HYPOCOERCIVITY & HYPOELLIPTICITY

Variations on a theme

Results and problems

Bismutfeast, May 31, 2013

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Common situation in dissipative PDE

- a "conservative" part + a "dissipative" part

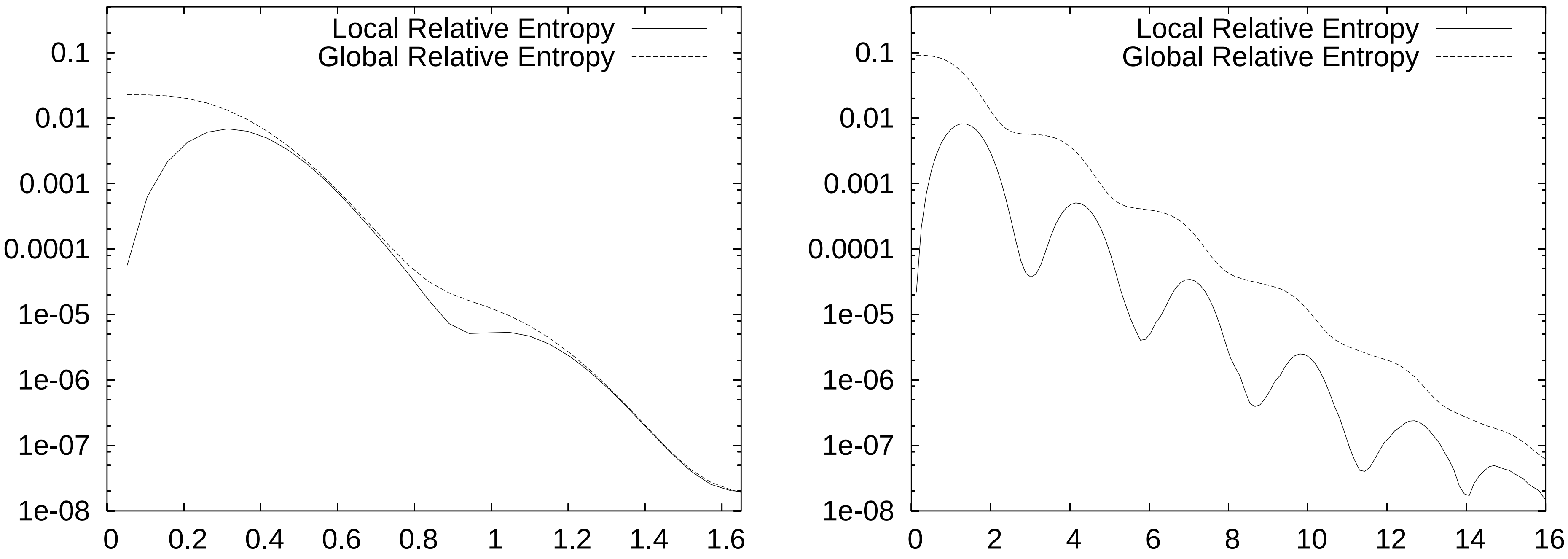
$\begin{cases} \text{- the conservative part alone does not induce relaxation} \\ \text{- the dissipative part is degenerate and not sufficient} \end{cases}$

..... but the combination of the two leads to relaxation.

Problem: How is the convergence as $t \to +\infty$? (how fast, etc.)

Numerical simulations (Filbet, around 2004)

Hydrodynamic approximation does not hold true in the large-time limit $\longrightarrow$ oscillations between "more hydrodynamic" and "more homogeneous" states (guessed by Desvillettes–V)



Two typical examples from kinetic theory

Unknown: $f(t, x, v) \geq 0$, $x \in \mathbb{R}^n$ (or $\Omega \subset \mathbb{R}^n$), $v \in \mathbb{R}^n$

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- the linear **kinetic Fokker–Planck equation**

$$\frac{d^2 X_t}{dt^2} = -\nabla V(X_t) - \frac{dX_t}{dt} + \sqrt{2}\,\frac{dB_t}{dt}$$

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- the linear **kinetic Fokker–Planck equation**

$$\textbf{(kFP)} \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (fv)$$

conservative diffusion/friction

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- the linear **kinetic Fokker–Planck equation**

$$\textbf{(kFP)} \qquad \frac{\partial f}{\partial t} \; + \; \underbrace{v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f}_{\text{conservative}} \; = \; \underbrace{\Delta_v f + \nabla_v \cdot (fv)}_{\text{diffusion/friction}}$$

- the **nonlinear Vlasov–Landau equation**

$$\textbf{(BE)} \qquad \frac{\partial f}{\partial t} \; + \; \underbrace{v \cdot \nabla_x f + F[f] \cdot \nabla_v f}_{\text{transport}} \; = \; \underbrace{Q_L(f,f)}_{\text{collisions}}$$

$v \cdot \nabla_x$: simple differential linear operator, "mixes" x, v

Q_L: complicated diffusion bilinear operator, acts only on v

- the **Boltzmann equation**

$$\textbf{(BE)} \qquad \frac{\partial f}{\partial t} + \underbrace{v \cdot \nabla_x f}_{\text{transport}} = \underbrace{Q(f,f)}_{\text{collisions}}$$

$v \cdot \nabla_x$: simple differential linear operator, "mixes" x, v

Q: complicated integral bilinear operator, acts only on v

Always the problem is to put the two properties together.

