Morse functions and families torsion

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Outline

1.1.1 Reidemeister-Franz Torsion

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Definition (Reidemeister '35, Franz '35)

$$\log \tau(M,F) = \frac{1}{2}\sum_{i=0}^{n}(-1)^n \log \det_{\mathrm{im}\,\delta_i}\left(\delta_i\delta_i^*\right)$$

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Application (Reidemeister '35, Franz '35)

τ distinguishes homotopy equivalent but non-homeomorphic lens spaces

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This definition is strictly analogous to the definition of $\tau(M;F)$

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Theorem (Cheeger '79, Müller '78)

$$\tau(M;F) = \mathcal{T}(M;F)$$

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Let $h\colon M \to \mathbb{R}$ be a Morse function with gradient field $\nabla^{TM}h$
Assume Smale transversality condition for $\nabla^{TM}h$
Then the unstable cells form a CW structure X on M
Call $(V,\delta) = \big(C^\bullet(X;F),\delta\big)$ the Thom-Smale complex of h

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The metric g^F at the critical points C induces a metric g^V on V
Define $\tau(M,F) = \tau\big(M,F;g^{TM},g^F,h\big)$ using (V,δ,g^V)

1.1.4 Bismut-Zhang Comparison Formula

Let $g^H_{L^2}$ denote the L^2-metric on $H = H^\bullet(M; F) \cong \ker \Delta$
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Let $e\big(TM, \nabla^{TM}\big)$ denote the Euler form of TM
Let δ_C denote the δ-distribution at the critical points C of h
Then the Mathai-Quillen current satisfies

$$d\big((\nabla^{TM}h)^*\psi(\nabla^{TM}, g^{TM})\big) = e\big(TM, \nabla^{TM}\big) - \delta_C$$

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Define a characteristic form of the flat vector bundle F by

$$\mathrm{ch}^o_1(F, g^F) = \mathrm{tr}\big(\omega^F\big) = \mathrm{tr}\big((g^F)^{-1} \nabla^F g^F\big) \in \Omega^1(M)$$

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Theorem (Bismut-Zhang '92)

$$\log \mathcal{T}(M, F; g^{TM}, g^F) - \log \tau(M, F; g^{TM}, g^F, h)$$
$$= \log \frac{\|\cdot\|_{\det H, g^H_{L^2}}}{\|\cdot\|_{\det H, g^H_V}} - \int_M \mathrm{ch}^o_1(F, g^F)\cdot(\nabla^{TM}h)^*\psi(\nabla^{TM}, g^{TM})$$

1.1.5 Bismut-Zhang Variational Formula

Let g^{TM} and g^{F} be parametrised by a manifold B
Get Euler class $e\big(TM, \nabla^{TM}\big) \in \Omega^{n}(M \times B; o(TM))$
Get $\mathrm{ch}_1^{o}\big(F, g^{F}\big) = \mathrm{tr}\big(\omega^{F}\big) \in \Omega^{1}(M \times B)$
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Theorem (Bismut-Zhang '92)

The Ray-Singer torsion depends on g^{TM} and g^F by

$$d\mathcal{T}(M, F; g^{TM}, g^F) = \int_M e(TM, \nabla^{TM})\, \mathrm{ch}_1^o(F, g^F) - \mathrm{ch}_1^o\big(H, g^H_{L^2}\big) \in \Omega^1(B)$$

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Remark

This looks like a family index theorem for the Euler operator

1.2.1 Characteristic Classes for Flat Vector Bundles

Let (F, ∇^F) be a flat vector bundle on M
If g^F is a metric on F, define the adjoint connection

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Definition (Kamber-Tondeur ’74, Bismut-Lott ’95)

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The class $\mathrm{ch}^o(F) = [\mathrm{ch}^o(F, g^F)]$ is independent of g^F
If g^F is parallel then $\mathrm{ch}^o\left(F, g^F\right) = 0$

