Random surfaces: real and imaginary

Jason Miller and Scott Sheffield

MIT

May 28, 2013

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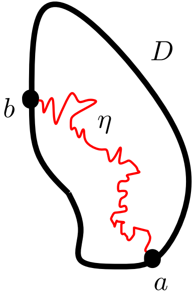
- 6. **What is a random surface? Liouville quantum gravity?**
- 7. **What is an imaginary surface? Surface with imaginary curvature?**
- 8. **What is a random imaginary surface? Liouville quantum gravity with extra *i* thrown in somewhere?**

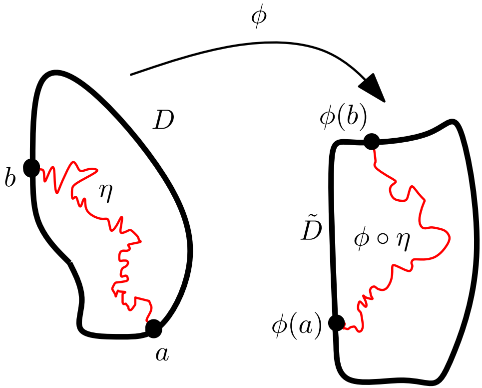
What is a random non-self-crossing path?

Given a simply connected planar domain D with boundary points a and b and a parameter $\kappa \in [0, \infty)$, the **Schramm-Loewner evolution** SLE_κ is a random non-self-crossing path in $\overline{D}$ from a to b.



The parameter κ roughly indicates how "windy" the path is. Would like to argue that SLE is in some sense the "canonical" random non-self-crossing path. What symmetries characterize SLE?

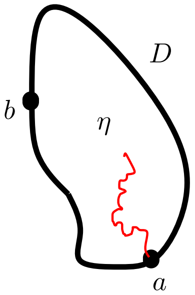
Conformal invariance



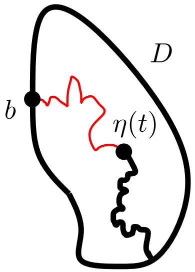
If ϕ conformally maps D to $\tilde{D}$ and η is an SLE_κ from a to b in D, then $\phi \circ \eta$ is an SLE_κ from $\phi(a)$ to $\phi(b)$ in $\tilde{D}$.

Markov Property

Given η up to a stopping time t...



law of remainder is SLE in $D \setminus \eta[0,t]$ from $\eta(t)$ to b.



Schramm-Loewner evolution (SLE)

- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.

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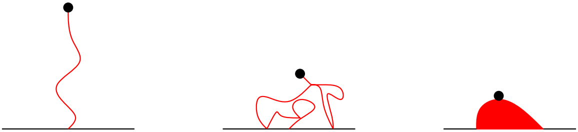
- **THEOREM [Oded Schramm]:** Conformal invariance and the Markov property completely determine the law of SLE, up to a single parameter which we denote by $\kappa \geq 0$.
- **Explicit construction:** An SLE path γ from 0 to ∞ in the complex upper half plane $\mathbb{H}$ can be defined in an interesting way: given path γ one can construct conformal maps $g_t : \mathbb{H} \setminus \gamma([0,t]) \to \mathbb{H}$ (normalized to look like identity near infinity, i.e., $\lim_{z\to\infty} g_t(z) - z = 0$). In SLE$_\kappa$, one defines g_t via an ODE (which makes sense for each fixed z):

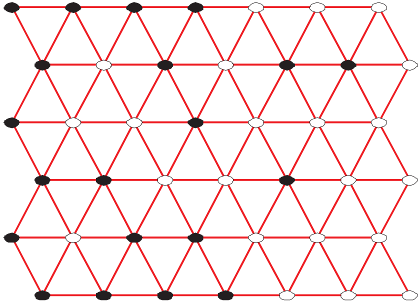
$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

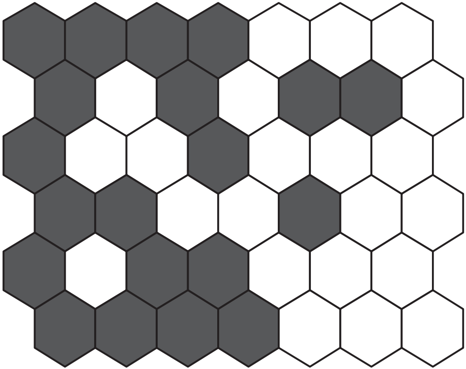
where $W_t = \sqrt{\kappa} B_t =_{LAW} B_{\kappa t}$ and B_t is ordinary Brownian motion.

SLE phases [Rohde, Schramm]



$\kappa \leq 4$ $\kappa \in (4, 8)$ $\kappa \geq 8$





Revisiting "What is a surface?": Riemann uniformization

Uniformization Every smooth simply connected Riemannian manifold $\mathcal{M}$ can be conformally mapped to either the unit disc $\mathbf{D}$, the complex plane $\mathbb{C}$, or the complex sphere $\mathbb{C} \cup \{\infty\}$.

Isothermal coordinates: $\mathcal{M}$ can be parameterized by points $z = x + iy$ in one of these spaces in such a way that the metric takes the form $e^{\lambda(z)}(dx^2 + dy^2)$ for some real-valued function λ. The (x, y) are called *isothermal coordinates* or *isothermal parameters* for $\mathcal{M}$.

Write D for the parameter space and suppose D is a simply connected bounded subdomain of $\mathbb{C}$ (which is conformally equivalent to $\mathbf{D}$ by the Riemann mapping theorem).

Isothermal coordinates

LENGTH of path in $\mathcal{M}$ parameterized by a smooth path P in D is $\int_P e^{\lambda(s)/2} ds$, where ds is the Euclidean length measure on D.

AREA of subset of $\mathcal{M}$ parameterized by a measurable subset A of D is $\int_A e^{\lambda(z)} dz$, where dz is Lebesgue measure on D.

GAUSSIAN CURVATURE DENSITY in D is $-\Delta\lambda$, i.e., if A is a measurable subset of the D, then the integral of the Gaussian curvature with respect to the portion of $\mathcal{M}$ parameterized by A is $\int_A -\Delta\lambda(z) dz$.

David Xianfeng Gu's conformal map images

Observation

- By Riemann uniformization, SLE can be defined on *any* simply connected Riemannian surface with boundary, not just a planar domain.

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- By Riemann uniformization, SLE can be defined on *any* simply connected Riemannian surface with boundary, not just a planar domain.
- Riemann uniformation lets us reduce problem of defining random manifold to problem of defining a random function from a planar domain to the reals. What is the most natural random (generalized) function from a planar domain to $\mathbb{R}$?

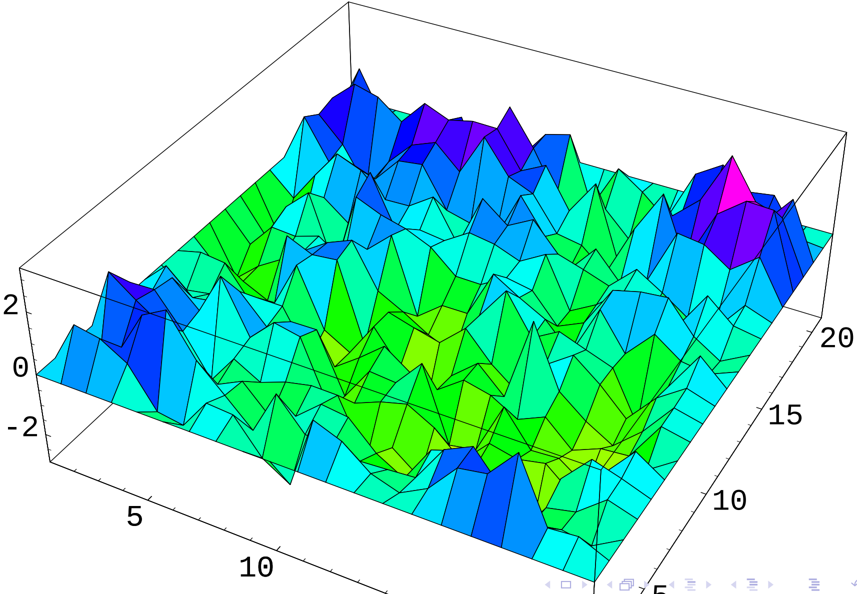
The discrete Gaussian free field

Let f and g be real functions defined on the vertices of a planar graph Λ. The **Dirichlet inner product** of f and g is given by

$$(f, g)_\nabla = \sum_{x \sim y} (f(x) - f(y))\,(g(x) - g(y))\,.$$

The value $H(f) = (f, f)_\nabla$ is called the **Dirichlet energy of f.** Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



The continuum Gaussian free field

is a "standard Gaussian" on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx dy.$$

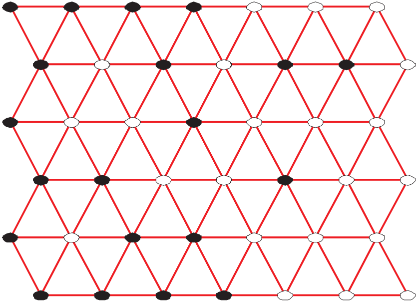
The GFF is the formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—inner products (h, ϕ) are well defined whenever ϕ is sufficiently smooth.

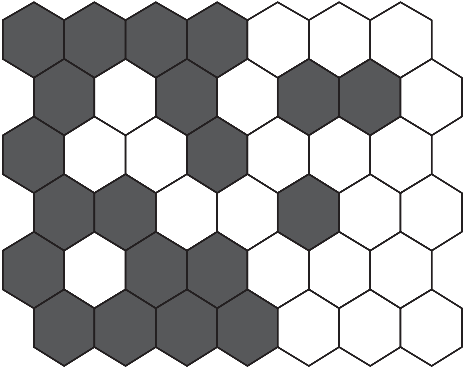
Some DGFF properties:

Zero boundary conditions: The Dirichlet form $(f, f)_\nabla$ is an inner product on the space of functions with zero boundary, and the DGFF is a standard Gaussian on this space.

Other boundary conditions: DGFF with boundary conditions f_0 is the same as DGFF with zero boundary conditions *plus* a deterministic function, which is the (discrete) harmonic interpolation of f_0 to Λ.

Markov property: Given the values of f on the boundary of a subgraph Λ' of Λ, the values of f on the remainder of Λ' have the law of a DGFF on Λ', with boundary condition given by the observed values of f on $\partial\Lambda'$.



