# Analytic torsion of locally symmetric spaces and cohomology of arithmetic groups


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Bismut Fest
Paris, May 27 – 30, 2013

# Prelude

“Mit Euch, Herr Doktor, zu spazieren
ist ehrenvoll und ist Gewinn”

*J.W. Goethe, Faust I*

It is an honor to walk out with you,
Doctor, and one I profit by

# Introduction

## Locally symmetric spaces

- $G$ semisimple real Lie group of non-compact type
- $K \subset G$ maximal compact subgroup
- $\widetilde{X} = G/K$ associated Riemannian symmetric space of non-positive curvature
- $\Gamma \subset G$ a lattice, i.e., discrete subgroup with $\mathrm{vol}(\Gamma\backslash G) < \infty$
- $X := \Gamma\backslash\widetilde{X}$ locally symmettric space.

A lattice $\Gamma$ is called arithmetic, if it is defined by "arithmetic conditions" like $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$.
More precisely: There is a semisimple algebraic group $\mathbf{G} \subset \mathrm{GL}_n$ which is defined over $\mathbb{Q}$ such that:

- $G = \mathbf{G}(\mathbb{R})$.
- $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and $\Gamma$ is commensurable with $\mathbf{G}(\mathbb{Z}) := \mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

Examples:

1. $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$. $G/K = \mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$.

$$\Gamma(N) := \left\{ \gamma \in \mathrm{SL}(2,\mathbb{Z}) \colon \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod (N) \right\}$$

   $X(N) = \Gamma(N)\backslash\mathbb{H}$ a modular surface.

2. $\mathbb{H}^n$ hyperbolic $n$-space

$$\mathbb{H}^n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n > 0\right\} \cong \mathrm{SO}^0(n,1)/\,\mathrm{SO}(n).$$

The invariant metric is given by

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

$\Gamma(N) \subset \mathrm{SO}^0(n,1;\mathbb{Z})$ torsion-free finite index subgroup. $\Gamma\backslash\mathbb{H}^n$ hyperbolic $n$-manifold.

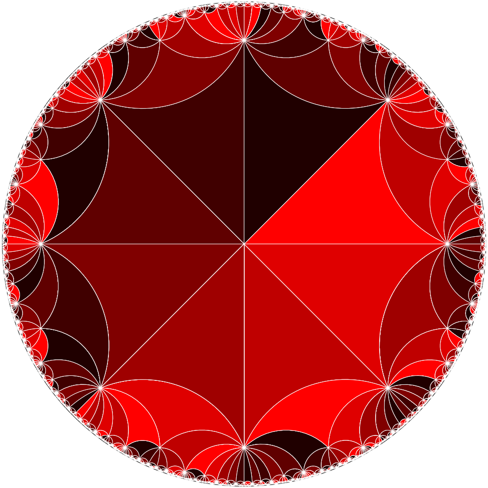
# 3.

$S$ space of positive definite $n \times n$-matrices of determinant 1.

$$S = \{ Y \in \mathrm{Mat}_n(\mathbb{R})\colon\ Y = Y^*,\ Y > 0,\ \det Y = 1 \} \cong \mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$$

- Invariant metric: $ds^2 = \mathrm{Tr}(Y^{-1} dY \cdot Y^{-1} dY)$.
- $\Gamma(N) \subset \mathrm{SL}(n, \mathbb{Z})$ principal congruence subgroup of level $N$. $X = \Gamma(N) \backslash S$.

surface of genus 2



Tesselation of the hyperbolic plane by fundamental domains of a Coxeter group (H. Koch, Bonn)

Analysis on locally symmetric spaces is closely related to representation theory, the theory of automorphic forms and number theory.
An important link between these fields is provided by the cohomology.

- $\rho\colon G \to \mathrm{GL}(V)$ finite-dimensional complex representation
- $H^j(\Gamma; V)$ cohomology of $\Gamma$ with coefficients in the $\Gamma$-module $V$.

