

The Atiyah–Singer Theorems: A Probabilistic Approach. I. The Index Theorem

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The Atiyah–Singer index theorem for classical elliptic complexes is proved by using probabilistic methods. The general idea is to use a probabilistic construction of the heat equation kernel, which permits the direct derivation of the index formulas, without using the theory of invariants of Gilkey. Stochastic calculus on the exterior algebra is then used to find the classical local formula for the index theorem. The Lefschetz fixed point formulas of Atiyah–Singer will be proved in Part II.

INTRODUCTION

The purpose of this paper is to give a direct proof of the index theorem of Atiyah–Singer [6] for classical elliptic complexes, by using a constructive version of the heat equation method, namely, by using probabilistic methods.

Before going into details, we first give a brief history of the heat equation method in the proof of the Index Theorem. As an alternative to the initial proofs by Atiyah–Singer of the Index Theorem which were topological (for the second proof see [6]), Atiyah–Bott [3] suggested the use of the ζ function of the considered elliptic operator, whose asymptotics could be studied by using the construction by Seeley [36] of the powers of an elliptic operator. Mc Kean–Singer [29] conjectured that for the de Rham complex and the operator $d + \delta$ acting on even and odd forms, the limit as $t \downarrow 0$ of the integral formula for the Euler number furnished by the heat equation method would exactly give the Chern–Gauss–Bonnet integrand. They could prove this only in dimension 2.

In a spectacular paper [33] Patodi showed that remarkable cancellations were indeed taking place, and that the conjecture of Mc Kean–Singer was true. In [19], Gilkey developed the theory of invariants, which permitted the elimination on a priori grounds of the unwanted terms involving higher order covariant derivatives of the curvature, this for the Euler number, and also for the Hirzebruch signature complex.

In [4], Atiyah–Bott–Patodi extended Gilkey’s method to elliptic complexes with coefficients in an auxiliary bundle, and treated the Index Theorem for a large class of elliptic operators. Also heat equations methods were used to give an analytic proof of the results of Atiyah–Bott [3], Atiyah–Singer [6] on Lefschetz fixed point formulas for elliptic complexes. For a complete review of these methods, we refer to Gilkey [21]. Still the proofs use the theory of invariants of Gilkey in more and more refined versions.

On the probability theory side, Malliavin [28] initiated the use of probabilistic methods in studying the cohomology of a manifold. In particular he systematically used a subordination procedure, which permitted the construction of the heat kernel on differential forms by means of the diffusion associated to the Laplace–Beltrami operator, i.e., the Brownian motion on the considered Riemannian manifold.

In [26, 27] Malliavin developed a stochastic calculus of variations, which permitted him to prove regularity results for the heat kernel of hypoelliptic second order differential operators, this by considering the heat kernel as the image of the standard Wiener measure by a stochastic differential equation. In particular Malliavin used integration by parts on Wiener space to prove these regularity results. The Malliavin calculus has been the object of several new developments. For a review and references, see [9].

Also the use of stochastic flows in stochastic analysis started by Malliavin and subsequently developed in various directions [7–9] (and references therein), [16] permitted to deal more and more with stochastic differential equations as if they were standard differential equations. In particular, it became possible to differentiate with respect to parameters not only in a L_2 sense, but also in more natural a.s. sense.

In our paper [10], we studied the problem of finding an asymptotic expression as $t \downarrow 0$ of the heat kernel semi-group associated to a second order hypoelliptic operator. The basic idea of [10] is to consider the heat kernel as the image of the Wiener measure by a stochastic differential equation, and to construct *explicitly* the corresponding disintegration of the standard Wiener measure on a finite codimensional fibration. The program of [10] was fully completed in the elliptic case. The main technique of [10] is the construction of an adequate orthogonal split of the Wiener measure, which is adapted to the geometry of the problem, in combination with the Malliavin calculus and the theory of large deviations (Ventcell–Freidlin [39], Varadhan [38]). To obtain geometrically invariant quantities, we used the construction by Malliavin [27], Eells–Elworthy [15, 16] of the Brownian motion on a Riemannian manifold. Also, we noted that the method of [10] could be used to construct the heat kernel on elliptic complexes, and that this construction had all the rotational invariance properties which were exploited in [4] for the proof of the Index Theorem.

In [2], Atiyah suggested an entirely different route for the proof of the Index Theorem for the spin complex and the corresponding Dirac operator on spin manifolds. Namely Witten [41] (see also [40]) had exhibited a natural closed 2-form on the loop space $\mathcal{L}(M)$ of a Riemannian manifold. Atiyah then noticed that the index formula for the Dirac operator, when expressed in terms of path integrals, could be written as an integral with respect to a formal exterior power of S . He then used the fact that S_1 has a natural "symplectic" action on $\mathcal{L}(M)$ (by time reparametrization of the loops) and that M is exactly the fixed point set of this action. By applying formally a cohomological formula of Duistermaat–Heckman [14] (proved in finite dimensions) and by renormalizing adequately the result of [14], he found the classical cohomological formula for the index of the Dirac operator. Also Professor Atiyah pointed out to us a recent paper by Getzler [44], in which the Index Theorem for the spin complex is proved by using pseudo-differential operators techniques.

Atiyah's conference (which took place at a Congress in honor of Schwartz) pushed us to test the methods of our paper [10] on the spin complex. The fact that we could obtain the Index Theorem for the spin complex was the starting point of this work.

This work consists of two parts. In part I, we prove the Atiyah–Singer Index Theorem for classical elliptic complexes. In part II (which will appear in a later issue of this journal) we prove the corresponding Lefschetz fixed point formulas of Atiyah–Bott [3] and Atiyah–Singer [6]

Part I contains the first three sections of the paper. In Section 1, basic facts concerning the spin representation, spinors and the Dirac operator are recalled from Atiyah [1], Atiyah–Bott [3], Atiyah–Bott–Patodi [4], Atiyah–Bott–Shapiro [5], Atiyah–Singer [6], Lichnerowicz [25]. We have tried to make precise the sign conventions which we shall later use. It should be pointed out that it is the very special expression of the square of the Dirac operator in term of the horizontal Laplacian which makes the spin complex so adapted to a probabilistic treatment.

In Section 2, we briefly construct the heat equation kernel for the Dirac operator, and apply our results of [10] to give an asymptotic expression of this heat kernel as an expectation of a t -depending random variable calculated on certain Brownian bridges.

In Section 3, the Atiyah–Singer Index Theorem is proved for twisted spin complexes, so that the limit of the integrand of the trace formula as $t \downarrow 0$ is proved to be exactly what should be expected (in the sense of McKean–Singer [29], Patodi [33], Atiyah–Bott–Patodi [4]). Note that in the whole paper, we use differential forms, and exceptionally cohomology classes. A typical feature of the method is that, to work with twisted spin complexes, we introduce an auxiliary Brownian motion which in some way correlates the auxiliary bundle and the diffusion on the manifold. We obtain

the limit of the integrand in the trace formula as an expectation over certain Brownian bridges. This expression is transformed into the classical expression of [4–6] by using stochastic calculus in the algebra of even forms. Certain results of Lévy are of critical importance in this section and in Section 4.

No systematic introduction to the index theorems of Atiyah–Singer has been attempted. We refer to [4, 21] for more details. Also the reader unfamiliar with stochastic calculus can consult [12] (and the references therein) [23, 30]. For an introduction to the Malliavin calculus we refer to [9]. Finally a careful look at the introduction of our paper [10] could ease the reading of this paper.

The results in this paper have been announced in [43].

I. SPIN REPRESENTATION AND THE DIRAC OPERATOR

In this section, we will recall some well-known facts concerning $SO(n)$, $\text{Spin}(n)$ (which is the double cover of $SO(n)$), spinors, spin-manifolds, and the Dirac operator with coefficients in an auxiliary bundle. Our main sources are Atiyah [1], Atiyah–Bott [3], Atiyah–Bott–Patodi [4], Atiyah–Bott–Shapiro [5], Atiyah–Singer [6], and Lichnerowicz [25].

We have tried to present in the simplest way as possible the $\text{Spin}(n)$ representation, and also to make explicit the sign conventions which we shall later use.

In 1.a, and following [1–6], the group $\text{Spin}(n)$ and its irreducible representations on the spaces S_+ , S_- of positive and negative spinors are introduced. Also if $S = S_+ \oplus S_-$, the complex exterior algebra $\bar{\Lambda}(R^n)$ is canonically identified to $S \otimes_{\mathbb{C}} S^*$ (where S^* is the dual of S).

In 1.b, following [3], trace formulas for the representation of $\text{Spin}(n)$ on S_+ , S_- are given. In 1.c the Lie algebra \mathcal{O} of $SO(n)$ is identified to $\Lambda^2(R^n)$, its action on S is described, and the Pfaffian is introduced. In 1.d, an essential differentiation formula is proved. Namely, if $n = 2l$, if $A \in \mathcal{O}$, if e^{tA} is calculated in $\text{Spin}(n)$, if $\chi_+(e^{tA})$, $\chi_-(e^{tA})$ are the traces of the action of e^{tA} on S_+ , S_- , then

$$\lim_{t \downarrow 0} \frac{\chi_+(e^{tA}) - \chi_-(e^{tA})}{t^l} = i^l \text{Pf } A. \tag{1.1}$$

This formula will play a key role in eliminating the singularity $1/t^l$ of the heat equation kernel.

In 1.e, and following [1–6], spin-manifolds and the Dirac operator on the spin complex are introduced. In 1.f the Dirac operator D acting on a twisted spin complex is also defined.

In 1.g D^2 is evaluated in terms of the horizontal Laplacian Δ^H , as in Lichnerowicz [25], Hitchin [22]. This explicit formula will be of utmost importance in the sequel. It is indeed because D^2 has a relatively simple expression in terms of Δ^H that probabilistic methods can be put at work in the index problem, independently of the fact that using twisted spin complexes, the other classical complexes can also be constructed, as in Atiyah–Bott–Patodi [4]

1.a. The Clifford Algebra and the Group $\text{Spin}(n)$

Let E be a real vector space of even dimension $n = 2l \geq 2$, which is oriented, and endowed with a positive definite inner product. We denote by cE the Clifford algebra over E , i.e., cE is the quotient of the full real tensor algebra over E modulo the ideal I generated by elements of the form $e \otimes e + \langle e, e \rangle$.

If e_1, \dots, e_n is an orthogonal base of E , cE is the free algebra generated over \mathbb{R} by a unit 1, and e_1, \dots, e_n with the defining relations

$$e_i^2 = -1; \quad e_i e_j + e_j e_i = 0, \quad i \neq j. \quad (1.2)$$

cE is a filtered algebra, and it canonically isomorphic (as a vector space) to the real exterior algebra $\Lambda(E)$; moreover the gradings of cE and $\Lambda(E)$ correspond.

$c_+ E$ and $c_- E$ denote the vector subspaces of cE generated by the even and odd products in cE . Clearly $cE = c_+ E \oplus c_- E$, $E \subset c_- E$, and $c_+ E$, $c_- E$ identify to $\Lambda^{\text{even}}(E)$, $\Lambda^{\text{odd}}(E)$ (which are the spaces of even and odd forms on E).

Let $x \rightarrow \bar{x}$ be the antiautomorphism of cE which sends $e_{j_1} \cdots e_{j_k}$ into $(-1)^k e_{j_k} \cdots e_{j_1}$. Denote by $c_* E$ the group of invertible elements of cE . $\text{Pin}(n)$ is the subgroup of the $x \in c_* E$ such that

$$xEx^{-1} \subset E, \quad \bar{x}x = 1. \quad (1.3)$$

$\text{Spin}(n)$ is the subgroup of $\text{Pin}(n)$,

$$\text{Spin}(n) = \text{Pin}(n) \cap c_+ E. \quad (1.4)$$

For any $x \in \text{Spin}(n)$, the transformation $\sigma(x)$ of E into E defined by

$$\sigma(x)e = xex^{-1} \quad (1.5)$$

is in $SO(n)$. The group homomorphism $\sigma: \text{Spin}(n) \rightarrow SO(n)$ is onto and the kernel of σ is generated by -1 . $\text{Spin}(n)$ is the double covering of $SO(n)$.

The complex algebra $\bar{c}E = cE \otimes_{\mathbb{R}} \mathbb{C}$ (which has complex dimension 2^n) can be identified to the full matrix algebra of a complex Hermitian vector space S , of dimension 2^l . S is called the space of spinors.

$\text{Pin}(n)$ acts unitarily and irreducibly on S . From now, we assume that the orthogonal base e_1, \dots, e_n is *oriented*. Set

$$\omega = e_1 \cdots e_n, \quad \tau = i^l \omega; \tag{1.6}$$

ω does not depend on the oriented base e_1, \dots, e_n . Moreover,

$$\omega^2 = (-1)^l, \quad \tau^2 = 1. \tag{1.7}$$

S_+, S_- denote the eigenspaces of τ

$$S_+ = \{s \in S; \tau s = s\}, \quad S_- = \{s \in S; \tau s = -s\}. \tag{1.8}$$

S_+ and S_- have complex dimension 2^{l-1} , are orthogonal in S so that

$$S = S_+ \oplus S_-. \tag{1.9}$$

S_+, S_- are the spaces of positive and negative spinors. If the orientation of E is changed, τ is changed into $-\tau$, so that S_+, S_- are interchanged.

Since τ commutes with the elements of $c_+ E$, $\text{Spin}(n)$ sends S_+ in S_+ and S_- in S_- . Moreover $\text{Spin}(n)$ acts unitarily and irreducibly on S_+ and S_- .

If $e \in E$, since $\tau e = -e\tau$, the Clifford multiplication operator $s \in S \rightarrow es \in S$ sends S_+ into S_- and S_- into S_+ . If $x \in \text{Spin}(n)$

$$x(es) = (xex^{-1})xs = \sigma(x) e \cdot xs. \tag{1.10}$$

The Clifford multiplication $E \otimes S \rightarrow S$ is then a homomorphism of $\text{Spin}(n)$ modules.

