First Steps in Model Theory

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This talk is given by a non-expert in model theory, and is intended for a mixed audience :

(i) for non-experts, will include naive comments;

(ii) for experts, who may have some interest in the feelings of mathematicians from the outside worlds confronted to model-theory.

Basic notions of model theory

- first order language $\mathfrak{L}$
- formulas and sentences σ of $\mathfrak{L}$
- structures M for $\mathfrak{L}$
- $M \models \sigma$, $M \models \{\sigma_i\}_{i \in I}$

Basic Theorems of Model theory

Theorem (Compactness – Gödel, Malcev)

Let Σ be a set of $\mathfrak{L}$-sentences.
If every finite subset of Σ has a model, then Σ has a model.

Theorem (Löwenheim, Skolem)

Let κ be an infinite cardinal $\geq \operatorname{card}(\mathfrak{L})$, $\mathcal{B}$ a $\mathfrak{L}$-structure, and X a subset of $|\mathcal{B}|$ such that

$$\operatorname{card}(X) \leq \kappa \leq \operatorname{card}(|B|).$$

There exists an elementary substructure $\mathcal{A}$ of $\mathcal{B}$ such that

$$\operatorname{card}(\mathcal{A}) = \kappa \ \textit{ and } \ X \subset \mathcal{A}.$$

Theorem (Upward Löwenheim – Skolem, Malcev)

Let Σ be a set of $\mathfrak{L}$-sentences.
If Σ has a model of infinite cardinal, then Σ has a model in any cardinal $\geq \operatorname{card}(L)$.

Moreover, will also discuss:

- Examples
- Types

Use of colors :

- Important words or data
- Formal symbols (*e.g.* $=$ is not the usual “equal” $=$)
- Naive remarks

Model theory and mathematical logic: What has model theory to say about foundational issues ?

Gap between first order mathematical constructions and “higher orders” ones.

On the other hand, model theory appears as some usual subfield of mathematics, that relies on standard mathematical techniques/formalism, and is developed in the standard formalism of “naive set theory”.

Today, adopt a pragmatic point of view:

Various issues that are supposed to be "problem-free" in a standard arithmetic or algebraic geometry seminar will be considered to be "problem-free" here as well. In particular, the following notions are supposed to be crystal clear :

- the set of "true" integers $\mathbb{N}$ and the elementary operations $+$ and $.$ on $\mathbb{N}$;
- variables, indeterminates.

The Axiom of Choice will be freely used too. Would you ever dare to contest the following facts ?

- Any non-empty affine scheme has a dense subspace of closed points.
- $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$.

A language $\mathfrak{L}$ is the data of :

- a set of function symbols $\mathfrak{F}$ and a positive integer n_f for every $f \in \mathfrak{F}$;
- a set of relation symbols $\mathfrak{R}$ and a positive integer n_R for every $R \in \mathfrak{R}$;
- a set of constant symbols $\mathfrak{C}$.

$$n_f, n_R =: \textit{arity}$$

Examples of languages

self-explanatory abusive notation

- $\mathfrak{L}_{\text{ring}} := \{+, -, ., 0, 1\}$

$$\mathfrak{F} = \{+, -, .\} \quad n_+ = n_. = 2, n_- = 1$$
$$\mathfrak{R} = \emptyset$$
$$\mathfrak{C} = \{0, 1\}.$$

- $\mathfrak{L}^w_{\text{ring}} := \{+, ., 0, 1\}$

$$\mathfrak{F} = \{+, .\} \quad n_* = 2$$
$$\mathfrak{R} = \emptyset$$
$$\mathfrak{C} = \{0, 1\}.$$

- $\mathfrak{L}_{\text{ordring}} := \{+, -, ., <, 0, 1\}$

$$\mathfrak{F} = \{+, -, .\}, \mathfrak{C} = \{0, 1\}$$
$$\mathfrak{R} = \{<\} \quad n_< = 2$$