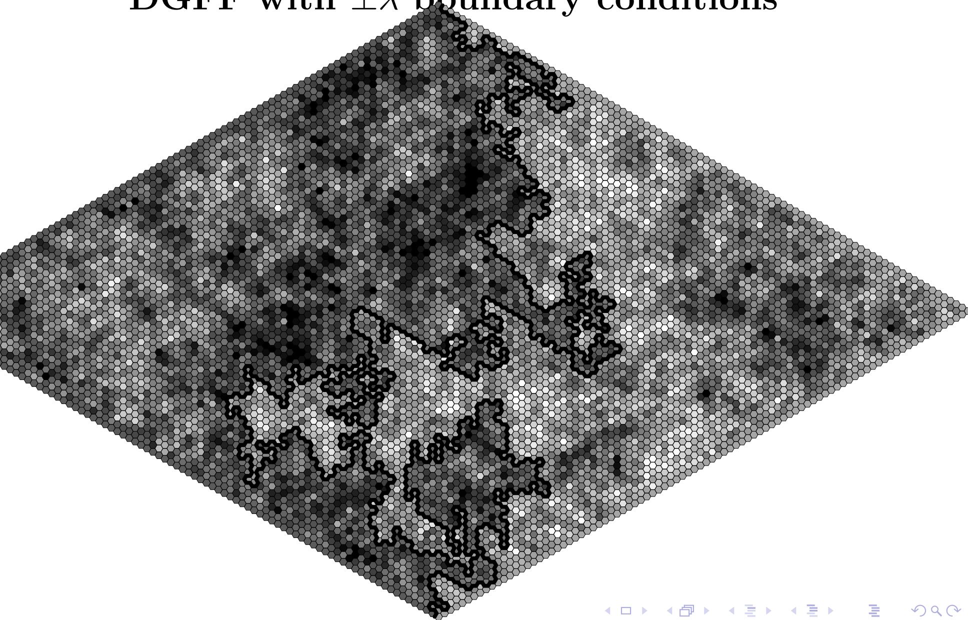


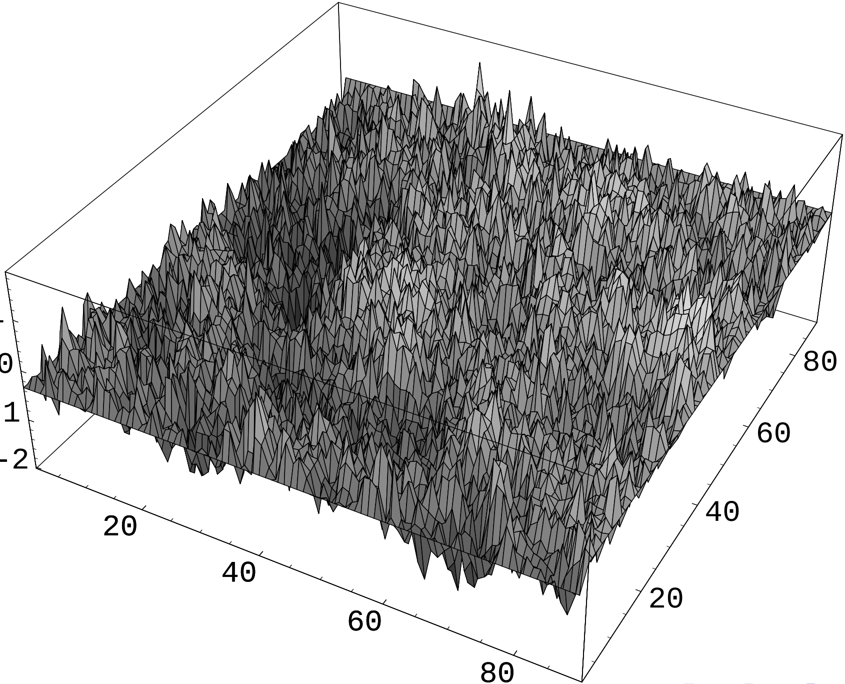
Scaling limit of zero-height contour line

Theorem (Schramm, S): If initial boundary heights are λ on one boundary arc and $-\lambda$ on the complementary arc, where λ is the constant $\sqrt{\frac{\pi}{8}}$, then the scaling limit of the zero-height interface (as the mesh size tends to zero) is SLE_4.

If the initial boundary heights are instead $-(1+a)\lambda$ and $(1+b)\lambda$, then as the mesh gets finer, the laws of the random paths described above converge to the law of $\mathrm{SLE}_{4,a,b}$.

DGFF with $\pm\lambda$ boundary conditions





0
1
-2
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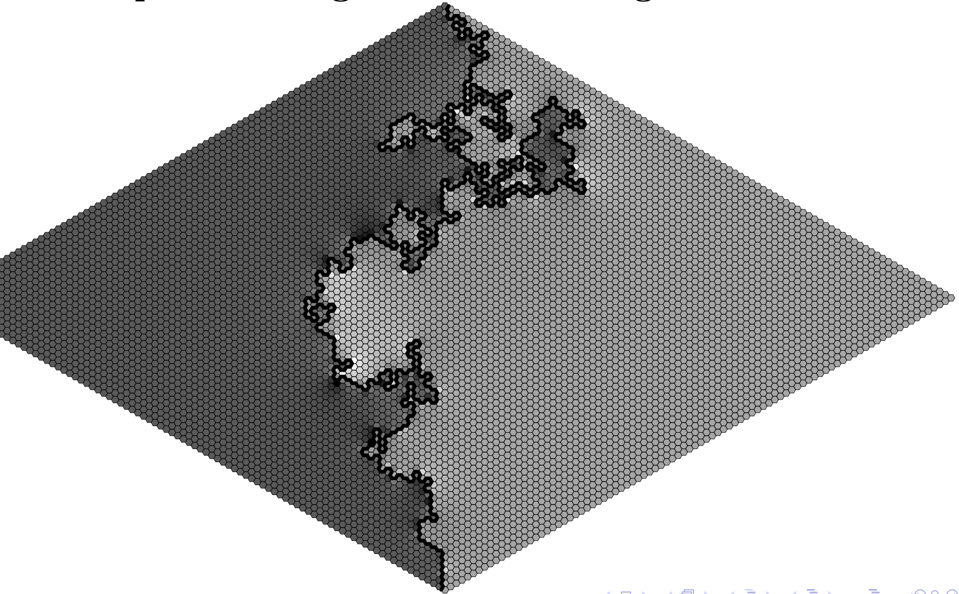
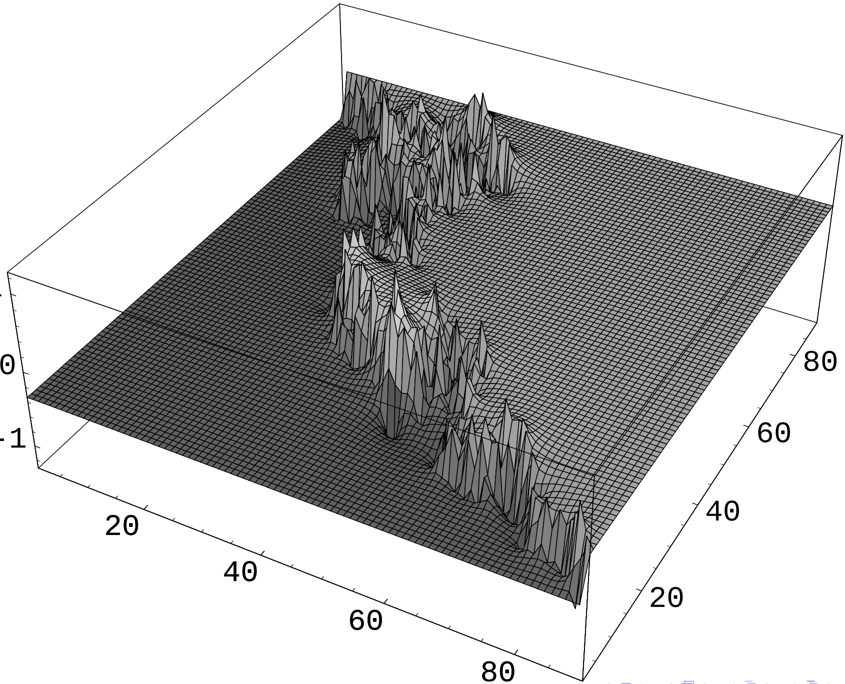
20
40
60
80

Expectations given values along interface





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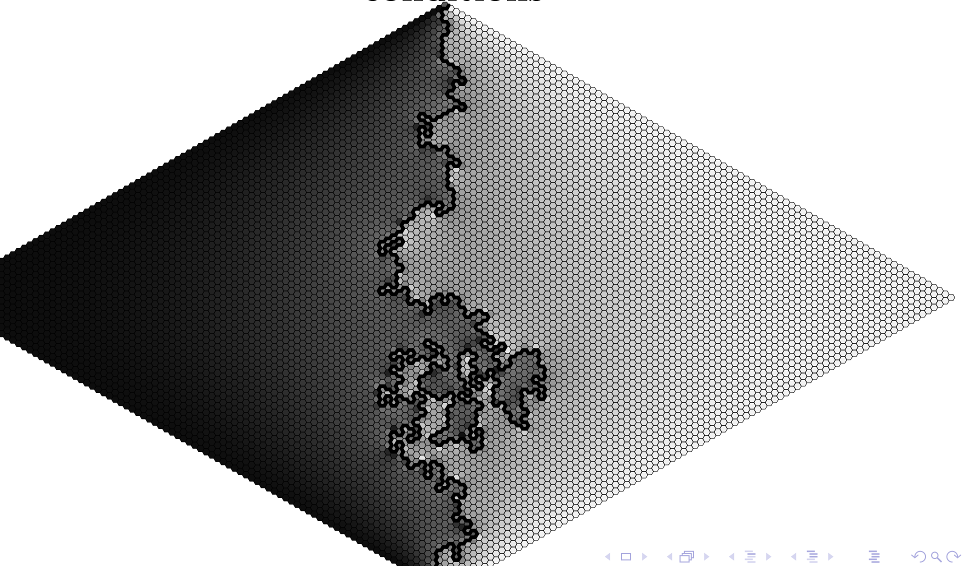
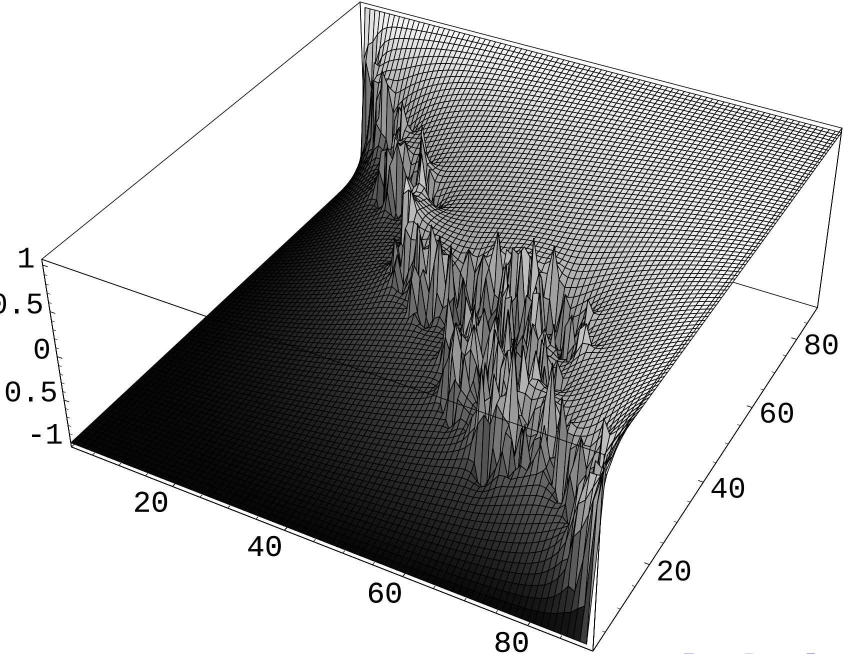
Expectations given interface, $\pm 3\lambda$ boundary conditions