Let S^*, S_+^*, S_-^* be the duals of S, S_+, S_- (i.e., the set of complex valued linear mappings on S, S_+, S_-). By [31, 14], since S, S_+, S_- are Hermitian, S^*, S_+^*, S_-^* are canonically isomorphic to the conjugate vector spaces $\bar{S}, \bar{S}_+, \bar{S}_-$.

$\text{Spin}(n)$ acts naturally on S^* . Namely, if $x \in \text{Spin}(n)$, if \tilde{x} is the transpose of x (which sends S^* into S^*), x acts on S^* by

$$s^* \in S^* \rightarrow \tilde{x}^{-1} s^* \in S^*. \tag{1.11}$$

Since $\bar{c}E$ is the full matrix algebra of S , we obviously have

$$\bar{c}E = S \otimes_C S^*. \tag{1.12}$$

Since $\bar{c}E$ and the complex exterior algebra $\bar{A}(E)$ are isomorphic, we have the identification

$$\bar{A}(E) = S \otimes_C S^*. \tag{1.13}$$

If τ' is the operator acting on $S \otimes_C S^*$

$$\tau' = \tau \otimes 1 \quad (1.14)$$

using the identification (1.13), τ' is exactly the signature operator of $\bar{A}(E)$ [1-6]. In fact if $*$ is the usual duality operator in $\bar{A}(E)$, τ' acts on $\bar{A}^p(E)$ (which is the space of p forms on E) by

$$\tau' = i^{p(p-1)+l} * . \quad (1.15)$$

If $\bar{A}_+(E)$, $\bar{A}_-(E)$ are the eigenspaces

$$\bar{A}_+(E) = \{\lambda \in \bar{A}(E); \tau'\lambda = \lambda\}, \quad \bar{A}_-(E) = \{\lambda \in \bar{A}(E); \tau'\lambda = -\lambda\}, \quad (1.16)$$

we have the identifications

$$\bar{A}_+(E) = S_+ \otimes S^*, \quad \bar{A}_-(E) = S_- \otimes S^*. \quad (1.17)$$

Moreover $\bar{c}_+(E) = c_+ E \otimes_R C$ is exactly the set of elements of $\bar{c}E$ which send S_+ into S_+ and S_- into S_- , i.e.,

$$\bar{c}_+ E = (S_+ \otimes_C S_+^*) \oplus (S_- \otimes_C S_-^*). \quad (1.18)$$

Similarly if $\bar{c}_- E = c_- E \otimes_R C$, then

$$\bar{c}_- E = (S_+ \otimes_C S_-^*) \oplus (S_- \otimes_C S_+^*).$$

We then have the identifications

$$\begin{aligned} \bar{A}^{\text{even}}(E) &= (S_+ \otimes_C S_+^*) \oplus (S_- \otimes_C S_-^*), \\ \bar{A}^{\text{odd}}(E) &= (S_+ \otimes_C S_-^*) \oplus (S_- \otimes_C S_+^*). \end{aligned} \quad (1.19)$$

$\text{Spin}(n)$ acts on $S \otimes S^*$ by inner automorphisms. Namely, if $x \in \text{Spin}(n)$, the action $\rho(x)$ of x on $S \otimes S^*$ is given by

$$\rho(x)(s \otimes s^*) = xs \otimes \tilde{x}^{-1}s^*. \quad (1.20)$$

Moreover $SO(n)$ acts naturally on $\bar{A}(E)$. If $y \in SO(n)$, let $\rho'(y)$ denote the action of y on $\bar{A}(E)$. It is then easy to verify that

$$\rho = \rho' \sigma. \quad (1.21)$$

1.b. Some Trace Formulas

Take $\theta_1, \dots, \theta_l \in R$. Set

$$x(\theta_1, \dots, \theta_l) = \prod_1^l \left(\cos \frac{\theta_j}{2} + \sin \frac{\theta_j}{2} e_{2j-1} e_{2j} \right). \quad (1.22)$$

Then $x(\theta_1, \dots, \theta_l) \in \text{Spin}(n)$, and moreover $x' = \sigma(x(\theta_1, \dots, \theta_l)) \in \text{SO}(n)$ is exactly given by

$$\begin{aligned} x'e_{2j-1} &= \cos \theta_j e_{2j-1} + \sin \theta_j e_{2j}, \\ x'e_{2j} &= -\sin \theta_j e_{2j-1} + \cos \theta_j e_{2j} \end{aligned} \tag{1.23}$$

(note that our sign conventions differ from [3, p. 482]).

Let e, e' be the identity mappings in $\text{SO}(n), \text{Spin}(n)$. Formula (1.22) permits us to lift explicitly a neighborhood of e in a neighborhood of e' , and more generally to lift any continuous curve in $\text{SO}(n)$ to a continuous curve in $\text{Spin}(n)$.

In particular let \mathcal{A} be the Lie algebra of $\text{SO}(n)$. \mathcal{A} is also the Lie algebra of $\text{Spin}(n)$. If $A \in \mathcal{A}$, (1.22) permits us to calculate explicitly $e^{tA} \in \text{Spin}(n)$.

DEFINITION 1.1. If $x \in \text{Spin}(n)$, $\chi_+(x)$ (resp. $\chi_-(x)$) denotes the trace of the action of x on S_+ (resp. S_-).

We then have the following result:

PROPOSITION 1.2. If $x = x(\theta_1, \dots, \theta_l)$ is given by (1.22), then

$$\begin{aligned} \chi_+(x) - \chi_-(x) &= \prod_1^l \left(e^{-\frac{i\theta_j}{2}} - e^{\frac{i\theta_j}{2}} \right), \\ \chi_+(x) + \chi_-(x) &= \prod_1^l \left(e^{-\frac{i\theta_j}{2}} + e^{\frac{i\theta_j}{2}} \right). \end{aligned} \tag{1.24}$$

Proof. The first line is proved in Atiyah–Bott [3, p. 484], with the observation that, as we have previously indicated, we have changed signs in (1.22) with respect to [3]. The second line is also obvious from the proof in [3]. ■

Remark 1. Recall that the action of $x \in \text{Spin}(n)$ on S^* is given by (1.11). If $\chi_+^*(x), \chi_-^*(x)$ are the traces of the action of x on S_+^*, S_-^* , by taking conjugates in (1.24), we find that if $x = x(\theta_1, \dots, \theta_l)$,

$$\begin{aligned} \chi_+^*(x) - \chi_-^*(x) &= \prod_1^l \left(e^{\frac{i\theta_j}{2}} - e^{-\frac{i\theta_j}{2}} \right), \\ \chi_+^*(x) + \chi_-^*(x) &= \prod_1^l \left(e^{\frac{i\theta_j}{2}} + e^{-\frac{i\theta_j}{2}} \right). \end{aligned} \tag{1.25}$$

From (1.24), (1.25), we see that

$$\begin{aligned}
 (\chi_+(x) - \chi_-(x))(\chi_+^*(x) + \chi_-^*(x)) &= \prod_1^l (e^{-i\theta_j} - e^{i\theta_j}), \\
 (\chi_+(x) - \chi_-(x))(\chi_+^*(x) - \chi_-^*(x)) &= 2^l \prod_1^l (1 - \cos \theta_j).
 \end{aligned}
 \tag{1.26}$$

The first line of (1.24) fits with the identifications (1.17), (1.21) and the trace formula in [6, p. 577] which expresses the difference of the traces of $\sigma(x)$ on $\bar{A}_+(E)$ and $\bar{A}_-(E)$ as being given by $\prod_1^l (e^{-i\theta_j} - e^{i\theta_j})$.

Similarly since

$$2^l \prod_1^l (1 - \cos \theta_j) = \det(e - \sigma(x)) \tag{1.27}$$

the second line of (1.26) fits with the identifications (1.19), (1.21) and the fact that the difference of the traces of $\rho'\sigma(x)$ on $\bar{A}^{\text{even}}(E)$ and $\bar{A}^{\text{odd}}(E)$ is exactly $\det(e - \sigma(x))$.

1.c. A Few Properties of the Lie Algebra \mathcal{O}

Recall that \mathcal{O} is exactly the set of real (n, n) antisymmetric matrices. In the sequel we identify $A \in \mathcal{O}$ to the element of $\mathcal{A}^2(E)$,

$$(X, Y) \rightarrow \langle X, AY \rangle. \tag{1.28}$$

If $A = (a_i^j)$, the associated element in $\mathcal{A}^2(E)$ is given by

$$\frac{1}{2} \sum_{i,j} a_j^i dx^i \wedge dx^j. \tag{1.29}$$

We will identify $A \in \mathcal{O}$ with (1.29). In particular, the exterior powers of A are well defined. We will note them $A^{\wedge 2}, \dots, A^{\wedge l}$.

Also remark that since $\text{Spin}(n)$ acts on S , its Lie algebra \mathcal{O} also acts on S . Namely, we have

PROPOSITION 1.3. *The action of $A = (a_i^j) \in \mathcal{O}$ on S is given by*

$$\frac{1}{4} \sum_{i,j} a_i^j e_i e_j. \tag{1.30}$$

Proof. Note that if $\theta_1, \dots, \theta_l \in R$,

$$\left[\frac{d}{ds} x(s\theta_1, \dots, s\theta_l) \right]_{s=0} = \frac{1}{2} \sum_j \theta_j e_{2j-1} e_{2j}.$$

Moreover, if

$$A = \sigma^* \left[\frac{d}{ds} x(s\theta_1, \dots, s\theta_l) \right]_{s=0}$$

by (1.23), A is the matrix made of the diagonal blocks $\begin{bmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{bmatrix}$. Formula (1.30) is then proved for such A . Since $A \in \mathcal{O}$ can always be expressed in this form on an (oriented) orthogonal base, the proposition is proved. ■

We now introduce the Pfaffian of $A \in \mathcal{O}$. By [37] the Pfaffian of A , which is denoted $\text{Pf } A$, is a polynomial function on \mathcal{O} such that

$$(\text{Pf } A)^2 = \det A. \tag{1.31}$$

We take the classical sign convention that if J is the (n, n) matrix whose diagonal entries are $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then

$$\text{Pf } J = 1. \tag{1.32}$$

We now have

PROPOSITION 1.4. *If $A \in \mathcal{O}$,*

$$\frac{A^{\wedge l}}{l!} = \text{Pf}(A) dx^1 \wedge \dots \wedge dx^n. \tag{1.33}$$

Proof. Equation (1.33) follows from the expression for $\text{Pf } A$ in [37, p. 420]. Equation (1.33) makes clear that if the orientation of E is changed, $\text{Pf } A$ is changed into $-\text{Pf } A$.

1.d. *A Basic Differentiation Formula*

Recall that e, e' are the identity elements in $SO(n), \text{Spin}(n)$.

THEOREM 1.5. *Let $t \rightarrow x_t$ be a continuous curve with values in $\text{Spin}(n)$, such that $x_0 = e'$ and which is C^1 at $t = 0$. If $A \in \mathcal{O}$ is defined by*

$$A = \frac{dx}{dt} \Big|_{t=0} \tag{1.34}$$

then

$$\lim_{t \uparrow \downarrow 0} \frac{\chi_+(x_t) - \chi_-(x_t)}{t^l} = i^l \text{Pf } A. \tag{1.35}$$

Proof. If $x = x(\theta_1, \dots, \theta_l)$ is given by (1.22), using (1.24), it is clear that

$$|\chi_+(x) - \chi_-(x)| = |\det(e - \sigma(x))|^{1/2}. \quad (1.36)$$

Equation (1.36) immediately extends to any $x \in \text{Spin}(n)$. It is then clear that

$$\lim_{t \downarrow 0} \frac{|\chi_+(x_t) - \chi_-(x_t)|}{t^l} = |\det A|^{1/2} = |\text{Pf } A|. \quad (1.37)$$

Since $\chi_+(x) - \chi_-(x)$ is a polynomial function of $x \in \text{Spin}(n)$ (considered as a linear mapping on S), (1.37) shows that the lhs of (1.35) exists and is equal to $c \text{Pf } A$, where c is a complex number such that $|c| = 1$.

If $x_t = x(t, \dots, t)$, using (1.24), it is clear that

$$\lim_{t \downarrow 0} \frac{\chi_+(x_t) - \chi_-(x_t)}{t^l} = (-i)^l. \quad (1.38)$$

Since in this case $\text{Pf } A = (-1)^l$, we find that $c = i^l$. ■

1.e. Spin Manifolds and the Dirac Operator

M is a C^∞ compact connected oriented Riemannian manifold, of even dimension $n = 2l$. TM denotes its tangent bundle, T^*M the cotangent bundle. TM and T^*M are identified by the metric.

In what follows, we will assume that R^n is the canonical n -dimensional Euclidean space R^n , and that R^n is oriented by its canonical base e_1, \dots, e_n .

N denotes the $SO(n)$ principal bundle of oriented orthogonal frames in TM . For every $x \in M$, the fiber N_x can be identified to the set of linear oriented isometries from R^n into $T_x M$. π denotes the projection $N \rightarrow M$.

We will assume that M is a spin-manifold, i.e., the $SO(n)$ principal bundle N lifts to a $\text{Spin}(n)$ principal bundle N' , so that the projection $N' \rightarrow_\sigma N$ induces the covering mapping $\text{Spin}(n) \rightarrow_\sigma SO(n)$ on each fiber.

The existence of a spin structure is equivalent to the vanishing of the second Stiefel–Whitney class $w_2(M)$ [31]. If $w_2(M) = 0$, the number of spin structures on M is the number of elements in $H^1(M, Z_2)$.

Note that on any manifold, a spin structure always exists locally. As in [4], this will allow us to work on manifolds without a spin structure.

F_+ , F_- are the complex vector bundles over M

$$\begin{aligned} F &= N' \times_{\text{Spin}(n)} S, \\ F_+ &= N' \times_{\text{Spin}(n)} S_+, \\ F_- &= N' \times_{\text{Spin}(n)} S_-. \end{aligned} \quad (1.39)$$

F, F_+, F_- are Hermitian vector bundles. Of course

$$F = F_+ \oplus F_- \tag{1.40}$$

and F_+ and F_- are orthogonal in F .