Collision operator

$$Q(f,f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} \Big[f(v')f(v'_*) - f(v)f(v_*) \Big] \, |v - v_*| \, d\sigma \, dv_*$$

$$(v, v_*) \xrightarrow{\text{collision}} (v', v'_*) \qquad \begin{cases} v' = (v+v_*)/2 \ + (|v-v_*|/2)\,\sigma \\ v'_* = (v+v_*)/2 - (|v-v_*|/2)\,\sigma \end{cases}$$

$Q(f,f) = 0$ if and only if f is hydrodynamic, i.e.

$$f(x,v) = M_{\rho u T} = \frac{\rho(x)\, e^{-\frac{|v-u(x)|^2}{2T(x)}}}{(2\pi T(x))^{N/2}}$$

Some difficulties

- For both (kFP) and (BE) the dissipative (collisional) part acts only on the variable v

$\Longrightarrow$ (BE) ceases to be dissipative on hydrodynamical states: $Q(M_{\rho\, u\, T}, M_{\rho\, u\, T}) = 0$

$\Longrightarrow$ (kFP) ceases to be dissipative on states $f(x,v) = \rho(x)\, e^{-|v|^2/2}$

- For Boltzmann, additional difficulties: nonlinearity, complexity, just one Lyapunov functional (entropy) and the understanding of its production is very tricky

....Anyway we need to cleverly use the conservative part

Grad's intuition

"the question is whether the deviation from a local Maxwellian, which is fed by molecular streaming in the presence of spatial inhomogeneity, is sufficiently strong to ultimately wipe out the inhomogeneity" (...)

"a valid proof of the approach to equilibrium in a spatially varying problem requires just the opposite of the procedure that is followed in a proof of the H-Theorem, viz., to show that the distribution function does not approach too closely to a local Maxwellian."

On Boltzmann's H Theorem (1965)

Incomplete bibliography for kinetic Fokker–Planck

Various convergence results by probabilistic methods (Wu, Rey-Bellet, Bakry–Cattiaux–Guillin, Mattingly–Stuart...)

Desvillettes–V (2001): convergence in $O(t^{-\infty})$ for $V \simeq a\,|x|^2$ at infinity and $f_0/f_\infty \in L^\infty$(!)

Exponential convergence for $f_0/f_\infty \in L^2(f_\infty)$: Hérau–Nier (2004), Helffer–Nier, Hérau....

Incomplete bibliography for Boltzmann

(With adequate boundary conditions)

Desvillettes–V (2005): **If** $f(t,x,v)$ is uniformly smooth and positive, then convergence like $O(t^{-\infty})$, and the decay bounds can be estimated from the smoothness and positivity bounds

Guo, Strain: $f_0(x,v)$ close enough to equilibrium $\Longrightarrow$ convergence to equilibrium like $O(e^{-\lambda t})$ (or $O(e^{-\lambda t^\gamma})$)

(Note: For Boltzmann equation in the large we don't know *any* estimate!)

The Desvillettes–V. method

Based on four first-order and second-order differential inequalities coupled by functional inequalities.

Uses

(a) Lower bound on the entropy production far from hydrodynamic states (information-theoretical input)

(b) Instability of the hydrodynamic approximation in presence of gradients (fluid mechanics input)

(c) Geometric inequalities (Poincaré, Korn)

(d) Study of the system of differential inequalities ("Gronwall" style)

+ a lot of interpolation (trade smoothness for exponents)

Hydrodynamic approximation (fluid mechanics)

$$f(t,x,v) \longrightarrow M^f = \rho \, \frac{e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{N/2}}$$

M^f = best approximation of f by a hydrodynamic state (“**projection**”)

$$\begin{cases} \rho(t,x) := \displaystyle\int f \, dv & \text{(density)} \\ u(t,x) := \dfrac{1}{\rho} \displaystyle\int f \, v \, dv & \text{(mean velocity)} \\ T(t,x) := \dfrac{1}{N\rho} \displaystyle\int f \, |v-u|^2 \, dv & \text{(temperature)} \end{cases}$$

Entropy production

$H(f) = \int f \log f \, dv \, dx$

$$D(f) = \frac{1}{4} \int \Big(f(v')f(v'_*) - f(v)f(v_*) \Big) \log \frac{f(v)f(v_*)}{f(v')f(v'_*)} \, |v - v_*| \, d\sigma \, dv \, dv_*$$