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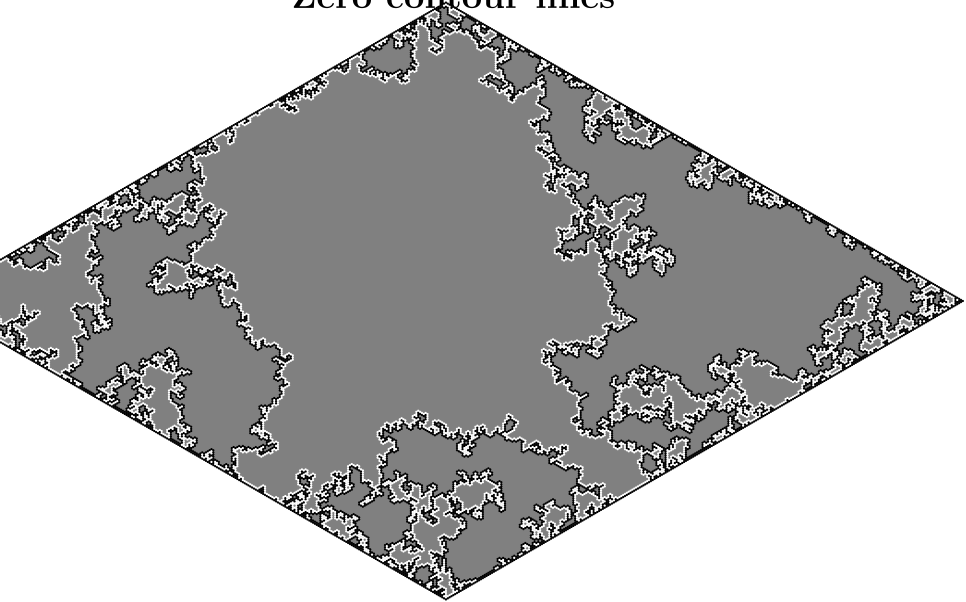
Zero contour lines



GFF References

- **The harmonic explorer and its convergence to SLE(4)**, Ann. Prob. [Schramm, S]
- **Local sets of the Gaussian free field, Parts I,II, and III**, Online lecture series: www.fields.utoronto.ca/audio/05-06 [S]
- **Contour lines of the two-dimensional discrete Gaussian free field**, Acta Math [Schramm, S]
- **A contour line of the continuum Gaussian free field**, PTRF [Schramm, S]

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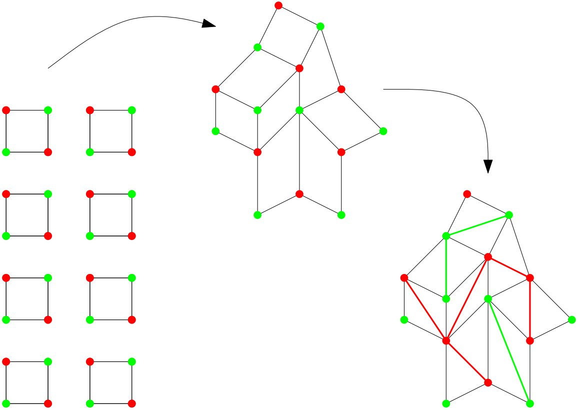
- **Discrete approach**: Glue together unit squares or unit triangles in a random fashion. (Random quadrangulations, random triangulations, random planar maps, random matrix models.)
- **Continuum approach**: As described above, use conformal maps to reduce to a problem of constructing a random real-valued function on a planar domain or a sphere. Using the Gaussian free field for the random function yields (critical) Liouville quantum gravity.

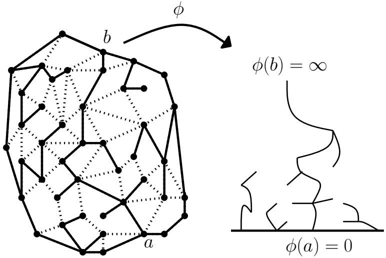
"There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitude are given by the sums over all possible surfaces with fixed boundary."

A.M. Polyakov, Moscow 1981

Discrete construction: gluing squares



Discrete uniformizing maps



Planar map with one-chord-wired spanning tree (solid edges), plus image under conformal map to $\mathbb{H}$ (sketch).

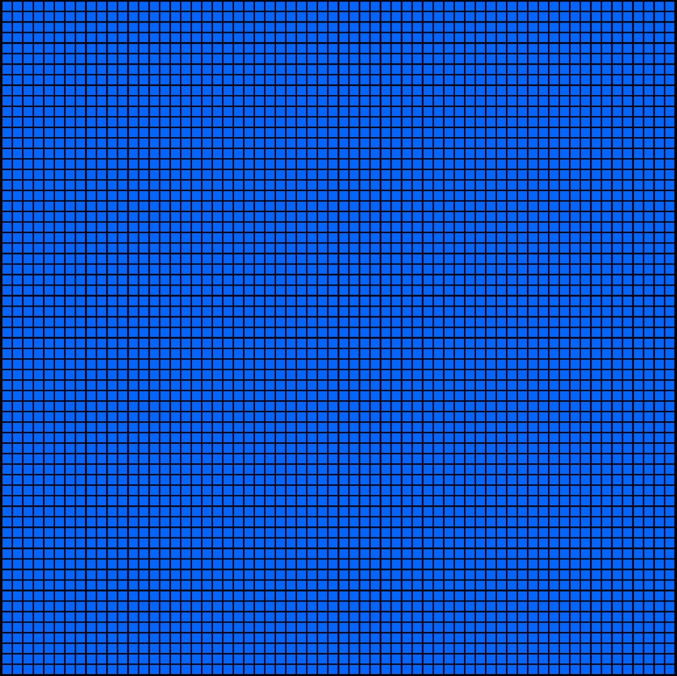
How about the continuum construction? Defining Liouville quantum gravity? Takes some thought because h is distribution not function.

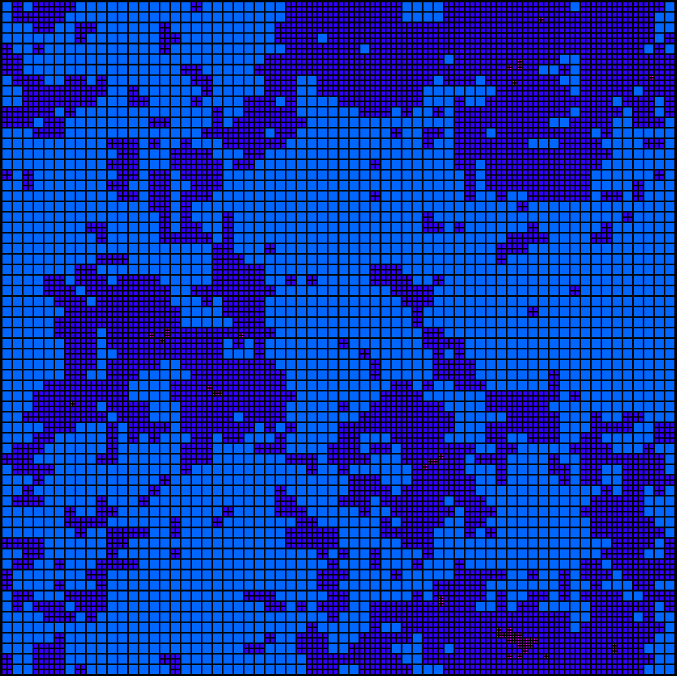
Constructing the random metric

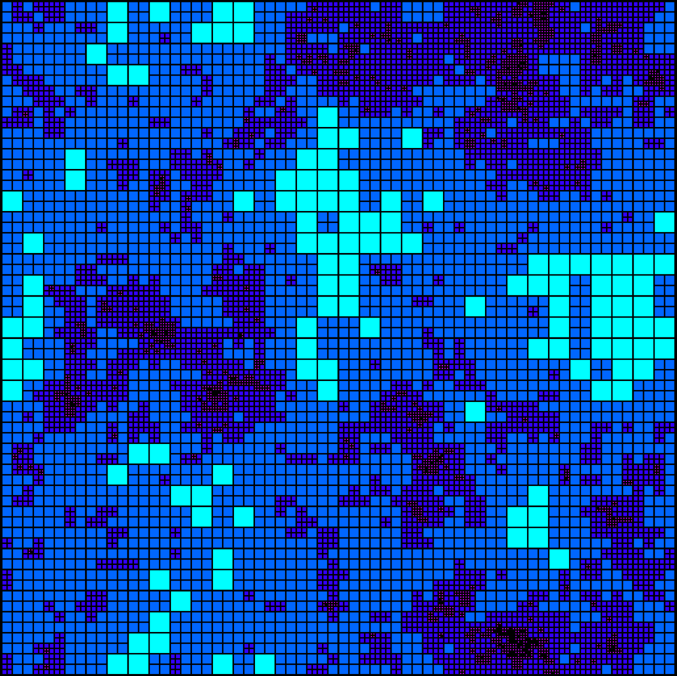
Let $h_\epsilon(z)$ denote the mean value of h on the circle of radius ϵ centered at z. This is almost surely a locally Hölder continuous function of (ϵ, z) on $(0, \infty) \times D$. For each fixed ϵ, consider the surface $\mathcal{M}_\epsilon$ parameterized by D with metric $e^{\gamma h_\epsilon(z)}(dx^2 + dy^2)$.

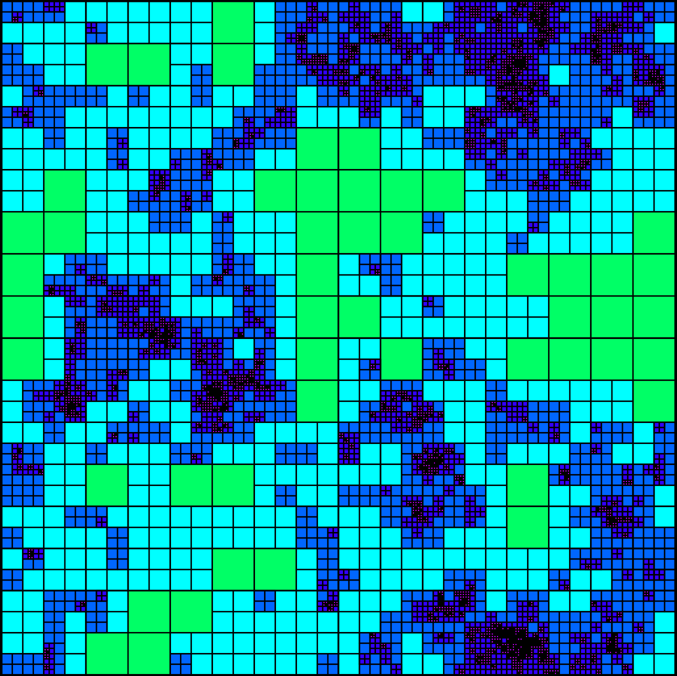
We define $\mathcal{M} = \lim_{\epsilon \to 0} \mathcal{M}_\epsilon$, but what does that mean?

