The Hodge Conjecture and arithmetic quotients of complex balls

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Ball quotients

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I will be interested in complex manifolds obtained as compact ball quotients:

$$S = S(\Gamma) = \Gamma \backslash \mathbf{B}^n$$

where $\Gamma \subset \mathrm{PU}(n,1)$ is a torsion free discrete subgroup.

The group

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transitively and by biholomorphisms; it preserves the Bergman (symmetric) metric. Denote by Ω the corresponding Kähler form.

Then $\frac{1}{2i\pi}\Omega$ induces a $(1,1)$-form on S which is the Chern form of the canonical fiber bundle. It follows from Kodaira's theorem that S is a complex projective manifold.

Arithmetic ball quotients

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Let E be a CM-field with totally real maximal subfield F, $[F : \mathbf{Q}] = d$. Let V be a non-degenerate anisotropic Hermitian E-vector space of dimension $n+1$. Set

$$G = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{U}(V).$$

Suppose that

$$G(\mathbf{R}) \cong \mathrm{U}(n,1) \times \mathrm{U}(n+1)^{d-1}.$$

Let $\Gamma \subset G^{\mathrm{ad}}(\mathbf{Q})$ be a torsion free congruence subgroup:

$$\Gamma = \Gamma_K := G^{\mathrm{ad}}(\mathbf{Q}) \cap K^{\mathrm{ad}}, \quad K \subset G(\mathbf{A}_f) \text{ compact-open.}$$

Then

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is a compact ball quotient.

Shimura-Deligne: S has a canonical model defined over a finite abelian extension of E.

Main theorem

A cohomology class on S is of level c if it is the pushforward of a cohomology class on a c-codimensional subvariety of S.

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Note that we have:

$$N^c H^k(S, \mathbf{Q}) \subset H^k(S, \mathbf{Q}) \cap \left(\bigoplus_{\substack{a+b=k \\ a,b \geq c}} H^{a,b}(S, \mathbf{C}) \right).$$

In particular $N^c H^k(S, \mathbf{Q}) = \{0\}$ if $k < 2c$.

Our main theorem is:

Theorem 1

Let k, $c \in \mathbf{N}$ s.t. $2k - c < n + 1$. Then we have:

$$N^c H^k(S, \mathbf{Q}) = H^k(S, \mathbf{Q}) \cap \left(\bigoplus_{\substack{a+b=k \\ a,b \geq c}} H^{a,b}(S, \mathbf{C}) \right).$$

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Remark. The RHS is not always a Hodge structure (Grothendieck) whereas the LHS is. So what we prove is a strong version of the generalized Hodge conjecture that cannot hold in general.

Theorem 1 indeed generalizes the Hodge conjecture: consider the case where $k = 2p$ and $c = p$. We have:

Corollary

Let $p \in [0, n] \backslash]\frac{n}{3}, \frac{2n}{3}[$. Then every Hodge class in $H^{2p}(S, \mathbf{Q})$ is a linear combination with rational coefficients of the cohomology classes of subvarieties.

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Proof.

By definition $N^p H^{2p}$ is the rational span of cohomology classes of subvarieties. Now:

$$2k - c = 4p - p = 3p < n + 1 \Leftrightarrow p \in [0, \frac{n}{3}].$$

The corollary therefore follows from Theorem 1 and Poincaré duality. □

Special cycles

Let $t \in \mathbf{N}^*$, $t \leq n$. To any totally positive subspace $U \subset V$ of dimension t we associate

$$H = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{U}(U^{\perp}) \subset \mathrm{Res}_{F/\mathbf{Q}}\mathrm{U}(V) = G.$$

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It corresponds a sub-ball

$$\mathbf{B}_H^{n-t} \subset \mathbf{B}^n \quad \text{(negative lines which lie in } U^\perp\text{).}$$

It is a complex (totally geodesic) submanifold of codimension t.

Let $\Gamma_U = \Gamma \cap H^{\mathrm{ad}}(\mathbf{Q})$. It corresponds an immersion

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- C_U algebraic cycle (defined over an abelian extension of E)
- $[C_U] \in H^{2t}(S, \mathbf{Q}) \cap H^{t,t}(S, \mathbf{C})$.