Best known lower bound

(precursors: Carlen–Carvalho, Toscani-V)

$D(f) \geq K_\varepsilon(f) \left[H(f) - H(M^f) \right]^{1+\varepsilon}$

$$K_\varepsilon(f) \text{ depends on } \begin{cases} \varepsilon \\ \text{regularity (high order Sobolev norms) of } f \\ \text{positivity: } f \geq K e^{-A|v|^q} \end{cases}$$

Rk: Inequality with $\varepsilon = 0$ (Cercignani conjecture) is false

....The system

$$
\begin{cases}
-\frac{d}{dt}[H(f)-H(M)] \geq K[H(f)-H(M^f)]^{1+\varepsilon} \\
\\
\frac{d^2}{dt^2}\|f-M^f_{\rho\,u\,T}\|^2_{L^2} \geq K\int_\Omega |\nabla T|^2\,dx \\
\qquad\qquad -\frac{C}{{\delta_1}^{1-\varepsilon}}\left(\|f-M^f_{\rho\,u\,T}\|^2_{L^2}\right)^{1-\varepsilon} - \delta_1[H(f)-H(M)] \\
\\
\frac{d^2}{dt^2}\|f-M^f_{\rho\,u\,\langle T\rangle}\|^2_{L^2} \geq K\int_\Omega |\nabla^{\mathrm{sym}} u|^2\,dx \\
\qquad\qquad -\frac{C}{{\delta_2}^{1-\varepsilon}}\left(\|f-M^f_{\rho\,u\,\langle T\rangle}\|^2_{L^2}\right)^{1-\varepsilon} - \delta_2[H(f)-H(M)] \\
\\
\frac{d^2}{dt^2}\|f-M^f_{\rho\,0\,1}\|^2_{L^2} \geq K\int_\Omega |\nabla \rho|^2\,dx \\
\qquad\qquad -\frac{C}{{\delta_3}^{1-\varepsilon}}\left(\|f-M^f_{\rho\,0\,1}\|^2_{L^2}\right)^{1-\varepsilon} - \delta_3[H(f)-H(M)]
\end{cases}
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.... $H(f(t)) - H(M) = O(t^{-1/200\,\varepsilon})$

Two goals which were subsequently pursued

(1) Find simpler (maybe less intuitive) methods

(2) Identify general structures gathering various models with a "degenerate diffusive part" and start a "toolbox"

$\Longrightarrow$ *Hypocoercivity* (*Memoirs AMS*, 2009)

Rather frequent situation:

- a “conservative” part + a “dissipative part”

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..... but the combination of the two leads to relaxation.

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..... but the combination of the two leads to relaxation.

Analogy with (parabolic) hypoellipticity theory

- a first-order part + a second-order part

$\begin{cases} \text{- the first-order part does not induce any regularization} \\ \text{- the second-order part is degenerate} \end{cases}$

..... but the combination leads to regularization

Denomination: hypocoercivity for the first situation (Gallay)

Recall: Hörmander's "$\sum X_i^2 + X_0$" theorem

$(a_i)_{0\leq i\leq m}$ smooth vector fields in dimension n

$(a_i) \leftrightarrow X_i = a_i \cdot \nabla$ differentiation operators in direction a_i

$$\frac{\partial u}{\partial t} = \Big(\sum_{i\geq 1} X_i^2 + X_0\Big) u$$

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Ellliptic case: $\operatorname{rank}((X_i)) = n \quad \Longrightarrow C^\infty$ regularization

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Still regularizing if

rank $((X_i), ([X_i, X_j]), ([[X_i, X_k], X_j]), \ldots) = n$

i.e. by taking a finite number of commutators involving the dissipative and the conservative part, one can generate all directions

The "$A^*A + B$" theorem

(abstract linear result in a Hilbert space)

$$A = (A_1, \ldots, A_m), \qquad B^* = -B \text{ in } \mathcal{H}, \qquad L = A^*A + B$$

$$C_0 = A, \quad [C_j, B] = C_{j+1} + R_{j+1} \quad (j \leq \mathrm{N_c}), \quad C_{\mathrm{N_c}+1} = 0,$$

$$\begin{cases} \text{(i) } [A, C_k] \text{ bounded relatively to } \{C_j\}_{0\leq j\leq k}, \{C_j A\}_{0\leq j\leq k-1} \\ \text{(ii) } [A^*, C_k] \text{ bounded relatively to } I, \{C_j\}_{0\leq j\leq k} \\ \text{(iii) } R_k \text{ bounded relatively to } \{C_j\}_{0\leq j\leq k-1}, \{C_j A\}_{0\leq j\leq k-1} \end{cases}$$

