On a canonical class of Green currents associated with the unit sections of abelian schemes

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Preamble: the exponential function and the Siegel functions

A **unit** is an algebraic integer whose inverse is also an algebraic integer.

A **cyclotomic unit** is an algebraic integer of the form

$$\boxed{1 - \exp(2i\pi \frac{k}{m})}$$

where $(k, n) = 1$ and m is a composite number.

This is the prime example of a unit, which is not a root of unity. Such units are built from the torsion points of the torus $\mathbb{G}_m$.

Elliptic units

Elliptic units are built from the torsion points of an elliptic curve with potential good reduction over an algebraic number field.

Let $E = \mathbb{C}/[1, \tau]$ be such a curve.

Let z be a point of order m of E, where m is composite.

The complex number

$$\boxed{e^{-z\cdot\mathrm{quasiperiod}(z)/2}\sigma(z)\Delta(\tau)^{\frac{1}{12}}} \quad (*)$$

is a unit, called the **elliptic unit** attached to z.

The function $(*)$ on $E(\mathbb{C})$ is usually called a **Siegel function.**

The Néron-Tate height

Why do Siegel functions give rise to units ?

To explain why they do so, the best way is to relate them to height functions.

Recall that the **Néron-Tate height** $\mathrm{NT}(\cdot)$ is the only height function associated with the origin of an elliptic curve, such that

$$m^2 \cdot \mathrm{NT}(P) = \mathrm{NT}(m \cdot P)$$

for all $m \geqslant 2$.

Formula for the Néron-Tate height

Let $E = \mathbb{C}/[1,\tau]$ be an elliptic curve with a model $\mathcal{E}$

$$y^2 = x^3 + Ax + B$$

over $\mathcal{O}_K$ (K a number field).
Let $P := (x, y) \in K^2$ be a point on E, which reduces into the smooth locus of $\mathcal{E}$.

Theorem (Tate)

$$\begin{aligned}\mathrm{NT}(P) &= \frac{1}{[K:\mathbb{Q}]}\Big[\ \log|N_{K/\mathbb{Q}}(16(4A^3+27B^2))| \\ &- \frac{1}{2}\log|N_{K/\mathbb{Q}}(\mathrm{Denominator}(x))| \\ &- \sum_{v\ \mathrm{ar.}} n_v \cdot \log|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v))/2}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|\ \Big]\end{aligned}$$

Mazur and Tate's refinement of the Néron-Tate height I

One cannot prove that Siegel functions give rise to units using the Néron-Tate height alone, because the latter involves an averaging.

Mazur and Tate constructed a refinement $\mathrm{MT}(\cdot)$ of the Néron-Tate height $\mathrm{NT}(\cdot)$.

The refined height $\mathrm{MT}(\cdot)$ has values in the group

$$\widehat{\mathrm{Cl}}(\mathcal{O}_K) := \Big(\bigoplus_{v \text{ n.}-\text{ar.}} \mathbb{Z} \bigoplus_{v \text{ ar.}} \mathbb{R} \Big) \,/\, \big\{ \oplus_{v \text{ n.}-\text{ar.}} v(k) \oplus_{v \text{ ar.}} \log|k|_v^{-2}, \ \ k \in K^* \big\}$$

which is a quotient of the idele class-group of K.

Mazur and Tate's refinement of the Néron-Tate height II

- The group $\widehat{\mathrm{Cl}}(\mathcal{O}_K)$ fits in a diagram

$$\begin{array}{ccccccccc} \mathcal{O}_K^* & \xrightarrow{reg} & \mathbb{R}^{\#(\mathrm{ar.\ v.})} & \longrightarrow & \widehat{\mathrm{Cl}}(\mathcal{O}_K) & \longrightarrow & \mathrm{Cl}(\mathcal{O}_K) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \widehat{\mathrm{Cl}}(\mathbb{Z}) & & & & \end{array}$$

  where the first row is exact and the map *reg* is $(-2) \times$ the Dirichlet regulator.
- $\mathrm{MT}(P)$ is mapped to $[K : \mathbb{Q}] \cdot \mathrm{NT}(P)$ by the map

$$\widehat{\mathrm{Cl}}(\mathcal{O}_K) \to \widehat{\mathrm{Cl}}(\mathbb{Z}) = \mathbb{R}.$$

Formula for the refined height

Just as the Néron-Tate height, the refined height $\mathrm{MT}(\cdot)$ has the property that

$$\mathrm{MT}(m \cdot P) = m^2 \cdot \mathrm{MT}(P).$$

Theorem (Mazur-Tate)

