(M_q,b)|\leq \beta_2 q^{n_2}$. We get

$$|P(M_q,b)|\leq \sum_{s\in R(M_q,b)} |Q(M_q,b,s)|\leq \beta_2 q^{n_2}\beta_1 q^{n_1},$$

so $\beta=\beta_1\beta_2$ works for P. □

References

[1] Salih Azgin, *Valued fields with contractive automorphism and Kaplansky fields*, J. Algebra **324**(10) (2010), 2757–2785 .

[2] Ehud Hrushovski, *The Elementary Theory of the Frobenius Automorphisms*, arXiv:math/0406514, 2004.