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+ **(H)** $\sum_{j=0}^{\mathrm{N_c}} C_j^* C_j$ is coercive

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$+$ **(H)** $\quad \sum_{j=0}^{\mathrm{N_c}} C_j^*C_j$ is coercive

Then $\left\| e^{-tL} \right\|_{\mathcal{H}^1\to\mathcal{H}^1} = O(e^{-\lambda t})$ $\qquad \|h\|^2_{\mathcal{H}^1}=\|h\|^2+\sum \|C_jh\|^2$

Application of the $A^*A + B$ theorem to kFP

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (f v)$$

$$h = f / f_\infty \in \mathcal{H} = L^2(f_\infty \, dx \, dv)$$

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Assumptions are satisfied as soon as

(a) $|\nabla^2 V| \leq \text{const.}\,(1 + |\nabla V|)$

(b) $e^{-V}\,dx$ satisfies a Poincaré inequality:

$$\int |\nabla_x h|^2 e^{-V}\,dx \geq K \int |h - \langle h \rangle|^2 \, e^{-V}\,dx$$

Then $\|h(t) - 1\| = O(e^{-\lambda t})$ constructive rate

(most general result at the time)

Core of the proof of $A^*A + B$

Introduce a Lyapunov functional: (say $N_c = 1$)

$$\mathcal{F}(h) = \|h\|^2 + a\|Ah\|^2 \qquad + c\|[A, B]h\|^2$$

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$$1 \gg a \gg b \gg c; \qquad b \ll \sqrt{ac}$$

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Key algebraic identity:

$$\left.\frac{d}{dt}\right|_B \langle Ah, [A, B]h \rangle = \langle ABh, [A, B]h \rangle + \langle Ah, [A, B]Bh \rangle$$

$$= \|[A, B]h\|^2 + \langle Ah, [[A, B], B]h \rangle$$

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Rks: • Not unrelated to tricks by Guo and Talay

• Often one can use similar Lyapunov functionals to **prove** Hörmander's hypoelliptic regularity bounds (Hérau, V)

$\Longrightarrow$ Global regularization theorems

Regularization from L^1 data

$$\partial_t f + v \cdot \nabla_x f - \nabla V \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (f v)$$

$$\int f_0(x,v)\,(1 + |x|^2 + |v|^2)\,dv\,dx < +\infty$$

Prove that $f_t \in H^k_x H^\ell_v(\mathbb{R}^n \times \mathbb{R}^n)$??

Note carefully: Because of the behavior at infinity, there is no regularization $L^1 \to L^2$!!

Global hypoellipticity (in 4 steps)

L^1 initial datum $\rightarrow$ (flat) Sobolev regularity

Step 1: Energy estimate

$$\mathcal{E}_m(f) = \sum_{3k+\ell \leq 3m} a_{k\ell} \int |\nabla_x^k \nabla_v^\ell f|^2 \, dx \, dv$$

$$\simeq \int |\nabla_x^m f|^2 + \int |\nabla_v^{3m} f|^2 + \int f^2$$

Choose ad hoc coefficients $a_{k\ell} \Longrightarrow$

Computation using integration by parts...

$$\frac{d\, \mathcal{E}_m(f)}{dt} \leq -K \int |\nabla_v^{3m+1} f|^2 \, dx \, dv + C\, \mathcal{E}_m(f)$$

Step 2: Mixed derivatives

$$\mathcal{M}_m(f) = \int \nabla_x^m f \cdot \nabla_x^{m-1} \nabla_v f \, dx \, dv \quad \left(= \int \nabla_x f \cdot \nabla_v f \ \text{ if } m = 1 \right)$$

.... Computation using commutator $[\nabla_v, v \cdot \nabla_x] = \nabla_x$

$$\frac{d\mathcal{M}_m}{dt} \leq -K \int |\nabla_x^m f|^2 \, dx \, dv \, + \, C \sum_{\substack{3k+\ell \leq 3m \\ k<m}} \int |\nabla_x^k \nabla_v^\ell f|^2$$

Step 3: Interpolation

Based on anisotropic Nash inequality

$$\int |D_x^\lambda D_v^\mu f|^2 \, dx \, dv$$

$$\leq C \left(\int |D_x^{\lambda'} f|^2 \, dx \, dv + \int |D_v^{\mu'} f|^2 \, dx \, dv \right)^{1-\theta} \left(\int f \right)^{\theta}$$

$$\theta = \frac{1 - \left(\frac{\lambda}{\lambda'} + \frac{\mu}{\mu'} \right)}{1 + \frac{n}{2} \left(\frac{1}{\lambda'} + \frac{1}{\mu'} \right)} \in (0, 1)$$

