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Géométrie des espaces des cycles : waist et graphes minimaux, 2010

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Chapitre 1

Introduction

1.1 Version francaise

La thése qui est présentée ici est composée de trois articles, intitulés "On Gromov's Waist of the Sphere Theorem", "A Lower Bound on the Waist of Unit Spheres of Uniformly Convex Normed Spaces" et "On the Maximum Number of Vertices of Minimal Embedded Graphs". Ces sujets se relient et font partie d'un même probleme qui appartient à la géométrie des espaces des cycles. Dans ce qui suit on essaie d'expliquer tout cela.

1.1.1 Quelques mots sur la concentration des mesures

Le phénomène de concentration de la mesure, découvert par P. Lévy et développé par V. Milman dans les années 70, est largement etudié de nos jours. Il trouve son origine dans des travaux où P. Lévy cherche à définir une moyenne pour les fonctions définies sur la sphère d'un espace de Hilbert de dimension infinie. Le mécanisme qui rend cela possible est déjà visible en dimension finie.

Théorème 1 *Soit μ la mesure riemanienne normalisée sur la sphère canonique $\mathbb{S}^n$. Soit $f : \mathbb{S}^n \to \mathbb{R}$ une fonction 1-lipschitzienne. Soit $\varepsilon > 0$. Il existe $m \in \mathbb{R}$ tel que*

$$\mu(\{|f - m| < \varepsilon\}) \geq 1 - \frac{\int_\varepsilon^{\pi/2} (\cos t)^{n-1} dt}{\int_0^{\pi/2} (\cos t)^{n-1} dt} \geq 1 - 2e^{-(n-1)\varepsilon^2/2}.$$

On peut prendre pour m la valeur médiane de f, i.e. le réel tel que $\mu(\{f \geq m\}) \geq 1/2$ et $\mu(\{f \geq m\}) \leq 1/2$. P. Lévy faisait tendre n vers l'infini, et pouvait déduire de ce théorème une notion de moyenne pour les fonctions 1-lipschitziennes définies sur la sphère d'un espace de Hilbert de dimension infinie.

P. Lévy a observé en outre que ce phénomène s'étend à d'autres hypersurfaces de l'espace euclidien. Cela suggère d'étudier les espaces dans lesquels il a lieu.

Définition 1.1.1 (mm-espace) *Le triplet $X = (X, d, \mu)$ s'appelle un* mm-espace *si X est un espace métrique polonais (i.e. X est complet et à base dénombrable), d est la distance définie sur X et μ est une mesure borélienne σ-finie sur X.*

Définition 1.1.2 (pm-espace) *Le triplet $X = (X, d, \mu)$ est un* pm-espace *si X est un mm-espace est de plus, la mesure μ est une mesure de probabilité.*

Définition 1.1.3 (Profil de concentration) *Soit $X = (X, d, \mu)$ un pm-espace. Le* profil de concentration *de (X, d, μ) est la plus petite fonction π sur $\mathbb{R}_+$ telle que, pour tout $\varepsilon > 0$ et pour toute fonction 1-lipschitzienne sur X, il existe $m \in \mathbb{R}$ tel que*

$$\mu(\{|f - m| > \varepsilon\}) \leq \pi(\varepsilon).$$

Définition 1.1.4 (Concentration gaussienne) *Soit $X = (X, d, \mu)$ un pm-espace. On dit que (X, d, μ) a une* concentration gaussienne *s'il existe des constantes C, $c > 0$ telles que son profil de concentration satisfait $\pi_{(X,d,\mu)}(\varepsilon) \leq C\, e^{-c\varepsilon^2/2}$.*

Remarque

Voici des exemples classiques possédant une concentration gaussienne.

- La mesure riemanienne normalisée sur la sphère canonique $\mathbb{S}^n$.
- La mesure riemanienne normalisée sur une variété riemanienne compacte et connexe ayant une courbure de Ricci positive.
- La mesure gaussienne sur l'espace euclidien $\mathbb{R}^n$.
- La mesure de comptage sur le cube de Hamming $\{0, 1\}^n$.

Définition 1.1.5 (Concentration exponentielle) *Soit $X = (X, d, \mu)$ un pm-espace. On dit que (X, d, μ) a une* concentration exponentielle *s'il existe des constantes C, $c > 0$ telles que son profil de concentration satisfait $\pi_{(X,d,\mu)}(\varepsilon) \leq C\, e^{-c\varepsilon}$.*

Remarque

Les espaces satisfaisant une concentration exponentielle en gros ne sont pas aussi bien concentrés que les précédents. Comme exemple important pour ce genre de concentration on peut donner les graphes expanseurs.

1.1.2 Lien entre le problème isopérimétrique et la concentration

Le problème isopérimétrique est un sujet très ancien. Pour les variétés riemaniennes, on peut formuler le problème isopérimétrique comme la recherche des ouverts relativement compacts de volume r tel que le volume du bord de ces ensembles soit minimum. Ou plus modestement, comme la recherche d'un minorant $I(r)$ pour le volume du bord des ensembles de volume r. Un argument simple dû à P. Lévy montre qu'étant donné une fonction minorante I, on peut minorer le volume du ε-voisinage tubulaire de tous les ensembles de volume r, pour tout $\varepsilon > 0$. Ceci garde un sens dans un mm-espace général. Nous considèrerons donc que, dans un mm-espace (X, d, μ) le problème isopérimétrique consiste, pour chaque r, $\varepsilon > 0$, à chercher, parmi tous les ouverts $A \subset X$ de volume égal à r, ceux qui minimisent le volume du ε-voisinage tubulaire.

Notation 1.1.1 (Voisinage Tubulaire) *Soit $X = (X, d, \mu)$ un mm-espace, Y un sous-espace de X, et soit $\varepsilon > 0$. Le* ε-voisinage tubulaire *de Y est défini et noté par*

$$Y + \varepsilon = \{x \in X \mid d(x, Y) \leq \varepsilon\}.$$

Définition 1.1.6 (Fonction isopérimétrique) *Soit (X, d, μ) un pm-espace. La* fonction isopérimétrique *est définie sur $\mathbb{R}_+$ par*

$$\alpha_{(X,d,\mu)}(r) = \sup\{1 - \mu(A + r) \mid A \subset X, \mu(A) \geq \frac{1}{2}\}.$$

Remarque.

La fonction isopérimétrique est appelée fonction de concentration par M. Ledoux. On préfère la nommer différemment pour distinguer isopérimétrie et concentration. La fonction isopérimétrique contrôle les invariants suivants.

- La premier valeur propre λ_1 du laplacien.
- La décroissance du noyau de chaleur et la probabilité de retour du mouvement brownien à son point de départ.
- Le transport optimal des mesures.

Pour en savoir plus, on pourra consulter [16].

Nous nous intéressons ici à la façon dont l'isopérimétrie est reliée à la concentration. L'isopérimétrie contrôle la concentration, mais la réciproque n'est pas rigoureusement vraie.

Proposition 1 *Isopérimétrie $\Rightarrow$ Concentration. En particulier pour tout $\varepsilon > 0$ on a*

$$\pi_{(X,d,\mu)}(\varepsilon) \leq 2\alpha_{(X,d,\mu)}$$

On prouvera cette proposition au chapitre 2.

1.1.3 Le waist, et en particulier le 1-waist

M. Gromov a introduit un invariant des mm-espaces, intermédiaire entre l'isopérimétrie et la concentration, nommé le 1-waist. La particularité de cet invariant c'est sa généralisation naturelle en (co)-dimension plus grande que 1, qui ouvre la porte au phénomène de concentration *topologique*.

Définition 1.1.7 (Waist d'un mm-espace, voir [8]) *Soit $X = (X, d, \mu)$ un mm-espace. Soit Z un espace topologique. Le* waist de X relatif à Z *est la plus grande fonction w sur $\mathbb{R}_+$ telle que pour toute application continue $f : X \to Z$, il existe un point $z \in Z$ tel que pour tout $\varepsilon > 0$,*

$$\mu(f^{-1}(z) + \varepsilon) \geq w(\varepsilon).$$

Définition 1.1.8 (1-Waist) *Le 1-waist de $X = (X, d, \mu)$ est le waist de X relatif à $Z = \mathbb{R}$.*

On vérifiera au chapitre 2 que

Proposition 2 *Soit X un mm-espace. On suppose que toutes les boules de X sont connexes. Alors*

Isopérimétrie $\Rightarrow$ 1-waist $\Rightarrow$ Concentration.

I.e. pour tout $\varepsilon > 0$ on a

$$1 - 2\alpha_{(X,d,\mu)}(\varepsilon) \leq w_{(X,d,\mu)}(\varepsilon) \leq 1 - \pi_{(X,d,\mu)}(\varepsilon).$$

Il y a aussi une réciproque.

Proposition 3 *1-waist $\Rightarrow$ Isopérimétrie : pour tout ouvert $A \subset X$ et pour tout $\varepsilon > 0$ on a*

$$\max\{\mu(A + \varepsilon), \mu(A^c + \varepsilon)\} \geq w(\varepsilon).$$

De même on vérifiera au chapitre 2 que le waist est bien un invariant des mm-espaces :

Proposition 4 *Soient (M, d, μ) et (M', d', μ') deux mm-espaces oú $dimM = dimM' = n$ et $n \geq k$. Soit $\phi : M \to M'$ une application c-Lipschitzienne oú $\phi_*(\mu) = \mu'$. Alors*

$$wst(M' \to \mathbb{R}^k, c\varepsilon) \geq wst(M \to \mathbb{R}^k, \varepsilon).$$

1.1.4 Le waist en codimension supérieure

La proposition 2 prouve que le phénomène de concentration est une consequence directe de l'inégalité isopérimétrique. L'inégalité isopérimétrique classique pour une variété riemannienne relie la mesure des ensembles (relativement) compacts à la mesure de leur bord. Ces inégalités sont en codimension 1, parce que la différence entre la dimension d'un ensemble et son bord est égal à 1.

F. Almgren, au début de sa carrière, en combinant les idées topologiques contenues dans le théorème de Dold-Thom et les idées analytique contenues dans les travaux de Federer et Fleming sur l'espace des courants intégraux a pu obtenir des inegalités de type isopérimétrique en codimension quelconque. Plus précisemment Almgren a trouvé une borne inférieure optimale sur le volume d'un k-cycle minimal ($k \leq n$) sur la sphère $\mathbb{S}^n$ voir [22], [7], [4]. Voici un corollaire obtenu à partir de la théorie d'Almgren-Morse

Théorème 2 *Soit $\mathbb{S}^n$ la sphère canonique de dimension n. Soit M une sous-variété minimale de dimension m ($m < n$) de $\mathbb{S}^n$. Alors*

$$vol(M) \geq vol(\mathbb{S}^m).$$

Quelques années plus tard, M. Gromov reprend ces idées et définit un invariant métrique associé aux espace métrique qui est l'invariant volume d'une application. A notre connaissance, ce fut F. Almgren qui a introduit dans la littérature les premiers inégalité de type isopérimétrique en codimension quelconque.

En introduisant l'invariant waist, Gromov relie, encore une fois, des idées de divers domaines des mathématiques. Avoir une inégalité de waist c'est à la fois avoir des informations sur le comportement de concentration d'un mm-espace et une information topologique quantitative sur l'espace des cycles, comme en donne la théorie de Morse.

Dans [8], Gromov prouve deux inégalités de waist optimales qu'on énonce maintenant.

Théorème 3 (Waist de la sphère) *Soit $f : \mathbb{S}^n \to \mathbb{R}^k$, une application continue de la sphère canonique dans l'espace Euclidien ou $k \leq n$. Il existe un point $z \in \mathbb{R}^k$ tel que pour tout $\varepsilon > 0$,*

$$vol(f^{-1}(z) + \varepsilon) \geq vol(\mathbb{S}^{n-k} + \varepsilon).$$

Ici, $\mathbb{S}^{n-k}$ désigne une sous-sphère totalement géodésique de codimension k.

Théorème 4 (Waist de l'espace gaussien) *Soit X un sous-ensemble convexe de $\mathbb{R}^n$ muni d'une mesure de probabilité log-concave μ. Soit $f : X \to \mathbb{R}^k$ une application continue. Alors, il existe un $z \in \mathbb{R}^k$ tel que pour tout $\varepsilon > 0$,*

$$\mu(f^{-1}(z) + \varepsilon) \geq (2\pi)^{-\frac{k}{2}} \int_{B(0,\varepsilon)} \exp -\frac{\|x\|^2}{2} dx^k.$$

On remarque que ces deux inégalités sont optimales (il suffit de prendre des projections linéaires).

1.1.5 Résultats

La majorité des trois années de préparation de cette thèse a été consacrée à étudier l'article [8], et notamment, à compléter les détails de la preuve du Théorème 3. Ce travail occupe entièrement le chapitre 3.

Dans [9], M. Gromov et V. Milman prouvent une inégalité isopérimétrique sur les sphères unités des espaces uniformément convexes en utilisant les techniques de localisation (décomposition en aiguilles). L'auteur a obtenu une minoration du waist, en codimension quelconque, pour les sphères unités des espaces uniformément convexes. C'est généralisation commune aux résultats de [9] et au Théorème 3.