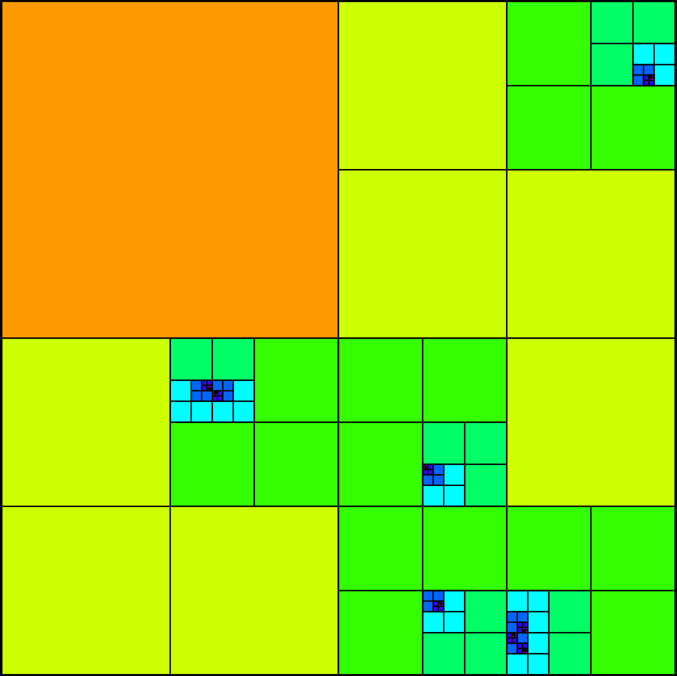
PROPOSITION: Fix $\gamma \in [0, 2)$ and define h, D, and μ_ϵ as above. Then it is almost surely the case that as $\epsilon \to 0$ along powers of two, the measures $\mu_\epsilon := \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dz$ converge weakly to a non-trivial limiting measure, which we denote by $\mu = \mu_h = e^{\gamma h(z)} dz$.











Knizhnik-Polyakov-Zamolodchikov (KPZ) Formula

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- Quantum exponent heuristically describes corresponding discrete models.
- Derived by KPZ in 1988, first compelling evidence of relationship between discrete and continuous models.

Knizhnik-Polyakov-Zamolodchikov (KPZ) Formula

- Number of size-δ Euclidean squares hit by fractal typically scales like power of δ, related to fractal dimension.
- In this case, number of quantum δ diadic squares hit scales like different power of δ.
- Exponents are related by a deterministic (but γ-dependent) formula called the KPZ formula.
- Quantum exponent heuristically describes corresponding discrete models.
- Derived by KPZ in 1988, first compelling evidence of relationship between discrete and continuous models.
- Recently proved rigorously [Duplantier, S]. Other forms of KPZ established by [Benjamini, Schramm], [Rhodes, Vargas], [Duplantier, Rhodes, Sheffield, Vargas].

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- Similarly, the boundary length ν_h is almost surely the image under ψ of the measure $\nu_{\tilde{h}}$.

Defining *quantum surfaces*

- **DEFINITION:** A **quantum surface** is an equivalence class of pairs (D, h) under the equivalence transformations $(D, h) \to (\psi^{-1} D, h \circ \psi + Q \log |\psi'|) = (\tilde{D}, \tilde{h})$.

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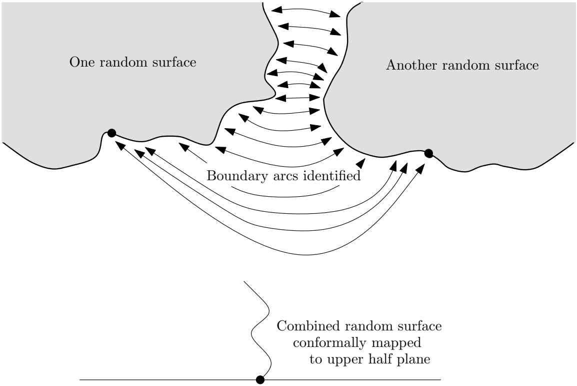
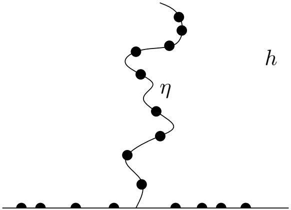
- Area, boundary length, and conformal structure are well defined for such surfaces.

Glue two random surfaces: interface is random path

Theorem [S.]: If you glue two appropriate independent random quantum surfaces along their boundaries (in a length preserving way) and conformally map the new surface you get back to the half plane, then the image of the interfaces becomes an SLE.



Stationarity and matching quantum lengths



- Sketch of interface path η with marks spaced at intervals of equal ν_h length.

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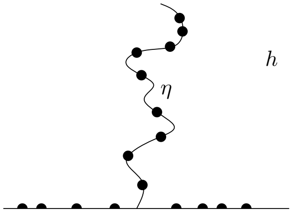

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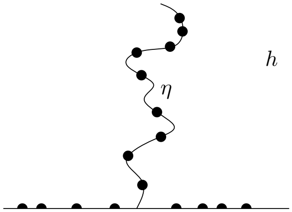

- Sketch of interface path η with marks spaced at intervals of equal ν_h length.
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- In this pair, h and η are (surprisingly) actually independent of each other.

Liouville quantum gravity References

- **Liouville quantum gravity and KPZ**, arXiv [Duplantier, S]
- **Duality and KPZ in Liouville quantum gravity**, PRL [Duplantier, S]
- **Conformal weldings of random surfaces: SLE and the quantum gravity zipper**, arXiv [S]
- **Schramm-Loewner evolution and Liouville quantum gravity**, PRL [Duplantier, S]

What is an imaginary surface?

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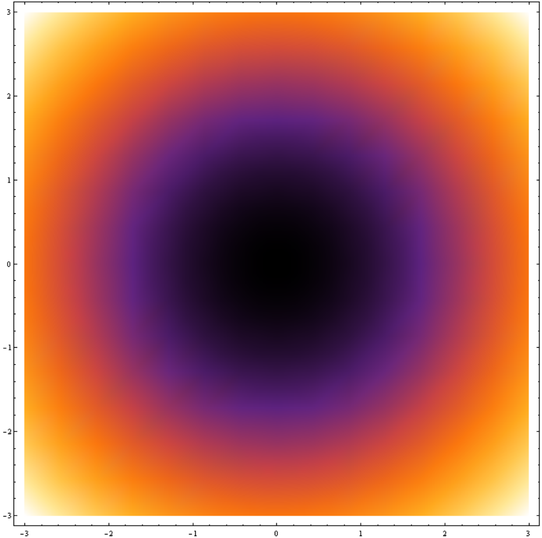

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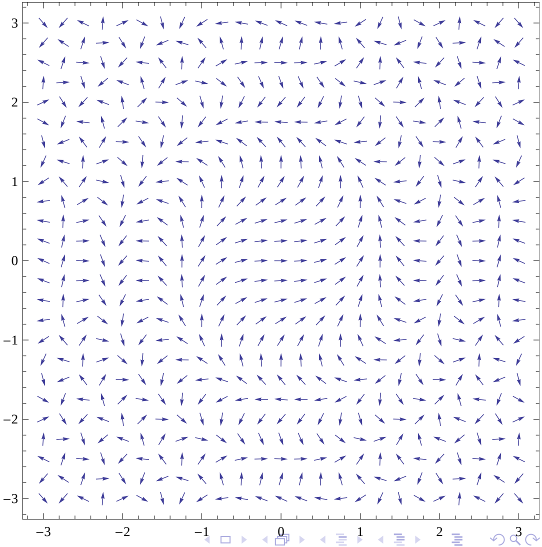

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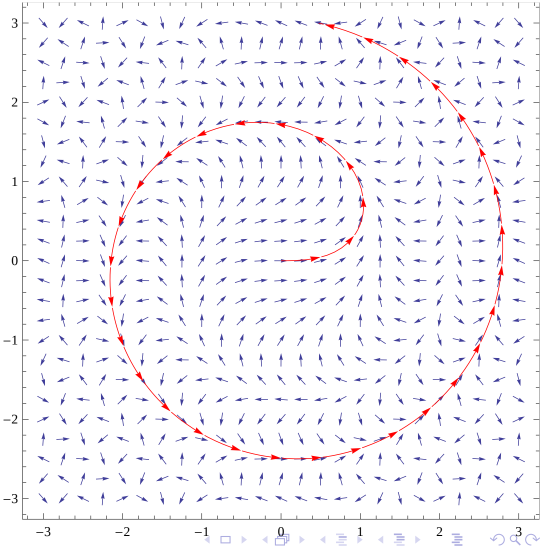

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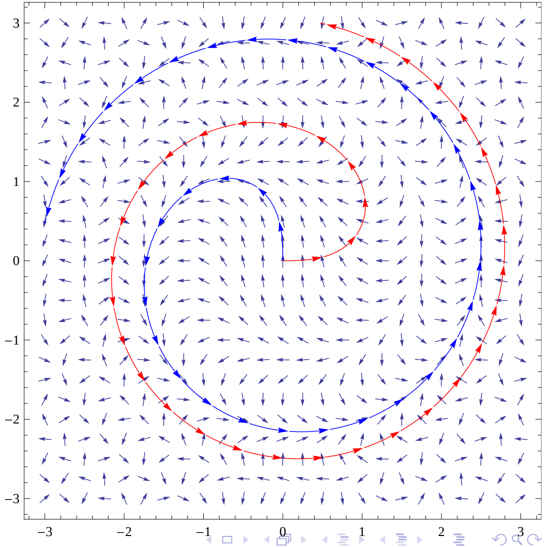

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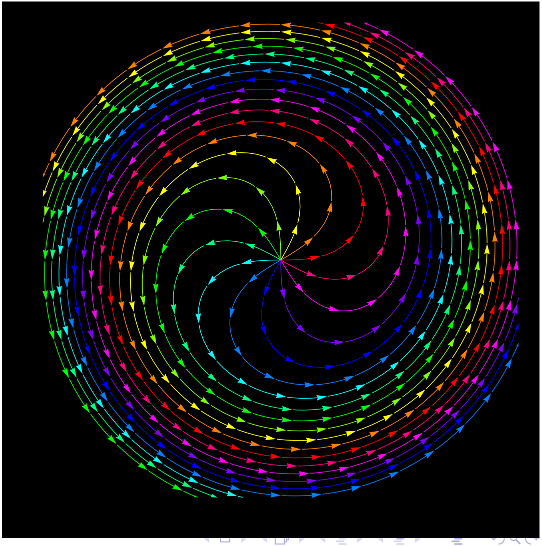