- $\mathfrak{L}_{\text{diffring}} := \{+, -, ., \delta, 0, 1\}$

$$\mathfrak{F} = \{+, -, ., \delta\} \quad n_\delta = 1$$

$$\mathfrak{R} = \emptyset$$

$$\mathfrak{C} = \{0, 1\}.$$

- Let k be a field. $\mathfrak{L}_{k-\text{vect}} := \{+, -, (\lambda_x)_{x \in k}\}$

$$\mathfrak{F} = \{+, -, (\lambda_x)_{x \in k}\} \quad n_+ = n_- = 2, n_{\lambda_x} = 1.$$

$$\mathfrak{R} = \emptyset$$

$$\mathfrak{C} = \{0, 1\}.$$

A $\mathfrak{L}$-structure $\mathcal{M}$ is the data of

- a non-empty set M, the underlying set $|\mathcal{M}|$ of $\mathcal{M}$;
- for any $f \in \mathfrak{F}$, a function

$$f^{\mathcal{M}} : M^{n_f} \longrightarrow M;$$

- for any R in $\mathfrak{R}$, a subset

$$R^{\mathcal{M}} \subset M^{n_R};$$

- for any $c \in \mathfrak{C}$, an element

$$c^{\mathcal{M}} \in M.$$

$*^{\mathcal{M}}$ =: interpretation of $*$ in$\mathcal{M}$

Examples of structures

1. Any ring $(A, +, .)$ defines a $\mathfrak{L}_{\text{ring}}$-structure $\mathcal{A}$:
   - $|\mathcal{A}| := A$
   - $+^{\mathcal{A}} := +$
   - $-^{\mathcal{A}} := -$
   - $.^{\mathcal{A}} := .$
   - $0^{\mathcal{A}} := 0_A$, $1^{\mathcal{A}} := 1_A$.

   Observe that $\mathcal{A}$ determines the "ring structure" of A.
2. Similarly any field $(K, +, .)$ defines a $\mathfrak{L}_{\text{ring}}$-structure $\mathcal{K}$, which conversely determines the field structure of K.
3. In the above examples, may replace $\mathfrak{L}_{\text{ring}}$ by $\mathfrak{L}^{w}_{\text{ring}}$.
   Observe however that any "semi-ring" (like $(\mathbb{N}, +.)$) defines a $\mathfrak{L}^{w}_{\text{ring}}$-structure, but in general not a $\mathfrak{L}_{\text{ring}}$-structure.

A structure, in the sense of model theory, is a rather poor thing.

In the definition of a $\mathfrak{L}$-structure, no conditions on the interpretation are required, contrary to the "usual" notion of structure in algebra, *à la* Bourbaki.

Formulas in the language $\mathfrak{L}$:= finite strings of symbols built using

1. the symbols of $\mathfrak{L}$, namely the elements of $\mathfrak{F} \cup \mathfrak{R} \cup \mathfrak{C}$;
2. variables (*aka* indeterminates) x_i, $i \in \mathbb{N}$;
3. =
4. Boolean symbols $\wedge$, $\vee$, $\neg$.
5. quantifiers $\forall$, $\exists$
6. (,) and

according to the "usual rules".

eg. $x_1 = x_2$ is a formula and $x_1 =$ is not.

Variants: may use only $\wedge$ and $\neg$, by replacing $\phi \vee \psi$ by $\neg(\phi \wedge \psi)$. May also use $\Rightarrow$, "defined" by $\phi \Rightarrow \psi = \neg(\phi \wedge \neg\psi)$.

Sentences are formulas with no free variables.

Formal inductive definitions

1. terms of $\mathfrak{L}$ are defined inductively (...) by the following rules
   - every variable x_i is a term;
   - every constant $c \in \mathfrak{C}$ is a term;
   - if $f \in \mathfrak{F}$ and if $t_1, \ldots, t_{n_f}$ are terms, then $f(t_1, \ldots, t_{n_f})$ is a term.
2. atomic formulas of $\mathfrak{L}$ are strings of symbols of the following form
   - $t_1 = t_2$ where t_1 and t_2 are terms;
   - $R(t_1, \ldots, t_{n_R})$ for any relation symbol $R \in \mathfrak{R}$ and any n_R-tuple $(t_1, \ldots, t_{n_R})$ of terms.
3. formulas of $\mathfrak{L}$ are defined inductively by the following rules :
   - all atomic formulas are formulas;
   - if ϕ is a formula, then $\neg\phi$ is a formula;
     if ϕ and ψ are formulas, then $\phi \wedge \psi$ and $\phi \vee \psi$ are formulas;
   - if ϕ is a formula and $i \in \mathbb{N}$, then $\exists x_i \phi$ and $\forall x_i \phi$ are formulas.

