

Groups interpretable in theories of fields

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Abstract

We survey some results on the structure of the groups which are definable in theories of fields involved in the applications of model theory to Diophantine geometry. We focus more particularly on separably closed fields of finite degree of imperfection.

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1. Introduction

In the last ten years, the model theory of fields has seen striking new developments, with applications in particular to differential algebra and Diophantine geometry. One of the main ingredients in these applications is the analysis of the structure of groups definable in fields with added “definable structure”.

Model theory studies structures with a family of distinguished subsets of their Cartesian products, the family of *definable* subsets, which is requested to be closed under finite Boolean operations and projections. In the case of algebraically closed fields, the definable sets are exactly the constructible sets in the Zariski topology (finite Boolean combinations of Zariski closed sets). If one considers fields which are not algebraically closed (for example, fields of positive characteristic which are separably closed and not perfect) or algebraically closed fields with new operators (differentially closed fields, fields with a generic automorphism), then the family of definable sets is much richer than the family of Zariski constructible sets. In each of the above cases, one can generalize the classical geometric notions, by using the tools developed by model theory (abstract notion of independence, of dimensions...). For example:

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1. One can define “good” topologies which strictly contain the Zariski topology.
2. Different notions of dimensions can be attached to definable sets (or infinite intersections of definable sets, which we call *infinitely definable*, or ∞ -*definable sets*). In the case of algebraically closed fields, all such possible notions of abstract dimension must coincide and be equal to the classical algebraic dimension. In the other cases, these dimensions may be different, some may take infinite ordinal values or may be defined only for some special classes of definable (or ∞ -definable) sets.
3. If K is any of the above mentioned fields, and if H is an algebraic group defined over K , then the group $H(K)$ of the K -rational points of H is a definable group. But there are “new” families of definable groups which are not of this form.

In fact, it is precisely the study of certain specific families of “new” definable groups of finite dimension which are at the center of the applications to Diophantine geometry. We will not attempt here to explain how the model theoretic analysis of the finite rank definable groups yields these applications. There have been in recent years many surveys and presentations of the subject to which we refer the reader (see for example, [4],[5], [14], [22] or [28]). We will come back to this subject, but very briefly, at the end in Section 3.5.

The first general question raised by the existence of these new definable groups is that of their relation to the classical algebraic groups. Remark that this question already makes sense in the context of “pure” algebraically closed fields, about the class of definable (= constructible) groups. In that case, it is true that any constructible group in an algebraically closed field K is constructibly isomorphic to the K -rational points of an algebraic group defined over K (see for example [3] or [23]).

Let us now consider briefly the case of a field K of characteristic $p > 0$ which is separably closed and not perfect. Then the class of constructible sets is no longer closed under projection and there are many definable groups which are not constructible, the most obvious one being K^p . There are also some groups which are proper intersections of infinite descending chains of definable groups: for example, $K^{p^\infty} (= \bigcap_n K^{p^n})$, the field of infinitely p -divisible elements of the multiplicative group, or $\bigcap_n p^n A(K)$, for A an Abelian variety defined over K .

It is nevertheless true, as we will see, that every definable group in K is definably isomorphic to the K -rational points of an algebraic group defined over K . Furthermore, as in the classical case of one-dimensional algebraic groups, it is possible to give a complete description, up to definable isomorphism, of the one-dimensional infinitely definable groups.

There are results of similar type for the other classes of enriched fields mentioned above. In this short paper, we will concentrate mainly on the case of separably closed fields (in Section 3.). Before this, in Section 2., we will only very briefly present the model theoretic setting for two other examples of “enriched” fields, in characteristic zero, differentially closed fields and generic difference fields. We hope this will give the reader an idea of what the common features and the differences might be in the model theoretic analysis of these different classes of fields.

Finally, there are of course many other classes of fields whose model theory has been extensively developed in the past years with many connections to algebra,

semi-algebraic or subanalytic geometry, and which we are not going to mention here: for example, valued fields, ordered fields, “o-minimal” expansions of the real field...