If $x \in M, e \in T_x M$, we may define the Clifford multiplication operator on $F_x: f \in F_x \rightarrow ef \in F_x$ which sends $F_{+,x}$ into $F_{-,x}$ and $F_{-,x}$ into $F_{+,x}$.

Also note that for every $x \in M$, the fiber N'_x can be identified to a set of unitary operators from S into F_x , which send S_+, S_- into $F_{+,x}F_{-,x}$.

Let θ be the R^n -valued 1-form on N such that if $u \in N, X \in T_x N$,

$$\theta(X) = u^{-1}\pi_* X. \tag{1.41}$$

Let ω be the \mathcal{O} -valued connection form for the Levi-Civita connection on N . Classically [24]

$$d\theta = -\omega\wedge\theta, \quad d\omega = -\omega\wedge\omega + \Omega, \tag{1.42}$$

where Ω is the equivariant representation of the curvature tensor R on M .

Since N' covers N , and since the Lie algebra of $\text{Spin}(n)$ is also \mathcal{O} , $\sigma_*\omega$ is a connection form on N' , which is associated to the natural lift of the Levi-Civita connection to N' . Since there is no risk of confusion, we will write ω instead of $\sigma_*\omega$. ∇ denotes the covariant differentiation operator for the (lifted) Levi-Civita connection. If G is a vector bundle over M , $\Gamma(G)$ is the vector space of C^∞ sections of G .

We now define the Dirac operator D [1-6].

DEFINITION 1.6. Take $f \in \Gamma(F)$. $Df \in \Gamma(F)$ is such that if $x \in M$, if e'_1, \dots, e'_n is an orthogonal base of $T_x M$, then

$$Df(x) = \sum_1^n e'_i(\nabla_{e'_i} f)(x). \tag{1.43}$$

In (1.43), e'_i acts on $(\nabla_{e'_i} f)(x)$ by Clifford multiplication. It is easy to check that the rhs of (1.43) does not depend on the base (e'_1, \dots, e'_n) . Clearly D sends $\Gamma(F_+)$ into $\Gamma(F_-)$ and $\Gamma(F_-)$ into $\Gamma(F_+)$.

Since F is a Hermitian bundle, $\Gamma(F)$ is naturally endowed with the Hermitian product

$$f, g \in \Gamma(F) \rightarrow \int_M \langle f(x), g(x) \rangle dx \tag{1.44}$$

(in (1.44), dx is the volume form in M). D is then a formally self-adjoint operator on $\Gamma(F)$.

1.f. The Dirac Operator with Coefficients in an Auxiliary Bundle

We do the same assumptions as in 1.e. Following [4], we now define the Dirac operator with coefficients in an auxiliary bundle. Let ξ be a k -dimensional complex Hermitian vector bundle. X denotes the $U(k)$ -principal bundle of unitary frames in ξ . By [24], we can find a connection on X .

If $\mathcal{Z}(k)$ is the Lie algebra of $U(k)$, let λ be the $\mathcal{Z}(k)$ -valued connection form on X . By [24], we know that

$$d\lambda = -\lambda A \lambda + A, \quad (1.45)$$

where A is the equivariant representation of the curvature tensor L .

$N' \boxtimes X$ denotes the $\text{Spin}(n) \times U(k)$ principal bundle whose base is M , and fiber at $x \in M$ is $N'_x \times X_x$. $N' \boxtimes X$ is naturally endowed with a connection whose connection form is (ω, λ) . We still note ∇ the covariant differentiation operator for this connection. Since F and ξ are Hermitian vector bundles, $F \otimes \xi$ is also a Hermitian vector bundle. It follows that there is a natural Hermitian product in $\Gamma(F \otimes \xi)$.

We now define the Dirac operator on $\Gamma(F \otimes \xi)$. Since there is no risk of confusion, it will still be noted D .

DEFINITION 1.7. D is the operator acting on $\Gamma(F \otimes \xi)$ which is such that if $f \in \Gamma(F)$, $g \in \Gamma(\xi)$, if $x \in M$, if e'_1, \dots, e'_n is an orthogonal base at x , then

$$D(f \otimes g)(x) = \sum_{i=1}^n (e'_i \nabla_{e'_i} f(x) \otimes g(x) + e'_i f(x) \otimes (\nabla_{e'_i} g)(x)). \quad (1.46)$$

D is also formally self-adjoint on $\Gamma(F \otimes \xi)$. It sends $\Gamma(F_+ \otimes \xi)$ in $\Gamma(F_- \otimes \xi)$ and $\Gamma(F_- \otimes \xi)$ in $\Gamma(F_+ \otimes \xi)$.

1.g. Evaluation of D^2 in Terms of the Horizontal Laplacian

For the sake of completeness, we now derive a well-known formula (Lichnerowicz [25], Hitchin [22]) expressing the operator D^2 in terms of the horizontal Laplacian Δ^H .

DEFINITION 1.8. Δ^H denotes the operator acting on $\Gamma(F \otimes \xi)$ which is such that if $x \in M$, if e'_1, \dots, e'_n is an orthogonal base of $T_x M$, if $h \in \Gamma(F \otimes \xi)$, then

$$(\Delta^H h)(x) = \sum_1^n (\nabla_{e'_i}^2 h)(x). \quad (1.47)$$

Recall that if $(e'_1(y), \dots, e'_n(y))$ is a smooth section of N defined on a neighborhood of x , then

$$(\nabla_{e'_i}^2 h)(x) = \nabla_{e'_i(x)} \nabla_{e'_i(x)} h(x) - (\nabla_{\nabla_{e'_i(x)} e'_i(x)} h)(x). \quad (1.48)$$

Let $K(x)$ be the scalar curvature of M at x . Recall that L is the curvature tensor of ξ .

We now have the following result [22, 25]:

THEOREM 1.9. *Take $h \in \Gamma(F \otimes \xi)$. If $x \in M$, if e'_1, \dots, e'_n is an orthogonal base of $T_x M$, then*

$$(D^2 h)(x) = -(\Delta^H h)(x) + \frac{K}{4}(x) h(x) + \frac{1}{2} \sum_{i,j} (e'_i e'_j \otimes L(e'_i, e'_j)) h(x). \quad (1.49)$$

Proof. Assume that $(e'_i(y), \dots, e'_n(y))$ is a smooth section of N defined on a neighborhood of $x \in M$. Using the defining relations (1.2), we see easily that if $f \in \Gamma(F)$, $g \in \Gamma(F)$,

$$\begin{aligned} D^2(f \otimes g)(x) &= -(\Delta^H f \otimes g)(x) - (f \otimes \Delta^H g)(x) - 2 \sum_1^n (\nabla_{e'_i} f)(x) \\ &\quad \otimes (\nabla_{e'_i} g)(x) + \frac{1}{2} \sum_{i,j} (e'_i e'_j (\nabla_{e'_i} \nabla_{e'_j} - \nabla_{e'_j} \nabla_{e'_i}) f) \otimes g(x) \\ &\quad + \frac{1}{2} \sum_{i,j} e'_i e'_j f \otimes (\nabla_{e'_i} \nabla_{e'_j} - \nabla_{e'_j} \nabla_{e'_i}) g(x). \end{aligned} \quad (1.50)$$

Now it is clear that

$$\begin{aligned} \Delta^H(f \otimes g)(x) &= (\Delta^H f \otimes g)(x) + (f \otimes \Delta^H g)(x) \\ &\quad + 2 \sum_1^n (\nabla_{e'_i} f \otimes \nabla_{e'_i} g)(x). \end{aligned} \quad (1.51)$$

If \bar{R} is the curvature tensor of the lifted Levi-Civita connection on F , obviously

$$\frac{1}{2} \sum_{i,j} e'_i e'_j (\nabla_{e'_i} \nabla_{e'_j} - \nabla_{e'_j} \nabla_{e'_i}) f(x) = \frac{1}{2} \sum_{i,j} e'_i e'_j \bar{R}(e'_i, e'_j) f(x). \quad (1.52)$$

By Proposition 1.3, we know that

$$\frac{1}{2} \sum_{i,j} e'_i e'_j \bar{R}(e'_i, e'_j) = \frac{1}{8} \sum_{i,j,k,m} \langle R(e'_i, e'_j) e'_k, e'_m \rangle e'_i e'_j e'_k e'_m. \quad (1.53)$$

Due to Bianchi's identity [24] and the defining relations (1.2), for one m the contribution of the terms where i, j, k are all distinct is 0. Using (1.2) again, we find that if S is the Ricci tensor, (1.53) is equal to

$$\frac{1}{4} \sum_{i,j,m} \langle R(e'_i, e'_j) e'_i, e'_m \rangle e'_j e'_m = -\frac{1}{4} \sum_{j,m} \langle S e'_j, e'_m \rangle e'_j e'_m. \quad (1.54)$$

Since S is symmetric, the contribution of the terms with $j \neq m$ is 0. Since K is the trace of S , we find using (1.2) that

$$\frac{1}{2} \sum_{i,j} e'_i e'_j \bar{R}(e'_i, e'_j) = \frac{K}{4}. \quad (1.55)$$

Finally

$$(\nabla_{e'_i} \nabla_{e'_j} - \nabla_{e'_j} \nabla_{e'_i}) g(x) = L(e'_i, e'_j) g(x). \quad (1.56)$$

Using (1.50), (1.55), (1.56), we see that (1.49) holds. ■

Remark 2. The key properties of the operator D^2 , which makes that the twisted spin complex lends itself to a relatively easy probabilistic treatment are

- (i) The operator $h \rightarrow Kh$ is in “diagonal” form.
- (ii) The operator $h \rightarrow \frac{1}{2} \sum_{i,j} e'_i e'_j \otimes L(e'_i, e'_j) h$, although not in diagonal form has a natural probabilistic interpretation.

Note that Δ^H and D^2 are both formally self-adjoint on $\Gamma(F \otimes \xi)$, and send $\Gamma(F_+ \otimes \xi)$ and $\Gamma(F_- \otimes \xi)$ into themselves.

Remark 3. Equations (1.17) and (1.19) show that the signature complex or the de Rham complex are sums of twisted spin complexes. By Atiyah–Bott–Patodi [4], the same observation holds for the $\bar{\partial}$ complex. It is a key observation in [4] that since the heat equation methods are local, to prove index formulas for complexes which are well defined on manifolds which are not necessarily spin, we only need to use a local spin structure which always exists. The proof of the Index Theorem for twisted spin complexes is then sufficient to obtain the Index Theorem for other classical complexes.

2. THE PARAMETRIX FOR THE HEAT KERNEL: A PROBABILISTIC CONSTRUCTION

In this section we use the results of our previous work [10] to give a probabilistic construction of a parametrix for the heat equation semi-group acting on $\Gamma(F \otimes \xi)$. The reader is referred to the introduction of [10] for more motivation and details on this construction.

In 2.a we give the main assumptions and notations. In 2.b the Brownian motion on the Riemannian manifold M is constructed using a technique of Malliavin [27], Eells–Elworthy [15]. Stochastic flows of diffeomorphisms of N , N' which are naturally associated to the Brownian motion are also

introduced. In 2.c the heat equation semi-group on $\Gamma(F \otimes \xi)$ is constructed using a subordination procedure. In 2.d we give an asymptotic expression as $t \downarrow 0$ of the heat equation kernel by using the results of [10]. This expression plays a key role in Sections 3 and 4.

2.a. *Assumptions and Notations*

We do the same assumptions as in 1.e, 1.f, and we will use the same notations as in Section 1. In particular M is still supposed to be a compact connected Riemannian spin manifold of dimension $n = 2l$. Recall that σ is the covering projection $N' \rightarrow N$, that π is the canonical projection $N \rightarrow M$. π' will be the canonical projection $N' \rightarrow M$, so that

$$\pi' = \pi \circ \sigma. \tag{2.1}$$

Also recall that we will write ω instead of $\sigma_* \omega$. (e_1, \dots, e_n) still denotes the canonical oriented base of R^n .

For $i = 1, \dots, n$, Y_i (resp. Y'_i) denotes the standard horizontal vector field on N (resp. N') defined by

$$\theta(Y_i) = e_i, \quad \omega(Y_i) = 0 \tag{2.2}$$

(resp.

$$\theta(Y'_i) = e_i, \quad \omega(Y'_i) = 0). \tag{2.2'}$$

Of course for $i = 1, \dots, n$,

$$Y_i = \sigma_* Y'_i. \tag{2.3}$$

Let \mathcal{L} (resp. \mathcal{L}') be the second-order differential operator on N (resp. N')

$$\mathcal{L} = \frac{1}{2} \sum_1^n Y_i^2 \tag{2.4}$$

(resp.

$$\mathcal{L}' = \frac{1}{2} \sum_1^n Y_i'^2). \tag{2.4'}$$

If Δ is the Laplace–Beltrami operator on M , it is well known (see [10]) that for $f \in C^\infty(M)$,

$$\mathcal{L}(f \circ \pi) = \frac{1}{2} (\Delta f) \cdot \pi \tag{2.5}$$

which implies

$$\mathcal{L}'(f \circ \pi') = \frac{1}{2} \Delta f \circ \pi'. \tag{2.6}$$

W denotes the probability space $\mathcal{E}(R^+; R^n)$, whose standard element is $w_t = (w_t^1, \dots, w_t^n)$. Let $\{F_t\}_{t \geq 0}$ be the canonical filtration of W , where F_t is defined by

$$F_t = \mathcal{B}(w_s \mid s \leq t). \quad (2.7)$$

P denotes the Wiener measure on W with $P[w_0 = 0] = 1$.

In the sequel, $\{F_t\}_{t \geq 0}$ will be made right-continuous and complete (for any of the considered probability measures) without further mention. dw denotes the differential of w in the sense of Stratonovitch, and δw the differential of w in the sense of Itô [7, 10, 12, 30].

Remark 1. Here is a word of advice for readers not familiar with Brownian motion. Although w_t is P a.s. nowhere differentiable (as a function of t), still the Stratonovitch differential dw behaves *formally* as the differential of a differentiable function. So in what follows, the reader can do as if w were differentiable, except when Itô differentials do appear.