$\lambda = \mu = 0, \;\; \lambda' = \mu' = 1 \longrightarrow$ usual Nash inequality in $\mathbb{R}^{2n}$

$$
\begin{cases}
K\left(\int |\nabla_x^m f|^2 + \int |\nabla_v^{3m} f|^2 + \int f^2\right) \le \mathcal{E}_m \le C\left(\quad \cdots \quad \right) \\[2em]
|\mathcal{M}_m| \le C\,\mathcal{E}_m^{1-\delta} \\[2em]
\dfrac{d\mathcal{E}_m}{dt} \le -K \int |\nabla_v^{3m+1} f|^2 + C\,\mathcal{E}_m \\[2em]
\int |\nabla_v^{3m} f|^2 + \int f^2 \le C\left(\int |\nabla_x^m f|^2 + \int |\nabla_v^{3m+1} f|^2\right)^{1-\theta} \\[2em]
\dfrac{d\mathcal{M}_m}{dt} \le -K \int |\nabla_x^m f|^2 + C\left(\int |\nabla_v^{3m} f|^2 + \int f^2\right)
\end{cases}
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K\left(\int |\nabla_x^m f|^2 + \int |\nabla_v^{3m} f|^2 + \int f^2\right) \leq \mathcal{E}_m \leq C\left(\quad \dots \quad \right) \\
|\mathcal{M}_m| \leq C\, \mathcal{E}_m^{1-\delta} \\
\dfrac{d\mathcal{E}_m}{dt} \leq -K \int |\nabla_v^{3m+1} f|^2 + C\, \mathcal{E}_m \\
\int |\nabla_v^{3m} f|^2 + \int f^2 \leq C\left(\int |\nabla_x^m f|^2 + \int |\nabla_v^{3m+1} f|^2\right)^{1-\theta} \\
\dfrac{d\mathcal{M}_m}{dt} \leq -K \int |\nabla_x^m f|^2 + C\left(\int |\nabla_v^{3m} f|^2 + \int f^2\right)
\end{cases}
$$

$$
\Longrightarrow \mathcal{E}_m(f) = O(t^{-1/\kappa}), \qquad \kappa = \min(\delta,\, \theta/(1-\theta))
$$

Regularization in Fisher information sense

$$\mathcal{F}(t, f_t) = \int f \log f + a\,t \int f\,|\nabla_v \log f|^2 + 2b\,t^2 \int f\,\langle \nabla_v \log f, \nabla_x \log f \rangle + c\,t^3 \int f\,|\nabla_x \log f|^2$$

(adaptation of a trick by Hérau)

..... compute .... choose coefficients well $\Longrightarrow$ $(d/dt)\mathcal{F} \leq 0$

$\Longrightarrow$

$$\int f_t |\nabla_{x,v} \log f_t|^2 \, dx\, dv \leq (C\,t^{-3}) \int f_0 \log f_0 \, dx\, dv$$

Discovery: Computations almost similar for h^2 or $h \log h$!

$$\partial_t h = A^* A h = \sum A_i^* A_i h$$

$$\frac{d}{dt}\int |Ch|^2 \, d\mu = -2\int |CAh|^2 \, d\mu - 2\int \langle [C, A^*]Ah, Ch\rangle \, d\mu$$
$$-2\int \langle CAh, [A, C]h\rangle \, d\mu$$

$$\frac{d}{dt}\int h\,|C\log h|^2 \, d\mu = -2\int h\,|CA\log h|^2 \, d\mu$$
$$-2\int h\,\langle [C, A^*]A\log h, C\log h\rangle \, d\mu$$
$$-2\int h\,\langle CA\log h, [A, C]\log h\rangle \, d\mu$$
$$-2\int h\sum_{ij}[A_i, C_j]^*\big((A_i\log h)\,(C_j\log h)\big)\, d\mu$$

(Compare with Bakry–Émery) Proof is by brute force.

Conclusion

Convergence in L^1

(a) $|\nabla^2 V| \leq C$

(b) log Sobolev inequality for $e^{-V}\,dx$ (e.g. $\nabla^2 V \geq \kappa\, I_n$ at ∞)

(c) $\int f_0(x,v)\,(1 + |v|^2 + |x|^2)\,dx\,dv < +\infty$

Then $f_t \to f_\infty$ in L^1 (and in entropy) exponentially fast

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Compare: Convergence in L^2

(a) $|\nabla^2 V| \le C\,(1+|\nabla V|)$

(b) Poincaré inequality for $e^{-V}\,dx$ (e.g. $\dfrac{|\nabla V|^2}{2} - \Delta V \underset{\infty}{\longrightarrow} \infty$)

(c) $f_0/f_\infty \in L^2(f_\infty)$

Then $f_t/f_\infty \to 1$ in $L^2(f_\infty)$ exponentially fast

Remark: Completely different approach to exponential convergence in (weak) L^1, by Hairer–Mattingly

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What about nonlinear models?