2. Two short examples

We will just very briefly describe the two characteristic zero examples mentioned above.

2.1. Differentially closed fields of characteristic zero

We consider a field K of characteristic zero, with a *derivation* δ , that is, an additive map from K to K which satisfies that for all x, y in K , $\delta(xy) = x\delta(y) + y\delta(x)$. We define the ring $K_\delta[X]$ of *differential polynomials* over K to be the ring of polynomials in infinitely many variables $K[X, \delta(X), \delta^2(X), \dots, \delta^n(X), \dots]$. The *order* of the differential polynomial $f(X)$ in $K_\delta[X]$ is -1 if $f \in K$ and otherwise the largest n such that $\delta^n(X)$ occurs in $f(X)$ with non zero coefficient. We say that K is *differentially closed* if for any non-constant differential polynomials $f(X)$ and $g(X)$, where the order of g is strictly less than the order of f , there is a z such that $f(z) = 0$ and $g(z) \neq 0$. In model theoretic terms, this means exactly that K is existentially closed.

From now on we suppose that (K, δ) is a large differentially closed field (a universal domain).

We say that $F \subseteq K^n$ is a δ -closed set, if there are $f_1, \dots, f_r \in K_\delta[X_1, \dots, X_n]$ such that $F = \{(a_1, \dots, a_n) \in K^n; f_1(a_1, \dots, a_n) = \dots = f_r(a_1, \dots, a_n) = 0\}$. The ring $K_\delta[X_1, \dots, X_n]$ is of course not Noetherian but the δ -closed sets (which correspond to radical differential ideals) form the closed sets of a Noetherian topology on K , the δ -topology.

We now consider the δ -constructible sets, that is, the finite Boolean combinations of δ -closed sets. This class is closed under projection (this is quantifier elimination for the theory), hence the definable sets (we call them δ -definable sets) are exactly the δ -constructible sets. To every δ -definable set one can associate a dimension (the Morley rank) which can take infinite countable ordinal values.

There are “new” definable groups, which are not of the form $H(K)$ for any algebraic group H . In particular, any $H(K)$ will have infinite dimension. In contrast, the *field of constants of K* , $\text{Cons}(K) = \{a \in K; \delta(a) = 0\}$, is a δ -closed set which is not constructible; it is an algebraically closed subfield of K and has dimension one. Nevertheless the following is true:

Proposition 1 ([21]) *Let G be a δ -definable group in K . Then there is an algebraic group H , defined over K , such that G is definably isomorphic to a (δ -definable) subgroup of $H(L)$.*

For the many more existing results on δ -definable groups, we refer the reader to [20], or from the differential algebra point of view, to [8].

2.2. Generic difference fields

We now consider an algebraically closed field K with an automorphism σ . We say that (K, σ) is a generic difference field if every difference equation which has a solution in an extension of K has a solution in K . The theory of generic difference fields has been extensively studied in [9] and [10].

Let us suppose that (K, σ) is a generic difference field in characteristic zero. We consider the ring of σ -polynomials,

$$K_\sigma[X_1, \dots, X_n] = K[X_1, \dots, X_n, \sigma(X_1), \dots, \sigma^2(X_1), \dots, \sigma^2(X_n), \dots].$$

We say that $F \subseteq K^n$ is a f σ -closed set if there are $f_1, \dots, f_r \in K_\sigma[X_1, \dots, X_n]$ such that $F = \{(a_1, \dots, a_n) \in K^n : f_1(a_1, \dots, a_n) = \dots = f_r(a_1, \dots, a_n) = 0\}$. The σ -closed sets form the closed sets of a Noetherian topology on K , the σ -topology. The class of σ -definable sets is the closure under finite Boolean operations and projections of the σ -closed sets.