Examples of formulas

1. In $\mathfrak{L}_{\text{ring}}$ and $\mathfrak{L}_{\text{ordring}}$, a term determines a polynomial in $\mathbb{Z}[x_i]_{i\in\mathbb{N}}$.
   In $\mathfrak{L}_{\text{diffring}}$, a term determines a differential polynomial in $\mathbb{Z}\{x_i\}_{i\in\mathbb{N}} := \mathbb{Z}[\delta^j v_i]_{(i,j)\in\mathbb{N}^2}$.

2. In $\mathfrak{L}_{\text{ring}}$, an atomic formula "is" a polynomial equation of the form

   $$P(x_i) = Q(x_i)$$

   for some "polynomials" P and Q with integer coefficients. In $\mathfrak{L}_{\text{ordring}}$, an atomic formula "is" a polynomial equation

   $$P(x_i) = Q(x_i)$$

   as above, or a polynomial inequation

   $$P(x_i) < Q(x_i).$$

   In $\mathfrak{L}_{\text{diffring}}$, an atomic formula "is" a polynomial differential equation.

Interpretation of terms

Let $\mathcal{M}$ be some $\mathfrak{L}$-structure.
For any finite subset $I \subset \mathbb{N}$ and any term t of $\mathfrak{L}$ involving only the variables $(x_i)_{i\in I}$, one attaches its interpretation in $\mathcal{M}$:

$$t^{\mathcal{M}} : M^I \longrightarrow M.$$

It is inductively defined using the following rules :

- if $I = \emptyset$ and $t = c \in \mathfrak{C}$, then $t^{\mathcal{M}} := c^{\mathcal{M}}$.
- if $t = x_j$ for some $j \in I$, then $t^{\mathcal{M}}(a_i)_{i\in I} = a_j$.
- if $f \in \mathfrak{F}$ and $t = f(s_1, \ldots, s_{n_f})$ for some terms $s_1, \ldots, s_{n_f}$, then

$$t^{\mathcal{M}} : M^I \xrightarrow{(s_j{}^{\mathcal{M}})} M^{n_f} \xrightarrow{f^{\mathcal{M}}} M.$$

Satisfaction of formulas

Consider some $\mathfrak{L}$-structure $\mathcal{M}$, a finite subset I of $\mathbb{N}$, some $\mathfrak{L}$-formula ϕ and $a \in M^I$.

$$\mathcal{M} \models \phi(a)$$

is defined inductively by the following rules:

1. atomic formulas
   - if $\phi = t_1 = t_2$, then $\mathcal{M} \models \phi(a)$ iff ${t_1}^{\mathcal{M}}(a) = {t_2}^{\mathcal{M}}(a)$.
   - if $\phi = R(t_1, \ldots, t_{n_f})$ for some symbol $R \in \mathfrak{R}$ and some n_R-tuple $(t_1, \ldots, t_{n_R})$ of terms, then $\mathcal{M} \models \phi$ iff $({t_1}^{\mathcal{M}}(a), \ldots, {t_{n_R}}^{\mathcal{M}}(a)) \in R^{\mathcal{M}}$.
2. Boolean operations
   - for any formula ϕ, $\mathcal{M} \models \neg\phi(a)$ iff $\neg(\mathcal{M} \models \phi(a))$;
   - for any two formulas ϕ and ψ,
     $\mathcal{M} \models \phi \wedge \psi(a)$ iff $\mathcal{M} \models \phi(a)$ and $\mathcal{M} \models \psi(a)$,
     $\mathcal{M} \models \phi \vee \psi(a)$ iff $\mathcal{M} \models \phi(a)$ or $\mathcal{M} \models \psi(a)$
3. quantifiers
   - if $\phi = \forall v \ \psi(v, x_i)_{i \in I}$,
     then $\mathcal{M} \models \phi(a)$ iff, for all $b \in M$, $\mathcal{M} \models \psi(b, a)$.
   - if $\phi = \exists v \ \psi(v, x_i)_{i \in I}$,
     then $\mathcal{M} \models \phi(a)$ iff there exists $b \in M$ such that $\mathcal{M} \models \psi(b, a)$.