We rather consider composite cycles: let $\beta \in \mathrm{Herm}_r(E)$ totally positive semidefinite. We set

$$\Omega_\beta = \left\{ \mathbf{x} = (x_1, \ldots, x_r) \in V^r \,\middle|\, \begin{array}{l} \frac{1}{2}((x_i, x_j))_{1,\ldots,r} = \beta \text{ and} \\ \dim \underbrace{\mathrm{Vect}_E(x_1, \ldots, x_r)}_{=U(\mathbf{x})} = \mathrm{rank}(\beta) \end{array} \right\}.$$

The group $\Gamma = \Gamma_K$ acts with finitely many orbits on Ω_β. Given a K-invariant $\varphi \in \mathcal{S}(V(\mathbf{A}_f)^r)$ we define

$$C_{\beta,\varphi} = \sum_{\substack{\mathbf{x} \in \Omega_\beta \\ \mathrm{mod}\ \Gamma}} \varphi(\mathbf{x}) C_{U(\mathbf{x})}.$$

A classical example

Consider the case where V is orthogonal and positive definite of dimension n over $F = \mathbf{Q}$, $\Gamma = \{1\}$ ($K = \prod_p \mathbf{Z}_p$) and $r = 1$.

If $\beta = \frac{k}{2} \in \mathrm{Sym}_1(\mathbf{Q})$ with $k \in \mathbf{N}$ and $\varphi = \mathbf{1}_{V(\prod_p \mathbf{Z}_p)}$. Then $C_{\beta,\varphi}$ is the "number"

$$r(k) = \{x \in \mathbf{Z}^n \mid x_1^2 + \ldots + x_n^2 = k\}.$$

Composite cycles behave well under congruence covers. If $\Gamma' = \Gamma_{K'}$ and $\Gamma = \Gamma_K$ with $K' \subset K$, consider the natural projection $p : S(\Gamma') \to S(\Gamma)$. We have:

$$p^*(C_{\beta,\varphi}) = C_{\beta,\varphi}.$$

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For all $\beta \gg 0$ we finally get a $G(\mathbf{A}_f)$-equivariant map

$$\begin{cases} \mathcal{S}(V(\mathbf{A}_f)^r)_{\mathbf{Q}} & \to & H^{2r}(\mathrm{Sh}(G), \mathbf{Q}) := \lim_{\substack{\to \\ \Gamma}} H^{2r}(S(\Gamma), \mathbf{Q}) \\ \varphi & \mapsto & [\beta, \varphi] := [C_{\beta,\varphi}]. \end{cases}$$

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We extend this construction to positive semidefinite β's by setting:

$$[\beta, \varphi] = L^{r-t}([C_{\beta,\varphi}]) \text{ where } t = \mathrm{rank}(\beta)$$

and L is the cup-product with the Lefschetz class Ω.

Definition

Let $SC^{2r}(\mathrm{Sh}(G)) \subset H^{2r}(\mathrm{Sh}(G), \mathbf{Q})$ be the subspace spanned by the classes $[\beta, \varphi]$.

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It is a rational subspace that is Hecke stable and it follows from Kudla-Millson that

$$SC^{\bullet}(\mathrm{Sh}(G)) := \bigoplus_{r} SC^{2r}(\mathrm{Sh}(G))$$

is a subring of $H^{\bullet}(\mathrm{Sh}(G), \mathbf{Q})$.

On the proof of Theorem 1

Our main technical result is the following:

Theorem 2

Let $a, b \in \mathbf{N}$ (with say $a \geq b$) and $3(a+b)+|a-b|<2(n+1)$.

- If $a > b$ (and $c = a - b$), then the natural cup-product map

$$SC^{2b}(\mathrm{Sh}(G)) \times H^{c,0}(\mathrm{Sh}(G), \mathbf{C}) \to H^{a,b}_{\mathrm{prim}}(\mathrm{Sh}(G), \mathbf{C})$$

  is surjective.

- If $a = b$, then the natural cup-product map

$$SC^{2(a-1)}(\mathrm{Sh}(G)) \times H^{1,1}(\mathrm{Sh}(G), \mathbf{C}) \to H^{a,a}_{\mathrm{prim}}(\mathrm{Sh}(G), \mathbf{C})$$

  is surjective.