Again there are “new” σ -definable groups. For example, the field $\text{Fix}(K) = \{a \in K : \sigma(a) = a\}$, the fixed field of σ in K , is a σ -closed set of dimension one.

Here the best result possible for arbitrary σ -definable groups is the following:

Proposition 2 ([18]) *Let G be a group definable in (K, σ) . Then there are an algebraic group H defined over K , a finite normal subgroup N_1 of G , a σ -definable subgroup H_1 of $H(K)$ and a finite normal subgroup N_2 of H_1 , such that G/N_1 and H_1/N_2 are σ -definably isomorphic.*

The analysis of groups of finite dimension is one of the main tools in Hrushovski’s proof of the Manin-Mumford conjecture in [15].

3. Separably closed fields of finite degree of imperfection

Separably closed fields are particularly interesting from the model theoretic point of view for many reasons, in addition to the fact that they form the framework for Hrushovski’s proof of the Mordell-Lang conjecture in characteristic p . Let us just mention one reason here: they are the only fields known to be stable and non superstable, and in fact it is conjectured that they are the only existing ones.

We will just focus on the main properties of the groups that are definable in a separably closed field of finite degree of imperfection, but we need first to introduce some notation and recall some basic facts (see [11]).

3.1. Some basic facts and notation

Let L be a separably closed field of characteristic $p > 0$ and of finite degree of imperfection which is not perfect, i.e., L has no proper separable algebraic extension, and $|L : L^p| = p^\nu$, with $0 < \nu$. In order to avoid confusion we denote the Cartesian product of k copies of L by $L^{\times k}$.

A subset $B = \{b_1, \dots, b_\nu\}$ of L is called a p -basis of L if the set of p -monomials of B , $\{M_j := \prod_{i=1}^\nu b_i^{j(i)} ; j \in p^\nu\}$ forms a linear basis of L over L^p . Each element x

in L can be written in a unique way as $x = \sum_{j \in p^\nu} x_j^p M_j$. **From now on we fix a p -basis B of L and the M_j 's, with $j \in p^\nu$, always denote the p -monomials of B .** We suppose that L is large (a universal domain, or in model theoretic terms, saturated) and we fix some small separably closed subfield K of L , containing B and of same degree of imperfection ν .

We let f_j denote the map which to x associates x_j . The x_j 's are called the *p -components of x of level one*. More generally, one can associate to x a tree of countable height indexed by $(p^\nu)^{<\omega}$, which we call the *tree of p -components of x* . For $\sigma \in (p^\nu)^{<\omega}$, we define x_σ by induction: $x_\emptyset = x$ and if $\tau \in (p^\nu)^n$, and $j \in p^\nu$, we let $x_{(\tau,j)}$ be equal to $f_j(x_\tau)$; $x_{(\tau,j)}$ is called a *p -component of x of level $n+1$* .

We will also use the notation $a_\infty := (a_\sigma)_{\sigma \in (p^\nu)^{<\omega}}$, for $a \in L$.

The ring $\mathbf{K}[X_\infty]$. $K[X_\infty]$ is the polynomial ring in countably many indeterminates indexed in a way which will allow the natural substitution by the p -components of elements: for X a single variable, $X_\infty := (X_\sigma)_{\sigma \in (p^\nu)^{<\omega}}$, and for $X = (Y_1, \dots, Y_k)$ a k -tuple of variables, $X_\infty := ((Y_1)_\infty, \dots, (Y_k)_\infty)$. The ring $K[X_\infty]$ is a countable union of Noetherian rings, hence each ideal is countably generated. We let $I^0(X)$ denote the ideal of $K[X_\infty]$ generated by the polynomials $X_\sigma - \sum_{j \in p^\nu} X_{(\sigma,j)}^p M_j$, $\sigma \in (p^\nu)^{<\omega}$.