2.b. Brownian Motion in M and the Associated Flow in N, N'

We now construct the Brownian motion on M using the method of Malliavin [27], Eells–Elworthy [15], i.e., considering its lift in N . For our special needs, we will also lift the Brownian motion to N' and also consider the associated stochastic flow on N' . For $u_0 \in N$ (resp. $u'_0 \in N'$), $t > 0$, consider the stochastic differential equation on $(W, \{F_s\}_{s \geq 0}, P)$,

$$du = \sum_{i=1}^n Y_i(u) \sqrt{t} \cdot dw^i, \quad u(0) = u_0 \quad (2.8)$$

(resp.

$$du' = \sum_{i=1}^n Y'_i(u') \sqrt{t} \cdot dw^i, \quad u'(0) = u'_0.) \quad (2.8')$$

Since N, N' are compact, (2.8), (2.8') have unique solutions. Using (2.5) (resp. (2.6)) it is easy to check that for one $t > 0$, if $x_s = \pi u_s$ (resp. $x_s = \pi' u'_s$) then the law of $x_{s,t}$ is the law of the Brownian motion on M starting at πu_0 (resp. $\pi' u'_0$). Conversely, u_s (resp. u'_s) is the horizontal lift of x_s in N (resp. in N') in the sense of [7, VIII].

The introduction of the parameter t in (2.8) (resp. (2.8')) will be justified when constructing the parametrix for the heat equation.

Using the results in Bismut [7, I], Elworthy [16], we now define

DEFINITION 2.1. $\psi_s(\sqrt{t} dw, \cdot)$ (resp. $\psi'_s(\sqrt{t} dw, \cdot)$) denotes the stochastic flow of C^∞ diffeomorphisms of N (resp. N') associated to equation (2.8) (resp. (2.8')).

By [7, 16], we know that we may assume that a.s., $\psi_s(\sqrt{t} dw, u_0)$ (resp. $\psi'_s(\sqrt{t} dw, u'_0)$) is jointly continuous in (s, t, u_0) (resp. (s, t, u'_0)), C^∞ in (\sqrt{t}, u_0) (resp. (\sqrt{t}, u'_0)) with derivatives in (\sqrt{t}, u_0) (resp. (\sqrt{t}, u'_0)) which are jointly continuous in (s, t, u_0) (resp. (s, t, u'_0)). Moreover, a.s., for any s, t , $\psi_s(\sqrt{t} dw, \cdot)$ (resp. $\psi'_s(\sqrt{t} dw, \cdot)$) is a C^∞ diffeomorphism of N (resp. N'). Finally if $u_0 \in N$ (resp. $u'_0 \in N'$), $\psi_s(\sqrt{t} dw, u_0)$ (resp. $\psi'_s(\sqrt{t} dw, u'_0)$) is the unique solution of (2.8) (resp. (2.8')).

Using (2.2), (2.2') it is clear that a.s.

$$\psi_s(\sqrt{t} dw, \cdot) \circ \sigma = \sigma \circ \psi'_s(\sqrt{t} dw, \cdot). \tag{2.9}$$

2.c. A Probabilistic Construction of the Heat Equation Semi-group on $\Gamma(F \otimes \xi)$

We now will explicitly construct the heat equation semi-group $e^{-tD^2/2}$ by using a subordination procedure already used by Malliavin [27, 28] for the Laplacian acting on 1-forms.

In what follows, to be entirely correct, we should have constructed horizontal stochastic flows on $N \boxtimes X$ or $N' \boxtimes X$ instead of just constructing them on N or N' . Since this does not create any difficulty, we will content ourselves with using parallel translation operators in the fibers of ξ along $x_s = \pi u_s$ or $x_s = \pi' u'_s$, which are well defined by [7, VIII].

$x_0 \in M$ is now fixed; u'_0 is one element of N'_{x_0} and u_0 is defined by $u_0 = \sigma u'_0$. For $t > 0$, set

$$\begin{aligned} u_s^{t'} &= \psi'_s(\sqrt{t} dw, u'_0), \\ u_s^t &= \sigma \psi'_s(\sqrt{t} dw, u'_0), \\ x_s^t &= \pi' \psi'_s(\sqrt{t} dw, u'_0). \end{aligned} \tag{2.10}$$

In what follows, the parallel translation operators are defined on any vector bundle to which the considered connection on $N' \boxtimes X$ applies.

DEFINITION 2.2. $\tau_s^{0,t}$ is the parallel translation operator along x_s^t from vector fibers over x_0 to vector fibers over x_s^t . $\tau_0^{s,t}$ is defined by

$$\tau_0^{s,t} = [\tau_s^{0,t}]^{-1}. \tag{2.11}$$

$\tau_s^{0,t}$ and $\tau_0^{s,t}$ exist a.s. for any $s > 0$ by [7, VIII]. Of course when acting on $T_{x_0} M$, we have

$$\tau_s^{0,t} = u_s^t u_0^{-1}. \tag{2.12}$$

Similarly, when acting on F_{x_0} , $\tau_s^{0,t}$ is given by

$$\tau_s^{0,t} = u_s^{t'} u_0'^{-1}. \tag{2.13}$$

We now define the subordinating process U_s^t :

DEFINITION 2.3. U_s^t is the process of linear mappings from the fiber $F_{x_0} \otimes \xi_{x_0}$ into itself defined by the differential equation

$$\begin{aligned} \frac{dU_s^t}{ds} &= -\frac{1}{2} t U_s^t \left(\sum_{i < j} (u_0 e_i)(u_0 e_j) \otimes \tau_0^{s,t} L_{x_s^t}(u_s^t e_i, u_s^t e_j) \right), \\ U_0^t &= I. \end{aligned} \quad (2.14)$$

Of course $u_s^t e_i = \tau_s^{0,t} u_0 e_i$. In (2.14) $(u_0 e_i)(u_0 e_j)$ acts on F_{x_0} by Clifford multiplication. Clearly U_s^t sends $F_{+,x_0} \otimes \xi_{x_0}$ into itself and $F_{-,x_0} \otimes \xi_{x_0}$ into itself. Recall that since D^2 is an elliptic second order differential operator, for $t > 0$, the operator $e^{-tD^2/2}$ is well defined and acts by a smooth kernel.

Namely, if dx is the Riemannian volume element of M , we now define:

DEFINITION 2.4. For $t > 0$, $P_t(x, y)$ denotes the smooth kernel on $\Gamma(F \otimes \xi)$, such that if $h \in \Gamma(F \otimes \xi)$, then for any $x \in M$

$$(e^{-tD^2/2} h)(x) = \int P_t(x, y) h(y) dy. \quad (2.15)$$

Of course for $x, y \in M$, $P_t(x, y)$ is a linear mapping from $(F \otimes \xi)_y$ into $(F \otimes \xi)_x$, which sends $(F_+ \otimes \xi)_y$ in $(F_+ \otimes \xi)_x$ and $(F_- \otimes \xi)_y$ in $(F_- \otimes \xi)_x$.

We now have the following matrix variant of a Feynman–Kac formula:

THEOREM 2.5. For any $h \in \Gamma(F \otimes \xi)$, then

$$e^{-tD^2/2} h(x_0) = E^P \left[\exp \left\{ -\frac{t}{8} \int_0^1 K(x_s^t) ds \right\} U_1^t \tau_0^{1,t} h(x_1^t) \right]. \quad (2.16)$$

Proof. We give a short and elementary proof. We first assume that $t = 1$. Set

$$V_s(x) = \int P_{1-s}(x, y) h(y) dy. \quad (2.17)$$

Clearly $V_s(x)$ is smooth in (s, x) , and moreover

$$\frac{\partial V_s}{\partial s} - \frac{1}{2} D^2 V_s = 0, \quad V_1(x) = h(x). \quad (2.18)$$

Now by [7, IX, Theorem 1.1], we know that for $t = 1$ (and omitting the superscripts t), we have

$$\begin{aligned} \tau_0^s V_s(x_s) &= V_0(x_0) + \int_0^s \tau_0^v \left(\frac{\partial V_v}{\partial v}(x_v) + \frac{1}{2} \Delta^H V_v(x_v) \right) dv \\ &\quad + \int_0^s \tau_0^v \nabla_{u_i e_i} V_v(x_v) \delta w_v^i. \end{aligned} \tag{2.19}$$

If R_s is defined by

$$R_s = \exp \left\{ -\frac{1}{8} \int_0^s K(x_v) dv \right\} U_s \tau_0^s \tag{2.20}$$

using Theorem 1.9 and Itô's formula, we see that

$$\begin{aligned} R_s V_s(x_s) &= V_0(x_0) + \int_0^s R_v \tau_0^v \left[\frac{\partial V_v}{\partial v}(x_v) - \frac{1}{2} D^2 V_v(x_v) \right] dv \\ &\quad + \int_0^s R_v \nabla_{u_i e_i} V_v(x_v) \delta w_v^i. \end{aligned} \tag{2.21}$$

Using (2.18), we see that $R_s V_s(x_s)$ is a martingale. It is then clear that

$$V_0(x_0) = E^P [R_1 V_1(x_1)] \tag{2.22}$$

or equivalently, using (2.18), we see that

$$V_0(x_0) = E^P [R_1 h(x_1)].$$

(2.15) is proved for $t = 1$. The proof immediately extends to a general $t > 0$. ■

2.d. *An Asymptotic Expression for $P_t(x_0, y_0)$*

We will here briefly summarize our results in [10] concerning the asymptotic behaviour as $t \downarrow 0$ of $P_t(x_0, y_0)$. Our work [10] was mostly devoted to studying the heat equation semi-group for the Laplace–Beltrami operator Δ , but as pointed out in [10, IV, Remark 7], the results of [10] immediately extend to the heat equation semi-group for \square or D^2 , since these are *subordinated* to the heat equation semigroup for Δ in a sense made clear in Theorem 2.5.

Also note that in [10] we only used the flow $\psi(\sqrt{t} dw, \cdot)$ on N , and *not* the flow $\psi'(\sqrt{t} dw, \cdot)$ on N' . However the proofs of [10] are absolutely not changed by lifting everything to N' , so that here we will freely use the results of [10] while changing ψ into ψ' .

Finally no justification will be given here for introducing such and such object. For motivation, we refer to the Introduction of [10], and for details, to Sections 2–4 in [10]. Let us just say that what we do in [10] is essentially to transfer what would be done trivially for smooth functions on a finite

dimensional space in an infinite dimensional context. In particular since we work with small time, we will be able to parametrize the Brownian bridges in the manifold M using standard Brownian bridges in the Euclidean space.

y_0 is another element of M . We will assume that x_0, y_0 are such that there is one single geodesic γ_s connecting x_0 and y_0 with $\gamma_0 = x_0, \gamma_1 = y_0$, and that moreover x_0, y_0 are non conjugate along γ . u'_0, u_0 are taken as before. $\lambda \in R^n$ is defined by

$$\lambda = u_0^{-1} \frac{d\gamma_0}{ds}. \quad (2.23)$$

f'_s is the flow of diffeomorphisms of N' associated to the (deterministic!) differential equation

$$du' = \sum_1^n Y'_i(u') \lambda^i ds. \quad (2.24)$$

Set

$$u'_s = f'_s(u'_0), \quad u_s = \sigma u'_s; \quad (2.25)$$

u'_s is exactly the parallel translation of u'_0 along the geodesic γ .

If g is a diffeomorphism of N' , we will note g^* its action on the tensor algebra of TN' . If K is a tensor field over N' , $(g^{*-1}K)(u')$ is the pull back at u' of $K(g(u'))$ by g^{*-1} .

DEFINITION 2.6. H is the Hilbert space

$$H = L_2([0, 1]; R^n). \quad (2.26)$$

H_1 is the subspace of H

$$H_1 = \left\{ v \in H; \pi'^* f'_1{}^* \int_0^1 (f'_s{}^{*-1} Y'_i)(u'_0) v'^i ds = 0 \right\}. \quad (2.27)$$

For $1 \leq i \leq n$, h_s^i is the continuous function defined on $[0, 1]$ with values in $T_{y_0}^*M$

$$h_s^i = \pi'^* f'_1{}^* (f'_s{}^{*-1} Y'_i)(u'_0). \quad (2.28)$$

H_2 is the n -dimensional subspace of H which is the image of $T_{y_0}^*M$ by the linear mapping ρ ,

$$q \in T_{y_0}^*M \rightarrow \rho(q) = (\langle q, h_s^1 \rangle, \dots, \langle q, h_s^n \rangle). \quad (2.29)$$

In the sequel T^*M and TM are identified by the metric.

We have the elementary

PROPOSITION 2.7. H_1 and H_2 are orthogonal in H , and moreover,

$$H = H_1 \oplus H_2. \tag{2.30}$$

Also $\lambda \in H_2$ and

$$\lambda = \rho \left(\frac{d\gamma_t}{ds} \right). \tag{2.31}$$

Proof. This easy result is proved in Theorem 4.4 and Remark 4.2 in [10]. ■

DEFINITION 2.8. P_1 denotes the Gaussian cylindrical measure on H_1 .

By Theorem 4.8 in [10], we know that we can define a continuous process w^1 with values in R^n such that the law of “ dw^1 ” is exactly P_1 . Since there is no risk a confusion, we will also call P_1 the law of w^1 on W . An elementary construction of P_1 is given in [10]. We first define:

DEFINITION 2.9. Q denotes the probability law on W of the R^n -valued Brownian bridge a_s ($0 \leq s \leq 1$), with $a_0 = a_1 = 0$.