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Remarks. 1. The symmetrical result with $a < b$ and $H^{c,0}$ replaced by $H^{0,c}$ holds as well.
2. When $a = b$, it is not true that $SC^{2a} \to H^{a,a}_{\mathrm{prim}}$ is surjective: totally geodesic codimension one cycles don't span.

We furthermore prove:

Proposition

If $a, b \in \mathbf{N}$ with $3(a+b) + |a-b| < 2(n+1)$ then there exists a rational subspace $Y \subset H^{a+b}(\mathrm{Sh}(G), \mathbf{Q})$ s.t.

$$Y \otimes_{\mathbf{Q}} \mathbf{C} = H^{a,b}(\mathrm{Sh}(G), \mathbf{C}) \oplus H^{b,a}(\mathrm{Sh}(G), \mathbf{C}).$$

Theorem 2 $\Rightarrow$ Theorem 1

It follows from the above Proposition and Theorem 2 that we have:

- if $a = b$ the image of the cup-product map

$$SC^{2(a-1)}(\mathrm{Sh}(G)) \times \left(H^{1,1}(\mathrm{Sh}(G), \mathbf{C}) \cap H^2(\mathrm{Sh}(G), \mathbf{Q})\right) \\ \to H^{a,a}(\mathrm{Sh}(G), \mathbf{C})$$

  spans the whole primitive subspace when $6a < 2(n+1)$ (i.e. $a < \frac{1}{3}(n+1)$).
  The Hodge Conjecture then follows from Lefschetz' $(1,1)$-Theorem.

- if $a \neq b$ cup-products of classes in $SC^{2\min(a,b)}(\mathrm{Sh}(G))$ and in

$$\left(H^{|a-b|,0}(\mathrm{Sh}(G),\mathbf{C}) \oplus H^{0,|a-b|}(\mathrm{Sh}(G),\mathbf{C})\right) \cap H^{|a-b|}(\mathrm{Sh}(G),\mathbf{Q})$$

span

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when $3(a+b) + |a-b| < 2(n+1)$.
This applies to

$$\bigoplus_{\substack{a+b=k \\ a,b \geq c}} H^{a,b}(\mathrm{Sh}(G),\mathbf{C})$$

as long as

$$3k + k - 2c < 2(n+1) \Leftrightarrow 2k - c < n+1.$$

Here $k = a+b$ and $|a-b|$ is maximal if say $b = c$ and $a = k - c$.

Theta series

Consider the split Hermitian space W of even dimension $2r$ over E. Let

$$G' = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{U}(W).$$

Given $\beta \geq 0$, it corresponds a Whittaker function $W_\beta(g')$ on $G'(\mathbf{R}) = \mathrm{U}(r, r)^d$.

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Example

In case $r = 1$ and $d = 1$, the symmetric space associated to $G'(\mathbf{R}) = \mathrm{U}(1,1)$ identifies with the Poincaré upper half plane $\mathbf{H}$ and the function W_β defines a function on $\mathbf{H}$ which turns out to be

$$\tau \mapsto e^{2i\pi\tau\beta}.$$

Kudla-Millson introduce the generating series

$$\theta_\varphi(g') = \sum_{\beta \geq 0} [\beta, \varphi] W_\beta(g')$$

with values in $H^{r,r}(\mathrm{Sh}(G), \mathbf{C})$.

A classical example

Consider again the case where V is orthogonal and positive definite of dimension n over $F = \mathbf{Q}$, $\Gamma = \{1\}$ ($K = \prod_p \mathbf{Z}_p$) and $r = 1$.

Then

$$\theta_\varphi(\tau) = \sum_{x \in \mathbf{Z}^n} e^{i\pi\tau(x,x)} = \sum_{k=0}^{+\infty} r(k) e^{i\pi\tau k}$$

is a classical θ-series.

$\rightarrow$ It is modular in τ.

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with values in $H^{r,r}(\mathrm{Sh}(G), \mathbf{C})$.

Kudla-Millson:

The map

$$g' \mapsto \theta_\varphi(g') \in H^{r,r}(\mathrm{Sh}(G), \mathbf{C})$$

defines an automorphic form.