3.2. The λ -topology

Given a set of polynomials S of $K[X_\infty]$, let $V(S) = \{a \in L^{\times k} : f(a_\infty) = 0 \text{ for all } f \in S\}$. Such a $V(S)$ is called *λ -closed* (with parameters in K or over K) in L .

Given $A \subseteq L^{\times k}$, we define its *canonical ideal* $I(A)$ over K , $I(A) := \{f \in K[X_\infty] : f(a_\infty) = 0 \text{ for all } a \in A\}$.

The λ -closed subsets of $L^{\times k}$ form the closed sets of the *λ -topology* on $L^{\times k}$. This topology is not Noetherian but is the limit of countably many Noetherian topologies.

Let C be a commutative K -algebra. An ideal I of C is *separable* if, for all $c_j \in C$, $j \in p^\nu$, if $\sum_{j \in p^\nu} c_j^p M_j \in I$, then each $c_j \in I$.

Fact 3 (“Nullstellensatz”) 1. *The map $A \mapsto I(A)$ induces a bijection between λ -closed subsets of the affine space L^k which are defined over K and ideals of $K[X_\infty]$ which are separable and contain $I^0(X)$. The inverse map is $I \mapsto V(I)$.*

Now for the basic properties of the first-order theory:

Fact 4 1. *The theory of separably closed fields of characteristic p , of degree of imperfection ν , and with p -basis $\{b_1, \dots, b_\nu\}$ is complete and admits elimination of quantifiers and elimination of imaginaries in the language*

$$\mathcal{L}_{p,\nu} = \{0, 1, +, -, \cdot\} \cup \{b_1, \dots, b_\nu\} \cup \{f_i; i \in p^\nu\}.$$

In particular, any definable set is λ -constructible, that is, a finite Boolean combinations of definable λ -closed sets. Remark that it is impossible to associate to an arbitrary definable set a well-behaved notion of dimension: indeed, such a dimension would need to be invariant under definable bijections, but for every n the

map λ_n , defined by $\lambda_n(x) := (x_\sigma)_{\sigma \in (p^\nu)^n}$, is a definable bijection between L and $L^{\times p^{\nu n}}$. But some ∞ -definable sets will have a well-defined dimension, for example the field $L^{p^\infty} := \bigcap_n L^{p^n}$, which is the biggest algebraically closed subfield of L , has dimension one. In fact, L^{p^∞} is the unique (up to definable isomorphism) infinitely definable field of dimension one ([19], [13]).

3.3. Definable groups

Again, amongst the definable groups, one finds the “classical” ones, that is groups of the form $H(L)$ for H any algebraic group defined over L . These groups have certain specific properties which are not true of all the definable groups in L . Recall that a definable subset X of G is said to be *generic* if G is covered by a finite number of translates of X , and an element of G is *generic for the group* if every definable set which contains it is generic. In an algebraic group, generics in the topological sense coincide with generics for the algebraic group. Recall also that a definable group is said to be *connected* if it has no proper definable subgroup of finite index, and *connected-by-finite* if it has a definable connected subgroup of finite index.

Proposition 5 ([6], [13]) *Let H be an algebraic group defined over K . Then $H(L)$ is connected-by-finite. If H is connected (hence irreducible as an algebraic group), then $H(L)$ is connected (and irreducible for the λ -topology) and if $a \in H(L)$ is a generic point, then the ideal $I(a) = \{f \in K[X_\infty] : f(a_\infty) = 0\}$ is minimal amongst the ideals $I(h)$, for $h \in H(L)$.*

The above says that in the group $H(L)$, the generics in the topological sense coincide with generics for the group. In an arbitrary group defined in L , this need not be the case. Consider the definable bijection f from L to L defined in the following way: if $x \in L \setminus L^p$, $f(x) = x^p$, if $x \in L^p \setminus L^{p^2}$, $f(x) = x^{1/p}$, if $x \in L^{p^2}$, $f(x) = x$. Transporting addition through f , one gets a group on L again, $G := (L, *)$, definably isomorphic to $(L, +)$, hence connected. The set L itself is of course λ -closed and irreducible with associated ideal $I(L) = I^0(X)$. The ideal associated to the (group) generic of $(L, *)$ is generated by $I^0(X)$ and $\{X_i = 0 : i \in p^\nu, i \neq 0\}$, and strictly contains $I^0(X)$.