Théorème 5 *Soit X un espace normé uniformément convexe de dimension finie égale à $n+1$. Soit $S(X)$ la sphère unité de X sur laquelle on utilise la distance induite par la norme de l'espace X et la mesure de probabilité cônique. Alors une borne inférieure pour le waist de $S(X)$ relatif à $\mathbb{R}^k$ est donné par*

$$w(\varepsilon) = \frac{1}{1 + (1 - 2\delta(\frac{\varepsilon}{2}))^{n-k}(k+1)^{k+1}\frac{F(k,\frac{\varepsilon}{2})}{G(k,\frac{\varepsilon}{2})}}$$

où $\delta(\varepsilon)$ est le module de convexité,

$$F(k,\varepsilon) = \int_{\psi_2(\varepsilon)}^{\frac{\pi}{2}} \sin(x)^{k-1}\, dx.$$

et

$$G(k,\varepsilon) = \int_0^{\psi_1(\varepsilon)} \sin(x)^{k-1}\, dx.$$

Et où

$$\psi_1(\varepsilon) = 2\arcsin(\frac{\varepsilon}{4\sqrt{k+1}})$$

et

$$\psi_2(\varepsilon) = 2\arcsin(\frac{\varepsilon}{2\sqrt{k+1}})$$

Le chapitre 4 sera consacré à la preuve complète de ce théorème et quelques détails complémentaires.

1.1.6 L'espace des cycles

Le waist appartient à la grande famille des invariants numériques fournis par l'étude variationnelle sur l'espace des cycles. On donne une définition pour l'espace des cycles.

Définition 1.1.9 (L'espace des cycles) *Soit X un espace métrique et soit G un groupe de coefficients. L'espace des k-cycles à coefficients dans G, noté $Z_G(k,n)$ est l'ensemble constitué des k-cycles singuliers lipschitziens à coefficients dans G. Un simplexe singulier lipschitzien $\sigma : \Delta \to X$ possède un volume. Si G est muni d'une norme, la masse d'un k-cycle $T = \sum g_i\sigma_i$ est*

$$M(T) = \sum |g_i| vol(\sigma_i),$$

et la norme $\flat$ est définie par

$$\flat(T) = \inf\{M(S) + M(R) \,|\, T = S + \partial R\}.$$

On munit $Z_G(k,n)$ de la topologie de la norme $\flat$.

On peut donner une autre définition pour l'invariant waist.

Définition 1.1.10 (Waist des mm-espace, le point de vue variationnel) *Soit X un mm-espace. Soit Z un espace topologique et soit F_z, $z \in Z$, une famille de cycles (sous-espaces) de X. Soit $w(\varepsilon)$ une fonction sur $\mathbb{R}_+$. On dit que le waist de X relatif à la famille des cycles F_z est au moins égal à $w(\varepsilon)$, et on écrit*

$$wst(X, F_z, \varepsilon) \geq w(\varepsilon),$$

s'il existe un $z \in Z$ tel que pour tout $\varepsilon > 0$,

$$\mu(F_z + \varepsilon) \geq w(\varepsilon).$$

La théorie de Morse classique étudie la fonctionnelle de longueur (et energie) sur l'espace (de dimension infinie) des lacets d'une variété riemannienne. La théorie d'Almgren-Morse étudie la fonctionnelle volume sur l'espace (de dimension infinie) des cycles. Récemment, dans [10] L. Guth a donné un nouveau point de vue. Il étudie la fonctionnelle volume d'Almgren sur l'algèbre de cohomologie de l'espace des cycles. Ainsi, pour chaque classe de cohomologie des espaces de cycles, il existe un problème variationnel (problème de Min-Max) associé à la classe de cohomologie. Pour le cas des coefficients dans $\mathbb{Z}_2$, L. Guth obtient des résultats (presque) optimaux. Pour le cas général, le problème est toujours ouvert (et loin d'être résolu). En remplaçant les volumes par les nombres $\mu(C + \varepsilon)$, on

étend le problème de L. Guth aux mm-espaces. Le problème étendu contient celui de minorer le k-waist.

Ici on s'interesse aux 1-cycles relatifs sur $(\mathbb{R}^2, A)$ où $A \subset \mathbb{R}^2$ est un ensemble fini de points du plan. La fonctionnelle est la fonctionnelle de longueur. Les points critiques de cette fonctionnelle sur l'espace des 1-cycles sont appelés *cycles minimaux attachés à A*. Plutôt qu'à la valeur de la longueur totale, on s'interesse à une quantité reliée, la complexité topologique des 1-cycles, qui est le nombre de sommets. Le problème de majorer le nombre de sommets des cycles minimaux attachés à un ensemble de points n'est pas facile, on n'a obtenu qu'un résultat partiel, une majoration du nombre de sommets d'un graphe minimal de degre 3 attaché à un nombre donn'é de points. En particulier on prouve le théorème suivant.

Théorème 6 *Le nombre maximum de sommets d'un graphe minimal 3-regulier sur le plan attaché à n points , noté $f_3(n)$, satisfait les égalités suivantes.*

- *Si* $n = 6k$,
$$f_3(n) = 6k^2 + 6k. \tag{1.1}$$
- *Si* $n = 6k + 1$,
$$f_3(n) = 6k^2 + 8k. \tag{1.2}$$
- *Si* $n = 6k + 2$,
$$f_3(n) = 6k^2 + 10k + 2. \tag{1.3}$$
- *Si* $n = 6k + 3$,
$$f_3(n) = 6k^2 + 12k + 4. \tag{1.4}$$
- *Si* $n = 6k + 4$,
$$f_3(n) = 6k^2 + 14k + 6. \tag{1.5}$$
- *Si* $n = 6k + 5$,
$$f_3(n) = 6k^2 + 16k + 8. \tag{1.6}$$

On remarque qu'un graphe minimal 3-régulier attaché à un nombre fini de points du plan peut être vu comme un $\mathbb{Z}_3$-cycle relatif, mais la réciproque n'est pas vraie. Le problème général reste toujours ouvert. Le chapitre 5 de cette thèse contient une preuve de ce théorème et des discussions et commentaires relatifs à ce problème. Il n'est pas inutile de remarquer que si on modifie la fonctionnelle de longueur, en mettant des poids sur les arêtes des graphes, alors il n'existe pas de borne sur le nombre maximal de sommets, comme W. Allard et F. Almgren l'ont montré. Ils ont construit une famille de graphes minimaux (avec poids) 3-réguliers attachés en 8 points, de longueurs bornées mais de nombres de sommets arbitrairement grands.

1.2 English version

The Thesis presented here is the concatenation of three articles entitled "On Gromov's Waist of the Sphere Theorem", "A Lower Bound on the Waist of Unit Spheres of Uniformly Convex Normed Spaces" and "On the Maximum Number of Vertices of Minimal Embedded Graphs". These subjects are related and are a part of the same problem which belongs to the geometry of the space of cycles. In what follows we will try to explain all these links. We begin with the recent concept of topological concentration.

1.2.1 Some Words on the Concentration of Measures

The phenomenon of concentration of measure, discovered by P. Lévy and developped by V. Milman in the 70's is widely studied nowadays. It has its origin in work where P. Lévy defines an expectation for functions defined on the unit sphere of an infinite dimensional Hilbert Space. The mechanism which enable this is already seen in finite dimensions.

Theorem 5 *Let μ be the normalized Riemannian measure defined on the canonical unit sphere $\mathbb{S}^n$. Let $f : \mathbb{S}^n \to \mathbb{R}$ be a 1-Lipschitz function. Let $\varepsilon > 0$. There exists a $m \in \mathbb{R}$ such that*

$$\mu(\{|f - m| < \varepsilon\}) \geq 1 - \frac{\int_\varepsilon^{\pi/2} (\cos t)^{n-1} dt}{\int_0^{\pi/2} (\cos t)^{n-1} dt} \geq 1 - 2e^{-(n-1)\varepsilon^2/2}.$$

We can take for m the value of the median of f, i.e. the real number such that $\mu(\{f \geq m\}) \geq 1/2$ and $\mu(\{f \geq m\}) \leq 1/2$. P. Lévy let n tend to infinity and deduced from this theorem a notion of mean value (or expectation) for 1-Lipschitz functions defined on the unit sphere of an infinite dimensional Hilbert Space.

P. Lévy made the observation that this phenomenon can be extended to other hypersurfaces of Euclidean space. This suggests to study the spaces on which this phenomenon can happen.

Definition 1.2.1 (mm-space) *The triple $X = (X, d, \mu)$ is called a* mm-space *if X is a polish metric space (i.e. X is complete and has countable basis), d is the distance defined on X and μ is a Borel σ-finite measure defined on X.*

Definition 1.2.2 (pm-space) *The triple $X = (X, d, \mu)$ is called a* pm-space *if X is a mm-space and μ is a probability measure.*

Definition 1.2.3 (The concentration profile) *Let $X = (X, d, \mu)$ be a pm-espace. The* concentration profile *of (X, d, μ) is the smallest function π on $\mathbb{R}_+$ such that for every $\varepsilon > 0$ and for every 1-Lipschitz function defined on X, there exists a $m \in \mathbb{R}$ such that*

$$\mu(\{|f - m| > \varepsilon\}) \leq \pi(\varepsilon).$$

Definition 1.2.4 (Guassien Concentration) *Let $X = (X, d, \mu)$ be a pm-space. We say that (X, d, μ) has a* Gaussian concentration *if there exist constants C, $c > 0$ such that the concentration profile satisfies $\pi_{(X,d,\mu)}(\varepsilon) \leq C\, e^{-c\varepsilon^2/2}$.*

Remark

Here are a few classical examples of spaces having Gaussian concentration.

- The normalized Riemannian measure defined on the canonical sphere $\mathbb{S}^n$.
- The normalized Riemannian measure defined on a connected compact Riemannian manifold with positive Ricci curvature.
- The Gaussian measure defined on Euclidean space $\mathbb{R}^n$.
- The counting measure defined on the Hamming cube $\{0, 1\}^n$.

Definition 1.2.5 (Exponential Concentration) *Let $X = (X, d, \mu)$ be a pm-space. We say that (X, d, μ) has an* exponential concentration *if there exist constants C, $c > 0$ such that the concentration profile satisfies $\pi_{(X,d,\mu)}(\varepsilon) \leq C\, e^{-c\varepsilon}$.*

Remark

The spaces satisfying exponential concentration are not as well concentrated as the previous examples. For an important class of examples we can give the class of expander graphs.

1.2.2 Relation Between Isoperimetric Problems and Concentration

The isoperimetric problem is a very ancient subject of study. For Riemannian manifolds, we can formulate the isoperimetric problem as the search for relatively compact open subsets of volume r minimizing the volume of the boundary. Or more modestly, the search for a lower bound $I(r)$ for the volume of the boundary of subsets of volume r. A simple argument due to P. Lévy shows if I is known, a lower bound for the volume of the ε-neighborhood of every subset having volume equal to r follows, for all $\varepsilon > 0$. This still makes sense in a general mm-space. Hence we consider that for a general mm-space (X, d, μ) the isoperimetric problem consists of the search, for every r, $\varepsilon > 0$, among all open subsets $A \subset X$ of volume equal to r, of those minimizing the volume of the ε-tubular neighborhood.

Notation 1.2.1 (Tubular Neighborhood) *Let $X = (X, d, \mu)$ be a mm-space, Y a subspace of X, and let $\varepsilon > 0$. The* ε-tubular neighborhood *of Y is defined and denoted by*

$$Y + \varepsilon = \{x \in X \mid d(x, Y) \leq \varepsilon\}.$$

Definition 1.2.6 (Isoperimetric Function) *Let (X, d, μ) be a pm-space. The* isoperimetric function *is defined on $\mathbb{R}_+$ by*

$$\alpha_{(X,d,\mu)}(r) = \sup\{1 - \mu(A + r) \mid A \subset X, \mu(A) \geq \frac{1}{2}\}.$$

Remark.

The isoperimetric function is called the concentration function by M. Ledoux. We prefer to name it differently to distinguish isoperimetry and concentration. The isoperimetric function controls the following invariants.

- The first (non-zero) eigenvalue λ_1 of the Laplacian.
- The decay of the heat kernel and the probability of return of Brownian motion to its departure point.
- The optimal transport of measures.

For more informations, one can consult [16].

Here we are interested in how isoperimetry is related to the concentration. Isoperimetry controls concentration but the converse is not rigorously true.

Proposition 6 *Isoperimetry $\Rightarrow$ Concentration. In particular for all $\varepsilon > 0$ we have*

$$\pi_{(X,d,\mu)}(\varepsilon) \leq 2\alpha_{(X,d,\mu)}$$

We prove this proposition in chapiter 2.

1.2.3 The waist, and in particular the 1-waist

M. Gromov introduced an invariant of mm-spaces, intermediate between isoperimetry and concentration, named the 1-waist. The particularity of this invariant is its natural generalization to higher (co)-dimensions. This opens the door to the *topological* concentration phenomenon.

Definition 1.2.7 (Waist of a mm-space, see [8]) *Let $X = (X, d, \mu)$ be a mm-space. Let Z be a topological space. The* waist of X relative to Z *is the largest function w defined on $\mathbb{R}_+$ such that for every continuous map $f : X \to Z$, there exists a point $z \in Z$ such that for every $\varepsilon > 0$,*

$$\mu(f^{-1}(z) + \varepsilon) \geq w(\varepsilon).$$

Definition 1.2.8 (1-Waist) *The 1-waist of $X = (X, d, \mu)$ is the waist of X relative to $Z = \mathbb{R}$.*

We will verify in chapter 2 that

Proposition 7 *Let X be an mm-space. Assume that all balls in X are connected. Then Isoperimetry $\Rightarrow$ 1-waist $\Rightarrow$ Concentration.*
I.e. for every $\varepsilon > 0$ we have

$$1 - 2\alpha_{(X,d,\mu)}(\varepsilon) \leq w_{(X,d,\mu)}(\varepsilon) \leq 1 - \pi_{(X,d,\mu)}(\varepsilon).$$

There exists a converse to the previous proposition.