It is known that under P , the law of $w_s - sw_1$ ($0 \leq s \leq 1$) is exactly Q . We now have

THEOREM 2.10. On (W, Q) consider the system calculated along $s \rightarrow u_s$

$$\begin{aligned} \alpha_s &= \int_0^s \Omega((u_t, \lambda)^*, (u_t, a_t)^*) dv, \\ w_s^1 &= a_s - \int_0^s \alpha_t \lambda dv. \end{aligned} \tag{2.32}$$

If K' is given by

$$K' = E^Q \exp \left\{ \int_0^1 \langle \Omega((u_s, a_s)^*, (u_s, \lambda)^*) \lambda, a_s \rangle ds - \frac{1}{2} \int_0^1 |\alpha_s \lambda|^2 ds \right\}, \tag{2.33}$$

then K' is $< +\infty$. If Q' is the probability measure

$$dQ'(a) = \frac{\exp \left\{ \int_0^1 \langle \Omega((u_s, a_s)^*, (u_s, \lambda)^*) \lambda, a_s \rangle ds - \frac{1}{2} \int_0^1 |\alpha_s \lambda|^2 ds \right\} dQ(a)}{K'} \tag{2.34}$$

under Q' , the law of w^1 is P_1 .

Proof. This is Theorem 4.10 in [10]. ■

Remark 2. The same observations as in Remark 1 apply to w^1 and a .

Theorem 2.10 shows in particular that under P_1 , w^1 is a semi-martingale.

DEFINITION 2.11. For $q \in T_{y_0}^*N$, $t > 0$, $v'_0 \in N'$ consider the stochastic differential equation on (W, P_1) ,

$$dv' = \sum_1^n Y'_i(v')(\sqrt{t} dw^{1,i} + \langle q, h^i \rangle ds), \quad v'(0) = v'_0. \quad (2.35)$$

$\psi'(\sqrt{t} dw^1, q, \cdot)$ denotes the stochastic flow of diffeomorphisms of N' associated to (2.35).

The fact that $\psi'(\sqrt{t} dw^1, q, \cdot)$ is well defined is shown in Section 4.d of [10]. Moreover P_1 a.s., we know by [10, Sections 3c and 4.d] that $\psi'_s(\sqrt{t} dw^1, q, v'_0)$ can be supposed to be jointly continuous in (s, \sqrt{t}, q, v'_0) , C^∞ in (\sqrt{t}, q, v'_0) with derivatives in (\sqrt{t}, q, v'_0) which are jointly continuous in (s, \sqrt{t}, q, v'_0) .

We will write $\psi'_s(q, v'_0)$ instead of $\psi'_s(0, q, v'_0)$ (which corresponds to the case where $t = 0$). Using (2.31), it is clear that for any s, v'_0

$$f'_s(v'_0) = \psi'_s \left(\frac{d\gamma_1}{ds}, v'_0 \right). \quad (2.36)$$

DEFINITION 2.12. C is the linear mapping from $T_{y_0}^*M$ into $T_{y_0}M$

$$\begin{aligned} q \in T_{y_0}^*M \rightarrow Cq &= \pi' * f_1' * \int_0^1 \sum_{i=1}^n (f_s'^{* -1} Y_i)(u'_0) \\ &\quad \times \langle q, \pi' * f_1' * (f_s'^{* -1} Y_i)(u'_0) \rangle ds. \end{aligned} \quad (2.37)$$

By [10, Theorem 4.4], C is invertible (this is a trivial result). Using (2.35), it is not difficult to see that

$$\pi' * \frac{\partial \psi'_1}{\partial q} \left(\frac{d\gamma_1}{ds}, u'_0 \right) = C. \quad (2.38)$$

Using the implicit function theorem, we know that for $\eta > 0$ small enough, for a.e. w^1 , for t small enough, the equation

$$\pi' \psi'_1 \left(\sqrt{t} dw^1, \frac{d\gamma_1}{ds} + q, u'_0 \right) = y_0; \quad |q| \leq \eta \quad (2.39)$$

has one unique solution $q(\sqrt{t} dw^1, y_0)$ which depends smoothly on \sqrt{t} . Set

$$q'(\sqrt{t} dw^1, y_0) = q(\sqrt{t} dw^1, y_0) + \frac{d\gamma_1}{ds}.$$

DEFINITION 2.13. $v^2(\sqrt{t} dw^1, y_0) \in H_2$ is defined by

$$v^2(\sqrt{t} dw^1, y_0) = \rho(q(\sqrt{t} dw^1, y_0)) \tag{2.40}$$

$C(\sqrt{t} dw^1, y_0)$ is the linear mapping from $T_{y_0}^*M$ into $T_{y_0}M$

$$q \in T_{y_0}^*M \rightarrow C(\sqrt{t} dw^1, y_0) q = \pi'^* \psi_1'^*(\sqrt{t} dw^1, q'(\sqrt{t} dw^1, y_0), u_0') \int_0^1 (\psi_s'^{*-1}(\sqrt{t} dw^1, q'(\sqrt{t} dw^1, y_0), \cdot) Y_i')(u_0') < q, \pi'^* f_1'^*(f_s'^{*-1} Y_i')(u_0') \rangle ds. \tag{2.41}$$

Of course $v^2(\sqrt{t} dw^1, y_0)$, $C(\sqrt{t} dw^1, y_0)$ are only defined for t small enough. Using (2.37), (2.41), and the invertibility of C , we know that for t small enough, $C(\sqrt{t} dw^1, y_0)$ is invertible.

DEFINITION 2.14. For $m \in N$, K'^m is the set of functions defined on $N' \times \{q \in T_{y_0}^*M, |q| \leq 1\}$ with values in N' which have m continuous derivatives (in all variables).

When endowed with the topology of uniform convergence of functions and their derivatives of order $\leq m$, K'^m is a metrizable space. Let d'^m be a distance in K'^m ; δ is a positive constant $< \eta$; g is a function in $C_b^\infty(R)$, which is ≥ 0 , equal to 1 for $|x| \leq \delta/2$, to 0 for $|x| \geq \delta$.

Set

$$G(\sqrt{t} dw^1) = g \left[\sup_{\substack{0 < s \leq 1 \\ t' \leq t}} d'^m \left[\psi_s' \left(\sqrt{t'} dw^1, \frac{d\gamma_1}{ds} + \cdot, \cdot \right), \psi_s' \left(\frac{d\gamma_1}{ds} + \cdot, \cdot \right) \right] \right]. \tag{2.42}$$

We will assume that $m \geq 4$, and that $\delta > 0$ has been chosen to be small enough so that if $G(\sqrt{t} dw^1) \neq 0$, for $t' \leq t$, $q(\sqrt{t'} dw^1, y_0)$, $v^2(\sqrt{t'} dw^1, y_0)$ are well defined and $\det C'(\sqrt{t} dw^1, y_0) \geq \alpha > 0$. The fact that this is indeed possible is an easy consequence of the implicit function theorem and is proved in [10, Section 4.e].

We will now use notations which are identical to those of Section 2.c, just replacing everywhere $\psi_s'(\sqrt{t} dw, u_0')$ by $\psi_s'(\sqrt{t} dw^1, q'(\sqrt{t} dw^1, y_0), u_0')$. In particular

$$\begin{aligned} u_s'^t &= \psi_s'(\sqrt{t} dw^1, q'(\sqrt{t} dw^1, y_0), u_0'), \\ u_s^t &= \sigma u_s'^t, \\ x_s^t &= \pi' u_s'^t. \end{aligned} \tag{2.43}$$

In the same way U_s^t is still defined by Eq. (2.14), but of course $\tau_0^{s,t}$,

$L_{x'_s}(u'_s e_i, u'_s e_j)$ are calculated using the new definition of x'_i . In fact there is no risk of confusion since only the results of this section will be used from now to the end of the paper. We will write

$$P_t(x_0, y_0) \equiv A_t(x_0, y_0) \quad (2.44)$$

if for any $k \in N$, as $t \downarrow 0$

$$|P_t(x_0, y_0) - A_t(x_0, y_0)| = e^{-d^2(x_0, y_0)/2t} o(t^k). \quad (2.45)$$

We now have the key result of [10]:

THEOREM 2.15. *The following relation holds*

$$\begin{aligned} P_t(x_0, y_0) &\equiv \frac{[\det C]^{1/2}}{(\sqrt{2\pi t})^n} \int_w \frac{\exp\{-\int_0^1 (\lambda + v_s^2(\sqrt{t} dw^1, y_0)^2/2t) ds\}}{\det C(\sqrt{t} dw^1, y_0)} \\ &\quad \times \exp\left\{-\frac{t}{8} \int_0^1 K(x'_s) ds\right\} U_1^t \tau_0^{t'} G(\sqrt{t} dw^1) \\ &\quad \times g(|q(\sqrt{t} dw^1, y_0)|) dP_1(w^1). \end{aligned} \quad (2.46)$$

Proof. This result is a consequence of Theorem 2.5, and of [10, Theorem 4.16 and its corollary] ■

Remark 3. When $G(\sqrt{t} dw^1) \neq 0$, everything is well defined in (2.46), and $\det C(\sqrt{t} dw^1, y_0)$ is $\geq a > 0$ so that the integral is $< +\infty$.

The results of [10] are much stronger. Indeed Theorem 4.21 in [10] guarantees that if we make *formally* $g = 1$, $G(\sqrt{t} dw^1) = 1$, and take the Taylor expansion at 0 in \sqrt{t} of the integrand in the rhs of (2.46), we exactly get the Taylor expansion of $P_t(x_0, y_0)$. G and g only serve as mollifiers, and have no influence on the final result. Finally note that the results of [10] are more precise, in the sense they give uniform estimates in (x_0, y_0, t) , which we will need in Section 4.

3. THE ATIYAH-SINGER THEOREM FOR TWISTED SPIN COMPLEXES

In this section, we will prove the Atiyah-Singer theorem for twisted spin complexes using the asymptotic representation of the heat equation semi-group which we gave in Theorem 2.15. This representation will allow us to identify by brute force the limit of the integrand in the trace formula giving the index of the Dirac operator, without using the theory of invariants of Gilkey [19-21].

In 3.a, we recall well-known results [4] expressing the index of the Dirac

operator D by means of an integral over M whose integrand $\mathcal{F}(t, x)$ is a simple functional of P_t . We give here no background material. The reader is referred to [4] for more details.

In 3.b, using Theorem 2.15, we give an asymptotic expression for $\mathcal{F}(t, x)$. To find the limit of $\mathcal{F}(t, x)$ as $t \downarrow 0$, we introduce in 3.c an auxiliary Brownian motion γ , with values in \mathcal{A} which somewhat “correlates” TM and ξ . Although γ does not have an obvious “physical” interpretation, it plays a crucial role in finding the asymptotics of certain traces using Theorem 1.5. The introduction of γ is typical of probabilistic methods, which by introducing more variables simplify the computations.

In 3.d, the limit $\mathcal{F}(x)$ of $\mathcal{F}(t, x)$ as $t \downarrow 0$ is expressed by means of an integral over Brownian motion. In 3.e, the Atiyah–Singer theorem is proved in the form given by Atiyah–Bott–Patodi [4]. Namely, the Brownian integrals are transformed using the Weil homomorphism. Stochastic calculus in the exterior algebra plays here a key role. Surprisingly enough, the factorization of the index formula for twisted spin complexes appears here as merely reflecting the independence of certain Brownian motions.

Finally in 3.f, using the index theorem of Atiyah–Singer, we show that if M has compact universal covering, in the trace formula obtained by the heat equation method, the contribution of non-0 homotopic paths is exactly 0 for any $t > 0$.

We suggest that the reader first considers the case where ξ is the trivial line bundle, for which the computations considerably simplify. At a formal level, there are certain similarities between the method of Atiyah [2] and what is done here, and also striking similarities with what is done in Duistermaat–Heckman [14] in a finite dimensional case. In fact Atiyah [2] extrapolated the results of [14] to an infinite dimensional situation to give a formal proof of the Index Theorem for the spin complex. We will come back to this in a later paper.

3.a. The Atiyah–Singer Theorem: The Heat Equation Approach

The assumptions and notations are the same as in Section 2. In particular M is still supposed to be a spin manifold of dimension $n = 2l$.

We now recall a few well-known facts on the Index Theorem of Atiyah–Singer and the heat equation method. We closely follow Atiyah–Bott–Patodi [4].

Recall that in Section 1.f, we have seen that D interchanges $\Gamma(F_+ \otimes \xi)$ and $\Gamma(F_- \otimes \xi)$. We will note D_+ the restriction of D to $\Gamma(F_+ \otimes \xi)$ and D_- the restriction of D to $\Gamma(F_- \otimes \xi)$. D_+ and D_- are formally adjoint to each other.

DEFINITION 3.1. The index of D_+ is the integer

$$\text{Ind } D_+ = \dim \text{Ker } D_+ - \dim \text{Ker } D_- . \tag{3.1}$$

For $x \in M$, if A is a linear operator sending $(F_+ \otimes \xi)_x$ (resp. $(F_- \otimes \xi)$) into itself, $\rho_+(A)$ (resp. $\rho_-(A)$) denotes the trace of A . In particular for any $x \in M$, $t > 0$, since $P_t(x, x)$ sends $(F_+ \otimes \xi)_x$ into itself and $(F_- \otimes \xi)_x$ into itself, $\rho_+(P_t(x, x))$ and $\rho_-(P_t(x, x))$ are both well defined.

DEFINITION 3.2. $\mathcal{F}(t, x)$ is defined by

$$\mathcal{F}(t, x) = \rho_+(P_t(x, x)) - \rho_-(P_t(x, x)). \quad (3.2)$$

THEOREM 3.3. For any $t > 0$,

$$\text{Ind } D_+ = \int_M \mathcal{F}(t, x) dx. \quad (3.3)$$

Proof. This is Theorem EIII in [4]. ■

The key fact is that as we shall later see

$$\mathcal{F}(x) = \lim_{t \downarrow 0} \mathcal{F}(t, x) \quad (3.4)$$

exists (this also follows from pseudo-differential operators techniques), and we shall explicitly identify $\mathcal{F}(x)$ using the stochastic calculus.