The refined height $\mathrm{MT}(P)$ of P is given by the formula

$$\bigoplus_{v \ \mathrm{n.-ar.}} \left(v(16(4A^3 + 27B^2)) - \frac{1}{2} v(\mathrm{Denominator}(x(P))) \right)$$

$$\bigoplus_{v \ \mathrm{ar.}} - \log \left| e^{-z(P_v)\mathrm{quasiperiod}(z(P_v)/2)} \sigma(z(P_v)) \Delta(\tau_v)^{\frac{1}{12}} \right|^2.$$

Interpretation of the formula of Mazur and Tate

- The term $v(16(4A^3+27B^2))$ comes from the bad reduction of E.
- The term $-\frac{1}{2}v(\mathrm{Denominator}(x(P)))$ is the intersection multiplicity of the section of $\mathcal{E}$ defined by P with the unit section of $\mathcal{E}$.
- The term $-\log|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v)/2)}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|^2$ is an archimedean intersection multiplicity and is best understood via Arakelov theory.

The refined height and elliptic units

Corollary (of the formula of Mazur and Tate)

Let P be an m-torsion point on an elliptic curve over K with good reduction everywhere. Suppose that P is defined over K. If m is composite, then

$$m^2 \cdot \bigoplus_{v \text{ ar.}} -\log|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v))/2}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|^2$$

lies in the image of the Dirichlet regulator map.

Hence

$$|e^{-z(P_v)\mathrm{quasiperiod}(z(P_v))/2}\sigma(z(P_v))\Delta(\tau_v)^{\frac{1}{12}}|$$

is a unit for each archimedean v.

Questions: do Siegel functions have analogs on any abelian scheme ?

How should the group $\widehat{\mathrm{Cl}}(\cdot)$ be defined on a higher-dimensional scheme ?

Is there a natural analog of $\mathrm{MT}(\cdot)$ on any abelian scheme ?

Can one generalize the formula of Mazur and Tate to any abelian scheme ?

We shall propose answers to these questions, which are based on Arakelov theory.

Generalization of the group $\widehat{\mathrm{Cl}}(\mathcal{O}_K)$, I

Let X be a regular scheme, which is of finite type over $\mathcal{O}_K$ (K a number field).

Gillet and Soulé defined the **arithmetic Chow group** $\widehat{\mathrm{CH}}^*(X)$.

If $X = \mathrm{Spec}\ \mathcal{O}_{\mathrm{K}}$, then $\widehat{\mathrm{CH}}^1(X) = \widehat{\mathrm{Cl}}(\mathcal{O}_K)$.

There is for any $g \geqslant 0$ an exact sequence

$$\underbrace{\mathrm{CH}^{g,g-1}(X)}_{\text{motivic coh. group}} \xrightarrow{\mathrm{cyc}_{\mathrm{an}}} \underbrace{\widetilde{A}^{g-1,g-1}(X_{\mathbb{R}})}_{\text{space of diff. forms}} \xrightarrow{a} \widehat{\mathrm{CH}}^g(X) \to \underbrace{\mathrm{CH}^g(X)}_{\text{Chow group}} \to 0$$

analogous to the sequence of Mazur and Tate for $\widehat{\mathrm{Cl}}(\mathcal{O}_K)$.

Generalization of the group $\widehat{\mathrm{Cl}}(\mathcal{O}_K)$, II

The arithmetic Chow group $\widehat{\mathrm{CH}}^*(X)$ is an extension of the ordinary Chow group, which includes differential geometric data on $X(\mathbb{C})$.

It is generated by pairs (Z, g_Z), where Z is a cycle on X and g_Z is a **Green current** for Z. By definition, such a g_Z has the property that

$$\frac{i}{2\pi}\partial\bar{\partial}g_Z + \underbrace{\delta_{Z(\mathbb{C})}}_{\text{Dirac current}} = \text{smooth current.}$$

For any hermitian vector bundle $\bar{E} := (E, h_{E(\mathbb{C})})$ on X, there are Chern classes $\hat{\mathrm{c}}^i(\bar{E}) \in \widehat{\mathrm{CH}}^i(X)$.

The arithmetic Chow groups are covariant for projective and generically smooth morphisms and contravariant for quasi-projective morphisms.

Higher analogs of the Siegel functions I

Let S be a regular scheme, which is of finite type over $\mathcal{O}_K$.

Let $\pi : \mathcal{A} \to S$ be an abelian scheme over S of relative dimension g.

We shall write $\mathcal{A}^\vee$ for the dual abelian scheme and $\mathcal{P}/\mathcal{A} \times_S \mathcal{A}^\vee$ for the Poincaré bundle.