This question of the uniqueness of the notion of generic is not the only one posing problems for arbitrary definable groups in L . For example, there is no reason, coming from general properties of stable (non superstable) theories, which a priori forces all these definable groups to be connected-by-finite.

Nevertheless, one can in fact show that the situation is as close to the classical one as it could be:

Proposition 6 [6] *Every definable group G in L is connected-by-finite and is definably isomorphic to the group of L -rational points of an algebraic group H defined over L .*

One more remark, in the case of algebraic groups, by Prop. 5, irreducibility transfers down to the set of L -rational points. But this is not the case for an arbitrary variety: if one considers for example the irreducible variety defined by the equation

$Y^{p^m}X + Z^{p^m} = 0$, for $m \geq 1$, then the λ -closed set $V(L)$ is no longer irreducible in the sense of the λ -topology.

3.4. Minimal groups

The previous result enables us to give a complete description of groups of dimension one, and more generally of some classes of commutative groups.

We say that an ∞ -definable set D is *minimal* if any definable subset of G is finite or co-finite. If D is actually definable, then we say that D is *strongly minimal*.

The minimal groups are exactly the connected groups of dimension (U-rank) equal to one. A minimal group must be commutative.

From the basic properties of commutative algebraic groups over an algebraically closed field of characteristic p and Proposition 6, one can deduce:

Lemma 7 *Let G be a minimal group ∞ -definable in L , then G has exponent p or G is divisible.*

We first consider the commutative groups of exponent p :

Proposition 8 [7] *Let G be a commutative ∞ -definable group of exponent p definable in L . Then G is definably isomorphic to a λ -closed subgroup of the additive group $(L, +)$. Furthermore, if G is definable, then it is definably isogenous to the group of L -rational points of a vector group.*

Note that even when G is connected it is not necessarily definably isomorphic to the group of rational points of a vector group.

Then we consider the commutative divisible groups, which we show to be exactly the ones that were considered by Hrushovski in [13]:

Proposition 9 [7] *1. Let G be any ∞ -definable commutative divisible group in L . Then G is definably isomorphic to some $p^\infty A(L) := \bigcap_n p^n A(L)$, for A a semi-Abelian variety defined over L .*

2. If A is a semi-Abelian variety defined over L , $p^\infty A(L)$, which is the maximal divisible subgroup of $A(L)$ is also the smallest ∞ -definable subgroup of $A(L)$ which is Zariski dense in A .

Finally, this analysis, together with some results from [11] and [13], yields the full description of minimal groups.

Before stating the actual result, let us give some last definitions. The group G is said to be of *linear type* if for every n , every definable subgroup of $G^{\times n}$ is a finite Boolean combination of translates of definable subgroups of $G^{\times n}$. We define the *transcendence rank over K* of a group G , defined over K , to be the maximum of $\{\text{tr.degree}(K(g_\infty), K) : g \in G\}$.

Proposition 10 *Let G be an ∞ -definable minimal group in L .*

1. *Either G is not of linear type and then,*
 - *G is definably isomorphic to the multiplicative group $((L^{p^\infty})^*, \cdot)$*
 - *or G is definably isomorphic to $E(L^{p^\infty})$ for E an elliptic curve defined over L^{p^∞}*

- or G is definably isogenous to $(L^{p^\infty}, +)$. (isogenous here cannot be replaced by isomorphic).
2. Or G is of linear type and then,
- G is divisible and G is definably isomorphic to $p^\infty A(L)$ for some simple Abelian variety A defined over K which is not isogenous to an Abelian variety defined over L^{p^∞} ,
 - or G is of exponent p and is definably isomorphic to a minimal λ -closed subgroup of $(L, +)$.