1.2.2 The Exterior Differential as a Superconnection

Let $p\colon E \to B$ be a smooth submersion with compact fibres M
Let $TM = \ker dp$ denote the vertical tangent bundle
Choose a horizontal subbundle $T^H E \cong p^* TB$ with

$$TE = TM \oplus T^H E$$

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This induces splittings

$$\begin{aligned} \Lambda^\bullet T^* E &\cong \Lambda^\bullet T^* M \mathbin{\hat{\otimes}} p^* \Lambda^\bullet TB \\ \text{and} \qquad \Omega^\bullet(E;F) &\cong \Omega^\bullet\big(B;\Omega^\bullet(E/B;F)\big) \end{aligned}$$

with $\Omega^\bullet(E/B;F) = p_* \Lambda^\bullet T^* M \to B$ a bundle of infinite rank

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The exterior differential d_E becomes a superconnection

$$\mathbb{A}' = d_E = d_M + \mathcal{L}_{\bar{\cdot}} - \iota_{[\bar{\cdot}, \bar{\cdot}]^{TM}}$$

This superconnection is flat because $(\mathbb{A}')^2 = d_E^2 = 0$

1.2.3 The Bismut Superconnection

Let N^M be the vertical number operator on $\Omega^\bullet(E/B; F)$ with

$$N^M|_{\Omega^k(E/B;F)} = k \text{ id}$$

Rescaling the superconnection $\mathbb{A}'$ gives $\mathbb{A}'_t = t^{N^M/2}\, \mathbb{A}'\, t^{-N^M/2}$

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These metrics induce an L^2-metric on $\Omega^\bullet(E/B; F) \to B$
Let $\mathbb{A}''_t$ be the adjoint superconnection of $\mathbb{A}'_t$ and put

$$\mathbb{X}_t = \mathbb{A}''_t - \mathbb{A}'_t \in \Omega^\bullet\big(B; \operatorname{End}\Omega^\bullet(E/B; F)\big)$$

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$\mathbb{A}_t = \frac{1}{2}\left(\mathbb{A}''_t + \mathbb{A}'_t\right)$ (almost) equals the Bismut superconnection
Its curvature is $\mathbb{A}_t^2 = -\mathbb{X}_t^2/4$.

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The part $\mathbb{X}_t^{[0]} \in \Omega^0\big(B; \mathrm{End}\,\Omega^\bullet(E/B;F)\big)$ of horizontal degree 0 is $\sqrt{t}\,(d^* - d)$, a fibrewise skewadjoint Hodge-Dirac operator

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Let $f(z) = z\,e^{z^2}$, then $f'(\mathbb{X}_t/2)$ is closely related to the heat operator used in Bismut's proof of the family index theorem

1.2.4 Bismut-Lott Torsion

Let $H = H^\bullet(E/B; F) \to B$ denote the fibrewise cohomology and

$$\chi(H) = \sum (-1)^i \operatorname{rk} H^i \;, \qquad \chi'(H) = \sum (-1)^i i \operatorname{rk} H^i$$

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The component $\mathcal{T}\big(T^H E, g^{TM}, g^F\big)^{[0]}$ of degree 0 represents (the logarithm of) the Ray-Singer torsion of the fibre.

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Computable for bundles with compact structure groups by [Bunke '99, '00] and [Bismut-G '04]

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$$d\mathcal{T}(T^H E, g^{TM}, g^F) = \int_{E/B} e(TM, \nabla^{TM})\ \mathrm{ch}^o(F, g^F) - \mathrm{ch}^o(H, g^H_{L^2}) \in \Omega^{\mathrm{odd}}(M)$$

Remark (Bismut-Lott '95)

Generalises the variational formula for the Ray-Singer torsion

1.2.5 Bismut-Lott Index Theorem

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Generalises the variational formula for the Ray-Singer torsion
Passing to cohomology, we get

$$\mathrm{ch}^o(H) = \int_{E/B} e(TM)\ \mathrm{ch}^o(F)$$

1.3.1 Algebraic K-Theory

Flat vector bundles on X are classified by homotopy classes of maps from X to $BGL\mathbb{C}_\delta$, that is, by $[X, BGL\mathbb{C}_\delta]$

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Similarly for ordinary cohomology classes,

$$tr_{BG} = \int_{E/B} e(TM) \cup \cdot : H^\bullet(E;\mathbb{R}) \to H^\bullet(B;\mathbb{R})$$

as in the Bismut-Lott index theorem

1.3.2 Dwyer-Weiss-Williams Index Theorem

Theorem (Dwyer-Weiss-Williams '03)

If $p\colon E \to B$ is a smooth fibre bundle with compact fibres, then

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If both sides vanish, one can define a Dwyer-Weiss-Williams Smooth Torsion.