Assume: $\Gamma$ is torsion free. Let $E_\rho \to \Gamma\backslash\widetilde{X}$ be the flat vector bundle associated to $\rho|_\Gamma$. Then

$$H^j(\Gamma; V) = H^j(\Gamma\backslash\widetilde{X}, E_\rho)$$

- Provides us with analytic tools to study $H^j(\Gamma; V)$.

## Eichler-Shimura isomorphism

$\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ congruence subgroup, torsion-free, $V_k := \mathrm{Sym}^k(\mathbb{C}^2)$, $\rho_k\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(V_k)$. We have

$$H^1(\Gamma\backslash\mathbb{H}; E_{k-2}) \cong S_k(\Gamma) \oplus \overline{S}_k(\Gamma) \oplus \mathrm{Eis}_k(\Gamma),$$

where $S_k(\Gamma)$ is the space of holomorphic weight $k$ cusp forms, and $\mathrm{Eis}_k(\Gamma)$ is the space of weight $k$ Eisenstein series.

## General case (Borel conjecture):

Subspace of automorphic forms

$$A(\Gamma, G) \subset C^\infty(\Gamma\backslash G)$$

(subspace of functions that are right $K$-finite, left $Z(\mathfrak{g})$-finite, and of moderate growth)

## Theorem (Franke)

$$H^*(\Gamma; V) \cong H^*(\mathfrak{g}, K; A(\Gamma, G) \otimes V)$$

- If $\Gamma$ is arithmetic, the groups $H^*(\Gamma; V)$ have an action of the Hecke operators which are defined algebraically.
- Eigenclasses are expected to correspond to Galois representations.

The de Rham cohomology of lattices has been studied to a great extend by many people. One important question is to determine the size of the cohomology groups. An example is the following theorem.

**Theorem (Lück, 1994)**
Let $\Gamma_N \subset \Gamma$ be a decreasing sequence of normal subgroups with $\cap_N \Gamma_N = \{1\}$. Then

$$\lim_{N\to\infty} \frac{\dim H_j(\Gamma_N; \mathbb{C})}{[\Gamma : \Gamma_N]} = b_j^{(2)},$$

where $b_j^{(2)}$ is the $L^2$-Betti number.

Generalization by Abert, Bergeron, Beringer, Gelander, Nikolov, Raimbault, and Samet

Concept of Benjamin-Schramm convergence: Let $(\Gamma_n)$ be a sequence of lattices in $G$. Let $X_n = \Gamma_n \backslash \widetilde{X}$. For $R > 0$ let

$$(X_n)_{<R} := \{x \in X_n \colon \mathrm{injrad}(x) < R\}.$$

$(X_n)$ BS-converge to $\widetilde{X}$, if for all $R > 0$ one has

$$\lim_{n\to\infty} \frac{\mathrm{vol}((X_n) < R)}{\mathrm{vol}(X_n)} = 0.$$

- BS-convergence allows much more general sequences of lattices $\Gamma_n$.

Let

$$\delta(G) := \mathrm{rank}_{\mathbb{C}}(G) - \mathrm{rank}_{\mathbb{C}}(K).$$

Then

$$b_j^{(2)} \neq 0 \Leftrightarrow \delta(G) = 0 \quad \text{and} \quad j = \frac{1}{2}\dim(G/K).$$

Examples:

- $\delta(G) = 0$ for $\mathrm{SU}_{p,q}$, $\mathrm{SO}_{p,q}$ ($pq$ even),
- $\delta(G) = 1$ for $\mathrm{SO}_{p,q}$ ($pq$ odd), $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_4(\mathbb{R})$,
- $\delta(G) \geq 2$ for $\mathrm{SL}_n(\mathbb{R})$, $n \geq 5$.

Growing local system

Let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a lattice. Let $V_n = \mathrm{Sym}^n(\mathbb{C}^2) \otimes \overline{\mathrm{Sym}^n(\mathbb{C}^2)}$.