In the divisible case G has finite transcendence rank; in the exponent p case, all transcendence ranks are possible.

The induced module-type structure on the minimal groups of exponent p and of linear type is analyzed in [2].

A short word about some of the tools involved in the proofs of Propositions 6 and 10: the proofs of 6, 1 and 2 all involve at some point the classical theorem of Weil’s constructing an algebraic group from a generic group law on a variety, or some generalizations of this theorem to an abstract model theoretic context. In the specific case of separably closed fields, another fundamental tool is the analysis of the properties of the Λ_n -functors, naturally associated to the maps λ_n : for each n , Λ_n is a covariant functor from the category of varieties V defined over K to itself, with the property that the L -rational points of the variety $\Lambda_n V$ are exactly the image by the map λ_n of the L -rational points of V . In the case of an algebraic group defined over K , Λ_1 is equal to the composition of the inverse of the Frobenius and of the classical Weil restriction of scalars functor from $K^{1/p}$ to K .

Finally, the way we have stated Proposition 10 uses the fact that if a minimal group is not of linear type, then it is non orthogonal to L^{p^∞} (and hence definably isogenous to the L^{p^∞} -rational points of some definable group over L^{p^∞}). The only known proof of this so far uses the powerful abstract machinery of Zariski structures from [16]. This dichotomy result, for the particular case of groups of the form $p^\infty A(L)$, is essential in Hrushovski’s proof of the Mordell-Lang conjecture in characteristic p , which is still the only existing proof for the general case. In a recent paper Pillay and Ziegler ([24]), show that, with some extra assumptions on A , one can replace in this proof the heavy Zariski structure argument by a much more elementary one. These extra assumptions are satisfied when A is an *ordinary* semi-Abelian variety (i.e. A has the maximum possible number of p^n -torsion points for every n), case which was already covered by previous non model-theoretic proofs (see [1]).

3.5. Final remarks and questions

As we have already mentioned earlier, the groups of finite dimension definable in these “enriched” theories of fields play a major role in the applications of model theory to Diophantine geometry. In the characteristic zero case, the relevant groups are the definable subgroups of the group of rational points of Abelian varieties in differentially closed fields (Mordell-Lang conjecture for function fields [13]), in generic

difference fields (the Manin-Mumford conjecture [15], [5] and the Tate-Voloch conjecture for semi-Abelian varieties defined over \mathbb{Q}_p [25], [26]). In the characteristic p case, the relevant groups are: the ∞ -definable divisible subgroups of the group of rational points of semi-Abelian varieties in separably closed fields (the Mordell-Lang conjecture for function fields [13]) and the definable subgroups of the additive groups in generic difference fields of characteristic p (Drinfeld modules [27]).

One should note that, in fact, separably closed fields are just another instance of a *field with extra operators* (derivations or automorphisms): one can equip any separably closed field L of finite degree of imperfection, with an infinite family of Hasse derivations in such a way that the resulting structure is bi-definably equivalent with L considered as a structure in the language described in section 3.2.. There are many interesting other possible types of “enriched” fields in this sense where the complete analysis of the model theoretic structure remains to be done.

Finally, one crucial step towards possible further applications of the fine study of finite rank definable sets to geometry would be an understanding of the structure induced on the so-called *trivial* or *disintegrated* definable (or infinitely definable) minimal sets, that is the minimal sets such that the induced pregeometry is disintegrated. This condition immediately rules out definable groups. The absence of any well-understood algebraic structure living on these “trivial” sets makes them very difficult to analyze. The only results obtained so far are in the context of differentially closed fields of characteristic 0: Hrushovski ([12]), building on some results of Jouanolou ([17]), showed that in any trivial strongly minimal set defined by a differential equation of order one, the induced pregeometry is locally finite. The question of whether this is true for higher order equations is still open.

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