1.4.1 Framed Functions

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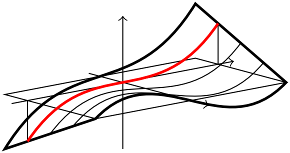
Definition (Igusa '87)

A generalised fibrewise Morse function is a function $h\colon M \to \mathbb{R}$ such that each fibrewise singularity is of Morse or of birth-death type



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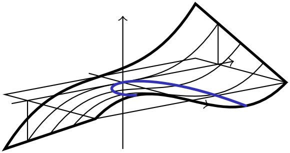

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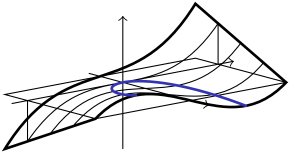

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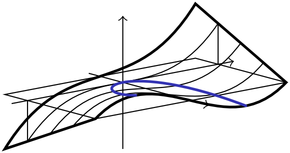

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Theorem (Igusa '87; Eliashberg-Mishachef '00)

Framed functions exist if $\dim M > \dim B$*.*
Generalised Morse functions always exist.

1.4.2 Igusa-Klein Torsion

Let $p\colon E \to B$ be a fibre bundle as above
Let $F \to E$ be flat, g^F parallel, $H = H^\bullet(E/B; F) = 0$
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Question: is $\tau(E/B, F)$ related to $\mathcal{T}(T^H E, g^{TM}, g^F)$?

1.5.1 Families of Thom-Smale Complexes

Let $p\colon E \to B$, F, $T^H E$, g^{TM} and g^F be as above
Let $h\colon E \to \mathbb{R}$ be a fibrewise Morse function
Assume that $\nabla^{TM} h$ satisfies Smale transversality on each fibre
Then the Thom-Smale complexes (V, δ) on the fibres
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Hence $A' = \delta + \nabla^V$ is a flat superconnection on $V \to B$.
Define $A'_t = \sqrt{t}\,\delta + \nabla^V$, $A''_t = \sqrt{t}\,\delta^* + \nabla^{V,*}$ and $X_t = A''_t - A'_t$

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$$T(A', g^V) = -\int_0^1 \left(\frac{4s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \int_0^\infty \Big(\operatorname{str}\Big(N^V f'(X_t/2)\Big) - \chi'(H) - \big(\chi'(V) - \chi'(H)\big) f'(\sqrt{-t/4})\Big) \frac{dt}{2t}\, ds$$

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This is analogous to the definition of $\mathcal{T}(T^H E, g^{TM}, g^F)$

1.5.2 A Comparison Formula

Let ζ denote the Riemann ζ-function
Define an additive characteristic class ${}^{0}J$ by

$$ {}^{0}J = \frac{1}{2} \sum_{k=1}^{\infty} \zeta'(-2k)\,\mathrm{ch}(\,\cdot\,)^{[4k]} \in H^{4\bullet}(\,\cdot\,;\mathbb{R}) $$

Let $\widetilde{\mathrm{ch}}^{o}$ denote the Chern-Simons form for ch^{o} such that

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This generalises the Bismut-Zhang comparison formula

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One can modify the smooth structure of $p\colon E \to B$ in such a way that $\xi_h^*\tau$ remains fixed and only $\hat{p}_* {}^0J(T^uM)$ changes

2.1.1 A more general Comparison Formula ?

Most $p\colon E \to B$ do not admit a fibrewise Morse function such that $\nabla^{TM} h$ satisfies Thom-Smale transversality on each fibre
Whenever $T(A', g^F)$ and $\tau(E/B; F)$ are both defined, they are zero