3.b. An Asymptotic Expression for $\mathcal{F}(t, x_0)$

We now select one $x_0 \in M$. We will use the results of Section 2.d with $y_0 = x_0$. In particular

$$\lambda = 0. \quad (3.5)$$

H_1 is here given by

$$H_1 = \left\{ v \in H; \int_0^1 v ds = 0 \right\} \quad (3.6)$$

and H_2 consists of the R^n -valued constant functions on $[0, 1]$. P_1 coincides with Q , i.e., w^1 is a standard Brownian bridge (with $w_0^1 = w_1^1 = 0$). C is the identity mapping of $T_{x_0}M$. ∂ will denote the differentiation operator with respect to \sqrt{t} , i.e.,

$$\partial = \frac{\partial}{\partial \sqrt{t}}. \quad (3.7)$$

Set

$$H(\sqrt{t} dw^1, x_0) = \frac{\exp\{-\int_0^1 (|v_s^2(\sqrt{t} dw^1, x_0)|^2 ds/2t) - (t \int_0^1 K(x_s^t) ds/8)\}}{\det C(\sqrt{t} dw^1, x_0)} \times G(\sqrt{t} dw^1) g(|q(\sqrt{t} dw^1, x_0)|). \quad (3.8)$$

Using formulas (4.177), (4.178) in [10] (which we shall reprove in (3.24)) we know that for $t = 0$,

$$v_s^2 = \partial v_s^2 = 0. \tag{3.9}$$

Using Taylor's formula it is then clear that as $t \downarrow 0$,

$$\int_0^1 \frac{|v_s^2(\sqrt{t} dw^1, x_0)|^2 ds}{2t} \rightarrow 0 \tag{3.10}$$

and so as $t \downarrow 0$,

$$H(\sqrt{t} dw^1, x_0) \rightarrow 1, \quad P_1 \text{ a.s.} \tag{3.11}$$

We now have

THEOREM 3.4. *As $t \downarrow 0$, for any $k \in N$,*

$$\begin{aligned} \mathcal{F}(t, x_0) = & \frac{1}{(\sqrt{2\pi t})^n} \left[\int_w [\rho_+(U_1^t \tau_0^{1,t}) - \rho_-(U_1^t \tau_0^{1,t})] \right. \\ & \left. \times H(\sqrt{t} dw^1, x_0) dP_1(w^1) + o(t^k) \right]. \end{aligned}$$

Proof. This is a trivial consequence of Theorem 2.15. ■

Remark 1. At this stage we have the crucial task of finding the asymptotics as $t \downarrow 0$ of

$$\frac{\rho_+(U_1^t \tau_0^{1,t}) - \rho_-(U_1^t \tau_0^{1,t})}{t^k}.$$

To do this we will introduce one auxiliary Brownian motion.

3.c. An Auxiliary Brownian Motion with Values in \mathcal{O}

If B is an operator acting on F_{+,x_0} (resp. F_{-,x_0}), $\chi_+(B)$ (resp. $\chi_-(B)$) denotes its trace on F_{+,x_0} (resp. F_{-,x_0}). We use here the same notation as in Definition 1.1, since there is no risk of confusion.

If B sends F_{+,x_0} into F_{+,x_0} and F_{-,x_0} into F_{-,x_0} , if B' acts on ξ_{x_0} and if $\text{Tr } B'$ is the trace of B' , we have the trivial

$$\rho_+(B \otimes B') - \rho_-(B \otimes B') = (\chi_+(B) - \chi_-(B)) \text{Tr } B'.$$

Now a quick look at Eq. (2.14) which gives U_1^t shows that the curvature tensor L of ξ is *connecting* ξ and TM so much that U_1^t does not have naturally such a product form. However, we will be able to express U_1^t as the

expectation (on an *auxiliary* probability space) of an operator in product form so that everything will (miraculously) simplify.

Recall that the Lie algebra \mathcal{O} is naturally endowed with a Euclidean scalar product which is the (normalized) Killing form

$$(A, B) \in \mathcal{O} \rightarrow -\frac{\text{Tr } AB}{2}. \quad (3.12)$$

Also $L^2(R^n)$ is naturally a Euclidean space. Under the identification (1.29) of \mathcal{O} and $L^2(R^n)$, the two scalar products coincide. We can then define Brownian motion with values in the Euclidean space \mathcal{O} . Namely, W' denotes the set $\mathcal{C}(R^+; \mathcal{O})$ whose standard element is $\gamma_s = (\gamma_{i,s}^j)$. $\{F'_t\}_{t \geq 0}$ is the canonical filtration of W' associated to the σ -fields

$$F'_t = \mathcal{B}(\gamma_s | s \leq t).$$

Using (1.29), we will also write

$$\gamma_s = \sum_{i < j} \gamma_{j,s}^i dx^i \wedge dx^j. \quad (3.13)$$

DEFINITION 3.5. P' denotes the Wiener measure on \mathcal{O} with $P'(\gamma_0 = 0) = 1$.

Under P' , if γ_s is given by (3.13), the $(\gamma_{j,s}^i)$ ($i < j$) are mutually independent standard Brownian motions. In the sequel, we will consider the probability space $(W \times W', P_1 \otimes P')$, so that w^1 and γ are independent. Of course (u^t, x^t, \dots) and γ are also independent.

In what follows, I will denote the identity mapping of the considered vector fiber.

DEFINITION 3.6. $V_s^{1,t}$ is the process of linear mappings of F_{x_0} into itself defined by the Itô equation

$$\begin{aligned} dV_s^{1,t} &= -\frac{t}{2} V_s^{1,t} \left(\sum_{i < j} (u_0 e_i)(u_0 e_j) \delta \gamma_{i,s}^j \right), \\ V_0^{1,t} &= I. \end{aligned} \quad (3.14)$$

$V_s^{2,t}$ is the process of linear mappings from ξ_{x_0} into itself defined by the Itô equation

$$\begin{aligned} dV_s^{2,t} &= V_s^{2,t} \left(\sum_{i < j} \tau_0^{s,t} L_{x_s^t}(u_s^t e_i, u_s^t e_j) \delta \gamma_{i,s}^j \right), \\ V_0^{2,t} &= I. \end{aligned} \quad (3.15)$$

Of course $V_s^{1,t}$ maps F_{+,x_0} into F_{+,x_0} and F_{-,x_0} into F_{-,x_0} .

In what follows $E^{P'}$ will denote the expectation operator with respect to γ , for every given w^1 . Of course we assume that for the considered w^1, t is small enough so that u_s^t, x_s^t, \dots , are well defined.

We now have

PROPOSITION 3.7. For P_1 a.e. w^1 , for $t > 0$ small enough

$$U_s^t = E^{P'} [V_s^{1,t} \otimes V_s^{2,t}]. \tag{3.16}$$

Proof. Using Itô's formula on (W', P') , we know that

$$\begin{aligned} d(V_s^{1,t} \otimes V_s^{2,t}) &= (V_s^{1,t} \otimes V_s^{2,t}) \left[\sum_{i < j} \left(-\frac{t}{2} (u_0 e_i)(u_0 e_j) \otimes I \right. \right. \\ &\quad \left. \left. + I \otimes \tau_0^{s,t} L_{x_s^t}(u_s^t e_i, u_s^t e_j) \right) \delta \gamma_{i,s}^t \right] \\ &\quad - \frac{t}{2} (V_s^{1,t} \otimes V_s^{2,t}) \left(\sum_{i < j} (u_0 e_i)(u_0 e_j) \otimes \tau_0^{s,t} \right. \\ &\quad \left. \times L_{x_s^t}(u_s^t e_i, u_s^t e_j) \right) ds \end{aligned} \tag{3.17}$$

$$V_0^{1,t} \otimes V_0^{2,t} = I.$$

Moreover, it is essentially trivial to prove that for any p with $1 \leq p < +\infty$, then $E^{P'}[\sup_{0 \leq s < 1} |V_s^{1,t}|^p]$ and $E^{P'}[\sup_{0 \leq s < 1} |V_s^{2,t}|^p]$ are $< +\infty$. It is then feasible to take expectations with respect to P' in (3.17), and so

$$\begin{aligned} E^{P'} [V_s^{1,t} \otimes V_s^{2,t}] &= I - \frac{t}{2} \int_0^s E^{P'} [V_h^{1,t} \otimes V_h^{2,t}] \\ &\quad \times \left(\sum_{i < j} (u_0 e_i)(u_0 e_j) \otimes \tau_0^{h,t} L_{x_h^t}(u_h^t e_i, u_h^t e_j) \right) dh \end{aligned} \tag{3.18}$$

From (3.18), (3.16) follows immediately. ■

We now have

THEOREM 3.8. As $t \downarrow 0$, for any $k \in N$

$$\begin{aligned} \mathcal{S}(t, x_0) &= \frac{1}{(\sqrt{2\pi t})^n} \left[\int_{w \times w'} [\chi_+(V_1^{1,t} \tau_0^{1,t}) - \chi_-(V_1^{1,t} \tau_0^{1,t})] \text{Tr } V_1^{2,t} \right. \\ &\quad \left. \times H(\sqrt{t} dw^1, x_0) dP_1(w^1) dP'(\gamma) + o(t^k) \right]. \end{aligned} \tag{3.19}$$

Proof. This is obvious by Theorem 3.4 and Proposition 3.7. ■

3.d. An Exact Expression for $\mathcal{F}(x_0)$ in Terms of w^1, γ

We now will give an exact expression of $\mathcal{F}(x_0)$ in terms of W^1, γ .

DEFINITION 3.9. V_s^2 is the process of linear mappings from ξ_{x_0} into itself defined by the Itô equation

$$dV_s^2 = V_s^2 \left(\sum_{i < j} L_{x_0}(u_0 e_i, u_0 e_j) \delta \gamma_{i,s}^j \right), \quad V_0^2 = I. \quad (3.20)$$

In (3.21), L is calculated at (the constant) x_0 . We know that for any p ($1 \leq p < +\infty$),

$$E^{P'} \left[\sup_{0 \leq s \leq 1} |V_s^2|^p \right] < +\infty. \quad (3.21)$$

We now have the crucial result.

THEOREM 3.10. As $t \downarrow 0$, $\mathcal{F}(t, x_0)$ has a limit $\mathcal{F}(x_0)$ given by

$$\begin{aligned} \mathcal{F}(x_0) = & \int_{W \times W'} (-i)^l \text{Pf} \left[\int_0^1 \frac{\Omega_{u_0}}{4\pi} ((u_0 dw_s^1)^*, (u_0 w_s^1)^*) + \frac{\gamma_1}{2\pi} \right] \\ & \times \text{Tr } V_1^2 dP_1(w^1) dP'(\gamma). \end{aligned} \quad (3.22)$$

Proof. Set

$$\theta_s^{(1)} = \theta(\partial u_s^t); \quad \omega_s^{(1)} = \omega(\partial u_s^t), \quad \omega_s^{(2)} = \partial \omega_s^{(1)}. \quad (3.23)$$

We will first calculate the quantities (3.23) at $t=0$. Using Eq. (1.42), we know that

$$\begin{aligned} d\theta_s^{(1)} &= dw^1 + \partial v^2 ds + \omega^{(1)}(\sqrt{t} dw^1 + v^2 ds); & \theta_0^{(1)} &= 0, \\ d\omega_s^{(1)} &= \Omega((u_s^t(\sqrt{t} dw^1 + v^2 ds))^*, (u_s^t \theta_s^{(1)})^*); & \omega_0^{(1)} &= 0. \end{aligned} \quad (3.24)$$

Since $\pi' u_1^t = \gamma_0$, we also have for t small enough

$$\theta_1^{(1)} = 0. \quad (3.25)$$

Using the definition (3.6) of H_1 , the fact that for $t=0, v^2=0$ and that “formally” $dw^1 \in H_1$, we find that for $t=0$,

$$\theta_s^{(1)} = w_s^1; \quad \omega_s^{(1)} = 0; \quad \partial v_s^2 = 0. \quad (3.26)$$

We now differentiate the second line of (3.24) to obtain an equation for $\omega_s^{(2)}$ when $t=0$. We get, using (3.26),

$$d\omega_s^{(2)} = \Omega_{u_0}((u_0 dw_s^1)^*, (u_0 w_s^1)^*); \quad \omega_0^{(2)} = 0 \quad (3.27)$$

and so

$$\omega_1^{(2)} = \int_0^1 \Omega_{u_0}((u_0 dw_s^1)^*, (u_0 w_s^1)^*). \tag{3.28}$$

We now show that

$$\lim_{t \downarrow 0} \frac{\chi_+(V_1^{1,t} \tau_0^{1,t}) - \chi_-(V_1^{1,t} \tau_0^{1,t})}{t^l} = (-i)^l Pf \left[\int_0^1 \frac{\Omega_{u_0}}{2} ((u_0 dw_s^1)^*, (u_0 w_s^1)^*) + \gamma_1 \right]. \tag{3.29}$$

We will use Theorem 1.5. First observe that for t small enough, $\tau_0^{1,t}$ is in $\text{Spin}(n)$ (considered as a set of linear mappings acting on F_{x_0}). However, in general, $V_1^{1,t}$ is *not* in $\text{Spin}(n)$. In fact observe that by Proposition 1.3 for $i < j$, $(u_0 e_i)(u_0 e_j)$ acts on F_{x_0} as an element of \mathcal{O} . However, (3.14) is an equation in the sense of Itô, so that $V_s^{1,t}$ does *not* remain in $\text{Spin}(n)$.