Proposition 8 *1-waist $\Rightarrow$ Isoperimetry : For all open subsets $A \subset X$ and for all $\varepsilon > 0$ we have*

$$\max\{\mu(A + \varepsilon), \mu(A^c + \varepsilon)\} \geq w(\varepsilon).$$

Again we will verify in chapter 2 that the waist is an invariant of mm-spaces :

Proposition 9 *Let (M, d, μ) and (M', d', μ') be two mm-spaces where $dimM = dimM' = n$ and $n \geq k$. Let $\phi : M \to M'$ be a c-Lipschitz map where $\phi_*(\mu) = \mu'$. Then*

$$wst(M' \to \mathbb{R}^k, c\varepsilon) \geq wst(M \to \mathbb{R}^k, \varepsilon).$$

1.2.4 The waist in higher codimension

Proposition 2 proves that the concentration phenomenon is a direct consequence of isoperimetric inequalities. The classical isoperimetric inequality for a Riemannian manifold relates the measure of (relatively) compact subsets to the measure of their boundaries. These inequalities are in codimension 1, as the difference between the dimension of a set and the dimension of its boundary is equal to 1.

F. Almgren, in the early days of his carreer, by combining topological ideas included in Dold-Thom's theorem and analytical ideas contained in the works of Federer and Fleming on the space of integral currents, has obtained isoperimetric type inequalities in arbitrary codimensions. More precisely Almgren found an optimal lower bound for the volume of a minimal k-cycle ($k \leq n$) on the sphere $\mathbb{S}^n$ see [22], [7], [4]. Here is a corollary obtained from Almgren-Morse theory.

Theorem 10 *Let $\mathbb{S}^n$ be the canonical sphere of dimension n. Let M be a minimal subvariety of dimension m ($m < n$) of $\mathbb{S}^n$. Then*

$$vol(M) \geq vol(\mathbb{S}^m).$$

To our knowledge, it was F. Almgren who introduced in the litterature the first isoperimetric type inequalities in arbitrary codimensions. Many years later, M. Gromov borrowed these ideas and defined an invariant associated to metric spaces which is the volume of maps.

By introducing the waist invariant, Gromov intertwines, once again, ideas coming from several different branches. To have a waist inequality is to have at the same time informations about the concentration aspect of a mm-space and quantitative topological informations on the space of cycles associated to the space, like in Morse theory.

In [8], Gromov proves two optimal waist inequalities which we state here.

Theorem 11 (Waist of the Sphere) *Let $f : \mathbb{S}^n \to \mathbb{R}^k$, be a continuous map of the canonical sphere to Euclidean space where $k \leq n$. There exists a point $z \in \mathbb{R}^k$ such that for every $\varepsilon > 0$,*

$$vol(f^{-1}(z) + \varepsilon) \geq vol(\mathbb{S}^{n-k} + \varepsilon).$$

Here, $\mathbb{S}^{n-k}$ represents a totally geodesic sub-sphere of codimension k.

Theorem 12 (Waist of Gaussian Space) *Let X be a convex subset of $\mathbb{R}^n$ equipped with a log-concave probability measure μ. Let $f : X \to \mathbb{R}^k$ be a continuous map. Then, there exists a $z \in \mathbb{R}^k$ such that for every $\varepsilon > 0$,*

$$\mu(f^{-1}(z) + \varepsilon) \geq (2\pi)^{-\frac{k}{2}} \int_{B(0,\varepsilon)} \exp -\frac{\|x\|^2}{2} dx^k.$$

We note that these two inequalities are optimal (equality is achieved for linear projections).

1.2.5 Results

The majority of the three years of the preparation of this thesis was devoted to the study of the paper [8], and notably to complete the details of the proof of Theorem 3. This work occupies the entire chapter 3.

In [9], M. Gromov and V. Milman prove an isoperimetric inequality on the unit spheres of uniformly convex spaces by using localisation techniques (needle decomposition). The author obtained a lower bound for the waist, in arbitrary codimensions, for the unit spheres of uniformly convex spaces. It is a common generalization to the results in [9] and of Theorem 3.

Theorem 13 *Let X be a uniformly convex normed space of finite dimension $n+1$. Let $S(X)$ be the unit sphere of X, for which the distance is induced from the norm of X. The measure defined on $S(X)$ is the conical probability measure. Then a lower bound for the waist of $S(X)$ relative to $\mathbb{R}^k$ is given by*

$$w(\varepsilon) = \frac{1}{1 + (1 - 2\delta(\frac{\varepsilon}{2}))^{n-k}(k+1)^{k+1}\frac{F(k,\frac{\varepsilon}{2})}{G(k,\frac{\varepsilon}{2})}}$$

where $\delta(\varepsilon)$ is the modulus of convexity,

$$F(k,\varepsilon) = \int_{\psi_2(\varepsilon)}^{\frac{\pi}{2}} \sin(x)^{k-1}\, dx.$$

and

$$G(k,\varepsilon) = \int_0^{\psi_1(\varepsilon)} \sin(x)^{k-1}\, dx.$$

And where

$$\psi_1(\varepsilon) = 2\arcsin(\frac{\varepsilon}{4\sqrt{k+1}})$$

and

$$\psi_2(\varepsilon) = 2\arcsin(\frac{\varepsilon}{2\sqrt{k+1}})$$

The chapter 4 is devoted to the complete proof of this Theorem and some complementary details.

1.2.6 The space of cycles

The waist belongs to the vast family of numerical invariants from the variational study on the space of cycles. We give a definition for the space of cycles.

Definition 1.2.9 (The space of cycles) *Let X be a metric space and let G be the group of coefficent. The space of k-cycles with coefficients in G, denoted by $Z_G(k,n)$ is the set constituted by the singular Lipschitz k-cycles with coefficients in G. A singular Lipschitz simplex $\sigma : \Delta \to X$ possesses a volume. If G is equipped with a norm, the mass of a k-cycle $T = \sum g_i\sigma_i$ is defined to be*

$$M(T) = \sum |g_i| vol(\sigma_i),$$

and the $\flat$ norm is defined by

$$\flat(T) = \inf\{M(S) + M(R) \,|\, T = S + \partial R\}.$$

We equip $Z_G(k,n)$ with the topology of the $\flat$ norm.

We can give another definition for the invariant waist.

Definition 1.2.10 (Waist of a mm-space, the variational viewpoint) *Let X be a mm-space. Let Z be a topological space and let F_z, $z \in Z$, be a family of cycles (subspaces) of X. Let $w(\varepsilon)$ be a function on $\mathbb{R}_+$. We say that the waist of X relative to the family of cycles F_z is at least equal to $w(\varepsilon)$, and we write*

$$wst(X, F_z, \varepsilon) \geq w(\varepsilon),$$

if there exists a $z \in Z$ such that for every $\varepsilon > 0$,

$$\mu(F_z + \varepsilon) \geq w(\varepsilon).$$

The classical Morse theory studies the length (energy) functional(s) on the (infinite dimensional) loop space of a Riemannian manifold. The Almgren-Morse theory studies the volume functional on the (infinite dimensional) space of cycles. Recently, in [10] L. Guth gave a new point of view. He studies the Almgren functional of volume on the cohomology algebra of the space of cycles. Thus, for every cohomology class of the space of cycles, there exists a variational problem (Min-Max problem) associated to the cohomology class. For the case where the coefficients belong to $\mathbb{Z}_2$, L. Guth obtains almost optimal results. For the general case the problem is still open (and far from being solved). Replacing the volumes by the numbers $\mu(C + \varepsilon)$, we extend Guth's problem to general mm-spaces. The extended problem contains the search for lower bounds on k-waist.

Here we are interested in relative 1-cycles on $(\mathbb{R}^2, A)$ where $A \subset \mathbb{R}^2$ is a finite set of points in the plane. The functional is the length functional. The critical points of this functional in the space of 1-cycles are called *minimal cycles attached to A*. Rather than the total length, we are interested in a quantity related to the topological complexity of the 1-cycles, which is the total number of vertices. The problem of giving an upper bound for the number of vertices of minimal cycles attached to a finite number of points of the plane is not easy, we only obtained a partial result, an upper bound to the number of vertices of 3-regular minimal graphs attached to a given number of points of the plane. In particular we prove the following theorem.

Theorem 14 *The maximum number of vertices of a 3-regular minimal graph attached to n points , denoted by $f_3(n)$, satisfies the following equalities.*

- *If* $n = 6k$,

$$f_3(n) = 6k^2 + 6k. \tag{1.7}$$

- *If* $n = 6k + 1$,

$$f_3(n) = 6k^2 + 8k. \tag{1.8}$$

- *If* $n = 6k + 2$,

$$f_3(n) = 6k^2 + 10k + 2. \tag{1.9}$$

- *If* $n = 6k + 3$,

$$f_3(n) = 6k^2 + 12k + 4. \tag{1.10}$$

- *If* $n = 6k + 4$,

$$f_3(n) = 6k^2 + 14k + 6. \tag{1.11}$$

- *If* $n = 6k + 5$,

$$f_3(n) = 6k^2 + 16k + 8. \tag{1.12}$$

We note that a 3-regular minimal graph attached to a finite number of points of the plane can be seen as a $\mathbb{Z}_3$-relatif cycle, but the converse is not true. The general problem remains open. Chapter 5 of this thesis contains a proof of this theorem and some complementary discussions and remarks. A slight change in the problem completely changes the answer. Assume that edges of the graphs are given weights. This slightly changes the notion of minimality. It turns out that the maximum number of vertices of minimal 3-regular graphs can be unbounded, as was demonstrated by W. Allard and F. Almgren. They constructed a (weighted) family of 3-regular minimal graphs attached to 8 points, with bounded length but arbitrary large number of vertices.

Chapitre 2

Le 1-waist et l'isopérimétrie

Dans ce chapitre, on montre comment le waist trouve sa place entre l'isopérimétrie et la concentration. Ceci donne une motivation pour cet invariant. Dans ce qui suit, pour des raisons techniques on suppose que toutes les boules soient connexes.

2.1 L'isopérimétrie contrôle le 1-waist et la concentration

Soit X un mm-espace. On suppose qu'on connaît la solution du problème isopérimétrique sur X. Alors on montre qu'on connaît le 1-waist de l'espace X.

Proposition 15 *Soit X un mm-espace. On suppose que toutes les boules de X sont connexes. Alors*
Isopérimétrie $\Rightarrow$ 1-waist $\Rightarrow$ Concentration. En particulier pour tout $\varepsilon > 0$ on a

$$1 - 2\alpha_{(X,d,\mu)}(\varepsilon) \leq w_{(X,d,\mu)}(\varepsilon) \leq 1 - \pi_{(X,d,\mu)}(\varepsilon).$$

Preuve On commence par l'implication évidente :

$$1 - waist \Rightarrow Concentration.$$

On suppose qu'on connaît une borne sur le 1-waist de l'espace X. Soit $f : X \to \mathbb{R}$ une fonction 1-Lipschitzienne. Comme les fonctions lipschitziennes sont continues, on peut appliquer la définition du 1-waist. Alors il existe un $m \in \mathbb{R}$ tel que pour tout ε

$$\mu(f^{-1}(m) + \varepsilon) \geq w_{(X,d,\mu)}(\varepsilon).$$

De plus (encore comme f est 1-lipschitzienne) on a

$$f^{-1}(m) + \varepsilon \subset f^{-1}(]m - \varepsilon, m + \varepsilon[)$$

et pour conclusion

$$\begin{aligned} \mu(f^{-1}(]m - \varepsilon, m + \varepsilon[) &\geq \mu(f^{-1}(m) + \varepsilon) \\ &\geq w_{(X,d,\mu)}(\varepsilon). \end{aligned}$$

On vient donc de prouver que le 1-waist implique la concentration et en particulier

$$w_{(X,d,\mu)}(\varepsilon) \leq 1 - \pi_{(X,d,\mu)}(\varepsilon).$$

On montre à présent l'implication moins évidente : Isopérimétrie $\Rightarrow$ 1-waist.

Soit $f : X \to \mathbb{R}$ une fonction continue. Soit m la médiane, i.e. $\mu(f \leq m) \geq \frac{1}{2}$ and $\mu(f \geq m) \geq \frac{1}{2}$ (la médiane existe toujours, elle peut ne pas être unique).

On definit les six ensembles $A = \{f \leq m\} + \varepsilon$, $B = \{f \geq m\} + \varepsilon$, $C = \{f < m\}$, $D = \{f > m\}$, $E = f^{-1}(m) + \varepsilon$, $F = f^{-1}(m)$.