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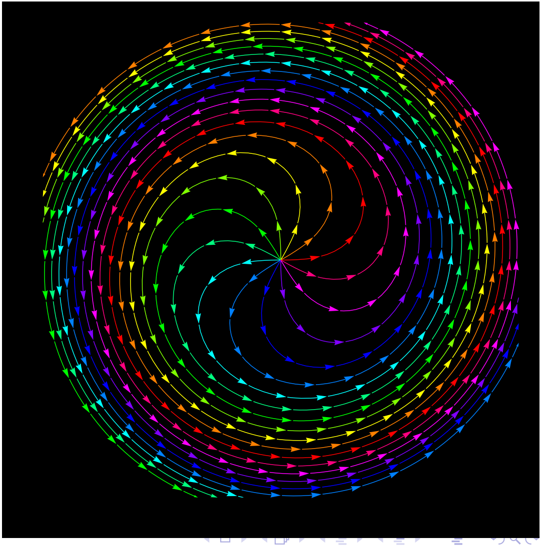
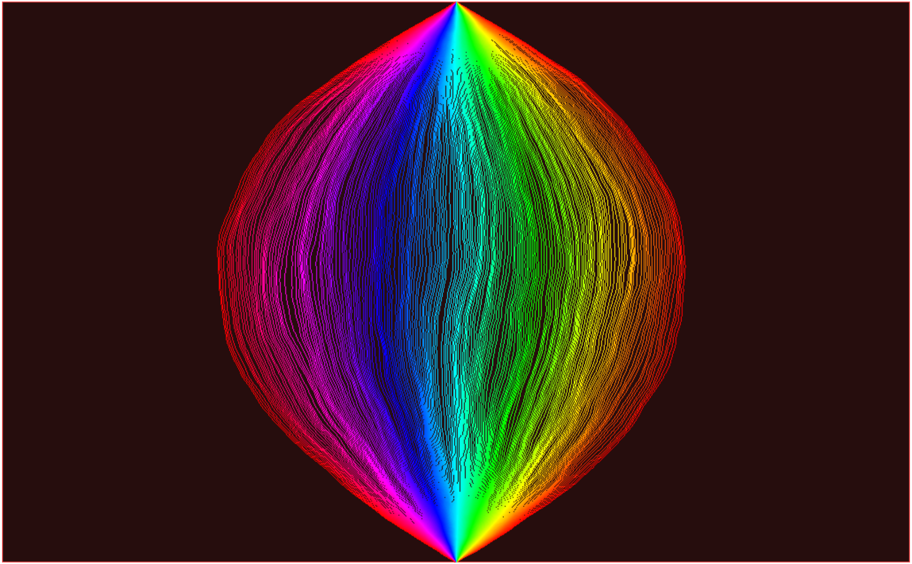
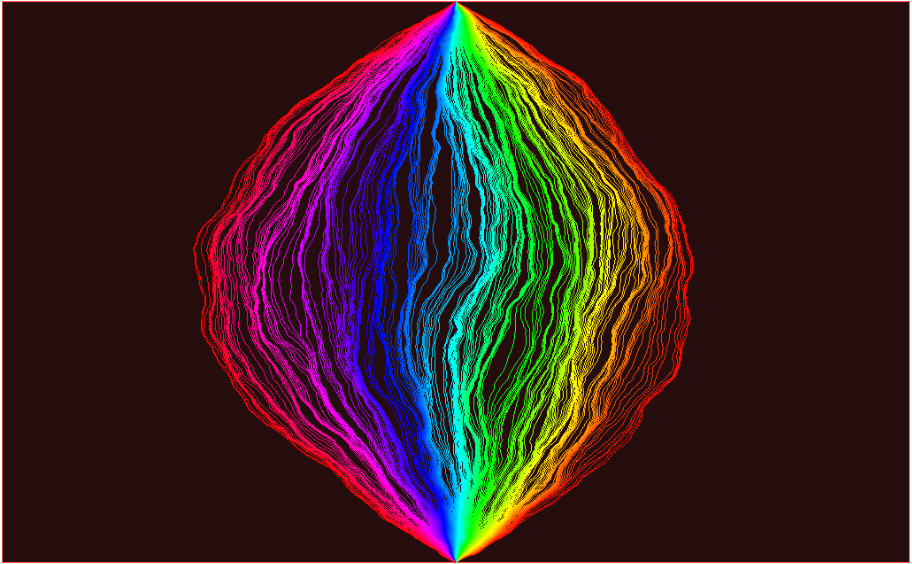


- **What is a random imaginary surface?**
  - *Take h to be a multiple of GFF.*
  - *Flow lines ("straight lines" on imaginary surface) turn out to be forms of SLE.*

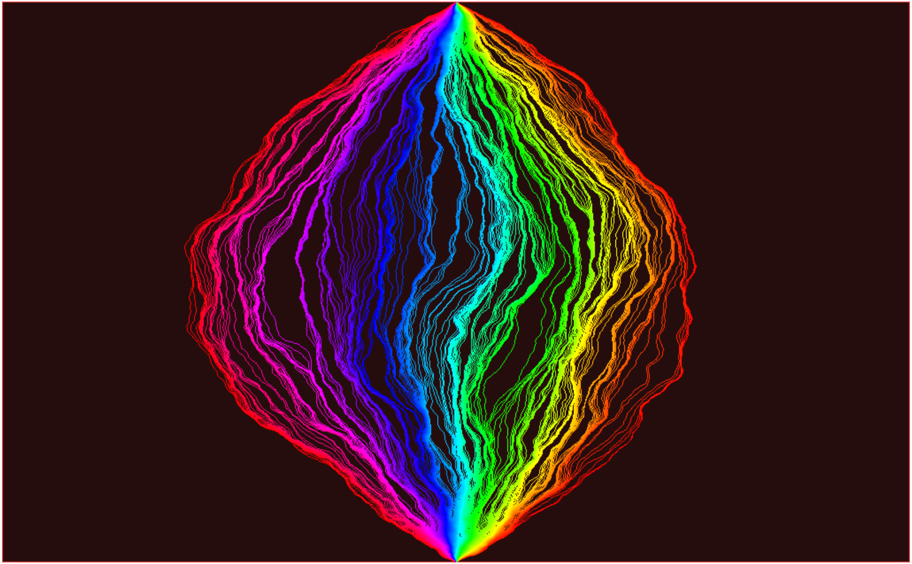
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 31.97$ $[\kappa = 1/256]$



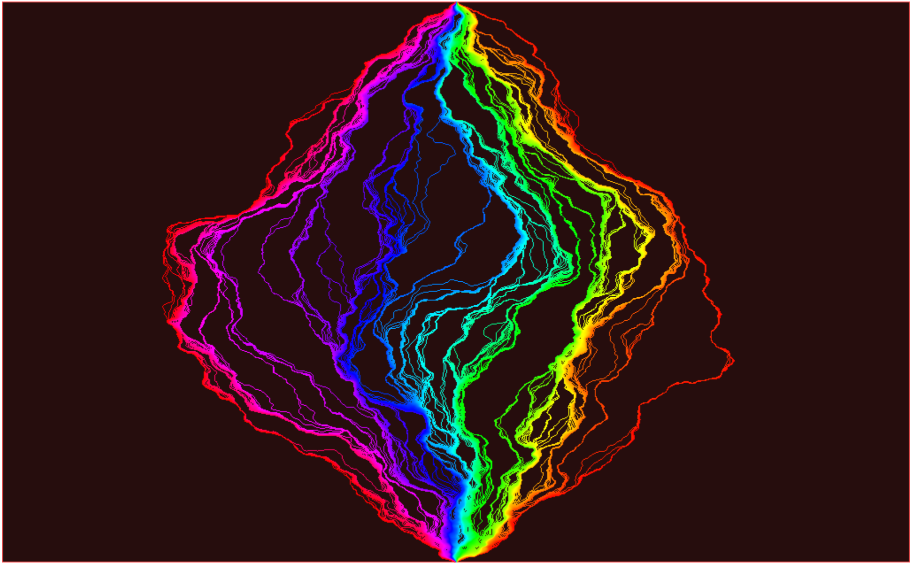
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 11.23$ [$\kappa = 1/32$]



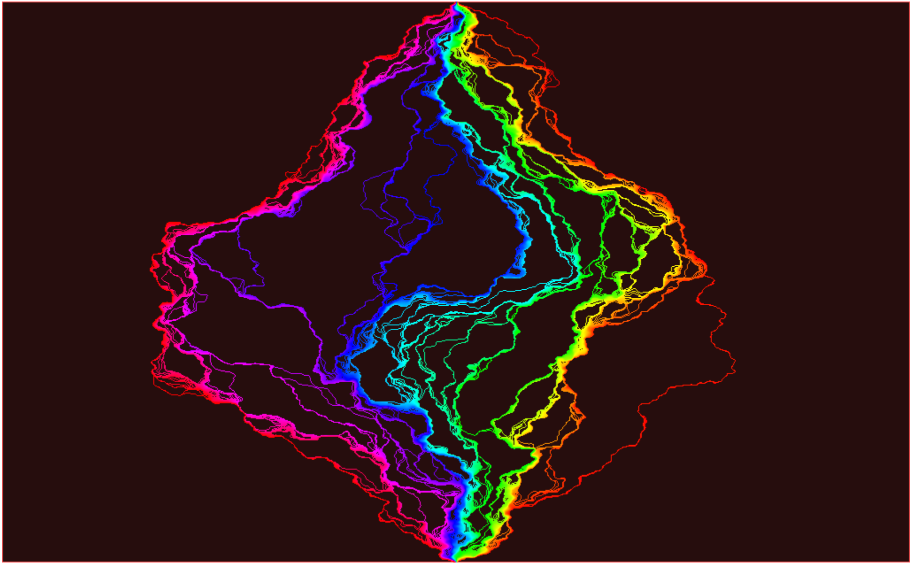
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 7.88$ $[\kappa = 1/16]$



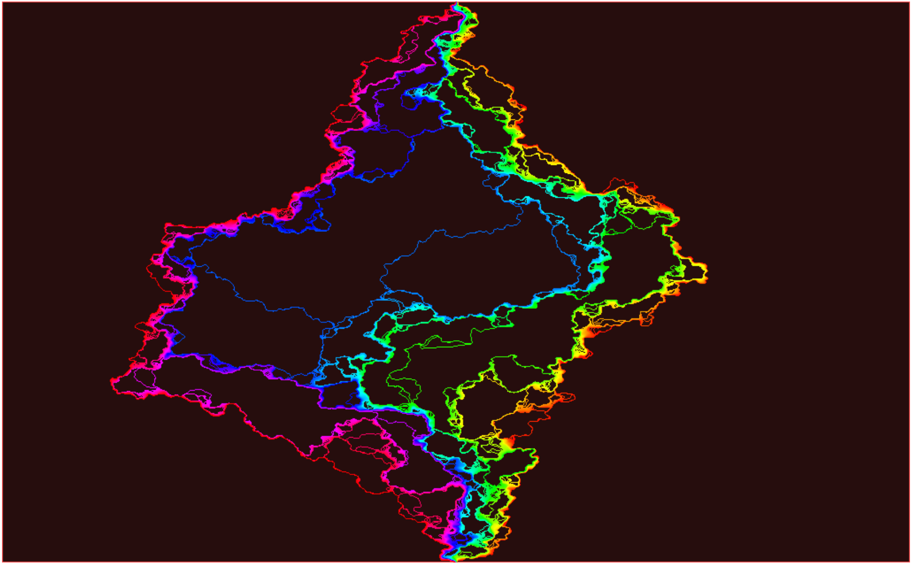
Rays of $e^{ih/\chi}$, h GFF, $\chi = 3.75$ [$\kappa = 1/4$]