The proof follows from a Poisson summation formula in $L^2(V(\mathbf{A})^r)$ applied to $\phi = \varphi_{\mathrm{KM}} \otimes \varphi$ with

$$\varphi_{\mathrm{KM}} \in \mathrm{Hom}_{\mathrm{U}(n)\times\mathrm{U}(1)}(\wedge^{r,r}\mathbf{C}^n, \mathcal{S}(V_\infty^r)) = \mathrm{Hom}_{\mathrm{U}(n,1)}(\Omega^{r,r}(\mathbf{B}^n), \mathcal{S}(V_\infty^r))$$

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It yields a lifting map

$$\{\text{automorphic forms on } G'\} \to H^{r,r}(\mathrm{Sh}(G), \mathbf{C})$$

whose image is $\mathrm{Span}\{[\beta, \varphi]\}$.

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I can now explain the main ideas of the proof of Theorem 2.

On the proof of Theorem 2

Decompose

$$H^\bullet(\mathrm{Sh}(G), \mathbf{C}) = \bigoplus_{\pi \text{ autom}} H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty) \otimes \pi_f .$$

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Decompose

$$H^\bullet(\mathrm{Sh}(G), \mathbf{C}) = \bigoplus_{\pi \text{ autom}} H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty) \otimes \pi_f.$$

We first prove:

Theorem 3

Let $a, b \in \mathbf{N}$ be s.t. $3(a+b) + |a-b| < 2(n+1)$. If π is s.t.

$$H^{a,b}(\mathfrak{g}, K_\infty; \pi_\infty) \neq \{0\},$$

then π is a θ-lift from a $\mathrm{U}(W)$ of signature (a, b) at infinity.

We then prove that if $\mathrm{U}(W)$ is quasi-split ($\Leftrightarrow$ W split) then any θ-lift belongs to the span of KM lifts.

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[As φ is arbitrary, this amounts to local (Archimedean) computations: show that one may reduce to $\varphi_\infty = \varphi_{\mathrm{KM}}$. This boils down to compute part of the $(\mathfrak{g}, K)$-cohomology of $\mathcal{S}(V_\infty^r)$.]

Theorem 2 then follows from the classification of Hermitian spaces:

- If $a > b$, then W decomposes as

$$W = W^{\text{split}}_{b,b} \oplus W_{a-b,0}.$$

- If $a = b$, then W decomposes as

$$W = W^{\text{split}}_{a,a} \text{ or } W = W^{\text{split}}_{a-1,a-1} \oplus W_{1,1}.$$

'Functoriality' of θ-lift then implies that

- $\theta(W) = \underbrace{\theta(W^{\text{split}}_{b,b})}_{\text{cycle}} \wedge \underbrace{\theta(W_{a-b,0})}_{\text{holom class of deg a}-\text{b}}$
- $\theta(W) = \underbrace{\theta(W^{\text{split}}_{a,a})}_{\text{cycle}}$ or $\underbrace{\theta(W^{\text{split}}_{a-1,a-1})}_{\text{cycle}} \wedge \underbrace{\theta(W_{1,1})}_{(1,1)-\text{class}}$.

On the proof of Theorem 3

There are two steps:

- If π is 'very non tempered' then π is a θ-lift.
  [The precise statement is the analogue for unitary groups of a theorem of Kudla-Rallis. The main ingredient of the proof is due to Ichino.]

On the proof of Theorem 3

There are two steps:

- If π is 'very non tempered' then π is a θ-lift.
  [The precise statement is the analogue for unitary groups of a theorem of Kudla-Rallis. The main ingredient of the proof is due to Ichino.]
- Arthur's classification of automorphic representations implies that *if an automorphic representation is very non tempered at one place then it is very non tempered everywhere.* We are therefore reduced to study the (non) temperedness of representations π_∞ s.t. $H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty) \neq \{0\}$.
  $\rightarrow$ condition $3(a+b) + |a-b| < 2(n+1)$.