Theorem (Finis, Grunewald, Tirao, 2008)

$$\dim H^1(\Gamma; V_n) = O\left(\frac{n^2}{\log n}\right).$$

What about torsion?

Let $M \subset V$ be a $\Gamma$-invariant lattice. Then $H^j(\Gamma; M)$ and $H_j(\Gamma; M)$ are finitely generated $\mathbb{Z}$-modules. If $\Gamma$ is torsion-free, let $\mathcal{M}$ be the local system of finite rank free $\mathbb{Z}$-modules over $X = \Gamma \backslash \widetilde{X}$, associated to $M$. Then

$$H^j(\Gamma; M) = H^j(X; \mathcal{M}).$$

Example: $\rho = 1$ is the trivial representation. Then we consider $H^j(\Gamma; \mathbb{C}) = H^j(X; \mathbb{C})$.

Question: Are there analogous results for $H^j(\Gamma; M)_{\mathrm{tors}}$ resp. $H_j(\Gamma; M)_{\mathrm{tors}}$?

Motivation: According to the Langlands program, torsion classes which are eigenclasses of Hecke operators are expected to correspond to Galois representations over finite fields.

- $\mathbf{G}$ a semisimple algebraic group over $\mathbb{Q}$, $G = \mathbf{G}(\mathbb{R})$.
- $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup such that $\Gamma\backslash G$ is compact.
- $\rho\colon \mathbf{G} \to \mathrm{GL}(V)$ a finite-dimensional rational representation, where $V$ is a $\mathbb{Q}$-vector space.
- $M \subset V$ a $\Gamma$-invariant lattice.
- $\Gamma_N \subset \Gamma$ a decreasing sequence of congruence subgroups with $\cap_N \Gamma_N = \{1\}$.

Conjecture (Bergeron, Venkatesh): There exists a constant $C_{G,M}$ such that

$$\lim_{N\to\infty} \frac{\log |H_j(\Gamma_N; M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_N]} = C_{G,M}\,\mathrm{vol}(\Gamma\backslash\widetilde{X}).$$

Moreover $C_{G,M} = 0$, unless $\delta(G) = 1$ and $j = \frac{\dim(\widetilde{X})-1}{2}$. In the latter case one has $C_{G,M} > 0$.

# Analytic torsion

Analytic torsion is an analytic tool to study torsion in the cohomology of arithmetic groups.

General set up:

- $(X, g)$ a compact Riemannian manifold of dimension $n$.
- $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$ finite-dimensional representation.
- $E_\rho \to X$ associated flat vector bundle.
- $h$ Hermitean fibre metric in $E_\rho$.

Let

$$\Delta_p(\rho)\colon \Lambda^p(X, E_\rho) \to \Lambda^p(X, E_\rho)$$

be the Laplace operator on $E_\rho$-valued $p$-forms.

- $\Delta_p(\rho)$ elliptic, self-adjoint, non-negative.

Spectrum of $\Delta_p(\rho)$: $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$.

Let

$$\zeta_p(s;\rho) := \sum_{j=1}^{\infty} \lambda_j^{-s}, \quad \mathrm{Re}(s) > n/2,$$

be the zeta function of $\Delta_p(\rho)$. $\zeta_p(s;\rho)$ admits meromorphic extension to $\mathbb{C}$, holomorphic at $s = 0$. Put

$$\det \Delta_p(\rho) = \exp\left( -\frac{d}{ds}\zeta_p(s;\rho)\Big|_{s=0} \right).$$

Ray-Singer analytic torsion:

$$T_X(\rho) := \prod_{j=1}^{n} (\det \Delta_p(\rho))^{(-1)^{p+1}p/2}.$$

- $T_X(\rho)$ depends on the metrics $g$ on $X$ and $h$ in $E_\rho$.