Par leur définition, on a

$$\begin{aligned} A &\subset C \cup F \cup (E - C \cup F) \\ B &\subset D \cup F \cup (E - D \cup F) \\ X &= C \cup D \cup F. \end{aligned}$$

En effet, si $x \in A$, il existe $y \in X$ tel que $f(y) \leq m$ et $d(x, y) < \varepsilon$. Si $f(x) \leq m$, alors $x \in C \cup F$. Si $f(x) > m$, par connexité de la boule $B(x, \varepsilon)$, il existe $z \in B(x, \varepsilon)$ tel que $f(z) = m$, d'où $x \in E$. La preuve est la même pour B.

Alors on a

$$\begin{aligned} \mu(E - C \cup F) &\geq \mu(A) - \mu(C) - \mu(F) \\ \mu(E - D \cup F) &\geq \mu(B) - \mu(D) - \mu(F) \\ \mu(E - F) &\geq \mu(A) + \mu(B) - (\mu(C) + \mu(D) + \mu(F)) - \mu(F) \\ \mu(E) &= \mu(E - F) + \mu(F) \geq \mu(A) + \mu(B) - 1. \end{aligned}$$

Mais l'inegalité isopérimétrique implique que pour tout ε,

$$\begin{aligned} \mu(\{f \leq m\} + \varepsilon) &\geq 1 - \alpha_{(X,d,\mu)}(\varepsilon) \\ \mu(\{f \geq m\} + \varepsilon) &\geq 1 - \alpha_{(X,d,\mu)}(\varepsilon), \end{aligned}$$

et donc on peut conclure

$$\begin{aligned}\mu(f^{-1}(m)+\varepsilon) &\geq \mu(\{f\leq m\}+\varepsilon)+\mu(\{f\geq m\}+\varepsilon)-1\\ &\geq 2-2\alpha_{(X,d,\mu)}(\varepsilon)-1\\ &\geq 1-2\alpha_{(X,d,\mu)}(\varepsilon).\end{aligned}$$

Donc on vient de prouver que $\text{waist}_{(X,d,\mu)}(\varepsilon)\geq 1-2\alpha_{(X,d,\mu)}(\varepsilon)$. Et le théorème est prouvé.

Verifions a présent que le waist est bien un invariant des mm-espaces :

Proposition 16 *Soient (M,d,μ) et (M',d',μ') deux mm-spaces oú $dimM = dimM' = n$ et $n\geq k$. Soit $\phi : M\to M'$ une application c-Lipschitz oú $\phi_*(\mu)=\mu'$. Alors*

$$wst(M'\to\mathbb{R}^k, c\varepsilon)\geq wst(M\to\mathbb{R}^k,\varepsilon).$$

Preuve Soit $f : M\to\mathbb{R}^k$ une application continue. Par définition du waist, il existe un $z\in\mathbb{R}^k$ tel que

$$\mu(f^{-1}(z)+\varepsilon)\geq wst(M\to\mathbb{R}^k,\varepsilon).$$

Comme f est c-Lipschitz,

$$\begin{aligned}\mu(f^{-1}(z)+\varepsilon) &= \mu'(\phi(f^{-1}(z)+\varepsilon))\\ &\leq \mu'(\phi(f^{-1}(z))+c\varepsilon)\\ &\leq wst(M'\to\mathbb{R}^k, c\varepsilon).\end{aligned}$$

Ainsi on obtient le résultat recherché.

2.2 Le 1-waist contrôle approximativement l'isopérimétrie

On vient de voir que l'isopérimétrie implique le 1-waist. On se pose la question réciproque et on cherche à savoir si l'inégalité de waist implique l'inégalité isopérimétrique.

Proposition 17 *Le 1-waist implique l'isopérimétrie avec une constante d'isopérimétrie presque optimale : pour tout ouvert relativement compact $A\subset X$ et pour tout $\varepsilon>0$ on a*

$$\max\{\mu(A+\varepsilon),\mu(A^c+\varepsilon)\}\geq w_{(X,d,\mu)}(\varepsilon).$$

Preuve.

Soit $A \subset X$. Soit $f_\gamma(x) = \min\{1, \frac{1}{\gamma}d(x,A)\}$. On applique la définition du 1-waist à cette fonction. Il existe $z \in \mathbb{R}$ tel que $\mu(f_\gamma^{-1}(z)+\varepsilon) \geq w_{(X,d,\mu)}(\varepsilon)$. Nécessairement, $z \in [0,1]$. Si $z=0$, $f_\gamma^{-1}(z) = A$, d'où $\mu(A+\varepsilon) \geq w_{(X,d,\mu)}(\varepsilon)$. Si $0 < z < 1$, $f_\gamma^{-1}(z) \subset A+\gamma$, d'où $f_\gamma^{-1}(z)+\varepsilon \subset A+\gamma+\varepsilon$, et

$$\mu(A+\gamma+\varepsilon) \geq w_{(X,d,\mu)}(\varepsilon).$$

Si $z=1$, $f_\gamma^{-1}(z)$ est le complémentaire de $A+\gamma$, il est contenu dans A^c, le complémentaire de A. Donc $f_\gamma^{-1}(z)+\varepsilon \subset A^c+\varepsilon$, et

$$\mu(A^c+\varepsilon) \geq w_{(X,d,\mu)}(\varepsilon).$$

En faisant tendre γ vers 0, il vient

$$\max\{\mu(A+\varepsilon), \mu(A^c+\varepsilon)\} \geq w_{(X,d,\mu)}(\varepsilon).$$

2.2.1 Discussion

On voit bien que cette inégalité est mauvaise lorsque ε tend vers 0. En revanche, elle est presque optimale lorsque $w(\varepsilon)$ est proche de 1, i.e. dans le régime de la concentration. On va illustrer ces remarques sur l'exemple concret de la sphère ronde, munie de sa mesure de probabilité naturelle μ. Par le théorème de Gromov, on connaît le 1-waist de la sphère qui est égal à $w_1(\varepsilon) = vol_n(\mathbb{S}^{n-1}+\varepsilon)$. La solution du problème isoperimétrique sur la sphère est bien connue (P. Lévy) : pour un ensemble A de volume donné, une calotte sphérique du même volume a un volume de bord plus petit (ou bien un volume de ε-voisinage plus petit) que A. En particulier, si $\mu(A) = \frac{1}{2}$,

$$\mu(A+\varepsilon) \geq \mu(S_+ + \varepsilon) = \frac{1}{2} + \frac{1}{2}w_1(\varepsilon).$$

Et donc dans ce cas particulier, l'inégalité de la proposition 17 compare $\frac{1}{2} + \frac{1}{2}w_1(\varepsilon)$ à $w_1(\varepsilon)$.

Lorsque n est fixé et ε tend vers 0, $\frac{1}{2} + \frac{1}{2}w_1(\varepsilon)$ tend vers $\frac{1}{2}$ alors que $w_1(\varepsilon)$ tend vers 0, c'est mauvais.

Lorsque ε est fixé et n tend vers l'infini, $w_1(\varepsilon)$ tend vers 1 et les deux membres de l'inégalité de la proposition 17 tendent vers 1 approximativement à la même vitesse.

Chapitre 3

Gromov's waist of the sphere theorem

3.1 Introduction

In this chapter we provide details of the proof of the following important theorem.

Theorem 18 (Gromov 2003) *Let $f : \mathbb{S}^n \to \mathbb{R}^k$ be a continuous map from the canonical unit n-sphere to a Euclidean space of dimension k where $k \leq n$. There exists a point $z \in \mathbb{R}^k$ such that the n-spherical volume of the ε- tubular neighborhood of $f^{-1}(z)$, denoted by $f^{-1}(z) + \varepsilon$ satisfies, for every $\varepsilon > 0$,*

$$vol_n(f^{-1}(z) + \varepsilon) \geq vol_n(S^{n-k} + \varepsilon).$$

Here $\mathbb{S}^{n-k}$ is the $(n-k)$-equatorial sphere of $\mathbb{S}^n$.

Clearly, the Min-Max quantity dealt with in Theorem 18 (supremum of volumes of ε-neighborhoods of fibers, minimized over all continuous maps to $\mathbb{R}^k$) makes sense for arbitrary metric-measure spaces. Let us call it *k-waist*. It indicates how big the space is in codimension k. One can see the waist as a generalization for the concentration of the measure phenomenon (which corresponds to $k = 1$). The generalization has a strong topological character which is absent from classical concentration.

M. Gromov has defined other metric measurements of k-dimensional size : *k-widths* are quantities which describe the thickness (diameter) of the space in codimension k and *k-volumes* of maps describe how big the k-codimensional Hausdorff measure of the fibers of a map can be.

The proof of Theorem 18 contains lots of interesting ideas from algebraic topology and measure theory. The first one is a generalization of the classical Borsuk-Ulam theorem.

3.2 A generalisation of the Borsuk-Ulam theorem

Let $k = n$ and $\varepsilon = \frac{\pi}{2}$ in Theorem 18. In other words, let $f : \mathbb{S}^n \to \mathbb{R}^n$ be a continuous map. Theorem 18 states the existence of a $z \in \mathbb{R}^n$ such that $vol_n(f^{-1}(z) + \pi/2) \geq vol_n(\{x, -x\} + \pi/2)$. But the right hand side of the inequality is equal the total volume of the sphere, so there is no choice for $f^{-1}(z)$ but to pass through two diametrally opposite points. We see that this particular case of the waist of the sphere theorem coincides with the classical Borsuk-Ulam theorem. So it is not a big surprise that the proof of the waist theorem relies on some algebraic topology arguments *à la* Borsuk-Ulam. We state first the classical Borsuk-Ulam theorem and then the generalization needed for the proof of Theorem 18.

3.2.1 The classical Borsuk-Ulam theorem

Theorem 19 *Let $f : \mathbb{S}^n \to \mathbb{R}^k$ $(k \leq n)$ be a continuous map from the n-sphere to Euclidean space of dimension k. There exists a partition of the sphere into two hemi-spheres and a point $z \in \mathbb{R}^k$ such that $f^{-1}(z)$ passes through the centers of both hemi-spheres.*

Remark :
It is clear that the centers of the two hemi-spheres are two diametrally opposite points of the sphere. We gave a slightly different formulation of the classical Borsuk-Ulam theorem which is better adapted to the generalization we will give later on.

Proof of Theorem 19
The map $x \mapsto g(x) = f(x) - f(-x)$ is a continuous map from $\mathbb{S}^n$ to $\mathbb{R}^k$. For every $i \in \{1, \cdots, k\}$,

$$g_i(x) = f_i(x) - f_i(-x) : \mathbb{S}^n \to \mathbb{R}$$

is a continuous function from $\mathbb{S}^n$ to $\mathbb{R}$. And by the definition of the map g we see that $\forall i \in \{1, \cdots, k\}$ we have $g_i(x) = -g_i(-x)$.

The canonical action of the group $\mathbb{Z}_2$ on the sphere $\mathbb{S}^n$ consists of sending every point to his diametrically opposite point. The quotient space is real projective space $\mathbb{R}P^n$. We can define an action of the group $\mathbb{Z}_2$ on $\mathbb{R}$ such that every point $x \in \mathbb{R}$ is sent to $-x$ by the non-trivial element of $\mathbb{Z}_2$. Hence, for every i, the function g_i is equivariant for the action of the group $\mathbb{Z}_2$. Such a function defines a continuous cross section of the tautological vector bundle over $\mathbb{R}P^n$. And so g defines a continuous cross section of Whitney sum of

k copies of the tautological vector bundle γ_n over $\mathbb{R}P^n$.

$$g : \mathbb{R}P^n \to E = \underbrace{\gamma_n \oplus \cdots \oplus \gamma_n}_{k}.$$

What remains to prove now is the existence of a zero for the continuous cross section g. For this, we refer to the theory of characteristic classes of vector bundles. In our case, as we are working with the actions of the group $\mathbb{Z}_2$, it is natural to use Stiefel-Whitney classes. The following classical result will be used here and later in this paper.

Lemma 3.2.1 *Let $\pi : E \to V$ be a real vector bundle of rank k over a manifold V. If the k-th Stiefel-Whitney class $w_k(E) \neq 0$, then every continuous cross section $s : V \to E$ has a zero.*

The cohomology ring of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}_2$ is $H^*(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2[a]/a^{n+1}$ where $a \in H^1(\mathbb{R}P^n, Z_2)$ is the generator of the first cohomology group. One of the axioms defining Stiefel-Whitney classes states that the total Stiefel-Whitney class $w = 1 + w_1 + \cdots + w_n$ is multiplicative under Whitney sums,

$$w(\xi \oplus \eta) = w(\xi) \smile w(\eta).$$

An other one states that $w(\gamma_n) = 1 + a$, see [21]. Thus

$$w(E) = (1 + a)^k = 1 + ka + \binom{k}{2} a^2 + \cdots + a^k,$$

and $w_k(E) = a^k$. As $k \leq n$, $a^k \neq 0$. So we proved that $w_k(E) \neq 0$. Lemma 3.2.1 implies that there exists a point $x \in \mathbb{R}P^n$ such that $g(x) = 0$. And the proof of the theorem follows.

Remark : One should think of $\mathbb{R}P^n$ as the space of unoriented partitions of the sphere into two hemi-spheres.

Other proofs of the Borsuk-Ulam theorem can be found in [18]. We gave here a proof which was best suited to Gromov's generalization.

3.2.2 The Gromov-Borsuk-Ulam theorem

We saw in the last section that the classical Borsuk-Ulam theorem proves the existence of a fiber passing through the center of two hemi-spheres. Gromov's generalization of Borsuk-Ulam consists of constructing a partition of the sphere into geodesically convex subsets of the sphere in order that there exists a fiber passing through the center points of

all the convex sets of the partition. A hemi-spheres has a natural center point. For more general convex sets, several notions of center can be used. The Gromov-Borsuk-Ulam theorem applies to a large class of notions of center.