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- Prove the comparison formula

$$\begin{aligned}\mathcal{T}\left(T^H E, g^{TM}, g^F\right) &= T(V, A', g^V) + \widetilde{\mathrm{ch}}^o\left(H, g^H_{L^2}, g^H_V\right)\\ &\quad + \int_{E/B} (\nabla^{TM} h)^* \psi(\nabla^{TM}, g^{TM}) \cdot \mathrm{ch}^o(F, g^F)\\ &\quad\quad + \hat{p}_* \, {}^0 J\left(T^s M - T^u M\right) \operatorname{rk} F\end{aligned}$$

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Two problems with generalised fibrewise Morse functions

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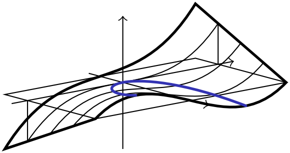

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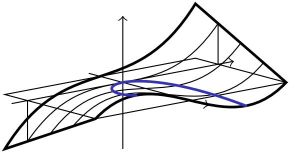

Birth-death singularities can be separated algebraically from the rest of the Thom-Smale complex such that

$$(V_+, A'_+, g^V_+) \cong (V_-, A'_-, g^V_-) \oplus \left(F \oplus F, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \nabla^{F\oplus F}, g^{F\oplus F}\right)$$

We call the right summand an elementary complex

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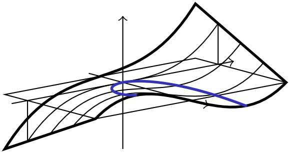
- lack of Smale transversality on some fibres
- birth-death singularities



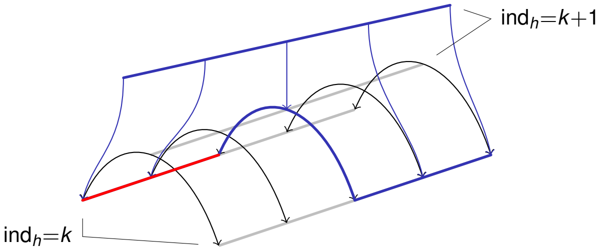
Birth-death singularities can be separated algebraically from the rest of the Thom-Smale complex such that

$$\left(V_+, A'_+, g^V_+\right) \cong \left(V_-, A'_-, g^V_-\right) \oplus \left(F \oplus F, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) + \nabla^{F \oplus F}, g^{F \oplus F}\right)$$

We call the right summand an elementary complex
It suffices to ensure that elementary complexes never contribute in the following constructions

2.1.3 Lack of Transversality

Non-transversality leads to varying Thom-Smale complexes



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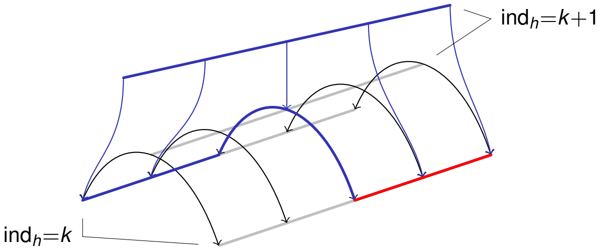

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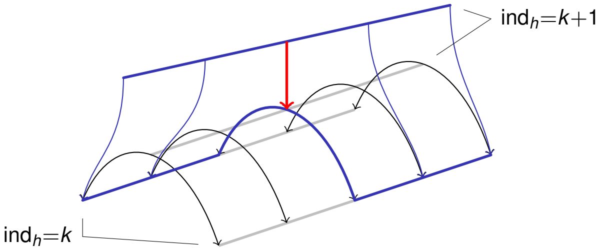

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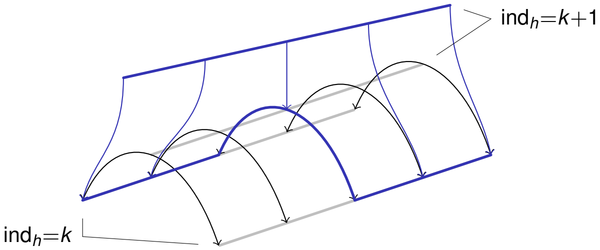

The exceptional flow line can be used to construct an isomorphism between the two Thom-Smale complexes