**Lemma:** If $\dim X$ is odd and $H^*(X; E_\rho) = 0$, then $T_X(\rho)$ is independent of $g$ and $h$.

**Topological counterpart:** Reidemeister torsion $\tau_X(\rho)$

- $\tau_X(\rho)$ is defined with the help of a triangulation $K$ of $X$.
- $\Lambda^*(X; E_\rho)$ is replaced by the twisted cochain complex $C^*(K; \rho)$.
- $\Delta_p(\rho)$ is replaced by the combinatorial Laplacian $\Delta_p^c(\rho)$.

$$\tau_X(\rho) := \prod_{j=1}^{n} \left(\det{}' \Delta_p^c(\rho)\right)^{(-1)^{p+1} p/2}.$$

Theorem (Cheeger, M., 1978)
$T_X(\rho) = \tau_X(\rho)$ for all unitary representations $\rho$ of $\pi_1(X)$.

Extension:

- M., 1992: $T_X(\rho) = \tau_X(\rho)$ for all unimodular representations ($\det \rho(\gamma) = 1$ for all $\gamma \in \pi_1(X)$).
- Bismut, Zhang, 1992: General case, new proof.

Corollary.
Assume that there exists a $\pi_1(X)$-invariant lattice $M \subset V_\rho$. Then

$$T_X(\rho) = R(\mathcal{M}) \cdot \prod_{p=0}^{n} |H^p(X; \mathcal{M})_{\mathrm{tors}}|^{(-1)^{p+1}},$$

where $R(\mathcal{M})$ is the regulator.

$$R(\mathcal{M}) = \prod_{p=0}^{n} R_p(\mathcal{M})^{(-1)^p}.$$

$R_p(\mathcal{M})$ is the covolume of the lattice $H^p(X;\mathcal{M})_{\mathrm{free}}$ in $H^p(X;\mathcal{M}\otimes\mathbb{R})$ with respect to the $L^2$ inner product induced by the Hodge isomorphism from $\mathcal{H}^p(X;E_\rho)$.

If $H^*(X;E_\rho)=0$ then $H^*(X;\mathcal{M})$ is finite and

$$T_X(\rho)=\prod_{p=0}^{n}|H^p(X;\mathcal{M})|^{(-1)^{p+1}}.$$

Example: Let $d\in\mathbb{Z}$, $d\neq 0$. Let $A\colon\mathbb{Z}\to\mathbb{Z}$ be defined by $A(n)=dn$. Let

$$C^*:0\to\mathbb{Z}\xrightarrow{A}\mathbb{Z}\to 0$$

Then

$$|\det A|=|d|=|H^1(C^*)|.$$

# Results

1. Sequences of coverings
   - $\widetilde{X} := G/K$, $\Gamma \subset G$ cocompact lattice, $X = \Gamma \backslash \widetilde{X}$.
   - $\Gamma_N \subset \Gamma$, $N \in \mathbb{N}$, a sequence of congruence subgroups. Let $X_N := \Gamma_N \backslash \widetilde{X}$. Then $X_N \to X$ is sequence of finite coverings.

Let $\rho \colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation, $E_\rho \to X_N$ flat bundle associated to $\rho|_{\Gamma_N}$, and $\Delta_{X_N,p}(\rho)$ the Laplacian on $E_\rho$-valued $p$-forms on $X_N$.
$\rho$ is called strongly acyclic, if there exists $c > 0$ such that

$$\mathrm{Spec}(\Delta_{X_N,p}(\rho)) \subset [c, \infty)$$

for all $N \in \mathbb{N}$ and $p = 0, \ldots, n$.

Proposition (Bergeron, Venkatesh): Strongly acyclic representations exist.

Example: The real represententation

$$\rho_{p,q} := \mathrm{Sym}^p(\mathbb{C}^2) \otimes \overline{\mathrm{Sym}^q(\mathbb{C}^2)}$$

of $\mathrm{SL}_2(\mathbb{C})$ is strongly acyclic if and only if $p \neq q$.