Models of a theory

Define a $\mathfrak{L}$-theory, or a set of axioms, as a set Σ of $\mathfrak{L}$-sentences. For any $\mathfrak{L}$-structure $\mathcal{M}$, define:

$$\mathcal{M} \models \Sigma \overset{\text{def}}{\iff} \forall \phi \in \Sigma, \mathcal{M} \models \phi.$$

"The $\mathfrak{L}$-structure $\mathcal{M}$ is a model of the $\mathfrak{L}$-theory Σ."

Examples

1. A ring "is" a $\mathfrak{L}_{\text{ring}}$-structure that satisfies the axioms of rings Σ_{ring}:
   - $\forall x_1 \forall x_2 (x_1 + x_2 = x_2 + x_1)$, $\forall x_1 \forall x_2 \forall x_3 ((x_1 + x_2) + x_3 = x_1 + (x_2 + x_3))$
   - $\forall x_1 \forall x_2 (x_1 . x_2 = x_2 . x_1)$, $\forall x_1 \forall x_2 \forall x_3 ((x_1 . x_2) . x_3 = x_1 . (x_2 . x_3))$
   - $\forall x_1 \forall x_2 \forall x_3 (x_1 . (x_2 + x_3) = x_1 . x_2 + x_1 . x_3)$
   - $\forall x_1 (0 + x_1 = x_1)$, $\forall x_1 (1 . x_1 = x_1)$
   - $\forall x_1 (-x_1 + x_1 = 0)$.
2. A field "is" a $\mathfrak{L}_{\text{ring}}$-structure that satisfies the axioms of fields:

$$\Sigma_{\text{field}} := \Sigma_{\text{ring}} \cup \{\forall x_1 (\neg (x_1 = 0) \Rightarrow \exists x_2 (x_1 . x_2 = 1))\}. \quad (1)$$

Examples

3. An algebraically closed field “is” a $\mathfrak{L}_{\text{ring}}$-structure that satisfies the axioms of algebraically closed fields:

$$\Sigma_{\text{ac}-\text{field}} :=$$

$$\Sigma_{\text{field}} \cup \{\forall x_1 \ldots \forall x_n \exists x_{n+1} (x_{n+1}^n + x_1 . x_{n+1}^{n-1} + \cdots + x_{n-1} . x_{n+1} + x_n = 0)\}_{n \in \mathbb{N}}$$

where

$$x^n := x.(x.(\cdots .x)) \cdots) \ (x \text{ occurs } n \text{ times}).$$

4. An algebraically closed field of characteristic zero “is” a $\mathfrak{L}_{\text{ring}}$-structure that satisfies the axioms

$$\Sigma_{\text{ac}-\text{field}} \cup \{\forall x_1 \neg (n \times x_1 = 0)\}_{n \in \mathbb{N}_{>0}}$$

where

$$n \times x := x + (x + (\cdots + x)) \cdots) \ (x \text{ occurs } n \text{ times}).$$

5. **Exercise**. Let k be a field. Write a set of “axioms” in $\mathfrak{L}_{k-\text{vect}}$, the model of which “are” k-vector spaces.

$$\{\{\mathfrak{L}-\text{structures}\}\} \stackrel{\models}{\longleftrightarrow} \{\mathfrak{L}-\text{sentences}\}$$

The satisfaction $\models$ establishes a kind of duality between the *class* of $\mathfrak{R}$-structures and the *set* of $\mathfrak{L}$-sentences.
May prefer to work with structures in some fixed *universe*, as in SGA 4.

Somewhat similar to diverse classical instances of duality ...