It is of the form $\mathrm{id} + a_1$, and a_1 decreases the value of h

A smoothing procedure produces a flat superconnection

$$A' = a_0(b) + \nabla^V + a_1(b)$$

2.1.4 The Thom-Smale superconnection

A small loop in B can lead to nontrivial holonomy in Aut V
However, non-transversal flow lines of codimension 2 give a cochain homotopy back to id_V

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Smoothing this gives a full-fledged flat superconnection

$$A' = a_0 + \left(\nabla^V + a_1\right) + a_2 + \dots, \qquad a_i \in \Omega^i\left(B; \mathrm{End}^{1-i} V\right)$$

The a_i strictly respect a local filtration on V induced by h

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Such superconnections are classified by $[B; Wh(\mathbb{C}, GL)]$
Restricting to fibrewise acyclic flat bundles F with parallel metric g^F leads to Igusa's $\left[B; Wh^h(\mathbb{C}, U)\right]$

2.2.1 Generalising Torsion Forms

Consider A'_t, A''_t and $X_t = A''_t - A'_t$ on V as above, with

$$X_t = t^{\frac{1}{2}} (a_0^* - a_0) + \left(\nabla^{V,*} + a_1^* - \nabla^V - a_1\right) + t^{-\frac{1}{2}} (a_2^* - a_2) + \dots$$

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Replace the integral over $[0, 1]$ in the construction of $T(A', g^F)$

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Works, but dependence on h prevents definition on $Wh(\mathbb{C}, GL)$

The following approach is equivalent, but independent of h

2.2.2 An Adapted Superconnection

For $s, t \in [0,1]$, we define a new superconnection

$$
{}^{2}A = t\Big((1-s)\,A' + s\,A''\Big) + (1-t)\Big((1-s)\,\nabla^{V} + s\,\nabla^{V,*}\Big) + (2s-1)\,N^{V}\,d\log t
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For $s = 0$, it strictly respects the local filtration by h
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For $t = 1$, it interpolates between A' and A''
For $t = 0$, it interpolates between ∇^V and $\nabla^{V,*}$
In particular

$$
\begin{aligned}
(2\pi i)^{\frac{1-N^B}{2}} \int_{s=0}^{1} \frac{1}{2}\,\mathrm{str}\Big(e^{-\,{}^2\!A^2|_{t=0}}\Big)& \\
= \pi i\,\widetilde{\mathrm{ch}}\big(F,\nabla^F,\nabla^{F,*}\big) &= \mathrm{ch}^o\big(\nabla^F, g^F\big)
\end{aligned}
$$

2.2.3 Torsion for General Thom-Smale Complexes

Let $p\colon E \to B$, $F \to E$, $h\colon E \to \mathbb{R}$ and g^V be as above
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\begin{aligned}
&T\big(V, A', g^V\big)\\
&\quad= -(2\pi i)^{-\frac{N^B}{2}} \int_0^1 \int_1^\infty \Big(\operatorname{str}\big(N^V f'\big(\sqrt{s(1-s)}\,X_t\big)\big) - \chi'(H)\\
&\qquad\qquad - \big(\chi'(V) - \chi'(H)\big)\, f'\big(\sqrt{-s(1-s)t}\,\big)\Big)\,\frac{dt}{2t}\,ds\\
&\qquad -(2\pi i)^{-\frac{N^B}{2}} \int_0^1 \int_0^1 \bigg(\frac{1}{2}\operatorname{str}_V\Big(e^{-{}^2\!A^2}\Big)\\
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\end{aligned}
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This form vanishes on elementary complexes
Hence it extends to a smooth torsion form on B

2.3.1 The Transgression Formula

Theorem

$$dT\big(V, A', g^V\big) = \mathrm{ch}^o(V, g^V) - \mathrm{ch}^o(H, g^H)$$

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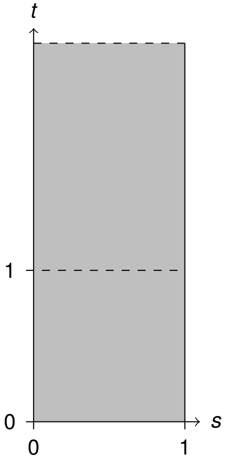