This bundle carries a natural hermitian metric and we write $\bar{\mathcal{P}}$ for the corresponding hermitian bundle.

We write S_0 (resp. $S_0^\vee$) for the unit section of $\mathcal{A}$ (resp. $\mathcal{A}^\vee$).

Higher analogs of the Siegel functions II

Theorem (existence and unicity of the current $\mathfrak{g}_{\mathcal{A}}$)

There is a real and conjugation invariant current $\mathfrak{g}_{\mathcal{A}}$ of type $(g-1, g-1)$ on $\mathcal{A}^\vee(\mathbb{C})$ with the following properties.

(a) *The current $\mathfrak{g}_{\mathcal{A}}$ is a Green current for the unit section $S_0^\vee$ of $\mathcal{A}^\vee$.*

(b) *We have $(S_0^\vee, \mathfrak{g}_{\mathcal{A}}) = (-1)^g p_{\mathcal{A}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}$ in $\widehat{\mathrm{CH}}^g(\mathcal{A}^\vee)_{\mathbb{Q}}$.*

(c) *The identity $\mathfrak{g}_{\mathcal{A}} = [n]_*\mathfrak{g}_{\mathcal{A}}$ holds for all $n \geqslant 2$.*

The current $\mathfrak{g}_{\mathcal{A}^\vee}$ is uniquely determined by these properties, up to currents of type $\partial(\cdot) + \bar{\partial}(\cdot)$.

Back to $g = 1$

Let $\sigma : S \to \mathcal{A}$ be a section. We have a pull-back map

$$\sigma^* : \widehat{\mathrm{CH}}^*(\mathcal{A}) \to \widehat{\mathrm{CH}}^*(S).$$

It can be shown that if $g = 1$, then

$$\mathrm{MT}(\sigma) = (-1)^g \sigma^*(p_{\mathcal{A}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}).$$

Furthermore, if $g = 1$ then

$$\mathfrak{g}_{\mathcal{A}} = -\log|e^{-z\cdot\mathrm{quasiperiod}(z)/2}\sigma(z)\Delta(\tau)^{\frac{1}{12}}|^2.$$

Algebraic properties of $\mathfrak{g}_{\mathcal{A}}$

By property (b) in the theorem, for all $n, g \geqslant 1$, we have

$$[n]^*(p_{\mathcal{A}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}) = p_{\mathcal{A}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}^{\otimes n})) = n^{2g} \cdot p_{\mathcal{A}^\vee,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}$$

and thus for any section $\sigma : S \to \mathcal{A}^\vee$ we have

$$([n] \cdot \sigma)^*(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}) = n^{2g} \cdot \sigma^*(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}).$$

In particular, if σ is a torsion section and $\sigma(S) \cap S_0^\vee = \emptyset$ we have

$$\sigma^*(\mathfrak{g}_{\mathcal{A}}) \in \mathrm{image}(\mathrm{cyc}_{\mathrm{an}}(\mathrm{CH}^{g,g-1}(S))_{\mathbb{Q}}.$$

This generalizes the theorem of Mazur and Tate and its Corollary.

Further properties of the current $\mathfrak{g}_{\mathcal{A}}$, I

Let $\mathcal{L}$ be a rigidified symmetric relatively ample line bundle on $\mathcal{A}$.

Endow $\mathcal{L}$ with the unique hermitian metric $h_{\mathcal{L}}$, which is compatible with the rigidification and whose curvature form is translation invariant on the fibres of $\mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$.

Let $\overline{\mathcal{L}} := (\mathcal{L}, h_{\mathcal{L}})$ be the resulting hermitian line bundle.

Let $\phi_{\mathcal{L}} : \mathcal{A} \to \mathcal{A}^{\vee}$ be the polarisation morphism induced by $\mathcal{L}$.

Further properties of the current $\mathfrak{g}_{\mathcal{A}}$, II

1. (distributivity) Let $\iota : \mathcal{A} \to \mathcal{B}$ be an isogeny of abelian schemes over S. Then the identity $\iota^{\vee}_{*}(\mathfrak{g}_{\mathcal{B}}) = \mathfrak{g}_{\mathcal{A}}$ holds.
2. ("formula of Beauville and Bloch") The equalities

$$(S_0^{\vee}, \mathfrak{g}_{\mathcal{A}}) = (-1)^g p_{\mathcal{A}^{\vee},*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}})) = \frac{1}{g!\sqrt{\deg(\phi_{\mathcal{L}})}} \phi_{\mathcal{L},*}(\widehat{c}_1(\overline{\mathcal{L}})^g)$$

are verified in $\widehat{\mathrm{CH}}^g(\mathcal{A}^{\vee})_{\mathbb{Q}}$.