Projective duality (Poncelet, Brianchon)

$$\check{\mathbb{P}}^N \overset{\perp}{\longleftrightarrow} \mathbb{P}^N$$

- $(\xi_0 : \cdots : \xi_N) \perp (x_0 : \cdots : x_N)$ iff $\sum_{i=0}^N \xi_i x_i = 0$;
- to L projective subspace in $\mathbb{P}^N$, associate
$$L^\perp := \{H \in \check{\mathbb{P}}^N \mid \forall x \in L, x \perp H\};$$
- to $\check{L}$ projective subspace in $\check{\mathbb{P}}^N$, associate
$$\check{L}^\perp := \{x \in \mathbb{P}^N \mid \forall H \in \check{L}, x \perp H\}.$$

Get inclusion reversing bijection :

$$\begin{array}{ccc} \{\text{linear subspaces of } \check{\mathbb{P}}^N\} & \overset{\sim}{\longleftrightarrow} & \{\text{linear subspaces of } \mathbb{P}^N\} \\ \check{L} & \longmapsto & \check{L}^\perp \\ L^\perp & \longleftarrow & L \end{array}$$

Duality in normed $\mathbb{R}$-vector spaces

$$E^{\vee} := \mathcal{L}(E, \mathbb{R}) \overset{\langle .,. \rangle}{\longleftrightarrow} E$$

- the setting is "less symmetric" than projective duality; in general, the inclusion $E \hookrightarrow E^{\vee\vee}$ is not onto !
- however the Theorem of Hahn-Banach asserts that $E^{\vee}$ is "large enough" in the following sense:

For any subset X of E and any x in E, we have

$$x \in \text{ closed convex hull of } X$$

iff

$$\forall \xi \in E^{\vee}, \xi(x) \in [\inf_X \xi, \sup_X \xi].$$

Vocabulary

$$\mathfrak{L}-\text{Structures} := \{\{\mathfrak{L}-\text{structures}\}\}$$

$$\mathfrak{L}-\text{Sentences} := \{\mathfrak{L}-\text{sentences}\}$$

- a $\mathfrak{L}$-theory := a subset Σ of $\mathfrak{L}$-Sentences. To Σ is attached

$$\Sigma^{\perp} := \{\{\mathcal{M} \in \mathfrak{L}-\text{Structure} \mid \forall \phi \in \Sigma, \mathcal{M} \models \phi\}\}.$$

$$\mathcal{M} \models \Sigma \overset{\text{def}}{\iff} \forall \phi \in \Sigma, \mathcal{M} \models \phi \iff \mathcal{M} \in \Sigma^{\perp}.$$

- to any $\mathfrak{L}$-structure $\mathcal{M}$ is attached its theory :

$$\text{Th}(\mathcal{M}) = \mathcal{M}^{\perp} := \{\phi \in \mathfrak{L}-\text{Sentences} \mid \forall \mathcal{M} \models \phi\}.$$

- if ϕ is an $\mathfrak{L}$-sentence and Σ some $\mathfrak{L}$-theory, then

$$\Sigma \models \phi \overset{\text{def}}{\iff} \phi \in \Sigma^{\perp\perp} \iff (\forall \mathcal{M} \in \mathfrak{L}\text{–Structures}, \mathcal{M} \models \Sigma \implies \mathcal{M} \models \phi).$$

"ϕ is a semantic consequence of Σ"

$$\Sigma \vdash \phi \overset{\text{def}}{\iff} \text{there exists a formal derivation of } \phi \text{ from } \Sigma$$

"ϕ is a syntactic consequence of Σ"

Formal derivation (Hilbert-style)

- ϕ formula, t term, v variable; $\phi(t/v)$:= result of replacing all occurrences of the *free* variable v by the term t — assuming that none of the variables in t occur as bound variable in ϕ.
- "Axioms"
  1. All tautologies: $P(\phi_1, \ldots, \phi_n)$, for P a "propositional function" which is identically true.
  2. All equality axioms (...).
  3. The formulas $(\forall v \phi) \Rightarrow \phi(t/v)$ and $\phi(t/v) \Rightarrow \exists v \phi(v)$
- Rules of inferences
  1. *Modus Ponens* — From $\phi \Rightarrow \psi$ and ϕ, infer ψ.
  2. *Generalization Rules* — If the variable v is not a free variable in the formula ϕ, from $\phi \Rightarrow \psi(v)$ infer $\phi \Rightarrow \forall y \psi(y)$, and from $\psi(v) \Rightarrow \phi$ infer $\exists y \phi(y) \Rightarrow \phi$.
- A formal proof of a formula ϕ from a set of sentences Σ is a finite sequence $\psi_1, \ldots, \psi_n = \phi$ of formulas, each of which is either an "axiom", or a member of Σ, or deduced from earlier ψ's by one of the rules of inferences.