Another framework: fully nonlinear equations

$\partial_t f + Bf = \mathcal{C} f$ in a scale of Banach spaces $(X^s)_{s\geq 0}$ that are in interpolation; X^0 is Hilbert

f is uniformly bounded in all X^s

$B, \mathcal{C}$ are Lipschitz $X^s \to X^{s'}$ (loss of index allowed)

B is "conservative" and $\mathcal{C}$ is "dissipative"

f_∞: stationary state

$\mathcal{E}$: (Lyapunov) functional $\geq K \, \|f - f_\infty\|^{2+\varepsilon}$

$(\Pi_j)_{1\leq j\leq J}$: nonlinear "projection" operators, twice differentiable $X^s \to X^{s'}$, $\qquad \Pi_j \circ \Pi_k = \Pi_{\max(j,k)}$

Assumptions (simplified)

$$\begin{cases} \mathcal{E}(f) - \mathcal{E}(\Pi_1 f) \geq K \, \| f - \Pi_1 f \|^{2+\varepsilon} \\ K \| \Pi_1 f - f_\infty \|^{2+\varepsilon} \leq \mathcal{E}(\Pi_1 f) - \mathcal{E}(f_\infty) \leq C \| \Pi_1 f - f_\infty \|^{2-\varepsilon} \end{cases}$$

$$\begin{cases} \mathcal{E}'(f) \cdot (Bf) = 0 \\ -\mathcal{E}'(f) \cdot (\mathcal{C} f) \geq K \big[\mathcal{E}(f) - \mathcal{E}(\Pi_1 f) \big]^{1+\varepsilon} \end{cases}$$

$$\forall j, \qquad \Pi_j(f_\infty) = f_\infty; \qquad \mathcal{C} \circ \Pi_j = 0 \qquad \Pi_{J+1} f = f_\infty$$

(H) $$\Big\| (\mathrm{Id} - \Pi_j)'_{\Pi_j f} \cdot (B \Pi_j f) \Big\|^2 \geq K_\varepsilon \big\| (\Pi_j - \Pi_{j+1}) f \big\|^{2+\varepsilon}$$

$$\Longrightarrow \| f - f_\infty \| = O(t^{-\infty}) \qquad \text{(quantitative)}$$

Application to the Boltzmann equation

In a nonaxisymmetric box: 3 projections:

$$\Pi_1 f = M_{\rho\, u\, T} \qquad \Pi_2 f = M_{\rho\, u\, \langle T \rangle} \qquad \Pi_3 f = M_{\rho\, 0\, 1} \qquad \Pi_4 f = f_\infty$$

For other boundary conditions, change the projectors.....

An unconditional nonlinear convergence result

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (f v)$$

$$F(t,x) = -\int \nabla W(x-y)\, f(t,y,w)\, dy\, dw$$

where $x \in \mathbb{T}^N$, $W \in C^2(\mathbb{T}^N)$, $\int W = 0$,

$\int f_0(x,v)\,(1+|v|^2)\,dx\,dv < +\infty$

$\max |W| < 1 \Longrightarrow$ unique equilibrium f_∞

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The general convergence theorem allows to prove

$\max |W| < 0.38 \Longrightarrow$ $\|f_t - f_\infty\| = O(t^{-\infty})$

The Lyapunov functional for the fully nonlinear case

$$\mathcal{L}(f) := \mathcal{E}(f) + \sum_{j=1}^{J} a_j \Big\langle (I - \Pi_j) f, (I - \Pi_j)'_f \cdot (Bf) \Big\rangle_{L^2}$$

$1 \gg a_1 \gg a_2 \ldots \gg a_J$ chosen recursively, depending on the smoothness bounds and how close to equilibrium

Remark: In practice, $\mathcal{L}$ might be quite complicated!!