A weak form of Arthur's endoscopic classification of automorphic representations of classical groups implies:

Weak base change

Let π be an irreducible automorphic representation of G which occurs as an irreducible subspace of $L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))$. Then, there exists a (unique) global representation $\Pi = \Pi_\Psi$ of $\mathrm{GL}(n+1, \mathbf{A}_E)$, induced from square integrable automorphic representations encoded in a parameter

$$\Psi = \mu_1 \boxtimes R_{m_1} \boxplus \ldots \boxplus \mu_r \boxtimes R_{m_r},$$

and a finite set of places S s.t. we have:

$$L^S(s, \pi_v) = L^S(s, \Pi_v).$$

Here each μ_j is an irreducible, unitary, cuspidal automorphic representation of $\mathrm{GL}(d_i)/F$, R_{m_j} is an irreducible representation of $\mathrm{SL}_2(\mathbf{C})$ of dimension m_j and $n+1 = m_1 d_1 + \ldots + m_r d_r$.

Given any $\mu_j \boxtimes R_{m_j}$ we form the induced representation

$$\mathrm{ind}(\mu_j|\cdot|_{\mathbf{A}_E}^{\frac{1}{2}(m_j-1)}, \mu_j|\cdot|_{\mathbf{A}_E}^{\frac{1}{2}(m_j-3)}, \ldots, \mu_j|\cdot|_{\mathbf{A}_E}^{\frac{1}{2}(1-m_j)})$$

(normalized induction from the standard parabolic subgroup of type $(d_j, \ldots, d_j)$). We then write Π_j for the unique irreducible quotient of this representation; it is a square integrable automorphic representation.

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We finally define Π_Ψ as the induced representation

$$\mathrm{ind}(\Pi_1 \otimes \ldots \otimes \Pi_r)$$

(normalized induction from the standard parabolic subgroup of type $(m_1 d_1, \ldots, m_r d_r)$).

The Euler product

$$L^S(s,\Pi_\psi) = \prod_{j=1}^{r}\prod_{v\notin S} L_v(s-\frac{m_j-1}{2},\mu_{j,v})\dots L_v(s-\frac{1-m_j}{2},\mu_{j,v})$$

is the product of partial L-functions of the square integrable automorphic representations associated to the parameters $\mu_j \boxtimes R_{m_j}$.

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is the product of partial L-functions of the square integrable automorphic representations associated to the parameters $\mu_j \boxtimes R_{m_j}$.

Example

If μ_j is a Dirichlet character then $L^S(s,\mu_j)$ is just its usual (partial) Dirichlet series. In particular if $\mu_j = 1$ then $L^S(s,\mu_j)$ is the (partial) Riemann zeta function; it has a pole in $s = 1$.

Kudla-Rallis, Ichino, ...

Let η be a character of $\mathbf{A}_E^\times/E^\times$. Assume that there exists some integer $u > 1$ such that the partial L-function $L^S(s, \pi \times \eta)$ is holomorphic in the half-plane $\mathrm{Re}(s) > \frac{1}{2}(u-1)$ and has a pole in $s = \frac{1}{2}(u-1)$.

Then there exists some $(n+1-u)$-dimensional Hermitian space over E such that π is in the image of the cuspidal theta correspondence from the group $\mathrm{U}(W)$.

The goal is to construct W such that

$$\Theta_{V \to W}(\pi) \neq 0. \quad (1)$$

The result then follows by duality.

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$$\Theta_{V \to W}(\pi) \neq 0. \tag{1}$$

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To prove (1) we take f in the space of π and compute the square of the Petersson norm of $\theta(f)$. The latter takes the rough form:

$$||\theta(f)||^2 = c\mathrm{Res}_{s=\frac{u}{2}} L^S(s + \frac{1}{2}, \pi \times \eta),$$

where c is a non-zero constant involving local coefficients of the oscillator representation and η some character.

If π_∞ is cohomological and is the Langlands' quotient of a standard representation with an exponent $(z/\bar{z})^{p/2}(z\bar{z})^{(u-1)/2}$, we have:

Lemma

If $3u > n + 1 + |p|$ then in the parameter Ψ some of the factor $\mu_j \boxtimes R_{m_j}$ is such that $m_j \geq u$ and the representation μ_j is a character.

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Vogan-Zuckerman: if π_∞ is cohomological of degree (a, b) then it is the Langlands' quotient of a standard representation with an exponent $(z/\bar{z})^{(b-a)/2}(z\bar{z})^{(n+1-(a+b)-1)/2}$. This yields the condition

$$3(n+1) - 3(a+b) > n+1+|a-b| \Leftrightarrow 3(a+b)+|a-b| < 2(n+1).$$

THE END