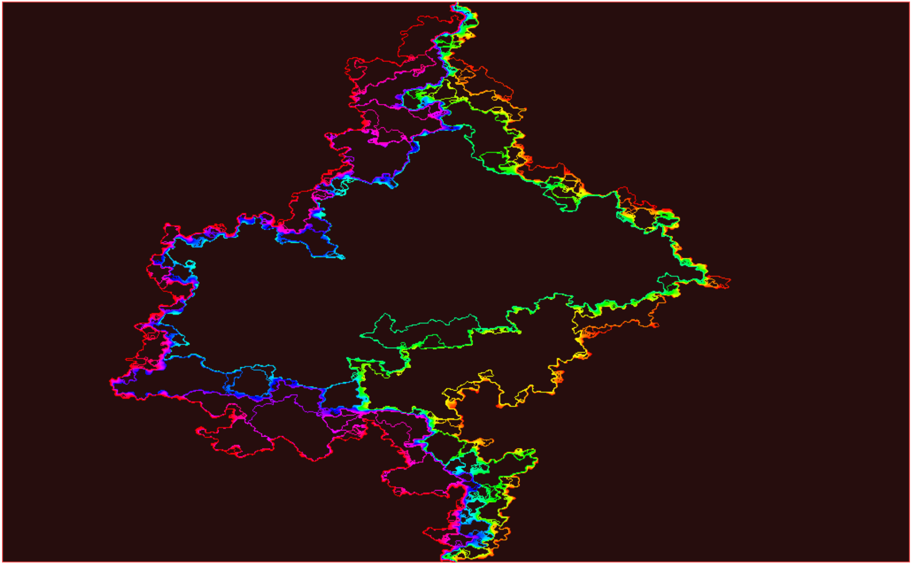
Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 2.47$ $[\kappa = 1/2]$



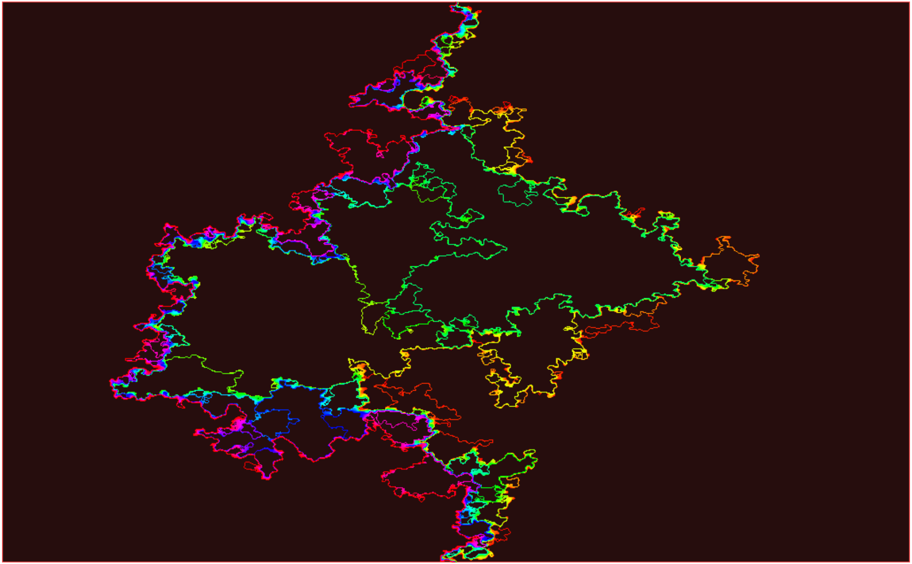
Rays of $e^{ih/\chi}$, h GFF, $\chi = 1.5$ $[\kappa = 1]$



Rays of $e^{ih/\chi}$, h GFF, $\chi \approx 1.02$ $[\kappa = 3/2]$



Rays of $e^{ih/\chi}$, h GFF, $\chi = 0.71$ $[\kappa = 2]$



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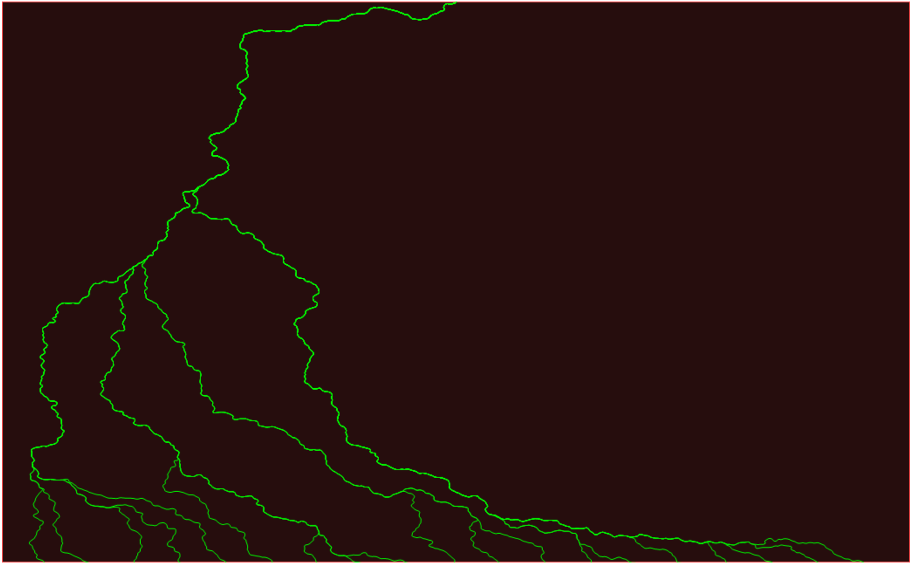

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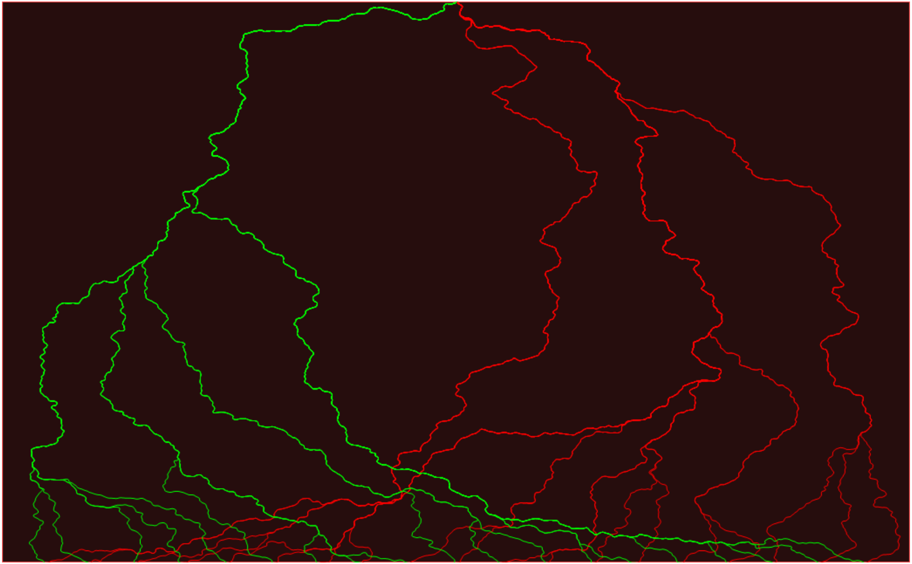

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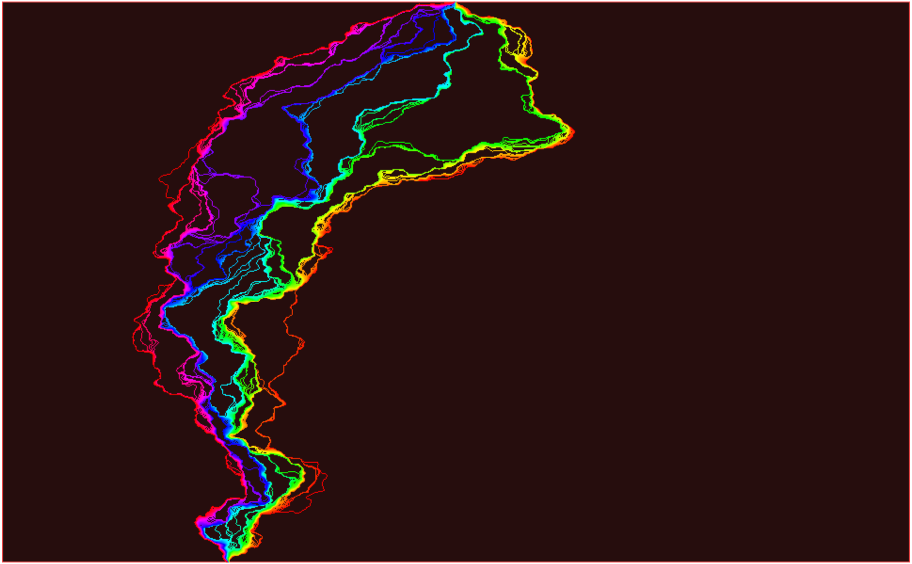

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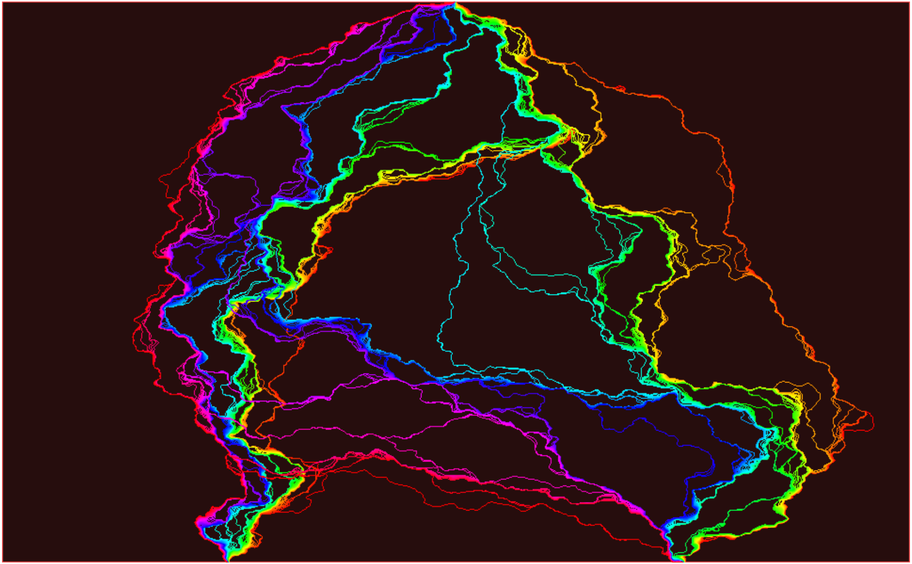
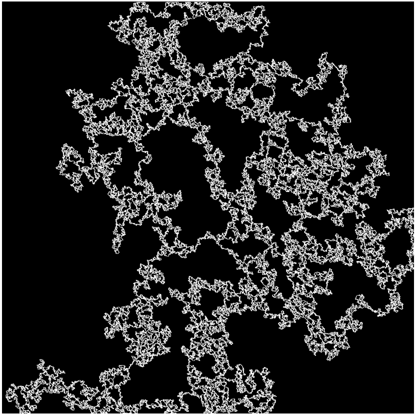


Rays of random imaginary surface: results

Existence and uniqueness of couplings (η, h) of a GFF h and $\eta \sim \mathrm{SLE}_\kappa$ are studied in the works of S., Schramm-S., Dubédat, and Izyurov-Kytöla. Dubédat shows that field determines path. Recent four-paper series by Miller, S. develops theory of surface rays in great detail, finds many applications to theory of SLE.

SLE Duality



The outer boundary of an $\mathrm{SLE}_{16/\kappa}$ process is described by a certain SLE_κ process for $\kappa \in (0,4)$.