$$
\begin{aligned}
\mathcal{L}_{\text{Boltz}}(f) = \quad & \int f \log f - a_1 \int (f - M^f_{\rho\,u\,T}) \cdot \Bigg(v \cdot \nabla_x f + M^f_{\rho\,u\,T} \Bigg\{ \left[\frac{\nabla u : D}{\rho T} + \frac{\nabla \cdot R}{\rho T} \right] \\
& + \frac{v-u}{\sqrt{T}} \cdot \left[-\left(\frac{N}{2} + 1 \right) \frac{\nabla T}{\sqrt{T}} - \frac{\nabla \cdot D}{\rho \sqrt{T}} \right] + \sum_{i<j} \left(\frac{v-u}{\sqrt{T}} \right)_i \left(\frac{v-u}{\sqrt{T}} \right)_j \left[\partial_{x_j} u_i + \partial_{x_i} u_j \right] \\
& + \sum_i \left(\frac{v_i - u_i}{\sqrt{T}} \right)^2 \left[\partial_{x_i} u_i - \frac{\nabla \cdot u}{N} - \frac{\nabla u : D}{N \rho T} - \frac{\nabla \cdot R}{N \rho T} \right] + \left| \frac{v-u}{\sqrt{T}} \right|^2 \left(\frac{v-u}{\sqrt{T}} \right) \cdot \frac{\nabla T}{2\sqrt{T}} \Bigg\} \Bigg) \\
& - a_2 \int (f - M^f_{\rho\,u\,\langle T \rangle}) \cdot \Bigg(v \cdot \nabla_x f + M^f_{\rho\,u\,\langle T \rangle} \Bigg\{ \left[-\nabla \cdot u + \frac{\int \nabla u : D}{\langle T \rangle_\rho} + \frac{\int \rho T \nabla \cdot u}{\langle T \rangle_\rho} \right] \\
& + \frac{v-u}{\sqrt{\langle T \rangle_\rho}} \cdot \left[-\frac{\nabla T}{\sqrt{\langle T \rangle_\rho}} - \left(\frac{T}{\sqrt{\langle T \rangle_\rho}} - \sqrt{\langle T \rangle_\rho} \right) \frac{\nabla \rho}{\rho} - \frac{\nabla \cdot D}{\rho \sqrt{\langle T \rangle_\rho}} \right] \\
& + \sum_{i<j} \left(\frac{v-u}{\sqrt{\langle T \rangle_\rho}} \right)_i \left(\frac{v-u}{\sqrt{\langle T \rangle_\rho}} \right)_j \left[\partial_{x_j} u_i + \partial_{x_i} u_j \right] \\
& + \sum_i \left(\frac{v_i - u_i}{\sqrt{\langle T \rangle_\rho}} \right)^2 \left[\partial_{x_i} u_i - \frac{1}{N \langle T \rangle_\rho} \int \nabla u : (D + \rho T I_N) \right] \Bigg\} \Bigg) \\
& - a_3 \int (f - M^f_{\rho\,0\,1}) \cdot \left(v \cdot \nabla_x f + M^f_{\rho\,0\,1} \left\{ -\frac{\nabla \cdot (\rho u)}{\rho} + v \cdot \frac{\nabla \rho}{\rho} \right\} \right)
\end{aligned}
$$

Further examples

The Landau–Lifschitz–Gilbert–Maxwell model

(by Capella, Loeschcke, Wachsmuth)

$$\begin{cases} \partial_t m = J(h-m) \\ \partial_t h = -\nabla \wedge \nabla \wedge h - J(h-m) \\ \nabla \cdot h = -\nabla \cdot m, \end{cases}$$

$m : \mathbb{R}^3 \to \mathbb{R}^2$ = linearized magnetization

$h : \mathbb{R}^3 \to \mathbb{R}^3$ = linearized magnetic field

$J[x_1, x_2, x_3] = [-x_2, x_1]$.

This is of the form $A^*A + B$ and general theorem applies after 3 commutators

2d incompressible flow

Model problem for the stability of Oseen vortices, studied by Gallagher, Gallay, Nier:

$$\mathcal{H} = L^2(\mathbb{R}; \mathbb{C}); \qquad L_\varepsilon = (-\partial_x^2 + x^2 + 1) + \frac{\mathbf{i}}{\varepsilon} f,$$

$$f(x) = \frac{(1 - e^{-|x|^2/4})}{|x|^2} \text{ (typically)}, \qquad \varepsilon \to 0$$

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Discovery: The antisymmetric part enhances the dissipation by a factor $\varepsilon^{-1/2}$

(or $\varepsilon^{-2/(4+k)}$ if f is Morse and decays like $|x|^{-k}$)

How to prove this??

- Hard approach: localize the spectrum by microlocal techniques and semiclassical asymptotics à la Hörmander, Sjöstrand, Zworski, Helffer, Nier.... Get
$\mathcal{R}e\, \sigma(L_\varepsilon) \geq K\, \varepsilon^{-1/2}$

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- Alternative approach: look at the evolution problem: search μ s.t. $\|e^{-tL}\| \leq C\,e^{-\mu t}$.

Set $A = \partial_x + x$, $B = (\mathrm{i}/\varepsilon) f$, then $L_\varepsilon = A^* A + B$

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Set $A = \partial_x + x$, $B = (\mathrm{i}/\varepsilon) f$, then $L_\varepsilon = A^* A + B$

$$\mathcal{L} = \int_{\mathbb{R}} \Big(\frac{|u|^2}{2} + \frac{a}{2} (|\partial_x u|^2 + x^2 |u|^2) + b\,\mathcal{R}e\,(\overline{u}\,\mathrm{i}\,f' \partial_x u) + \frac{c}{2} f'^2\,|u|^2 \Big)$$

$$A^* A + [A, B]^* [A, B] = \widetilde{L}_\varepsilon = -\partial_x^2 + x^2 + 1 + \frac{1}{\varepsilon^2}\,f'(x)^2$$

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$$\ldots\ldots.\ \|e^{-tL_\varepsilon}\| \leq C\, \exp(-\varepsilon^{-1/2} t)$$

Other hypocoercivity results

Sometimes commutators are **not** well-behaved, neither fractional powers

$\Longrightarrow$ Play with projections...