Theorem (Bergeron, Venkatesh, 2009): Let $\rho\colon G \to \mathrm{GL}(V)$ be strongly acyclic. Let $\Gamma_N$ be sequence of congruence subgroups of $\Gamma$ for which the injectivity radius of $X_N = \Gamma_N \backslash \widetilde{X}$ goes to infinity. Then

$$\lim_{N\to\infty} \frac{\log T_{X_N}(\rho)}{[\Gamma : \Gamma_N]} = \log T_X^{(2)}(\rho),$$

where $T_X^{(2)}(\rho)$ is the $L^2$-torsion of $X$.

Since $\widetilde{X}$ is homogeneous, we have

$$\log T_X^{(2)}(\rho) = \mathrm{vol}(X) t_{\widetilde{X}}^{(2)}(\rho),$$

where $t_{\widetilde{X}}^{(2)}(\rho)$ is a constant which depends only on $\widetilde{X}$ and $\rho$ and is given by the Plancherel formula.

Let $\rho\colon G \to \mathrm{GL}(V)$ be an arithmetic, strongly acyclic module, i.e., $\rho$ is strongly acyclic and there exists a $\Gamma$-invariant lattice $M \subset V$.

- Can be obtained from a rational representation $\rho\colon \mathbf{G} \to \mathrm{GL}(M \otimes \mathbb{Q})$.

Then

$$\lim_{N\to\infty} \sum_{p=0}^{n} (-1)^{p+1} \frac{\log |H^p(\Gamma_N; M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_N]} = \mathrm{vol}(X) t^{(2)}_{\widetilde{X}}(\rho).$$

If $\delta(G) = 1$, we have $t^{(2)}_{\widetilde{X}}(\rho) \neq 0$. Then $\dim \widetilde{X}$ is odd. It follows that

$$\liminf_{N} \sum_{p} \frac{\log |H^p(\Gamma_N; M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_N]} \geq C_{G,M} \,\mathrm{vol}(X),$$

where $p$ is taken over integers with the same parity as $\frac{\dim X - 1}{2}$ and $C_{G,M} > 0$.

Example: $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$. Let $F$ be an imaginary quadratic number field and $D$ a quaternion division algebra over $F$. Let $\mathbf{G} := \mathrm{SL}_1(D)$. Then $\mathbf{G}$ is an algebraic group over $F$ which is an inner form of $\mathrm{SL}_2/F$. So

$$\mathbf{G}(F) = D^1 = \{x \in D\colon N(x) = 1\}, \quad \mathbf{G}(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C})$$

Let $\mathfrak{o} \subset D$ be an order in $D$. Then $\mathfrak{o}^1 = \mathfrak{o} \cap D^1$ corresponds to a cocompact arithmetic subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$.
Each even symmetric power $\mathrm{Sym}^{2k}(\mathbb{C}^2)$ contains a $\Gamma$-invariant lattice $M_{2k}$ and is strongly acyclic.

Corollary (Bergeron, Venkatesh):
Let $\Gamma_N \subset \Gamma$ be a decreasing sequnece of congruence subgroups with $\cap_N \Gamma_N = \{1\}$. Then there is $C_k > 0$ such that

$$\lim_{N\to\infty} \frac{\log |H_1(\Gamma_N, M_{2k})|}{[\Gamma : \Gamma_N]} = C_k \,\mathrm{vol}(\Gamma\backslash\mathbb{H}^3).$$

# 2. Sequences of representations

Now we consider the opposite case. We fix $\Gamma$ and vary the representation.

a) Hyperbolic 3-manifolds.