- Predicted by Duplantier

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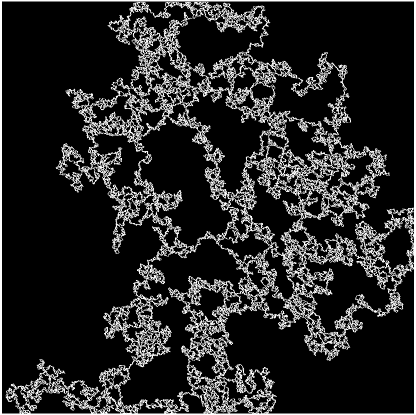

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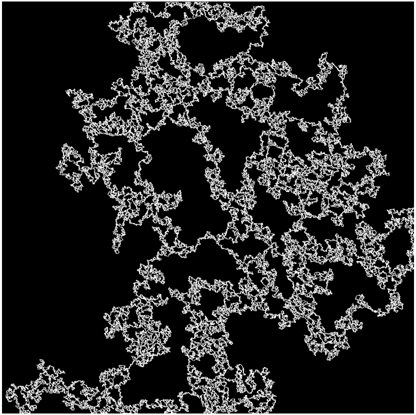

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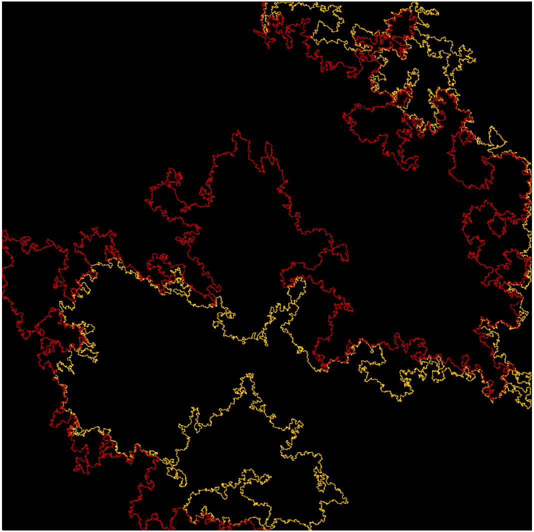
- Proved in various forms by Zhan and Dubédat

Duality in the Imaginary Geometry: the SLE Light Cone



Flow lines with fixed angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

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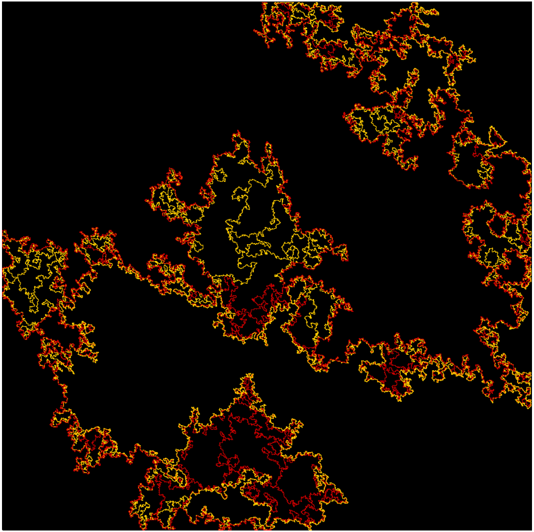

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; one direction change.

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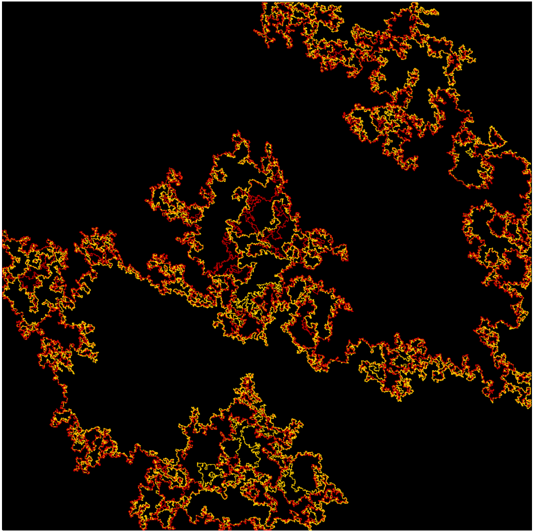

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; two direction changes.

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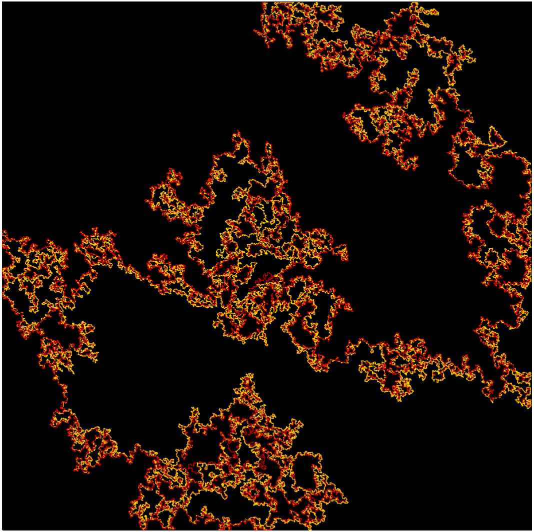

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; three direction changes.

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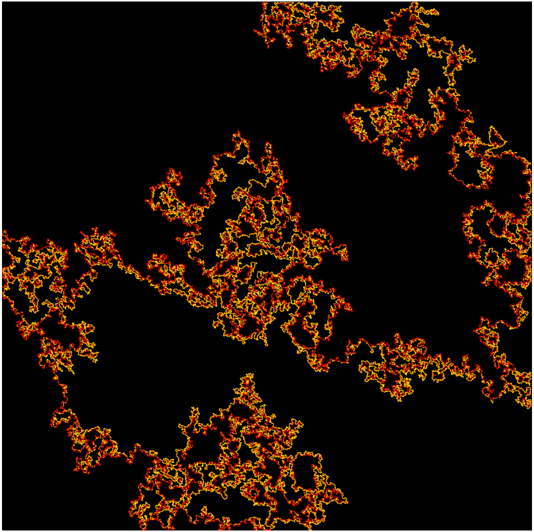

Flow lines with angle $\frac{\pi}{2}$ and $-\frac{\pi}{2}$; four direction changes.

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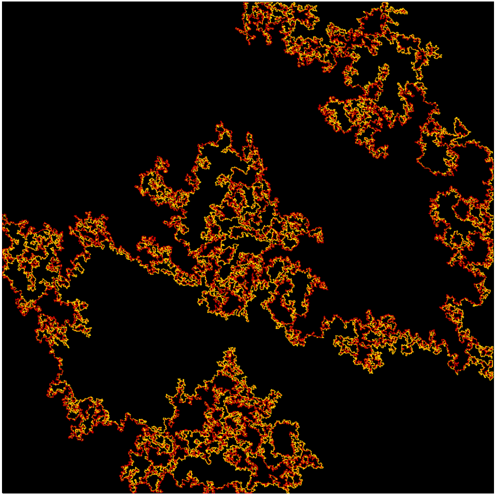
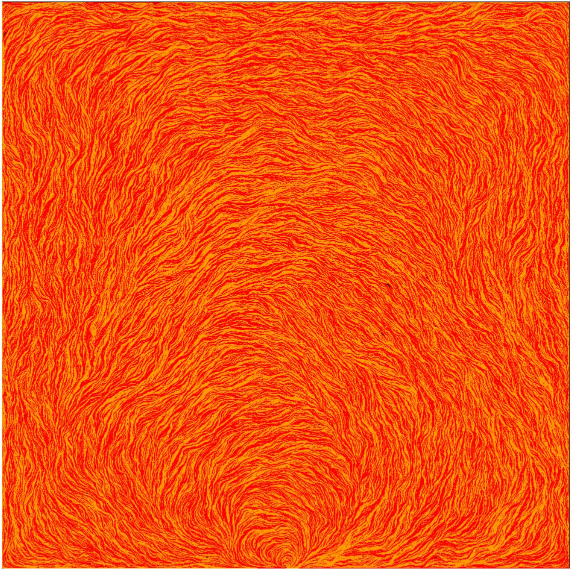
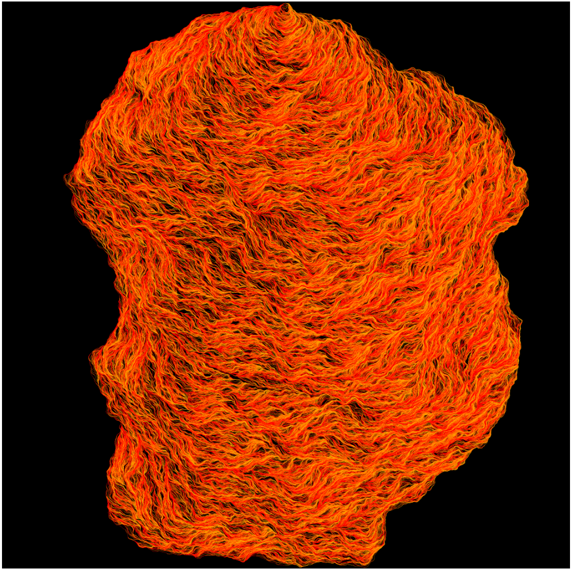


Theorem (Miller, S.): The set of all point accessible by SLE_κ flow lines ($\kappa \in (0,4)$) with angles restricted in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is an $\mathrm{SLE}_{16/\kappa}$ process.

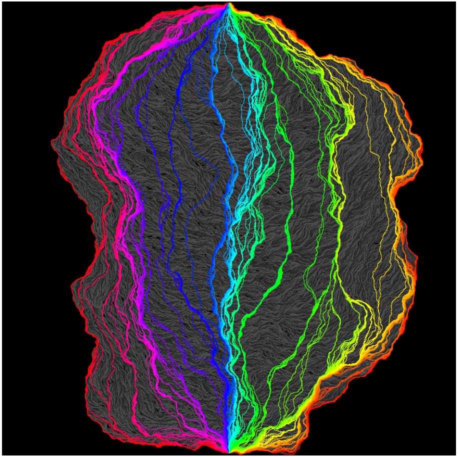
SLE_{128} Light Cone



$\mathrm{SLE}_{64}(32;32)$ Light Cone



The SLE_κ fan



Theorem (Miller, S.) The fan is a strict subset of the light cone: the probability that the fan contains $\eta'(\tau')$ for any η' stopping time τ' is zero.

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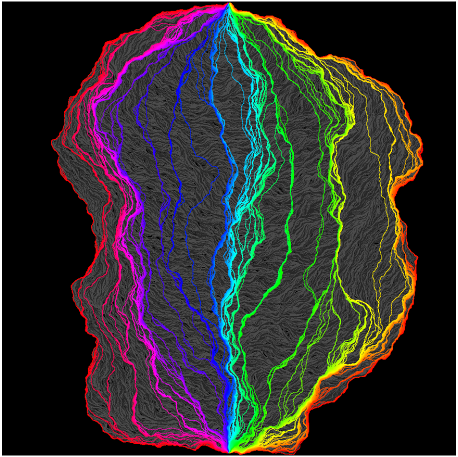

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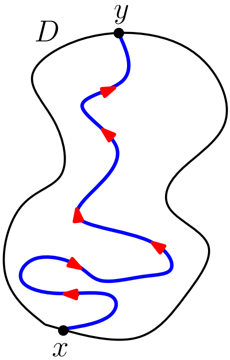
Theorem (Miller, S.) The fan is a deterministic function of η'

Reversibility

- An SLE_κ η from x to y is said to be **reversible** if the time-reversal of η (parameterized in the reverse direction) has the law of an SLE_κ from y to x.