Definition 3.2.1 *Say a subset S of the sphere $\mathbb{S}^n$ is* convex *if S is contained in a hemi-sphere and the cone on S with vertex at the origin is convex in $\mathbb{R}^{n+1}$. Let $\mathbb{O}$ be the space of all open convex subsets of $\mathbb{S}^n$. The topology on the space $\mathbb{O}$ is defined by the Hausdorff distance between convex sets. A* center*map is a continuous map from $\mathbb{O}$ to $\mathbb{S}^n$.*

Remark The center of a convex set is not necessarily contained in the convex set itself. From now on, until further mention, we will fix a center map $c_.$.

Theorem 20 (Gromov 2003) *Let $f : \mathbb{S}^n \to \mathbb{R}^k$ $(k \leq n)$ be a continuous map from the n-sphere to Euclidean space of dimension k. For every $i \in \mathbb{N}$, there exists a partition of the sphere $\mathbb{S}^n$ into 2^i open convex sets $\{S_i\}$ of equal volumes $(= Vol(S^n)/2^i)$ and such that all the center points $c_.(S_i)$ of the elements of partition have the same image in $\mathbb{R}^k$.*

Remark For $i = 1$ and for a convenient choice of the center map $c_.$, we find Theorem 19. So this theorem can be seen as a generalisation of the classical Borsuk-Ulam theorem. But even for $i = 1$ this theorem tells more than the classical Borsuk-Ulam theorem as there exists an infinite choice for the center map which won't coincide with the geometrical center of hemi-spheres.

We saw in the last section that the space of unoriented partitions of the sphere into two hemi-spheres is identified with the real projectif space. But what can we say for the space of partitions of the sphere for $i \geq 2$?
The space of partitions into 2^i open convex sets of the sphere is an infinite dimensional space, we will define a finite dimensional subspace of the general space of partitions which will have very satisfying topological properties and will be easy to study. This finite dimensional subspace will be sufficient for the proof of the theorem 20.

3.2.3 The space of partitions of S^n

In this section, we define in an algorithmic way, a finite dimensional space which will be a subspace of the space of partitions of the sphere into 2^i open convex sets, for every natural number i.

We consider the following algorithm :

- First step. Divide S^n by an oriented hyperplane into two equal hemi-spheres. The halving procedure is done by choosing a unit vector v in $\mathbb{R}^{n+1}$, the two hemi-spheres are $H_v^+ = \{x \in \mathbb{S}^n \quad (x.v) \geq 0\}$ and $H_v^- = \{x \in \mathbb{S}^n \quad (x.v) \leq 0\}$. The hemi-spheres are ordered and oriented by the vector v.

- Inductive step. Divide every convex set obtained in the $(i-1)$-th step of the algorithm into two convex sets by an oriented hyperplane.

After i repetitions of the above algorithm, the sphere will be partitioned into 2^i convex sets. Some might be empty. In order to have 2^i convex sets we need

$$1 + 2 + 2^2 + \cdots + 2^{i-1} = 2^i - 1$$

hyperplanes (hyperspheres).

Definition 3.2.2 *The* space of i-step oriented partitions of S^n *is*

$$P_i = \underbrace{S^n \times \cdots \times S^n}_{2^i-1}.$$

The index set $\{1, \ldots, 2^i - 1\}$ *is viewed as the set of internal nodes of a rooted binary tree of depth* $i+1$. *Pieces of the partition correspond to leaves of the tree. Indeed, following the downward path connecting the root to a leaf, one meets nodes, i.e. unit vectors* $v_1, \ldots, v_i$, *and edges which tell whether one must use* $H^+_{v_j}$ *or* $H^+_{v_j}$. *The piece is the intersection* $\bigcap_{j=1}^{i} H_j^{\pm}$ *(eventually empty).*

Next we want to define the space of unoriented partitions. Since the partition is defined in terms of paths connecting the root to leaves in a rooted tree, automorphisms of the rooted tree will permute points of P_i which define the same unoriented partition. In the last section we saw an example for the case $i = 1$. Two diametrally opposite points of the sphere define the same unoriented partition into hemi-spheres. For $i \geq 1$ things are more complicated. We give here another example.

Example 3.2.1 *Let* $i = 2$, *we consider the space* $P_2 = S^n \times S^n \times S^n$ *of oriented partition of the sphere into* 4 *convex sets.*

Let $(x, y, z) \in P_2$. x is the first hyperplane cutting the sphere into two equal hemi-spheres, and defines the first step of the algorithm. At the second step, y cuts the hemi-sphere pointed by x into two convex pieces and z cuts the hemi-sphere pointed by $-x$ into two convex pieces, providing the 4 convex pieces of the partition defined by (x, y, z). Consider the point $w = (x, -y, z)$ of P_2, we want to compare the partition defined by this point with the partition defined by $u = (x, y, z)$. x defines the same first cut in both partitions. $-y$ and y define the same hyperplane and they both cut the hemi-sphere pointed by x. At last, the hyperplane defined by z cuts the hemi-sphere pointed by $-x$. Hence the two partitions defined by u and w are considered as the same unoriented partition. With the

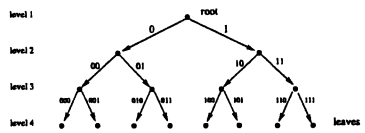


FIGURE 3.1 – The binary tree T_3

same argument, we can easily check that the 8 following points of P_2 define the same partition

$$\{(x,y,z),(x,-y,z),(x,y,-z),(x,-y,-z),(-x,z,y),(-x,-z,y)(-x,z,-y),(-x,-z,-y)\}.$$

We define the space Q_2 as the quotient of P_2 by the equivalence relation defined by identifying the 8 points of the above set. Q_2 is hence the space of unoriented partitions into 4 convex sets defined by the above algorithm.

In the next subsections we will explore the space Q_i for all i.

3.2.4 The binary tree T_i

We saw in Example 3.2.1 that the space of oriented partitions defined as a product of some $\mathbb{S}^n$ is larger than the space of unoriented partitions. On our way to define the space of unoriented partitions, let us describe in more detail the tree structure briefly alluded to in Definition 3.2.2. We index the $2^i - 1$ coordinates in P_i by the internal nodes (i.e. vertices which are not leaves) of an oriented binary tree of depth i, which we denote by T_i. The edges are downwards oriented and indexed by strings of 0 and 1, as shown on Figure 3.1.

Let $p = (v_n)_{n \text{ internal node}} \in P_i$. The unit vector v_n attached to the internal node n is thought of as on oriented hypersphere. To the two edges emanating from n correspond hemi-spheres : the hemi-sphere to which v_n points for the left edge (whose index ends with 0), the hemi-sphere to which $-v_n$ points for the right edge (whose index ends with 1).

3.2.5 $Aut(T_i)$

For understanding the structure of the group of automorphism $Aut(T_i)$ of the binary tree T_i we need the following definition.

Definition 3.2.3 (Wreath product) *Let G be a group which acts on a set I. Let H be any group. Denote by H^I the group of maps $I \to H$. The wreath product of G and H, denoted by $G \wr H$, is the semi-direct product of the group H^I by G,*

$$G \wr H = H^I \rtimes_\phi G,$$

where the action ϕ of G on H^I is the left action by permuting factors,

$$(g \cdot h)(f) = h(g^{-1} \cdot f)).$$

The automorphism group of a graph $\mathcal{G}$ is the set of bijections of the set of vertices such that the adjacency relationship between the vertices is respected. In other words, an automorphism of the graph $\mathcal{G}$ is a bijection σ such that for every edge $e = uv$ where u and v are vertices of the graph, $\sigma(u)\sigma(v)$ is an edge of $\mathcal{G}$ (denoted by $\sigma(e)$).

Lemma 3.2.2 *For every $i \in \mathbb{N}$ we have*

$$Aut(T_i) = Aut(T_{i-1}) \wr \mathbb{Z}_2.$$

Proof of the Lemma

$G = Aut(T_{i-1})$ identifies with the subgroup of $Aut(T_i)$ which does not change the last bit in the string associated to an edge. This gives a permutation action of $Aut(T_{i-1})$ on the set I of $i-1$-st level vertices of T_i. Note that I has 2^{i-1} elements. It is this action which defines the wreath product. One can also view $K = (\mathbb{Z}_2)^I$ as the set of elements of $Aut(T_i)$ which fix all internal nodes. It is a normal subgroup. Indeed, any automorphism of a rooted tree permutes internal nodes. Given a leaf ℓ attached to an internal node n, denote by $b(\ell)$ denote the last bit in the string associated to the edge $n\ell$. Then $k \in (\mathbb{Z}_2)^I$ acts on leaf ℓ as follows : if $k(n) = 0$, $k(\ell) = \ell$. Otherwise, $k(\ell)$ is the other leaf attached to n. In other words, $b(k(\ell)) = b(\ell) + h(n)$.

Let $g \in G$ and $k \in K$. Then $b(g^{-1}(\ell)) = b(\ell)$, $b(kg^{-1}(\ell)) = b(\ell) + k(g^{-1}(n))$, $b(gkg^{-1}(\ell)) = b(\ell) + k(g^{-1}(n))$. This shows that $gkg^{-1} = g \cdot k$ in H. Therefore the map $(k, g) \to kg \in Aut(T_i)$ defines a group homomorphism $K \rtimes_\phi G \to Aut(T_i)$. It is one to one, since any element of $Aut(T_i)$ coincides on internal nodes with a unique $g \in G$, and the remaining switches of leaves can be achieved by postcomposing with a unique element of K. Thus we get an isomorphism $K \rtimes_\phi G \simeq Aut(T_i)$, and the proof of the Lemma follows.

From Lemma 3.2.1 we see that the automorphism group of the graph T_i is formed by i iterated wreath products of $\mathbb{Z}_2$ (be aware that the wreath product is not associative). And that $Aut(T_i)$ has cardinality equal to 2^{2^i-1}.

3.2.6 Unoriented partitions

In general, If G acts on a set I and H acts on a set F, $G \wr H$ acts on the set F^I of maps $I \to F$ as follows. If $k \in H^I$, $z \in F^I$, $g \in G$ and $v \in I$,

$$kg(z)(v) = k(v) \cdot Z(g^{-1} \cdot v).$$

Definition 3.2.4 *$Aut(T_i)$ acts on P_i as follows. Elements of $Aut(T_{i-1})$ permute internal nodes, and so act by permuting the factors. If I denotes the set of nodes of level i, elements of $K = (\mathbb{Z}_2)^I$ act on factors, with the generator indexed by v acting by $x \to -x$ on the corresponding sphere factor.*

Similarly, $Aut(T_i)$ acts on $(\mathbb{R}^k)^I$.

Note that since the $\mathbb{Z}_2$ action on the sphere is free, the former action on P_i is free.

Definition 3.2.5 *We define the* space of i-step unoriented partitions *of the sphere as the quotient space*

$$Q_i = P_i / Aut(T_i).$$

We have enough information to give the proof of the Gromov-Borsuk-Ulam theorem.

3.2.7 Proof of Theorem 20

Let f be a continuous map from $\mathbb{S}^n$ to $\mathbb{R}^k$. Let $i \in \mathbb{N}$ be fixed and let $p \in P_i$. p is a sequence of $2^i - 1$ points of $\mathbb{S}^n$ that define a partition of the sphere into 2^i open convex sets. We represent the coordinates of p by the vertices of a rooted binary tree T_i of depth i embedded in the plane. The 2^{i-1} last coordinates of p are the 2^{i-1} hyperplanes of the last step of the algorithm. To each hyperplane $p_{.i.}$ belonging to the last 2^{i-1} vertices of the tree, we associate the open convex set which corresponds to the left edge outgoing from the vertex $p_{.i.}$. Hence we obtain a bijection between the 2^{i-1} last vertices of the tree and the left edges outgoing from each vertex. We denote this correspondance by $h_{.i.} \to S_{.i.}$ and we define the two following maps.

$$v(h_{.i.}) = vol_n(S_{.i.}),$$

$$\varphi(h_{.i.}) = v(h_{.i.}) f(c_{.}(S_{.i.})),$$

where we remind that $c_{.}$ is the continuous center map that is supposed to be fixed.

These two maps are defined only for the hyperplanes of the ith step. We extend these two maps to all the hyperplanes (vertices) of T_i in the following way. Let $h_{.j.}$ be a

hyperplane of the jth step of the algorithm (a vertex of level j of T_i). Let $T_{h_{.j.}} \subseteq T_i$ be the rooted binary subtree of T_i whose root corresponds to $h_{.j.}$ and the edges are all the edges of T_i which belongs to the subtree $T_{h_{.j.}}$. We consider the hyperplanes of the last level of the subtree $T_{h_{.j.}}$ and we define the two following maps,

$$v(h_{.j.}) = \sum_{h_{.i.} \in T_{h_{.j.}}} v(h_{.i.}),$$

$$\varphi(h_{.j.}) = \sum_{h_{.i.} \in T_{h_{.j.}}} \varphi(h_{.i.}).$$

Here, the sum is taken over all the vertices of level i of the subtree corresponding to a vertex of level j.