- a contradiction is a sentence of the form $\phi \wedge \neg\phi$.
- a set Σ of formulas is consistent iff no contradiction may be formally derived from Σ.
  Observe :
  1. for any two sentences ϕ and ψ, $\phi \wedge \neg\phi$ may be derived from $\psi \wedge \neg\psi$.
  2. for any consistent set of formulas Σ and any formula ϕ, either $\Sigma \cup \{\phi\}$ or $\Sigma \cup \{\neg\phi\}$ is consistent.

Completeness and compactness

Theorem (Completeness – Gödel)

For any language $\mathfrak{L}$, any set Σ of $\mathfrak{L}$-sentences and any $\mathfrak{L}$-sentence ϕ,

$$\Sigma \vdash \phi \Longleftrightarrow \Sigma \models \phi.$$

The implication $\Longrightarrow$ is "clear".

The point of the Completeness Theorem is the converse implication $\Longleftarrow$. It follows from the following statement, applied to $\Sigma \cup \{\neg\phi\}$:

Theorem (Completeness – second version)

For any language $\mathfrak{L}$ and any set Σ of $\mathfrak{L}$-sentences, if no contradiction may be formally derived from Σ, then there exists some model $\mathcal{M}$ of Σ, namely some $\mathfrak{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \Sigma$.

The property, for a theory Σ to be consistent — that is, to have no contradiction as a formal consequence — is clearly of finite character. In particular the Completeness Theorem immediately implies :

Theorem (Compactness – Gödel, Malcev)

Let Σ be a set of $\mathfrak{L}$-sentences. If every finite subset of Σ has a model, then Σ has a model.

About the proof of Gödel's Completeness Theorem

Consider Σ a consistent $\mathfrak{L}$-theory.
We want to construct some model $\mathcal{M}$ for Σ !

Observation:
Let $\mathcal{N}$ be some $\mathfrak{L}$-structure.
Consider a "set of new constants" C and a surjection

$$\begin{array}{ccc} C & \xrightarrow{\text{onto}} & |\mathcal{N}| \\ c & \longmapsto & n_c, \end{array}$$

and the new language $\mathfrak{L} \cup C$. From the $\mathfrak{L}$-structure $\mathcal{N}$, we deduce a $\mathfrak{L} \cup C$-structure $\mathcal{N}^*$ defined by

- $|\mathcal{N}^*| := |\mathcal{N}|$
- $R^{\mathcal{N}^*} := R^{\mathcal{N}}, \ldots$
- for any $c \in C$, $c^{\mathcal{N}^*} := n_c$.

The complete diagram of the $\mathfrak{L}$-structure $\mathcal{N}$ is $\mathrm{Th}_{\mathfrak{L}\cup C}\mathcal{N}^*$.
It is easily seen to be a *complete Henkin theory* in the following sense:

- A $\mathfrak{L}\cup C$-theory Σ^+ is called a Henkin theory with set of constants C if, for every $\mathfrak{L}\cup C$-formula $\phi(x)$ there is a constant c in C such that $(\exists x\phi(x) \Rightarrow \phi(c)) \in \Sigma^+$.
- a $\mathfrak{K}$-theory $\tilde{\Sigma}$ is called complete if it is consistent and if, for every $\mathfrak{K}$-sentence ϕ, either $\phi \in \tilde{\Sigma}$ or $\neg\phi \in \tilde{\Sigma}$.