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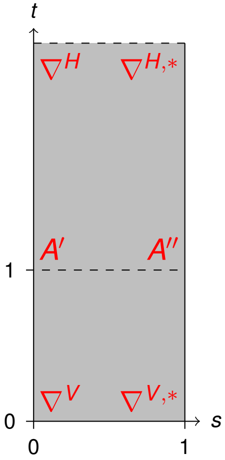

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The integrand (up to correction terms) is the Chern form of a superconnection on V that interpolates between the superconnections in the corners of the two rectangles

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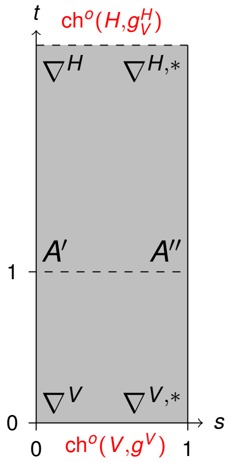

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The Statement follows from the fibrewise Stokes theorem ☐

2.3.2 Another Proof of the Bismut-Lott Theorem

Theorem

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\begin{aligned}
d\Big(T(V, A', g^V) + \widetilde{\mathrm{ch}}^o(H, g^H_{L^2}, g^H_V) \\
+ \int_{E/B} (\nabla^{TM} h)^* \psi(\nabla^{TM}, g^{TM}) \cdot \mathrm{ch}^o(F, g^F) \Big) \\
= \int_{E/B} e(TM, \nabla^{TM})\ \mathrm{ch}^o(F, g^F) - \mathrm{ch}^o(H, g^H_{L^2})
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Passing to cohomology, we see as before that

$$\mathrm{ch}^o(H) = \int_{E/B} e(TM)\ \mathrm{ch}^o(F) = tr_{BG}\,\mathrm{ch}^o(F)$$

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Proof.

This follows using the properties of $\widetilde{\mathrm{ch}}^o(H,g^H_{L^2},g^H_V)$ and of the Mathai-Quillen current $\psi(\nabla^{TM},g^{TM})$

2.3.3 A Comparison Formula ?

Comparing the left hand sides of the two versions of the Bismut-Lott index theorem, we could guess a relation between $\mathcal{T}(T^H E, g^{TM}, g^F)$ and $T(V, A', g^V)$

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Conjecture

$$\begin{aligned}\mathcal{T}(T^H E, g^{TM}, g^F) &= T(V, A', g^V) + \widetilde{\mathrm{ch}}^o(H, g^H_{L^2}, g^H_V) \\ &\quad + \int_{E/B} (\nabla^{TM} h)^* \psi(\nabla^{TM}, g^{TM}) \cdot \mathrm{ch}^o(F, g^F) \\ &\qquad + \hat{p}_* \, {}^0J\big(T^s M - T^u M\big) \ \mathrm{rk}\, F\end{aligned}$$

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The proof should use Witten deformation and an adapted quasi-isomorphism

$$I \colon \big(\Omega^\bullet(B; \Omega^\bullet(E/B; F)), \mathbb{A}'\big) \to \big(\Omega^\bullet(B, V), A'\big)$$

2.4.1 Cohomological Invariants

Assume that F carries a parallel decreasing filtration $\mathcal{F}F$ and that all subquotients $\mathcal{F}^k F/\mathcal{F}^{k+1}F$ carry parallel metrics

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Definition

Assume that both F and H are filtered as above. Then put

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If h is framed, put $T(E/B; F) = T(E/B, F, h)$ independent of h

2.4.2 A Cheeger-Müller Type Theorem

Theorem

If h is framed and both torsions below are defined, then

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Theorem (Igusa '08)

Each torsion invariant for families with $H^\bullet(E/B;\mathbb{C}) \to B$ trivial (or, more generally, unipotent) that satisfies the additivity and the transfer axiom is a linear combination of $\tau(E/B,\mathbb{C})$ and the generalised Miller-Morita-Mumford class $tr_{BG}{}^0J(TM)$

Thanks for your attention