Then we define a map $F : P_i \to (\mathbb{R}^{k+1})^{2^i-1}$ which is given by

$$F : \{h_{.j.}\} \to \{v(h_{.j.}) - v(-h_{.j.}), \varphi(h_{.j.}) - \varphi(-h_{.j.})\}.$$

Since the construction only depends on the tree structure, F is $Aut(T_i)$ -equivariant for the actions of $Aut(T_i)$ on P_i and $(\mathbb{R}^{k+1})^{2^i-1}$. F defines a continuous cross section of the vector bundle

$$(P_i \times (\mathbb{R}^{k+1})^{2^i-1})/Aut(T_i) \to Q_i = P_i/Aut(T_i).$$

The point is to show that this section vanishes. In view of Lemma 3.2.1, the following characteristic class computation completes the proof of the Gromov-Borsuk-Ulam theorem.

Lemma 3.2.3 *The top Stiefel-Whitney class of $L_i = (P_i \times (\mathbb{R}^{k+1})^{2^i-1})/Aut(T_i)$ does not vanish.*

Proof of the Lemma.

As the action of $Aut(T_i)$ on both P_i and $(\mathbb{R}^{k+1})^{2^i-1}$ is defined in an inductive way, it is natural to prove this lemma by induction.

Since P_i splits as a product $P_{i-1} \times (S^n)^{2^{i-1}}$ in a $Aut(T_{i-1})$-invariant manner, one gets a map $p_i : Q_i \to Q_{i-1}$ which is a fiber bundle with fiber $(\mathbb{R}P^n)^{2^{i-1}}$. Furthermore, by definition 3.2.4 $Aut(T_{i-1})$ acts on the last 2^{i-1} factors $\mathbb{R}^{k+1}$ in the same way that it acts on the last 2^{i-1} factors $(S^n)^{2^{i-1}}$ of $P_{i-1} \times (S^n)^{2^{i-1}}$. On each fiber, the restriction of the bundle L_i is the sum of a trivial bundle and of the bundle

$$\underbrace{(\gamma_n)^{k+1} \oplus \cdots \oplus (\gamma_n)^{k+1}}_{2^{i-1}}$$

over $(\mathbb{R}P^n)^{2^{i-1}}$. This implies that there exists a vectorbundle α_n on Q_i whose restriction to fibers are isomorphic to γ_n, such that

$$L_i = p_i^* L_{i-1} \oplus \underbrace{(\alpha_n)^{k+1} \oplus \cdots \oplus (\alpha_n)^{k+1}}_{2^{i-1}}.$$

Thus $w(L_i) = p_i^* w(L_{i-1}) \smile w(\alpha_n)^{(k+1)2^{i-1}}$. In particular, the top-dimensional components multiply,

$$w_{top}(L_i) = p_i^* w_{top}(L_{i-1}) \smile w_1(\alpha_n)^{(k+1)2^{i-1}}.$$

By induction on i, we can assume that $w_{top}(L_{i-1}) \neq 0$. This implies that $w_{top}(L_i) \neq 0$.

3.3 Pancakes

Using the Gromov-Borsuk-Ulam theorem, we ideally would like to construct an infinite partition of the sphere which will have some desired properties. We know that for any continuous map from the sphere to a Euclidean space of smaller dimension, and for every natural number i, there exists a partition of the sphere into 2^i open convex sets of equal volumes and a fiber passing through the center of the convex sets of the partition. Since the volumes of the pieces of the partition tend to zero, we will have in the limit an infinite partition by convex subsets of smaller dimension. The purpose of this section is to analyse the dimension of the convex subsets when i tends to infinity. How small the convex subsets can be and how can we control the dimension of the convex subsets of the partition? In [9], a similar problem was considered where the sphere was sent to a two-dimensional Euclidean space and where the authors proved the existence of an infinite partition of the sphere by convex subsets of dimension 1 using Borsuk-Ulam theorem. Here we follow the same line of ideas and by using the Gromov-Borsuk-Ulam theorem we prove the existence of an infinite partition of the sphere by convex subsets of at most dimension equal to k.

Definition 3.3.1 *Let S be an open convex subset of $\mathbb{S}^n$, S is called an (k, ε)-pancake if there exists a convex set S_π of dimension k such that every point of S is at distance at most ε from S_π.*

Remark (k, ε)-pancakes, are used to control the dimensional size of open convex sets. For big enough ε we can say that all open convex sets are pancakes. The interest of the above definition is when ε is very small. In this case for a convex set to be a pancake would mean to be very close to a k-dimensional convex set and hence it would mean that the

pancake has very small widths in $n-k$ directions orthogonal to the convex of dimension k.

The typical example in Euclidean space are the rectangles, where for a rectangle of dimension n to be a k pancake would mean that the size of $n-k$ sides of the rectangle are very small.

Here is an improvement on Theorem 20.

Theorem 21 *Let $f : \mathbb{S}^n \to \mathbb{R}^k$ be a continuous map. For all $\varepsilon > 0$, there exists an integer i_0 such that for all $i \geq i_0$ there exists a finite partition of $\mathbb{S}^n$ into 2^i open convex subsets such that :*

I. Every convex subset of the partition is a (k, ε)-pancake.

II. The centers of all convex subsets of the partition have the same image in $\mathbb{R}^k$.

III. All convex subsets of the partition have the same volume.

Proof of the Theorem

In the proof of Theorem 20, there was no restriction on the choice of the hyperplanes cutting the sphere. The idea of the proof of this theorem is to take a parametrized choice for the sequence of hyperplanes used to cut the sphere.

We suppose that the sphere is cut into two equal pieces and the two center points have the same image in $\mathbb{R}^k$. Let S_+ be a hemi-sphere. We suppose that $\mathbb{S}^n$ is the unit sphere of $\mathbb{R}^{n+1}$, the boundary of the unit ball. Let L be a plane of dimension $n-k-1$ passing through the origin in $\mathbb{R}^{n+1}$. Obviously L intersects S_+ and the intersection locus is a half $n-k-2$-sphere. Let $L^\perp$ be the orthogonal to L which we identify to a $\mathbb{R}^{k+2}$. By orthogonally projecting $\mathbb{R}^{n+1}$ onto $L^\perp$, every unit vector in S^{k+1} defines a hyperplane (of dimension n), which contains L. So we can parametrize the hyperplanes (of dimension n) which contain L by a sphere S^{k+1}.

We remember that the cutting hyperplanes of Theorem 20 are indexed by their orthogonal unit vector, the idea now is to use Theorem 20 by choosing every unit vector orthogonal to a hypersphere in a S^{k+1}. As the dimension of the range is equal to k, we can apply Theorem 20 to the $2^i - 1$ cartesian product of S^{k+1} for every natural number i. In this case for every i, we obtain a partition of the sphere into 2^i open convex subsets of same volume, and such that in every previous step $j \leq i$, the unit vectors orthogonal to hyperplanes corresponding to this step belong to one S^{k+1}.

Lemma 3.3.1 *For all $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ and a sequence $L_1, L_2, \ldots, L_N$ of $(n-k-1)$-dimensional planes such that for every ball of radius ε in $\mathbb{S}^{k+1}$, there exists at least one L_j which contains a point of that ball.*

Remark

If $k = 1$, this lemma is equivalent to the existence of an ε-net. For $k \geq 1$ the lemma defines roughly speaking an ε-net in dimension k.

Proof of the Lemma

Let $Gr(n-k-1, n+1)$ denote the Grassmannian of $(n-k-1)$-planes in $\mathbb{R}^{n+1}$. Let $L \in Gr(n-k-1, n+1)$. Let $V(L)$ be the set of all $L' \in Gr(n-k-1, n+1)$ such that L' cuts the ball $B(x, \varepsilon) \cap \mathbb{S}^{k+1}$. Hence $V(L)$ is a neighbourhood of L in $Gr(n-k-1, n+1)$.

The collection of $V(L)$'s defines an open covering of $Gr(n-k-1, n+1)$. By compactness, there exists a finite sub-covering and so a finite family of planes $L_1, \ldots L_N$ such that the $V(L_j)$ cover the Grassmannian and the proof of the lemma follows.

Lemma 3.3.1 lets us control the *l-widths* for $l \geq k$ of pieces of the partition. Let S_π be a piece of partition and let a $(k+2)$-dimensional plane passing through the origin which cuts S_π. By lemma 3.3.1 and the choice of the L_i, we can conclude that there does not exist any ball of radius δ of $\mathbb{S}^{k+1}$ in $S_\pi \cap S^{k+1}$. Indeed, if there exists a ball of radius δ in the intersection, then there exists a plane L_j which passes through a point of this ball and hence a hyperplane H_j containing L_j which would cut the convex by passing through the intersecting point and this is not possible because otherwise the convex would be cut in the direction of H_j.

We now prove that for ζ small enough, all the S_π are (ζ, k)-pancakes.

Lemma 3.3.2 *For all $\zeta > 0$, there exists $\varepsilon > 0$ such that if C is a convex set such that for every sphere $\mathbb{S}^{k+1}$, C does not contain any ball of radius $\frac{\varepsilon}{4}$ of $\mathbb{S}^{k+1}$, then C is a (ζ, k)-pancake.*

Proof of the Lemma

By contradiction. If not, there exists a $\zeta > 0$, there exists a sequence of convex sets C_m which do not contain any ball of dimension $k+1$ and of radius $\varepsilon_m = \frac{1}{m}$ and which are not (ζ, k)-pancakes. Let $C = \lim C_{m_j}$ where C_{m_j} is a subsequence of the sequence C_m. Then C does not contain any ball of dimension $k+1$.

Indeed $C = \lim C_{m_j}$, then for all $a, b, c \in C$ there exists $a_{m_j}, b_{m_j}, c_{m_j} \in C_{m_j}$ such that the sequences $a_{m_j} \to a$, $b_{m_j} \to b$, $c_{m_j} \to c$. By convexity, the convex hull of the three points $a_{m_j}, b_{m_j}, c_{m_j} : Conv(a_{m_j}, b_{m_j}, c_{m_j}) \subset C_{m_j}$. But there exists $d_{m_j} \in C_{m_j}$ such that

$$B(d_{m_j}, \varepsilon/16) \subset Conv(a_{m_j}, b_{m_j}, c_{m_j})$$

and so

$$B(d, \varepsilon/16) \subset Conv(a, b, c).$$

Hence $dim(C) \leq k$. Therefore for m big enough $d_H(C_{m_j}, C) \leq \zeta$ and this is a contradiction. This proof by contradiction uses Blaschke's selection principle .

This completes the proof of Theorem 21.

3.4 Convexely derived measures on the sphere

3.4.1 Definition

Remember that $\mathbb{S}^n$ is the boundary of the unit ball centered at the origin of $\mathbb{R}^{n+1}$. On $\mathbb{R}^{n+1}$ the Lebesgue measure m_{n+1} is defined. We can define the (normalized) Riemannian measure on $\mathbb{S}^n$ as follows. Let H be a measurable subset of $\mathbb{S}^n$. We define the set $co(H)$ by :

$$co(H) = \{\bigcup tH | 0 \leq t \leq 1\}.$$

The set $co(H)$ is the cone centered at the origin of $\mathbb{R}^{n+1}$ over H. $co(H) \subseteq \mathbb{R}^{n+1}$. We set

$$\mu_n(H) = \frac{m_{n+1}(co(H))}{m_{n+1}(B_{n+1}(0,1))}.$$

μ_n is the normalised Riemannian measure on the sphere $\mathbb{S}^n$.

Definition 3.4.1 *A convexely derived measure on $\mathbb{S}^n$ (resp. $\mathbb{R}^n$) is a limit of a vaguely converging sequence of probability measures of the form $\mu_i = \frac{vol|S_i}{vol(S_i)}$, where S_i are open convex sets.*

Remark. The support of a convexely derived measure is a convex set.

In [1] and [9], the authors use concavity properties of density functions of convexely derived measures on Euclidean convex sets. Here we need also some sort of concavity properties for the density of convexely derived measures defined on convex sets of the sphere. Our approach will be to use Euclidean convex geometry by taking the cones over convex sets of the sphere and reduce spherical problems to Euclidean problems.

We begin by giving the following

Definition 3.4.2 *A real function f defined on an interval of length less that 2π is called* sin-*concave, if, when transported by a unit speed paramatrization of the unit circle, it can be extended to a 1-homogeneous and concave function on a convex cone of $\mathbb{R}^2$.*

This definition provides a family of example of sin-concave functions. Indeed one way of obtaining a sin-concave function is to consider a concave and 1-homogeneous function on $\mathbb{R}^2$ and restrict it to $\mathbb{S}^1$.

Example 3.4.1 *The linear function $f(x,y) = y$ is 1-homogeneous and concave on $\mathbb{R}^2$. By restricting this function to the unit circle we obtain the well known function* $\sin(t)$. *So the sine function is* sin-*concave.*

Definition 3.4.3 *A nonnegative real function f is called* $\sin^k$-*concave if the function $f^{\frac{1}{k}}$ is* sin-*concave.*

The next lemma provides a familly of examples of $\sin^k$-concave functions for k greater than 1. This family will be all we need in this paper.