- $X = \Gamma\backslash\mathbb{H}^3$, $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, a compact oriented hyperebolic 3-manifold.
- For $m \in \mathbb{N}$, let

$$\tau(m) := \mathrm{Sym}^m\colon\ \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathrm{Sym}^m(\mathbb{C}^2))$$

  be the $m$-th symmetric power of the standard representation of $\mathrm{SL}(2,\mathbb{C})$.
- Let $T_X(\tau(m))$ be the analytic torsion w.r.t. the representation $\tau(m)|_\Gamma$ of $\Gamma$.

Theorem (M., 2012)

As $m \to \infty$, we have

$$-\log T_X(\tau(m)) = \frac{Vol(X)}{4\pi} m^2 + O(m).$$

Corollary

The set $\{\tau_X(\tau(m))\colon m \in \mathbb{N}\}$ determines $\mathrm{vol}(X)$.

Now let $\Gamma$ be an arithmetic group derived from a quaternion devision algebra over a imaginary quadratic field. Then for every $k \in \mathbb{N}$, there exists a $\Gamma$-invariant lattice $M_{2k} \subset \mathrm{Sym}^{2k}(\mathbb{C}^2)$. Note that $H^*(X; \mathcal{M}_{2k})$ is finite abelian.

Theorem (Marshall, M., 2012)

For every choice of $\Gamma$-stable lattices $M_{2k}$ in $\mathrm{Sym}^{2k}(\mathbb{C}^2)$ one has

$$\lim_{k\to\infty} \frac{\log |H^2(X; \mathcal{M}_{2k})|}{k^2} = \frac{2}{\pi} \mathrm{vol}(X).$$

Furthermore, for $p = 1, 3$ one has

$$\log |H^p(X; \mathcal{M}_{2k})| \ll k \log k$$

uniformly over all choices of lattices $M_{2k}$.

Equivalently:

$$\lim_{k \to \infty} \frac{\log |H_1(\Gamma; M_{2k})|}{k^2} = \frac{2}{\pi} \operatorname{vol}(X).$$

# Higher dimensions

- $X = \Gamma \backslash G/K$ compact locally symmetric manifold.
- $\mathfrak{g}$ Lie algebra of $G$, $\mathfrak{h} \subset \mathfrak{g}$ fundamental Cartan subalgebra.
- $U$ compact real form of $G_{\mathbb{C}}$ such that $\mathfrak{h}_{\mathbb{C}}$ is the complexification of $\mathfrak{u}$.
- Rep($G$) (resp. *Rep*($U$)) irreducible finite-dimensional complex representations of $G$ (resp. $U$). Then Rep($G$) $\cong$ Rep($U$).
- $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ highest weight, analytically integral w.r.t. $U$.
- $\tau_\lambda \in$ Rep($G$) irreducibel representation corresponding to the representation of $U$ with highest weight $\lambda$.
- $\theta\colon G \to G$ Cartan involution w.r.t. $K$.
- $\lambda_\theta$ highest weight of $\tau_\lambda \circ \theta$.

Theorem (Bismut-Ma-Zhang, M.-Pfaff, 2012)
Let $\dim G/K$ be even or let $\delta(G) \neq 1$. Then $T_X(\tau) = 1$ for all finite-dimensional representations $\tau$ of $G$.

- $\dim(X)$ odd and $\delta(G) = 1$.
- $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ a highest weight with $\lambda_\theta \neq \lambda$.
- For $m \in \mathbb{N}$ let $\tau_\lambda(m)$ be the irreducible representation of $G$ with highest weight $m\lambda$.

Theorem (M.-Pfaff, Bismut-Ma-Zhang, 2012) There exist constants $c > 0$ and $C_{\widetilde{X}} \neq 0$, and a polynomial $P_\lambda(m)$, which depends on $\lambda$, such that

$$\log T_X(\tau_\lambda(m)) = C_{\widetilde{X}} \operatorname{vol}(X) \cdot P_\lambda(m) + O\left(e^{-cm}\right)$$

as $m \to \infty$. Furthermore, there is a constant $C_\lambda > 0$ such that

$$P_\lambda(m) = C_\lambda \cdot m \dim(\tau_\lambda(m)) + R_\lambda(m),$$

with $R_\lambda(m)$ of lower order.