However, consider the equation in the sense of Stratonovitch

$$\begin{aligned} dV_s^{1,t} &= -\frac{t}{2} V_s^{1,t} \left(\sum_{i < j} (u_0 e_i)(u_0 e_j) d\gamma_{i,s}^j \right), \\ V_0^{1,t} &= I. \end{aligned} \tag{3.30}$$

Now for every s , $V_s^{1,t} \in \text{Spin}(n)$. Moreover, if we write (3.14) as an equation in the sense of Stratonovitch, we get

$$\begin{aligned} dV_s^{1,t} &= -\frac{t}{2} V_s^{1,t} \left(\sum_{i < j} (u_0 e_i)(u_0 e_j) d\gamma_{i,s}^j \right) \\ &\quad - \frac{t^2}{8} V_s^{1,t} \left(\sum_{i < j} ((u_0 e_i)(u_0 e_j))^2 \right) ds, \\ V_0^{1,t} &= I. \end{aligned} \tag{3.31}$$

Using the defining relations (1.2), we see that for $i < j$,

$$((u_0 e_i)(u_0 e_j))^2 = -1 \tag{3.32}$$

so that (3.31) writes

$$\begin{aligned} dV_s^{1,t} &= -\frac{t}{2} V_s^{1,t} \left(\sum_{i < j} (u_0 e_i)(u_0 e_j) d\gamma_{i,s}^j \right) + \frac{t^2}{8} V_s^{1,t} \frac{n(n-1)}{2} ds, \\ V_0^{1,t} &= I. \end{aligned} \tag{3.33}$$

From (3.31), (3.33) it is then obvious that

$$V_s^{1,t} = \exp\left(\frac{n(n-1)}{16}st^2\right) V_s'' \quad (3.34)$$

and so

$$\frac{\chi_+(V_1^{1,t}\tau_0^{1,t}) - \chi_-(V_1^{1,t}\tau_0^{1,t})}{t^l} = \exp\frac{n(n-1)}{16}t^2 \frac{\chi_+(V_1''\tau_0^{1,t}) - \chi_-(V_1''\tau_0^{1,t})}{t^l}. \quad (3.35)$$

Now in the rhs of (3.35), $V_1''\tau_0^{1,t} \in \text{Spin}(n)$.

We now show that Theorem 1.5 is applicable to the curve $t \rightarrow V_1''\tau_0^{1,t} \in \text{Spin}(n)$.

Using (3.23), (3.26) and Taylor's formula it is clear that, by still identifying the Lie algebra of $\text{Spin}(n)$ to \mathcal{A} , if $\omega_1^{(2)}$ is still given by (3.28), we have for $t \downarrow 0$,

$$\tau_1^{0,t} = u_0' \left(e' + \frac{t}{2} \omega_1^{(2)} + o(t) \right) u_0'^{-1} \quad (3.36)$$

so that

$$\tau_0^{1,t} = u_0' \left(e' - \frac{t}{2} \omega_1^{(2)} + o(t) \right) u_0'^{-1}. \quad (3.37)$$

Moreover, it is clear that

$$\frac{dV_1''}{dt} t = 0 = -\frac{1}{2} \sum_{i < j} (u_0 e_i)(u_0 e_j) \gamma_{i,j}' \quad (3.38)$$

Using Proposition 1.3 which gives the action of $A \in \mathcal{A}$ on S , we find that as $t \downarrow 0$,

$$V_1'' = u_0'(e' - t\gamma_1 + o(t)) u_0'^{-1}. \quad (3.39)$$

Using (3.37), (3.39) it is then clear that as $t \downarrow 0$,

$$V_1''\tau_0^{1,t} = u_0' \left(e' - t \left(\frac{\omega_1^{(2)}}{2} + \gamma_1 \right) + o(t) \right) u_0'^{-1}. \quad (3.40)$$

Theorem 1.5 now tells us that

$$\lim_{t \downarrow 0} \frac{\chi_+(V_1''\tau_0^{1,t}) - \chi_-(V_1''\tau_0^{1,t})}{t^l} = (-i)^l \text{Pf} \left[\frac{\omega_1^{(2)}}{2} + \gamma_1 \right]. \quad (3.41)$$

Using (3.28), (3.35), and (3.41), (3.29) is proved.

Moreover, it is clear that as $t \downarrow 0$,

$$V_1^{2,t} \rightarrow V_1^2 \quad \text{a.s.} \tag{3.42}$$

Finally by (3.11) as $t \downarrow 0$,

$$H(\sqrt{t} dw^1, x_0) \rightarrow 1. \tag{3.43}$$

To prove (3.22), we only need to show that we can take the obvious limit under the expectation sign in (3.19). To see this we proceed as in the proof of Theorem 4.18 in [10]. Recall that we can choose $\delta > 0$ when defining $g, G(\sqrt{t} dw^1)$. Now by the proof of Theorem 4.18 in [10], we know that if $\delta > 0$ is small enough, for one $p > 1$,

$$\int |H(\sqrt{t} dw^1, x_0)|^p dP_1(w^1) \tag{3.44}$$

remains uniformly bounded as $t \downarrow 0$. In the sequel, we assume that such a $\delta > 0$ has been chosen. We know that if $G(\sqrt{t} dw^1) \neq 0$, $q(\sqrt{t'} dw^1, x_0)$ is well defined for any $t' \leq t$. Using Taylor's formula, we find that there exists $h \leq t$ depending on t, w^1 such that if $G(\sqrt{t} dw^1) \neq 0$,

$$\begin{aligned} \frac{\chi_+(V_1^{1,t} \tau_0^{1,t}) - \chi_-(V_1^{1,t} \tau_0^{1,t})}{t^l} &= (-i)^l \text{Pf} \left[\frac{\omega_1^{(2)}}{2} + \gamma_l \right] \\ &+ \frac{t^{1/2}}{(2l+1)!} \partial^{2l+1} [\chi_+(V_1^{1,h} \tau_0^{1,h}) - \chi_-(V_1^{1,h} \tau_0^{1,h})] \end{aligned} \tag{3.45}$$

(recall that we are using Taylor's formula in the variable \sqrt{t}).

Now using the method of the proof of Theorem 4.18 in [10] it is not hard to prove that for any $q \geq 1$,

$$E^{P \otimes P'} [|\partial^{2l+1} (\chi_+(V_1^{1,h} \tau_0^{1,h}) - \chi_-(V_1^{1,h} \tau_0^{1,h}))|^q G(\sqrt{t} dw^1)] \tag{3.46}$$

remains uniformly bounded as $t \downarrow 0$.

Using (3.44), (3.46), we see that there is uniform integrability in the rhs of (3.19). Equation (3.22) is proved. ■

Remark 2. The reader should now see the origin of the remarkable "cancellations" detected by Patodi [33] in his proof of the Index Theorem for the de Rham complex: it is essentially the fact that the energy functional $\int_0^1 |v^2(\sqrt{t} dw^1, x_0)|^2 ds/2t$ and the Malliavin covariance matrix $C(\sqrt{t} dw^1, x_0)$ both give a trivial contribution in formula (3.22) which makes that no covariant derivative of the curvature tensors R or L appears.

3.e. *Stochastic Calculus in the Exterior Algebra: The Atiyah–Singer Theorem for Spin-manifolds*

We will now transform the expression (3.22) in order to obtain $\text{Ind } D_+$ as the integral of a n -form over M . A word of warning is here necessary. Except when we explicitly indicate it, we will always talk about *differential forms*, and only exceptionally about cohomology classes, since what we want to do is identify the *exact* limit $\mathcal{F}(x_0)$.

Similarly we will use the Weil homomorphism [24, 37] as an homomorphism of the complex commutative algebra of certain invariant polynomials into the complex commutative algebra of *differential forms* \mathcal{H} defined by

$$\mathcal{H} = \bigoplus_0^l \Gamma(\bar{A}^{2p}(M)). \tag{3.47}$$

Of course \mathcal{H} is an algebra for the exterior product \wedge . In particular, we will talk about Chern forms, Pontryagin forms, etc. In the usual terminology of [14, 37] the Weil homomorphism takes polynomials into cohomology classes. Of course the theory of characteristic classes [31] shows that the cohomology class of the considered differential forms does *not* depend on the metric, the connection..., but we will (almost) forget about this.

First, we will transform slightly the expression (3.22). r_0 is now an element of the fiber X_{x_0} (recall that X is the bundle of unitary frames in ξ).

DEFINITION 3.11. $V_s'^2$ is the process of linear mappings from C^k into itself defined by the Itô equation

$$dV_s'^2 = V_s'^2 \left(\sum_{i < j} A_{r_0}((u_0 e_i)^*, (u_0 e_j)^*) \delta \gamma_{i,s}^j \right); \quad V_0'^2 = I. \tag{3.48}$$

DEFINITION 3.12. η denotes the canonical Riemannian n -differential form which defines the orientation of M .

Of course u_0 sends the form $dx^1 \wedge \dots \wedge dx^n$ into $\eta(x_0)$.

Finally recall that for any $x_0 \in M$, complex antisymmetric matrices acting on $T_{x_0}M$ identify to $\bar{A}^2(T_{x_0}M)$ as in (1.28), (1.29), and so their exterior powers are well defined as in Section 1.c.

We first give a slightly different form of Theorem 3.9.

PROPOSITION 3.13. *The following equality holds*

$$\begin{aligned} \mathcal{F}(x_0) \eta(x_0) = & \int_{w \times w'} \frac{1}{l!} \left[-i \int_0^1 \frac{R}{4\pi} (u_0 dw_s^1, u_0 w_s^1) - i \frac{u_0 \gamma_1 u_0^{-1}}{2\pi} \right]^{\wedge l} \\ & \times \text{Tr } V_1'^2 dP_1(w^1) dP'(y). \end{aligned} \tag{3.49}$$

Proof. Since A is the equivariant representation of L , it is clear that

$$\text{Tr } V_1^2 = \text{Tr } V_1'^2. \quad (3.50)$$

Using Proposition 1.4 and Theorem 3.9, we know that

$$\begin{aligned} \mathcal{F}(x_0) dx^1 \wedge \cdots \wedge dx^n &= \int_{w \times w'} \frac{1}{l!} \left[-i \int_0^1 \frac{\Omega_{u_0}}{4\pi} ((u_0 dw_s^1)^*, (u_0 w_s^1)^*) - i \frac{\gamma_1}{2\pi} \right]^{\wedge l} \\ &\times \text{Tr } V_1'^2 dP_1(w^1) dP'(\gamma). \end{aligned} \quad (3.51)$$

Since $\eta(x_0)$ is the image of $dx^1 \wedge \cdots \wedge dx^n$ by u_0 , and since Ω is the equivariant representation of the curvature tensor R , (3.49) is now obvious. ■

To evaluate (3.49), we introduce

DEFINITION 3.14. If $A \in \bar{A}^2(T_{x_0}M)$, $\exp^{\wedge} A \in \bigoplus_0^l \bar{A}^{2p}(T_{x_0}M)$ is defined by

$$\exp^{\wedge} A = 1 + A + \frac{A^{\wedge 2}}{2!} + \cdots + \frac{A^{\wedge l}}{l!}. \quad (3.52)$$

Of course for $l' > l$ $A^{\wedge l'} = 0$. If $B, B' \in \bigoplus_0^l \bar{A}^{2p}(T_{x_0}M)$; we will write

$$B \stackrel{\#}{=} B'$$

if B and B' have the same components in $\bar{A}^n(T_{x_0}M)$. We now have the identity:

THEOREM 3.15. For any $x_0 \in M$

$$\begin{aligned} \mathcal{F}(x_0) \eta(x_0) &\stackrel{\#}{=} \int_w \exp^{\wedge} \left\{ -\frac{i}{4\pi} \int_0^1 R(u_0 dw_s^1, u_0 w_s^1) \right\} dP_1(w^1) \\ &\wedge \int_{w'} \exp^{\wedge} \left\{ -\frac{i}{2\pi} u_0 \gamma_1 u_0^{-1} \right\} \text{Tr } V_1'^2 dP'(\gamma). \end{aligned} \quad (3.53)$$

Proof. By Proposition 3.13 and Definition 3.14, it is clear that

$$\begin{aligned} \mathcal{F}(x_0) \eta(x_0) &\stackrel{\#}{=} \int_{w \times w'} \exp^{\wedge} \left[-i \int_0^1 \frac{R}{4\pi} (u_0 dw_s^1, u_0 w_s^1) - i \frac{u_0 \gamma_1 u_0^{-1}}{2\pi} \right] \\ &\times \text{Tr } V_1'^2 dP_1(w^1) dP'(\gamma). \end{aligned} \quad (3.55)$$

Now recall that \mathcal{H} is commutative, so that

$$\begin{aligned} & \exp^{\wedge} \left[-i \int_0^1 \frac{R}{4\pi} (u_0 dw_s^1, u_0 w_s^1) - i \frac{u_0 \gamma_1 u_0^{-1}}{2\pi} \right] \\ &= \exp^{\wedge} \left[-i \int_0^1 \frac{R}{4\pi} (u_0 dw_s^1, u_0 w_s^1) \right] \wedge \exp^{\wedge} \left[-i \frac{u_0 \gamma_1 u_0^{-1}}{2\pi} \right]. \end{aligned} \tag{3.56}$$

Using (3.55), (3.56) and the fact that $V_1'^2$ only depends on γ , (3.53) is now obvious. ■

Remark 3. At this stage, we already see that F and ξ have separated, i.e., we must now evaluate *separately*

$$\int_w \exp^{\wedge} \left\{ \frac{-i}{4\pi} \int_0^1 R(u_0 dw_s^1, u_0 w_s^1) \right\} dP_1(w^1) \tag{3.57}$$

and

$$\int_{w'} \exp^{\wedge} \left\{ \frac{-i}{2\pi} u_0 \gamma_1 u_0^{-1} \right\} \text{Tr } V_1'^2 dP'(\gamma). \tag{3.58}$$

We first identify (3.57). Recall [24–37] that the Weil homomorphism φ associated to the real Euclidean vector bundle TM endowed with the Levi-Civita connection sends the algebra \mathcal{P}_0 of complex $\text{ad-}O(n)$ invariant polynomials over the Lie algebra \mathcal{O} into \mathcal{H} .

PROPOSITION 3.16. $\int_w \exp^{\wedge} \left\{ -i/4\pi \int_0^1 R(u_0 dw_s^1, u_0 w_s^1) \right\} dP_1(w^1)$ is the image by the Weil homomorphism φ of

$$A \in \mathcal{O} \rightarrow \int_w \exp \left\{ \frac{-i}{4\pi} \int_0^1 \langle Aw_s^1, dw_s^1 \rangle \right\} dP_1(w^1). \tag{3.59}$$

Proof. Recall that by (1.28), $\int_0^1 R(u_0 dw_s^1, u_0 w_s^1)$ identifies to the element of $A^2(T_{x_0}M)$, $X, Y \in T_{x_0}M \rightarrow \int_0^1 \langle X, R(u_0 dw_s^1, u_0 w_s^1) Y \rangle$. Classically if $X, Y \in T_{x_0}M$,

$$\int_0^1 \langle X, R(u_0 dw_s^1, u_0 w_s^1) Y \rangle = \int_0^1 \langle R(X, Y) u_0 w_s^1, u_0 dw_s^1 \rangle. \tag{3.60}$$

Recalling that π is the canonical projection $N \rightarrow M$, it is clear that the 2-form on $T_{u_0}M$,

$$X', Y' \in T_{u_0}N \rightarrow \int_0^1 \langle \Omega_{u_0}(X', Y') w_s^1, dw_s^1 \rangle, \tag{3.61}$$

projects on M as the 2-form (3.60).