Lemma 3.4.1 *Let S be a geodesically convex set of dimension k of the sphere $\mathbb{S}^n$ with $k \leq n$. Let μ be a convexely derived measure defined on S (with respect to the normalized Riemannian measure on the sphere). Then μ is a probability measure having a continuous density f with respect of the canonical Riemannian measure on $\mathbb{S}^k$ restricted to S. Furthermore the function f is* $\sin^{n-k}$-*concave on every geodesic arc contained in S.*

Proof of the Lemma

Let S_i be a sequence of open convex subsets of $\mathbb{S}^n$ which Hausdorff converges to S, where S is a convex subset of dimension k of the sphere. For every i we define the convex cone over S_i and denote it (as we saw in the beginning of this section) by $co(S_i)$. Then the sequence of open convex cones $co(S_i)$ (of dimension $(n+1)$), Hausdorff converges to the convex subset $co(S)$ (of dimension $(k+1)$). Then the sequence of normalised (probability) measures $\mu'_i = \frac{m_{n+1}|co(S_i)}{m_{n+1}(co(S_i))}$ vaguely converges to a probability measure μ' on $co(S)$. The measure μ' is convexely derived from the sequence of probability measures μ'_i. We know from [9] that the measure μ' admits a density function with respect to the $(k+1)$-dimensional Lebesgue measure and $d\mu' = F dm_{k+1}$, where F is a $(n-k)$-concave function.

Lemma 3.4.2 *The measure μ' is $(n+1)$-homogeneous and the function F is $(n-k)$-homogenous. Which means for every $t \in [0,1]$ and every Borel set A, $\mu'(tA) = t^{n+1}\mu'(A)$ and for every $x \in S$, $F(tx) = t^{n-k}F(x)$.*

Proof of the Lemma

The measure μ' is convexely derived from the normalized $(n+1)$-dimensional Lebesgue measure m_{n+1}. m_{n+1} is $(n+1)$-homogeneous and so will be for μ'.
As $d\mu' = F dm_{k+1}$ and from the fact that μ' is $(n+1)$-homogeneous and m_{k+1} is $(k+1)$-homogeneous, the function F turns out to be $(n-k)$-homogeneous and the proof of the Lemma follows.

It is then clear that the convexely derived measure μ on S admits a continuous density function with respect to the canonical Riemannian measure of dimension k. We take two

points x and y on S, take the geodesic arc σ joining x and y. We take the cone over σ which is a subset of dimension 2 of $co(S)$. We take the restriction of the function F on $co(\sigma)$. We claim that the restriction of a $(n-k)$-concave function which is also $(n-k)$-homogeneous on σ (considered as a subset of $\mathbb{S}^1$) is $\sin^{n-k}$-concave. As F is $(n-k)$-concave, then $F^{1/(n-k)}$ is a concave function which is also 1-homogeneous (as $F(tx) = t^{n-k}F(x)$ then $F^{1/(n-k)}(tx) = tF^{1/(n-k)}(x)$). Then by the previous Lemma $f^{1/(n-k)}$ is sin-concave and then f is $\sin^{n-k}$-concave by definition. And the proof of the main Lemma follows.

3.4.2 More properties of sin-concave functions

Lemma 3.4.3 *Let f be a $\sin^k$-concave function defined on a closed interval of $\mathbb{R}$, then f admits only one maximum point. Morever f does not have any local minima.*

Proof of the Lemma
We put $g = f^{1/k}$. g is sin-concave. There exists a 1-homogeneous and concave function G such that $G|S = g$. Suppose g has two maxima denoted by x_1 and x_2. $[x_1, x_2]$ is the segment joining these two points in $\mathbb{R}^2$. By concavity property we know that $G(\frac{x_1+x_2}{2}) \geq g(x_1) = g(x_2)$. The point $x' = \frac{x_1+x_2}{2}/|\frac{x_1+x_1}{2}| \in S$. As G is 1-homogeneous we have $g(x') = G(\frac{x_1+x_2}{2})/|\frac{x_1+x_2}{2}|$ and as $|\frac{x_1+x_2}{2}| \leq 1$ then we have $g(x') \geq G(\frac{x_1+x_2}{2}) \geq g(x_1) = g(x_2)$ and this is a contradiction. Hence every $\sin^k$-function admit at most one maximum point.
Suppose g has a local minimum at y. By elementary geometry we know that there exist two points $x_1, x_2 \in S$ such that $y = \frac{x_1+x_2}{2}/|\frac{x_1+x_2}{2}|$. By the same argument as above we deduce that $g(y) \geq Min\{x_1, x_2\}$ and this is a contradiction. And the proof of the lemma follows.

Lemma 3.4.4 *Let f be a continuous function defined on the interval $[-a, a]$. Assume that*

- *$f_{|[-a,0]}$ and $f_{|[0,a]}$ are concave.*
- *the left and right derivative of f at 0 satisfy*

$$f'(0-) \geq f'(0+).$$

Then f is concave on the full interval $[-a, a]$.

Proof of the Lemma
Up to adding a linear function one can assume that $f'(0-) \geq 0 \geq f'(0+)$. Then f is nondecreasing on $[-a, 0]$ and nonincreasing on $[0, a]$. For $x \in [-a, 0]$, let g_x be an affine function such that $g_x(x) = f(x)$ and $g_x \geq f$ on $[-a, 0]$. Then g_x is non increasing, thus, for $t \in [0, a]$, $g_x(t) \geq g_x(0) \geq f(0) \geq f(t)$. This shows that $g_x \geq f$ on $[-a, a]$. A similar

argument applies for $x \in [0, a]$, and show that f is the minimum of a family of affine functions, therefore f is concave on $[-a, a]$.

Lemma 3.4.5 *Let f be a* sin*-concave function on an interval containing* 0*, which achieves its maximum at* 0*. Let $g(t) = f(|t|)$. Then g is* sin*-concave.*

Proof of the Lemma

View f and g as functions on an arc of the unit circle in the plane containing $(1, 0)$. Let F and G denote the 1-homogeneous extensions of f and g to a plane sector C containing the half line $\{(x, 0) \mid x > 0\}$. Then $G(x, y) = F(x, |y|)$ on C. Let $t \mapsto c(t) = (x + \alpha t, \beta t)$, $t \in [-a, a]$, be a parametrization of a line segment contained in C. Then $h(t) = G(c(t))$ is continuous, concave on $[-a, 0]$ and $[0, a]$. Assume that $\beta > 0$ and $x > 0$. The left and right derivatives of h at $t = 0$ are equal to

$$\begin{aligned} h'(0-) &= \alpha f(0) + x\beta g'(0-) = \alpha f(0) + x\beta f'(0-), \\ h'(0+) &= \alpha f(0) + x\beta g'(0+) = \alpha f(0) - x\beta f'(0-). \end{aligned}$$

By assumption, $f'(0-) \geq 0$, thus $h'(0-) \geq h'(0+)$. Lemma 3.4.4 implies that h is concave. This shows that G is concave, and g is sin-concave.

Lemma 3.4.6 *Let $0 < \varepsilon < \pi/2$. Let $\tau > \varepsilon$. Let f be a nonnegative* $\sin^k$*-concave function on $[0, \tau]$, which attains its maximum at* 0*. Let $h(t) = c\cos^k(t)$ where c is chosen such that $f(\varepsilon) = h(\varepsilon)$. Then*

$$\begin{cases} f(x) \geq h(x) & \text{for } x \in [0, \varepsilon], \\ f(x) \leq h(x) & \text{for } x \in [\varepsilon, \tau]. \end{cases}$$

In particular, $\tau \leq \pi/2$.

Proof of the Lemma

Without loss of generality, we can assume that $k = 1$. Define $g(t) = f(|t|)$. View g and h as functions on an arc S of length $\min\{\pi, 2\tau\}$ of the unit circle. Let G and H denote the 1-homogeneous extensions of g and h to the plane sector $C = co(S)$. Then $H(x, y) = cx$ on C. According to Lemma 3.4.5, G is concave on C, and so is $G - H$. By construction, $G - H$ vanishes both at $p = (\cos(\varepsilon), \sin(\varepsilon))$ and at $q = (\cos(\varepsilon), -\sin(\varepsilon))$. Since $G - H$ is concave, $G - H \geq 0$ on the line segment $[p, q]$, and $G - H \leq 0$ on the remainder of $C \cap D$ where D denotes the line through p and q. Since $G - H$ is 1-homogeneous, $G - H \geq 0$ on the sector delimited by the half lines $\mathbb{R}_+ q$ and $\mathbb{R}_+ p$, and $G - H \leq 0$ on the remainder of C. This shows that $g \geq h$ on $[-\varepsilon, \varepsilon]$ and $g \leq h$ on $[\varepsilon, \min\{\pi/2, \tau\}]$. Assume that $\tau > \pi/2$. Then $f(\pi/2) = g(\pi/2) = h(\pi/2) = 0$, so that $\pi/2$ is a local minimum of f. This contradicts Lemma 3.4.3. Therefore $\tau \leq \pi/2$.

Lemma 3.4.7 *Let $\tau > 0$. Let f be a nonzero nonnegative $\sin^k$-concave function on $[0, \tau]$, which attains its maximum at 0. Then $\tau \leq \pi/2$ and for all $\alpha \geq 0$ and $\varepsilon \leq \pi/2$,*

$$\frac{\int_0^{\min\{\varepsilon,\tau\}} f(t)\sin^\alpha(t)\,dt}{\int_0^\tau f(t)\sin^\alpha(t)\,dt} \geq \frac{\int_0^\varepsilon \cos^k(t)\sin^\alpha(t)\,dt}{\int_0^{\pi/2}\cos^k(t)\sin^\alpha(t)\,dt}.$$

Proof of the Lemma

If $\epsilon \geq \tau$, the left hand side equals 1, which is obviously larger than the right hand side. Otherwise, set

$$v = \frac{\int_0^\varepsilon \cos^k(t)\sin^\alpha(t)\,dt}{\int_\varepsilon^{\pi/2}\cos^k(t)\sin^\alpha(t)\,dt}.$$

Choose $c > 0$ such that $h(t) = c\cos^k(t)$ satisfies $f(\varepsilon) = h(\varepsilon)$. From Lemma 3.4.6, $\tau \leq \pi/2$, $f \geq h$ on $[0, \varepsilon]$, $f \leq h$ on $[\varepsilon, \tau]$, thus

$$\begin{aligned}
\int_0^\varepsilon f(t)\sin^\alpha(t)\,dt &\geq \int_0^\varepsilon h(t)\sin^\alpha(t)\,dt \\
&= c\int_0^\varepsilon \cos^k(t)\sin^\alpha(t)\,dt \\
&= cv\int_\varepsilon^{\pi/2}\cos^k(t)\sin^\alpha(t)\,dt \\
&\geq v\int_\varepsilon^\tau h(t)\sin^\alpha(t)\,dt \\
&\geq v\int_\varepsilon^\tau f(t)\sin^\alpha(t)\,dt.
\end{aligned}$$

Thus

$$(1+v)\int_0^\varepsilon f(t)\sin^\alpha(t)\,dt \geq v\int_0^\tau f(t)\sin^\alpha(t)\,dt,$$

i.e.

$$\frac{\int_0^\varepsilon f(t)\sin^\alpha(t)\,dt}{\int_0^\tau f(t)\sin^\alpha(t)\,dt} \geq \frac{v}{1+v} = \frac{\int_0^\varepsilon \cos^k(t)\sin^\alpha(t)\,dt}{\int_\varepsilon^{\pi/2}\cos^k(t)\sin^\alpha(t)\,dt}.$$

The result of Lemma 3.4.7 is very important for the estimation of the waist, as we will see in the next section.

3.4.3 Lower bound for the measure of balls

Notation 3.4.8 *Let μ be a convexely derived measure supported on a convex set of dimension $k < n$. We denote by $M_0(\mu)$ the unique point where its density with respect to Lebesgue k-dilensional measure achieves its maximum.*

What we need is a lower bound for $\mu(B(M_0(\mu), \varepsilon))$. This lower bound is provided in the following Lemma.

Lemma 3.4.9 *Let μ be a convexely derived measure supported on a convex set S of dimension $k < n$. Then*

$$\mu(B(M_0(\mu), \varepsilon)) \geq \frac{\int_0^\varepsilon \cos^{n-k}(t) \sin^{k-1}(t)\, dt}{\int_0^{\pi/2} \cos^{n-k}(t) \sin^{k-1}(t)\, dt}.$$

Proof of the Lemma

We use polar coordinates $(t, u) \mapsto \phi(t, u) = \exp_{M_0(\mu)}(tu)$ centered at $M_0(\mu)$ on the k-sphere containing $S : t \in [0, \pi]$, $u \in \mathbb{S}^{k-1}$. By convexity of S, there exists a nonnegative function τ on $\mathbb{S}^{k-1}$ such that

$$\phi^{-1}(S) = \{(t, u) \,|\, 0 \leq t \leq \tau_u\},$$

and

$$\phi^{-1}(B(M_0(\mu), \varepsilon)) = \{(t, u) \,|\, 0 \leq t \leq \min\{\varepsilon, \tau_u\}\},$$

The convexely derived probability measure on S is $d\mu = f\, dv$, where $dv = \sin^{k-1}(t)\, dt\, du$ and dt is the Lebesgue measure on $[0, \pi]$, du is the $(k-1)$-dimensional canonical Riemannian measure of $\mathbb{S}^{k-1}$.