- Gives a complete asymptotic expansion.

The theorem follows from

Proposition (Bismut-Ma-Zhang, M.-Pfaff, 2012)

$$\log T_X(\tau_\lambda(m)) = \log T_X^{(2)}(\tau(m)) + O\left(e^{-cm}\right).$$

- B-M-Z studied this in the more general context of analytic torsion forms on arbitrary compact manifolds

Application: $\widetilde{X} = \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$, $X = \Gamma\backslash\widetilde{X}$.
$\omega_i$, $i = 1, 2$, fundamental weights. non-invariant under $\theta$. $\tau_i(m)$ irreducible representations with heighest weight $m\omega_i$. Then

$$\log T_X(\tau_i(m)) = \frac{2\pi \operatorname{vol}(X)}{9 \operatorname{vol}(\widetilde{X}_d)} m^3 + O(m^2)$$

Let $\Gamma \subset \mathrm{SL}(3,\mathbb{R})$ be derived from a 9-dimensional division algebra over $\mathbb{Q}$. Let $M_{i,m} \subset V_{\tau_i(m)}$, $i = 1, 2$, $m \in \mathbb{N}$, be a $\Gamma$-invariant lattice. Then

$$\liminf_m \sum_{j=0}^{2} \frac{\log |H^{2j+1}(\Gamma; M_{i,m})|}{m^3} \geq \frac{2\pi}{9\,\mathrm{vol}(\widetilde{X}_d)} \mathrm{vol}(X).$$

Conjecture

$$\lim_{m\to\infty} \frac{\log |H^3(\Gamma; M_{i,m})|}{m^3} = \frac{2\pi}{9\,\mathrm{vol}(\widetilde{X}_d)} \mathrm{vol}(X).$$

$$\log |H^j(\Gamma; M_{i,m})| = o(m^3), \quad j \neq 3.$$

- Similar results for $\widetilde{X} = \mathrm{SO}(p,q)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$, $p, q$ odd.

# Methods

Given $\tau \in \text{Rep}(G)$, let $\Delta_p(\tau)$ be the Laplace operator on $\Lambda^p(X; E_\tau)$, where $E_\tau$ is the flat bundle associated to $\tau|_\Gamma$. A key ingrdient of the proof is the following lemma.

**Lemma**

Let $\lambda \in \mathfrak{h}_\mathbb{C}^*$ be a highest weight with $\lambda \neq \lambda_\theta$. There exist $C_1, C_2 > 0$ such that

$$\Delta_p(\tau_\lambda(m)) \geq C_1 m^2 - C_2, \quad m \in \mathbb{N}.$$

**Proof.**

Let $\tau\,\text{Rep}(G)$. Let $\nu_p := \Lambda^p \operatorname{Ad}_\mathfrak{p}^* \colon K \to \text{GL}(\Lambda^p \mathfrak{p}^*)$, where $\mathfrak{p} = \mathfrak{g}/\mathfrak{k} \cong T_{x_0}(\widetilde{X})$. By Kuga's lemma one has

$$\Delta_p(\tau) = \nabla^*\nabla + \tau(m)(\Omega) - (\nu_p \otimes \tau(m))(\Omega_K),$$

where $\Omega$ and $\Omega_K$ are the Casimir elements of $G$ and $K$, resp.