Step 1. *For a suitable choice of C, Σ is contained in some consistent Henkin theory.*
Observe that: *If $\phi(x)$ is a a formula and c some new constant, then $\Sigma \cup \{(\exists x\phi(x) \Rightarrow \phi(c))\}$ is consistent.*
(Exercise on "formal derivation" !)
and use "induction"...

Step 2. *Every consistent $\mathfrak{K}$-theory Σ^+ is contained in some complete $\mathfrak{K}$-theory Σ^*.*

Recall that, *for any consistent set of formulas Σ and any formula ϕ, either $\Sigma \cup \{\phi\}$ or $\Sigma \cup \{\neg\phi\}$ is consistent*, and use (transfinite) induction.

Step 3. *If Σ^* is a complete Henkin $\mathfrak{L} \cup C$-theory with set of constants C, then there exists a $\mathfrak{L} \cup C$-structure $\mathcal{M}^*$ such that*

$$\mathcal{M}^* \models_{\mathfrak{L} \cup C} \Sigma^*$$

and

$$\begin{array}{ccc} C & \xrightarrow{\text{onto}} & |\mathcal{M}^*| \\ c & \longmapsto & c^{\mathcal{M}^*}. \end{array}$$

Actually, $\mathcal{M}^*$ is unique up to canonical isomorphism and Σ^* is its complete diagram. The set $|\mathcal{M}^*|$ may be constructed as the quotient $C/\sim$ where $c \sim c' \overset{\text{def}}{\Longleftrightarrow} c = c' \in \Sigma^*$.

Comments

Compare with the proof of the Hahn-Banach Theorem :

– simple elementary step(s) + (transfinite) induction;

– in real life, the details of the proof do not really matter;

– under suitable countability assumptions, the "full" Axiom of Choice is not required, but only sequences of successive choices are used.

That is the case here if $\mathfrak{L}$ is (at most) countable, and the proof then establishes the existence of some (at most) countable model $\mathcal{M}$.

More vocabulary

Two theories Σ_1 and Σ_2 are equivalent if they have the same models. This holds iff, for every $\phi_1 \in \Sigma_1$ and ϕ_2 in Σ_2, $\Sigma_1 \vdash \phi_2$ and $\Sigma_2 \vdash \phi_1$.

If $\mathcal{K}$ is a class of structures, we define its theory:

$$\mathrm{Th}(\mathcal{K}) = \mathcal{K}^{\perp} := \bigcap_{\mathcal{M} \in \mathcal{K}} \mathrm{Th}(\mathcal{M}).$$

$\mathcal{K}$ is said to be an elementary class when it is the class of models $\Sigma^{\perp}$ of some theory Σ. Then this holds for $\Sigma := \mathrm{Th}(\mathcal{K})$, and a set of sentences which is equivalent to $\mathrm{Th}(\mathcal{K})$ is called a set of axioms for $\mathcal{K}$.

A theory Σ is complete iff it is equivalent to the theory $\mathrm{Th}(\mathcal{M})$ of some structure $\mathcal{M}$, or equivalently, when its is equivalent to a theory complete in the above sense.

Two structures $\mathcal{M}$ and $\mathcal{M}'$ are elementarily equivalent iff they have the same theories : $\mathcal{M} \equiv \mathcal{M}'$ iff $\mathcal{M}^{\perp} = \mathcal{M}'^{\perp}$.

Examples

1. The class of fields of positive characteristic — seen as $\mathfrak{L}_{\text{ring}}$-structures — is *not* an elementary class.

2. There exists a *countable* ordered field $(F, <)$ which is elementarily equivalent — as an $\mathfrak{L}_{\text{ordring}}$-structure — to $(\mathbb{R}, <)$.

3. Any two algebraically closed fields K_1 and K_2 of the same characteristic — seen as $\mathfrak{L}_{\text{ring}}$-structures — are elementarily equivalent.
[Hint : use Chevalley's Constructibility theorem for schemes of finite type over $\mathbb{Z}$, and the Nullstellensatz in the following form : *If k is a field and K an algebraically closed extension of k, a k-scheme of finite type X is non-empty iff $X(K)$ is not empty.*]
As a consequence, the theory of algebraically closed fields of some fixed characteristic is complete.