- Linearized Boltzmann in the torus: Mouhot–Neumann

Dolbeault–Mouhot–Schmeiser (2013)

$$\frac{\partial f}{\partial t} + Tf = Lf, \qquad \Pi = \mathrm{proj}_{\ker L}$$

$$L^* = L, \quad -\langle Lf, f\rangle \geq \lambda \|(I - \Pi)f\|^2$$

$$T^* = -T, \quad \|T\Pi f\|^2 \geq \lambda \|\Pi f\|^2$$

$\Pi T \Pi = 0$ + some technical conditions (boundedness...)

$$A := \Big(1 + (T\Pi)^*(T\Pi)\Big)^{-1} (T\Pi)^*$$

Then $\|f\|^2 + \varepsilon \langle Af, f\rangle \simeq \|f\|^2$ is a contracting norm

Some mysteries?

Qualitative understanding of the role of confinement

$\partial_t h + v \cdot \nabla_x h = \Delta_v h - v \cdot \nabla_v h$

in $L^2(a\mathbb{T}^d_x \times \mathbb{R}^d_v; e^{-|v|^2/2}\, dv\, dx)$

- Spectrum is real (!) — physical implications
- Bottom of spectrum is equal to λ_{OU} for a small

$\partial_t h + v \cdot \nabla_x h - \omega x \cdot \nabla_v h = \Delta_v h - v \cdot \nabla_v h$ in $L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v; e^{-(\omega|x|^2+|v|^2)/2}\, dv\, dx)$

- Bottom of spectrum is $\dfrac{\lambda}{2}$ + im.part as $\omega \to \infty$
- "Best" choice is $\omega = 1/2$

Large dimensions: Models of heat conduction

$$\begin{cases} \ddot{q}_j = -\nabla V(q_j) - \nabla W(q_j - q_{j-1}) + \nabla W(q_{j+1} - q_j) \\ \dfrac{d\ell}{dt} = -\gamma \ell + \delta q_0 - \kappa_\ell \dfrac{dw}{dt} \\ \ddot{q}_0 = -\nabla V(q_0) + \nabla W(1_1 - q_0) + \ell \end{cases}$$

Spectral properties as $N \to \infty$?

- Does the stationary μ satisfy a Poincaré inequality?
- Meaningful asymptotics? Better work in entropy?
- Estimates as $N \to \infty$??

$\longrightarrow$ For a related model with weak coupling, Liverani–Olla prove the hydrodynamic limit toward diffusion model using both hypoelliptic (sum of squares) and hypocoercive ($A^*A + B$) methods.

Riemannian case

$\partial_t f + \xi f = \Delta_V f - v \cdot \nabla_V f$

Set in tangent or cotangent formalism

Use ∇_V, ∇_H in place of ∇_v, ∇_x

- Regularization: L^2, L^1: same as in flat space
- Hypocoercivity in L^2: ok
- Hypocoercivity in L^1: ?? The problem is that

$[d_H, \xi] = O(|v|^2)\nabla_V$ cannot be easily controlled in entropy estimates

$\int |v|^2 \, |\nabla_v \log h|^2 \, h \, d\mu \leq$

$C \left(\int |\nabla_v \log h|^2 \, h \, d\mu + \int |\nabla_v^2 \log h|^2 \, h \, d\mu \right)$ does **NOT** hold

Weakly diffusive Landau damping in low regularity?

$$\boxed{\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = \varepsilon \, Q_L(f, f)}$$

$$= \varepsilon \, \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \frac{\Pi_{(v-v_*)^\perp}}{|v-v_*|} \Big(f(v_*) \nabla_v f(v) - f(v) \nabla_v f(v_*) \Big) \, dv_* \right\}$$

- $\varepsilon = \log \Lambda / (2\pi\Lambda) \simeq 10^{-2} \to 10^{-30} \ll 1$

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- Expect: regularize in $\mathcal{G}^\nu$, like $O(\exp(\varepsilon t)^{-\nu/(2-\nu)})$ in v, maybe $O(\exp(\varepsilon^\nu (\varepsilon t)^{-3\nu/(2-3\nu)}))$ in x

- Expect: homogenize at least as fast as VP (diffusion; stability of homogeneity), i.e. $O(\exp -t^\nu)$

- Deduce: damping on time scale $\varepsilon^{-\zeta} \ll \varepsilon^{-1}$, while entropy increase is $O(\varepsilon^{1-\zeta}) \ll 1$

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- Heuristics: $\zeta \simeq 8/9$... (1/6 without x-smoothness)