Using (3.60), (3.61), to prove the proposition, we only need to show that (3.59) is $\text{ad} - O(n)$ invariant. Now if $a \in O(n)$,

$$\begin{aligned} & \int_w \exp \left\{ -\frac{i}{4\pi} \int_0^1 \langle (\text{ad } a A) w_s^1, dw_s^1 \rangle \right\} dP_1(w^1) \\ &= \int_w \exp \left\{ \frac{-i}{4\pi} \int_0^1 \langle Aa^{-1}w_s^1, a^{-1} dw_s^1 \rangle \right\} dP_1(w^1). \end{aligned} \quad (3.62)$$

Since P_1 is invariant by $O(n)$, (3.62) is also equal to

$$\int \exp \left\{ \frac{-i}{4\pi} \int_0^1 \langle Aw_s^1, dw_s^1 \rangle \right\} dP_1(w^1). \quad (3.63)$$

The proposition is proved. ■

We now evaluate (3.59). For one $A \in \mathcal{A}$, we can find an oriented orthogonal base of R^n such that in this base A writes

$$2\pi \begin{bmatrix} 0 & x_1 & 0 & & \\ -x_1 & 0 & & \ddots & \vdots \\ 0 & \ddots & \ddots & & 0 \\ \vdots & \ddots & 0 & 0 & x_l \\ 0 & & 0 & -x_l & 0 \end{bmatrix}. \quad (3.64)$$

The signs and the normalization coefficients are chosen to be compatible with the usual conventions in the theory of characteristic classes.

We now have

THEOREM 3.17. *The following equality holds*

$$\int_w \exp \left\{ \frac{-i}{4\pi} \int_0^1 \langle Aw_s^1, dw_s^1 \rangle \right\} dP_1(w^1) = \prod_1^l \frac{x_i/2}{\text{sh}(x_i/2)}. \quad (3.65)$$

Proof. Since P_1 is invariant under $O(n)$, we may and we will assume that A is exactly given by (3.64). Then

$$\frac{1}{4\pi} \int_0^1 \langle Aw_s^1, dw_s^1 \rangle = \sum_{i=1}^l \frac{x_i}{2} \int_0^1 (w^{1,2i} dw^{1,2i-1} - w^{1,2i-1} dw^{1,2i}). \quad (3.66)$$

Now be a well-known result of Lévy (see Yor [42]) we know that for any $\sigma \in R, i \in 1 \dots l$,

$$\int_w \exp \left\{ -i\sigma \int_0^1 (w^{1,2i} dw^{1,2i-1} - w^{1,2i-1} dw^{1,2i}) \right\} dP_1(w^1) = \frac{\sigma}{\text{sh}\sigma}. \quad (3.67)$$

Equation (3.65) follows from the independence of $(w^{1,2i}, w^{1,2i-1})$ for $i = 1 \dots l$. ■

We now identify (3.58). Let φ' be the Weil homomorphism associated to the Hermitian bundle ξ and the considered unitary connection in ξ , which sends the algebra of $\text{ad} - U(k)$ invariant polynomials over the Lie algebra $\mathcal{U}(k)$ into \mathcal{H} .

We now prove

THEOREM 3.18. $\int_w \exp^{\wedge} \{ (-i/2\pi) u_0 \gamma_1 u_0^{-1} \} \text{Tr } V_1'^2 dP'(\gamma)$ is the Chern character $\text{ch } \xi$ of ξ , i.e., the image by φ' of

$$B \in \mathcal{U}(k) \rightarrow \text{Tr} \exp \frac{-B}{2i\pi}. \tag{3.68}$$

Proof. Clearly, we have

$$\exp^{\wedge} \left\{ -\frac{i}{2\pi} u_0 \gamma_1 u_0^{-1} \right\} = u_0 \left(\exp^{\wedge} \frac{-i}{2\pi} \gamma_1 \right) u_0^{-1}. \tag{3.69}$$

Recall that by (1.29), γ_s is equal to

$$\gamma_s = - \sum_{i < j} \gamma_{i,s}^j dx^i \wedge dx^j. \tag{3.70}$$

We now use Itô's formula for the process $\exp^{\wedge}(-i/2\pi) \gamma_s$. Since $\bigoplus_0^l \bar{A}^{2p}(R^n)$ is a commutative algebra, the usual rules of the Itô calculus apply. Using (3.70), we get

$$\begin{aligned} \exp^{\wedge} \frac{-i}{2\pi} \gamma_s &= 1 - \frac{i}{2\pi} \int_0^s \exp^{\wedge} \left(\frac{-i}{2\pi} \gamma_h \right) \wedge \delta \gamma_h \\ &\quad - \frac{1}{8\pi^2} \int_0^s \exp^{\wedge} \left(\frac{-i}{2\pi} \gamma_s \right) \wedge dx^i \wedge dx^j \wedge dx^i \wedge dx^j \end{aligned} \tag{3.71}$$

(the last term is characteristic of Itô's formula). Now it is clear that

$$dx^i \wedge dx^j \wedge dx^i \wedge dx^i = 0 \tag{3.72}$$

so that (3.71) writes

$$\exp^{\wedge} \frac{-i}{2\pi} \gamma_s = 1 - \frac{i}{2\pi} \int_0^s \exp^{\wedge} \left(\frac{-i}{2\pi} \gamma_h \right) \delta \gamma_h. \tag{3.73}$$

Using (3.48), (3.70), (3.73), and Itô's formula, we get

$$\begin{aligned} \exp^\wedge \left(\frac{-i}{2\pi} \gamma_s \right) \otimes V_s'^2 &= 1 \otimes I + \int_0^s \left(\exp^\wedge \left(\frac{-i}{2\pi} \gamma_h \right) \otimes V_h'^2 \right) \left(\frac{-i}{2\pi} \delta\gamma_h \otimes I + 1 \right. \\ &\quad \left. \otimes \sum_{i < j} A_{r_0}((u_0 e_i)^*, (u_0 e_j)^*) \delta\gamma_{i,s}^j \right) \\ &\quad + \frac{i}{2\pi} \int_0^s \left(\exp^\wedge \left(\frac{-i}{2\pi} \gamma_h \right) \otimes V_h'^2 \right) \left(\sum_{i < j} dx^i \wedge dx^j \right. \\ &\quad \left. \otimes A_{r_0}((u_0 e_i)^*, (u_0 e_j)^*) \right) dh. \end{aligned} \quad (3.74)$$

Of course in (3.74), the first components in tensor products multiply by the exterior product \wedge . Moreover, it is classical that for any $p > 1$,

$$E^{P'} \left[\sup_{0 \leq s \leq 1} |V_s'^2|^p \right] < +\infty.$$

Taking expectations in (3.74), we find that

$$\begin{aligned} E^{P'} \exp^\wedge \left(\frac{-i}{2\pi} \gamma_s \right) \otimes V_s'^2 &= 1 \otimes I + \int_0^s \left(E^{P'} \exp^\wedge \left(\frac{-i}{2\pi} \gamma_h \right) \otimes V_h'^2 \right) \\ &\quad \times \frac{i}{2\pi} \left(\sum_{i < j} dx^i \wedge dx^j \otimes A_{r_0}((u_0 e_i)^*, (u_0 e_j)^*) \right) dh. \end{aligned} \quad (3.75)$$

By (3.75), it is now clear that

$$E^{P'} \exp^\wedge \left(\frac{-i}{2\pi} \gamma_s \right) \otimes V_s'^2 = \exp \frac{i}{2\pi} \left[\sum_{i < j} (dx^i \wedge dx^j) \otimes A_{r_0}((u_0 e_i)^*, (u_0 e_j)^*) s \right] \quad (3.76)$$

(the exponential in the rhs is calculated with the obvious multiplication rules).

Using (3.69), (3.76) we immediately see that

$$\begin{aligned} E^{P'} \exp^\wedge \left(\frac{-i}{2\pi} u_0 \gamma_1 u_0^{-1} \right) \text{Tr } V_1'^2 \\ = \text{Tr} \exp \frac{i}{2\pi} \left[\sum_{i < j} A_{r_0}((u_0 e_i)^*, (u_0 e_j)^*) (u_0 dx^i) \wedge (u_0 dx^j) \right] \end{aligned} \quad (3.77)$$

(where $u_0 dx^i$ is the image of dx^i by u_0). Now as a $\mathcal{H}(k)$ valued 2-differential form on $T_{r_0} X$, it is clear that A_{r_0} projects on $T_{x_0} M$ as the 2-form

$$\sum_{i < j} A_{r_0}((u_0 e_i)^*, (u_0 e_j)^*) u_0 dx^i \wedge u_0 dx^j \quad (3.78)$$

so that if π'' is the projection $X \rightarrow M$

$$\pi''^* \left[E^{p'} \exp^{\wedge} \left(\frac{-i}{2\pi} u_0 \gamma_1 u_0^{-1} \right) \text{Tr } V_1'^2 \right] = \text{Tr} \exp \left[\frac{i}{2\pi} A_{r_0} \right]. \quad (3.79)$$

The theorem is proved. ■

Remark 4. Let us again insist here that $\text{ch } \xi$ is an element of \mathcal{H} and is *not* considered as a cohomology class.

Also note that we use the sign conventions of [24], while Atiyah–Bott Patodi [4] use the conventions of [31]. For this sign question, see [31, p. 304].

Using Theorem 3.15, Proposition 3.16, Theorems 3.17, 3.18, we finally get the theorem of Atiyah–Singer [4, 6].

THEOREM 3.19. *The following equality holds:*

$$\mathcal{F} \eta \stackrel{n}{=} \prod_1^l \frac{x_i/2}{\text{sh}(x_i/2)} (TM) \wedge \text{ch } \xi, \quad (3.80)$$

where in (3.80), the symmetric functions in x_1^2, \dots, x_l^2 should be replaced by the Pontryagin forms of TM .

Remark 5. For the applications of Theorem 3.19 to other classical indexes, see [4], and Remark 3 in Section 1.

3.f. A Remark on the Path Integral Representation of the Index

Assume that M is an even dimensional connected compact Riemannian manifold. We will assume that M is spin, but the argument which follows also holds for manifolds which are not spin.

For $t > 0$, $x \in M$, let $Q^{t,x}$ be the Brownian bridge on the time interval $[0, t]$, i.e., the Brownian motion x_s ($0 \leq s \leq t$) with $x_0 = x$, conditional on $x_t = x$ (for the precise definition, see [10]).

Let $p_t(x, y)$ be the heat equation kernel for the Laplace–Beltrami operator of M . By using Theorem 2.5, we know that

$$\mathcal{F}(t, x) = p_t(x, x) E^{Q^{t,x}} S^{t,x}, \quad (3.81)$$

where $S^{t,x}$ is a random variable depending on x .

Using (3.3), we see that

$$\text{Ind } D_+ = \int_M E^{Q^{t,x}} [S^{t,x}] p_t(x, x) dx. \quad (3.82)$$

Naturally, in the Index Theorem, t has the interpretation of a homotopy

parameter since what we do is contracting the loop space on constant loops. However, there is a topological obstruction which is the homotopy group.

DEFINITION 3.20. For $x \in M$, $A_x^{0,t}$ (resp. $A_x^{1,t}$) denotes the set of loops $s \in [0, t] \rightarrow \gamma_s$ with $\gamma_0 = \gamma_t = x$ which are homotopic (resp. nonhomotopic) to the constant loop $s \in [0, t] \rightarrow x$.

We now have the following result:

THEOREM 3.21. Assume that the universal covering M' of M is compact. Then for any $t > 0$,

$$\int_M E^{Q^{t,x}} [1_{x \in \Lambda_x^{1,t}} S^{t,x}] p_t(x, x) dx = 0 \tag{3.83}$$

and so

$$\text{Ind } D_+ = \int_M E^{Q^{t,x}} [1_{x \in \Lambda_x^{0,t}} S^{t,x}] p_t(x, x) dx. \tag{3.84}$$

Proof. Equation (3.80) shows that $\text{Ind } D_+$ is the integral of a n differential form ω on M . Let D'_+ be the corresponding operator on M' , and k be the canonical projection $M' \rightarrow M$. Clearly $\text{Ind } D'_+$ is the integral of the form $k_* \omega$ because of the universal expression (3.80).

Now if h is the number of elements of the homotopy group of M , we have classically

$$\int_{M'} k_* \omega = h \int_M \omega \tag{3.85}$$

so that

$$\text{Ind } D'_+ = h [\text{Ind } D_+]. \tag{3.86}$$

Now formula (3.82) applied to $\text{Ind } D'_+$ shows without difficulty that

$$\text{Ind } D'_+ = h \int_M E^{Q^{t,x}} [S^{t,x} 1_{x \in \Lambda_x^{0,t}}] p_t(x, x) dx \tag{3.87}$$

essentially because loops in M' project on M as 0-homotopic loops, the factor h coming from the summation along the fibers of M' . Comparing (3.84), (3.86), (3.87), we find that (3.83) and (3.84) hold. ■

Remark 6. General heat equation methods show that (3.83) is asymptotically 0 as $t \downarrow 0$. The fact it is exactly 0 seems to be a consequence of the Index Theorem.

The previous result shows that t can be legitimately considered as a homotopy parameter.

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Note added in proof. The following very interesting reference should also be included: L. ALVAREZ-GAUMÉ, Supersymmetry and the Atiyah–Singer Index Theorem. *Commun. Math. Phys.* **90** (1983), 161–173.

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