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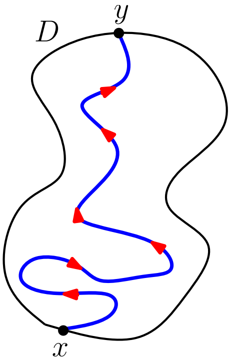

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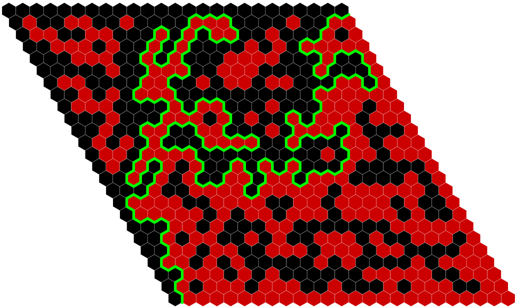
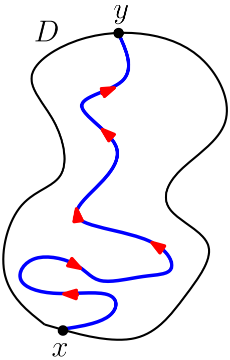
- Not obvious from the definition of SLE.
- Holds for $\kappa = 2, 3, 4, 16/3, 6, 8$ since for these values it is the scaling limit of discrete models with reversibility built in.





Many of the random interfaces which are known or believed to converge to SLE are reversible, in the sense that the reversed path has the same law as the original path (with respect to a slightly modified setup). This motivates the following problem from [76].

Problem 7.3. *Let γ be the chordal* SLE_κ *path, where $\kappa \leq 8$. Prove that up to reparametrization, the image of γ under inversion in the unit circle (that is, the map $z \mapsto 1/\bar{z}$) has the same law as γ itself.*

Oded Schramm, 2006 ICM proceedings

Reversibility for $\kappa \in (0, 4]$

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Non-boundary intersecting $\mathrm{SLE}_\kappa(\rho)$ *is reversible for* $\kappa \in (0,4]$.

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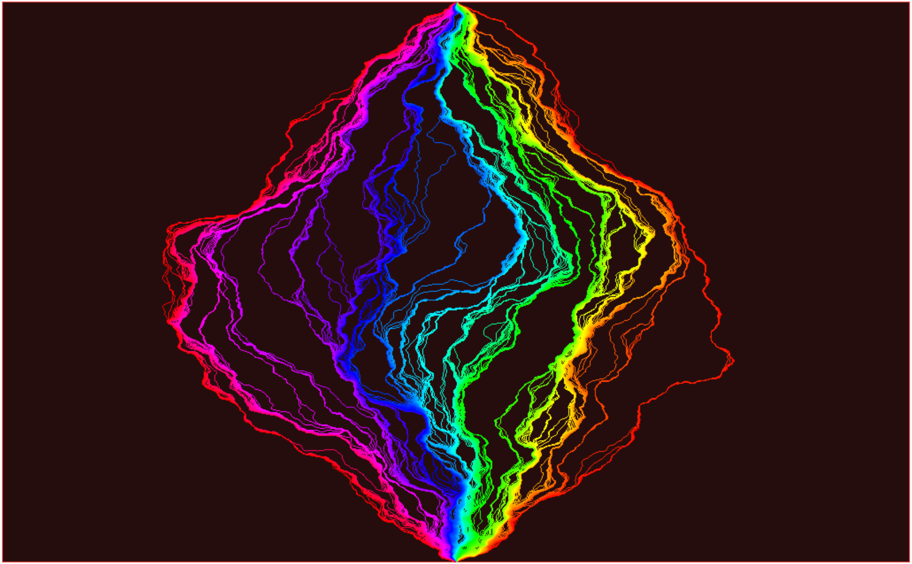
Theorem (Miller, S.)

$\mathrm{SLE}_\kappa(\rho_1; \rho_2)$ *processes are reversible for* $\kappa \in (0, 4]$*, even when they intersect the boundary.*

- Based on imaginary geometry techniques
- Independent proof for SLE_κ, $\kappa \in (0, 4]$
- Description of the time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes

Corollary: the fan is "reversible"



Reversibility of SLE_κ for $\kappa \in (4,8)$

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More generally, $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ *processes are reversible for* $\rho_1,\rho_2 \geq \frac{\kappa}{2} - 4$ *and are non-reversible if* $\min(\rho_1,\rho_2) < \frac{\kappa}{2} - 4$.

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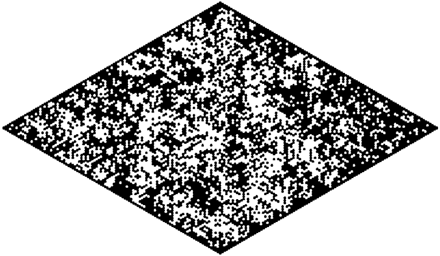
More generally, $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ *processes are reversible for* $\rho_1,\rho_2 \geq \frac{\kappa}{2}-4$ *and are non-reversible if* $\min(\rho_1,\rho_2) < \frac{\kappa}{2}-4$.

- $\frac{\kappa}{2}-4$ is the critical threshold for these processes to be boundary filling

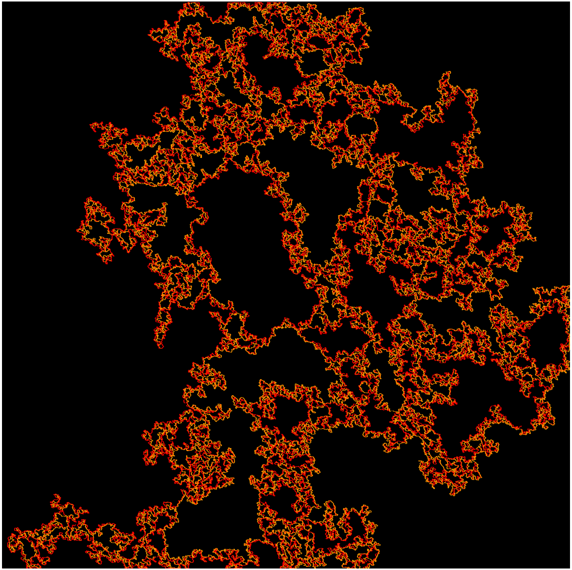
($\mathrm{CLE}_{4.5}$ simulation)



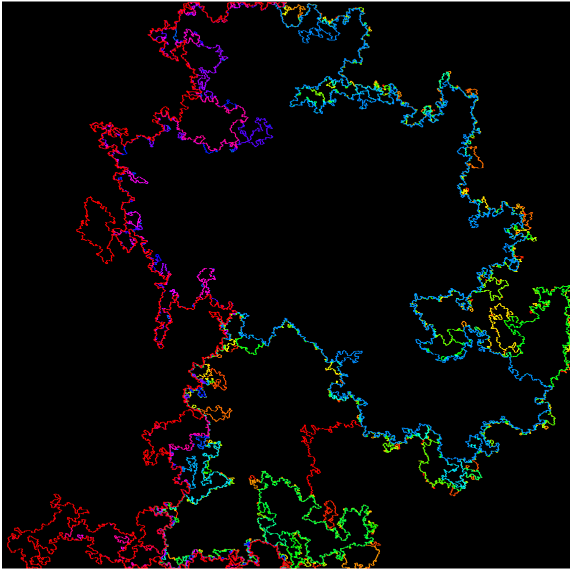
Important consequence:

- The CLE_κ processes (loop version of SLE_κ) are well defined for $\kappa \in (4,8)$.
- (Recently proved by S. and Werner for $\kappa \in (8/3,4]$ using loop soups).

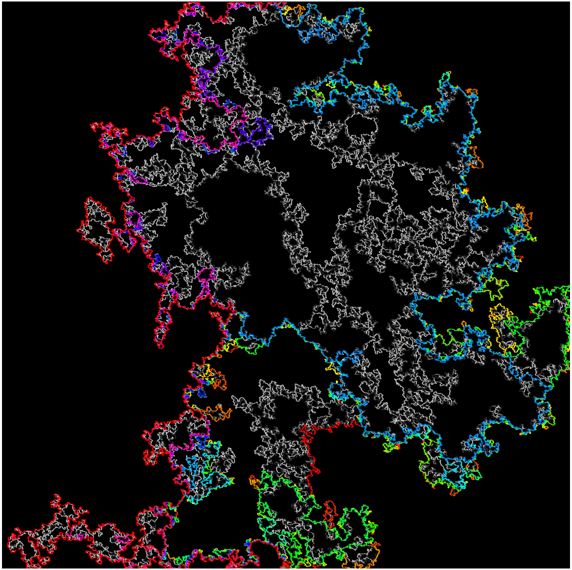
SLE_6 is reversible

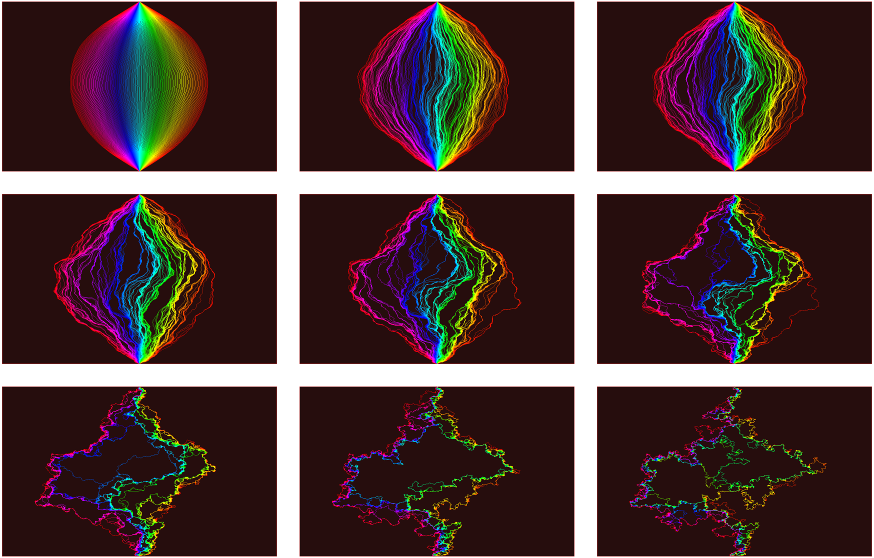


The $\mathrm{SLE}_{8/3}$ fan is reversible



The SLE_6 and the $\mathrm{SLE}_{8/3}$ fan are not jointly reversible





Imaginary Geometry References

- **Imaginary geometry I: Interacting SLEs**, arXiv.org [Miller, S]
- **Imaginary geometry II: reversibility of SLE**$_\kappa(\rho_1;\rho)2$ **for** $\kappa \in (0,4)$, arXiv.org [Miller, S]
- **Imaginary geometry III: reversibility of SLE**$_\kappa$ **for** $\kappa \in (4,8)$, arXiv.org [Miller, S]
- **Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees**, arXiv.org [Miller, S]