Let $\tau \in \mathrm{Rep}(G)$ with highest weight $\lambda \neq \lambda_\theta$. Then

$$\mathrm{Tr}\left(e^{-t\Delta_p(\tau)}\right) = O(e^{-ct})$$

as $t \to \infty$. Thus

$$\zeta_p(s;\rho) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}\, dt.$$

Put

$$K(t,\tau) := \frac{1}{2} \sum_{p=1}^{n} (-1)^p p \,\mathrm{Tr}\left(e^{-t\Delta_p(\tau)}\right).$$

Then

$$\log T_X(\tau) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)} \int_0^\infty K(t,\tau) t^{s-1}\, dt\right)\Bigg|_{s=0}.$$

Now let $\tau_\lambda(m) \in \mathrm{Rep}(G)$ with highest weight $m\lambda$. Since $\tau_\lambda(m)$ is acyclic and $\dim X$ is odd, $T_X(\tau_\lambda(m))$ is metric independent. So we can rescale the metric or, equivalently, replace $\Delta_p(\tau_\lambda(m))$ by $\frac{1}{m}\Delta_p(\tau_\lambda(m))$. Then

$$\log T_X(\tau_\lambda(m)) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1} K\left(\frac{t}{m}, \tau_\lambda(m)\right)\, dt\right)\bigg|_{s=0} + \int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau_\lambda(m)\right)\, dt.$$

The lemma implies

$$\int_1^\infty t^{-1} K\left(\frac{t}{m}, \tau_\lambda(m)\right)\, dt = O\left(e^{-cm}\right)$$

as $m \to \infty$.

To deal with the first term, we apply the Selberg trace formula. There exists a smooth $K$-finite function $k_t^{\tau_\lambda(m)}$ which belongs to Harish-Chandra's Schwartz space $\mathcal{C}(G)$ such that

$$K(t, \tau_\lambda(m)) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} k_t^{\tau_\lambda(m)}(g^{-1}\gamma g)\, d\dot{g}.$$

- contribution of the $\gamma \neq 1$ is $O(e^{-cm})$.
- contribution of the identity is

$$\mathrm{vol}(X) t_{\widetilde{X}}^{(2)}(\tau_\lambda(m)) + O\left(e^{-cm}\right),$$

where

$$t_{\widetilde{X}}^{(2)}(\tau_\lambda(m)) = \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty k_t^{\tau_\lambda(m)}(1) t^{s-1}\, dt\right)\Bigg|_{s=0}$$

and $\log T_X^{(2)}(\tau_\lambda(m)) := \mathrm{vol}(X) t_{\widetilde{X}}^{(2)}(\tau_\lambda(m))$ is the $L^2$-torsion.

# Jacquet-Langlands correspondence

A consequence of the conjectures of Langlands is that the integral homology of arithmetic groups for different inner forms of the same group is related in a non-trival way.
Calegari-Venkatesh proved a numerical form of the Jacquet-Langlands correspondence in the torsion setting.
Relationship between $H_1(\Gamma)_{\mathrm{tors}}$ and $H_1(\Gamma')_{\mathrm{tors}}$ for certain incommensurable lattices $\Gamma, \Gamma' \subset \mathrm{SL}(2, \mathbb{C})$.

# The finite volume case

Standard arithmetic groups like $\mathrm{SL}(2,\mathbb{Z}[i]) \subset \mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$ are not cocompact. Extension to these groups is very desirable.

- If $\Gamma\backslash G/K$ is not compact, but has finite volume, then the Laplace operators have non-empty continuous spectrum
- The zeta function can not be defined in the usual way.
- Regularization of the trace of the heat operator is necessary.

$$\zeta_p(s;\rho) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left(e^{-t\Delta_p(\tau)}\right) t^{s-1}\, dt.$$

J. Raimbault, 2012 Case of Bianchi groups. $F$ imaginary quadratic number field, $\mathcal{O}_F$ ring of integers. $\Gamma_N \subset \mathrm{SL}(2,\mathcal{O}_F)$ sequence of congrunece subgroups.

M.-Pfaff, 2013 Hyperbolic manifolds of finite volume.

# Problems

- Extend the results to arbitrary flat bundles, especially the trivial one.
- Relation between analytic and topological torsion
- Study the regulator
- Finite volume and higher rank case. The Arthur trace formula will be one of the main tools.