3. If S is projective over $\mathcal{O}_K$, then the condition

$$(S_0^{\vee}, \mathfrak{g}_{\mathcal{A}}) = (-1)^g p_{\mathcal{A}^{\vee},*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{(g)}$$

(ie (b) in the last theorem) can be replaced by the weaker

$$\frac{i}{2\pi}\partial\bar{\partial}\mathfrak{g}_{\mathcal{A}} + \delta_{S_0^{\vee}(\mathbb{C})} = (-1)^g p_{\mathcal{A}^{\vee},*}(\mathrm{ch}(\overline{\mathcal{P}}))^{(g)}.$$

Spectral interpretation of the current $\mathfrak{g}_{\mathcal{A}}$, I

Part of the current $\mathfrak{g}_{\mathcal{A}^\vee}$ can be interpreted as the Bismut-Köhler higher analytic torsion form of the Poincaré bundle.

Let λ be a $(1,1)$-form on $\mathcal{A}(\mathbb{C})$ defining a Kähler fibration structure on the fibration $\mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$.

We suppose that λ is translation invariant on the fibres of the map $\mathcal{A}(\mathbb{C}) \to S(\mathbb{C})$ as well as conjugation invariant.

We shall write

$$T(\lambda, \overline{\mathcal{P}}^0) \in \widetilde{A}(\mathcal{A}^\vee \backslash S_0^\vee) := \bigoplus_{p \geqslant 0} \widetilde{A}^{p,p}(\mathcal{A}^\vee \backslash S_0^\vee)$$

for the Bismut-Köhler higher analytic torsion form of $\overline{\mathcal{P}}$ restricted to the fibration

$$\mathcal{A}(\mathbb{C}) \times_{S(\mathbb{C})} (\mathcal{A}^\vee(\mathbb{C}) \backslash S_0^\vee(\mathbb{C})) \longrightarrow \mathcal{A}^\vee(\mathbb{C}) \backslash S_0^\vee(\mathbb{C}).$$

Spectral interpretation of the current $\mathfrak{g}_{\mathcal{A}}$, II

Let $\bar{\Omega}$ be the sheaf of differentials of $\mathcal{A}$, endowed with the metric induced by the Kähler fibration. Let $\epsilon : S \to \mathcal{A}$ be the unit section.

Theorem

The equality

$$\mathfrak{g}_{\mathcal{A}}|_{\mathcal{A}^\vee(\mathbb{C})\setminus S_0^\vee(\mathbb{C})} = \mathrm{Td}(\epsilon^*\overline{\Omega}) \cdot T(\lambda, \overline{\mathcal{P}}^0)$$

holds on $\mathcal{A}^\vee \setminus S_0^\vee$.

In particular $T(\lambda, \overline{\mathcal{P}}^0)^{(g-1)}$ does not depend on λ.

This theorem specialises to the **second Kronecker limit formula**, when $g = 1$.

Fields of definition

Let

$$N_{2g} := 2 \cdot \mathrm{denominator}\,[(-1)^{g+1} B_{2g}/(2g)],$$

where B_{2g} is the $2g$-th Bernoulli number. Recall that

$$\sum_{t \geqslant 1} B_t \frac{u^t}{t!} := \frac{u}{\exp(u) - 1}.$$

Theorem

Suppose that $\sigma : S \to \mathcal{A}^\vee$ is an n-torsion section, such that $\sigma(S) \cap S_0^\vee = \emptyset$. Then

$$2g \cdot n \cdot N_{2g} \cdot \sigma^* T(\lambda, \overline{\mathcal{P}}^0) \in \mathrm{image}(\mathrm{reg}_{\mathrm{an}}(K_1(S))).$$

In particular, $48 \cdot \sigma^* T(\lambda, \overline{\mathcal{P}}^0) \in \mathrm{image}(\mathrm{reg}_{\mathrm{Dirichlet}}(\mathcal{O}_S^*))$ if $g = 1$.

Further topics

- There should be a connection between $\mathfrak{g}_{\mathcal{A}^\vee}$ and the Hodge realisation of the abelian polylogarithm (work in progress by G. Kings and D. R.).
- If $\dim(S) \leqslant 1$ then $\mathfrak{g}_{\mathcal{A}}$ is the canonical harmonic Green current associated with $S_0^\vee$ and the above results (aside from the spectral interpretation) are contained in the work of K. Künnemann.