We shall denote abusively $f \circ \phi(t, u)$ by $f(t, u)$. Hence

$$\mu(B(M_0(\mu), \varepsilon)) = \int_{0 \leq t \leq \min\{\varepsilon, \tau_u\}} f(t, u) \sin^{k-1}(t)\, dt\, du.$$

Here we can apply Lemma 3.4.7. Let

$$w = \frac{\int_0^\varepsilon \cos^{n-k}(t) \sin^{k-1}(t)\, dt}{\int_0^{\pi/2} \cos^{n-k}(t) \sin^{k-1}(t)\, dt}.$$

We know that for every $u \in \mathbb{S}^{k-1}$, $t \mapsto f(t, u)$ is a $\sin^{n-k}$-concave function on $[0, \tau_u]$. Therefore $\tau_u \leq \pi/2$ and

$$\int_0^{\min\{\varepsilon, \tau_u\}} f(t, u) \sin^{k-1}(t)\, dt \geq w \int_0^{\tau_u} f(t, u) \sin^{k-1}(t)\, dt$$

Integrating over $\mathbb{S}^{k-1}$ yields

$$\begin{aligned}\mu(B(M_0(\mu),\varepsilon)) &\geq w\int_{\mathbb{S}^{k-1}}\int_0^{\tau_u} f(t,u)\sin^{k-1}(t)\,dt\,du \\ &= w\mu(S) = w,\end{aligned}$$

since μ is a probability measure.

Lemma 3.4.10 *Let $\mathbb{S}^{n-k}$ be an equatorial $(n-k)$-dimensional sphere in $\mathbb{S}^n$ then*

$$\frac{vol_n(\mathbb{S}^{n-k}+\varepsilon)}{vol_n(\mathbb{S}^n)} = \frac{\int_0^{\varepsilon}\cos^{n-k}(t)\sin^{k-1}(t)\,dt}{\int_0^{\pi/2}\cos^{n-k}(t)\sin^{k-1}(t)\,dt}.$$

Proof of the Lemma

Let $\mathbb{S}^{n-k}$ be an equatorial sphere. Let take the distance function from $\mathbb{S}^{n-k}$, $d(x) = d(x,\mathbb{S}^{n-k}) : \mathbb{S}^n \to \mathbb{R}$. The pushforward measure is equal $\gamma(n)\cos^{n-k}(t)\sin^{k-1}\,dt$, and the proof of the Lemma follows.

3.5 Infinite partitions

Definition 3.5.1 (space of convexely derived measures) *Let $\mathcal{MC}^n$ denote the set of probability measures on $\mathbb{S}^n$ of the form $\mu_S = vol_{|S}/vol(S)$ where $S \subset \mathbb{S}^n$ is open and convex. The space $\mathcal{MC}$ of* convexely derived probability measures *on $\mathbb{S}^n$ is the vague closure of $\mathcal{MC}^n$.*

It is a compact metrizable topological space.

Lemma 3.5.1 *For all open convex sets $S \subset \mathbb{S}^n$ and all $x \in S$,*

$$\frac{vol(S\cap B(x,r))}{vol(S)} \geq \frac{vol(B(x,r))}{vol(\mathbb{S}^n)}.$$

Proof.

Apply Bishop-Gromov's inequality in Riemannian geometry. In this special case ($\mathbb{S}^n$ has constant curvature 1), it states that the ratio

$$\frac{vol(S\cap B(x,r))}{vol(B(x,r))}$$

is a nonincreasing function of r. It follows that

$$\frac{vol(S \cap B(x,r))}{vol(B(x,r))} \geq \frac{vol(S)}{vol(\mathbb{S}^n)}.$$

This inequality extends to all convexely derived measures, thanks to the following Lemma.

Lemma 3.5.2 (See [14]). *Let μ_i be a sequence of positive Radon measures on a locally compact space X which vaguely converges to a positive Radon measure μ. Then for every relatively compact subset $A \subset X$ such that $\mu(\partial A) = 0$,*

$$\lim_{i \to \infty} \mu_i(A) = \mu(A).$$

Corollary 3.5.3 *For all measures $\mu \in \mathcal{MC}$ and all $x \in$ support(μ),*

$$\mu(S \cap B(x,r)) \geq \frac{vol(B(x,r))}{vol(\mathbb{S}^n)}.$$

Proof.

Let $\mu = \lim \mu_{S_j}$. Up to extracting a subsequence, one can assume that S_j Hausdorff converges to a compact convex set S. Then support$(\mu) \subset S$. Indeed, if $x \notin S$, there exists $r > 0$ such that $S \cap B(x,r) = \emptyset$. Let f be a continuous function on $\mathbb{S}^n$, supported in $B(x, r/2)$. Then for j large enough, $S_j \cap B(x, r/2) = \emptyset$, $\int f \, d\mu_{S_j} = 0$, so $\int f \, d\mu = 0$, showing that $x \notin$ support(μ).

If μ is a Dirac measure, then the inequality trivially holds. Otherwise, let $x \in$ support(μ). There exist $x_j \in$ support(μ_j) such that x_j tend to x. Since μ gives no measure to boundaries of metric balls, Lemma 3.5.2 applies, and the inequality of Lemma 3.5.1 passes to the limit.

Lemma 3.5.4 *Let Comp$(\mathbb{S}^n)$ denote the space of compact subsets of $\mathbb{S}^n$ equipped with Hausdorff distance. The map* support $: \mathcal{MC} \to$ *Comp$(\mathbb{S}^n)$ which maps a measure to its support is continuous.*

Proof.

Let $\mu_j \in \mathcal{MC}$ converge to μ. One can assume that $S_j =$ support(μ_j) converge to a compact set S. We saw in the proof of Corollary 3.5.3 that support$(\mu) \subset S$. To prove the opposite inclusion, let us define, for $r > 0$ and $x \in \mathbb{S}^n$,

$$f_{r,x}(y) = \begin{cases} 1 & \text{if } d(y,x) < \frac{r}{2}, \\ 2 - 2\frac{d(y,x)}{r} & \text{if } \frac{r}{2} \leq d(y,x) < r, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in S$. Let $x_j \in S_j$ converge to x. According to Lemma 3.5.3, if $d(x_j, x) < r/4$,

$$\int f_{x,r}(y)\, d\mu_j(y) \geq const.r^n,$$

i.e. $\int f_{x,r}\, d\mu_j$ does not tend to 0. It follows that $\int f_{x,r}\, d\mu > 0$, and x belongs to support(μ). This shows that support is a continuous map on $\mathcal{MC}$.

The support of a convexely derived probability measure is a closed convex set, it has a dimension.

Notation 3.5.5 *$\mathcal{MC}^k$ denotes the set of convexely derived probability measures whose support has dimension k, $\mathcal{MC}^{\leq k} = \bigcup_{\ell=0}^{k} \mathcal{MC}^k$, $\mathcal{MC}^+ = \mathcal{MC} \setminus \mathcal{MC}^0$. For $\rho > 0$, $\mathcal{MC}_\rho$ denotes the set of convexely derived probability measures whose support has diameter $\geq \rho$.*

Lemma 3.5.6 *As r tends to 0, $\mu(B(x,r))$ tends to 0 uniformly on $\mathcal{MC}_\rho \times \mathbb{S}^n$.*

Proof.

Since we deal with small radii, we can make computations as if the sphere were flat, i.e. let $\mathbb{S}^n = \mathbb{R}^n$. We can assume that ρ is very small as well. Let μ be a convexely derived measure supported by a k-dimensional convex set S, let $x \in \mathbb{R}^n$ and $B = S \cap B(x,r)$. Since S has diameter at least ρ, there is a point y at distance at least $\rho/2$ of x. Up to a translation, we can assume that y is the origin of $\mathbb{R}^k$. Let ϕ be the density of μ. Then $\phi^{1/(n-k)}$ is concave. Thus, for $x' \in B$ and $\lambda \in]0,1[$,

$$\phi(\lambda x) \geq \lambda^{n-k}\phi(x).$$

Changing variables gives

$$\begin{aligned}
\mu(\lambda B) &= \int_{\lambda B} \phi(z)\, dz \\
&= \lambda^k \int_B \phi(\lambda z)\, dz \\
&\geq \lambda^n \int_B \phi(z)\, dz \\
&= \lambda^n \mu(B).
\end{aligned}$$

If N is an integer such that $N \leq \rho/4r$, then one can choose N values of λ between $1/2$ and 1 leading to disjoint subsets λB of S, and this yields

$$1 = \mu(S) \geq N(\frac{1}{2})^n \mu(B),$$

i.e.

$$\mu(B) \leq 2^n/N \simeq \text{const.}\, r/\rho.$$

Lemma 3.5.7 *The function $(\mu, x, r) \mapsto \mu(B(x,r))$ is continuous on $\mathcal{MC}_+ \times \mathbb{S}^n \times [0, \pi/2)$.*

Proof.
We remind the following well known

Lemma 3.5.8 (Dini) *Let X be compact, $f_j : X \to \mathbb{R}$ be a increasing (resp. decreasing) sequence of continuous functions, i.e for $i \leq i'$, $f_i \leq f_{i'}$ (resp $f_i \geq f_{i'}$). If the sequence f_i is pointwise convergent then it is uniformly convergent.*

Fix $\rho > 0$. Let $X = \mathcal{MC}_\rho \times \mathbb{S}^n$. Let (μ_j, x_j) converge to (μ, x). By the symmetry of the sphere, we can choose a sequence ϕ_j such that for every j, $\phi_j \in Iso(\mathbb{S}^n)$ in such a way that ϕ_j uniformly converges to the identity and for every $j \in \mathbb{N}$ we have $\phi_j(x_j) = x$. Hence $\mu_j(B(x_j, r)) = (\phi_{j*}\mu_j)(B(x,r))$. For every $f \in C^0(\mathbb{S}^n)$,

$$\|f \circ \phi_j - f\|_\infty \underset{j\to\infty}{\longrightarrow} 0,$$

thus

$$\int_{\mathbb{S}^n} (f \circ \phi_j - f) d\mu_j \underset{j\to\infty}{\longrightarrow} 0,$$

and

$$\begin{aligned} \lim_{j\to\infty} \int_{\mathbb{S}^n} f d\phi_{j*}\mu_j &= \lim_{j\to\infty} \int_{\mathbb{S}^n} f \circ \phi_j d\mu_j \\ &= \int_{\mathbb{S}^n} f d\mu, \end{aligned}$$

i.e. the sequence $\phi_{j*}\mu_j$ converges vaguely to μ. For every $r < \frac{\pi}{2}$ and $\mu \in \mathcal{MC}_\rho$, $\mu(\partial B(x,r)) = 0$, thus Lemma 3.5.2 applies and we conclude that $\mu_j(B(x_j, r))$ tends to $\mu(B(x,r))$. This proves that for every $r \in [0, \pi/2)$, $\mu(B(x,r))$ is a continuous function of (μ, x).

In general, for an increasing sequence of sets A_j, $\mu(\bigcup A_j) = \lim_j \mu(A_j)$. This shows that for fixed (x, μ),

$$\lim_{r'\to r, r'<r} \mu(B(x,r')) = \mu(B(x,r)), \quad \lim_{r'\to r, r'>r} \mu(B(x,r')) = \mu(\overline{B(x,r)}).$$

Again, since $\mu(\partial B(x,r)) = 0$, $\mu(B(x,r)$ depends continuously on r. Dini's Lemma implies that the function $v_r : (\mu, x) \mapsto \mu(B(x,r))$ varies continuously with r in $C^0(\mathcal{MC}_\rho \times \mathbb{S}^n)$.

If $\mu_j \to \mu$, $x_j \to x$ and $r_j \to r$,

$$\lim_{j\to\infty} \mu_j(B(x_j, r_j)) = \lim_{j\to\infty} v_{r_j}(\mu_j, x_j) = v_r(\mu, x).$$

Hence the continuity of $(\mu, x, r) \to \mu(B(x,r))$ on $\mathcal{MC}_\rho \times \mathbb{S}^n \times [0, \pi/2)$ and the proof of the Lemma follows.

Definition 3.5.2 (limits of finite convex partitions) *Let Π be a finite convex partition of $\mathbb{S}^n$. We view it as an atomic probability measure $m(\Pi)$ on $\mathcal{MC}$ as follows : for each piece S of Π, let $\mu_S = vol_{|S}/vol(S)$ be the normalized volume of S. Then set*

$$m(\Pi) = \sum_{\text{pieces } S} \frac{vol(S)}{vol(\mathbb{S}^n)} \delta_{\mu_S}.$$

We define the space of *(infinite)* convex partitions $\mathcal{CP}$ *as the vague closure of the image of the map m in the space $\mathcal{P}(\mathcal{MC})$ of probability measures on the space of convexely derived measures. The subset $\mathcal{CP}^{\leq k}$ of convex partitions of dimension $\leq k$, consists of elements of $\mathcal{CP}$ which are supported on the subset $\mathcal{MC}^{\leq k}$ of convexely derived measures with support of dimension at most k.*

Note that $\mathcal{CP}$ is compact and $\mathcal{CP}^{\leq k}$ is closed in it. Measures in the support of a convex partition can be thought of as the pieces of the partition.

Lemma 3.5.9 (desintegration formula) *Let $A \subset \mathbb{S}^n$ be a set such that the intersection of ∂A with every ℓ-dimensional subsphere has vanishing ℓ-dimensional volume, for all ℓ, $0 < \ell < n$. Let $\Pi \in \mathcal{CP}$. Assume that $\Pi(\mathcal{MC}^0) = 0$. Then*

$$\frac{vol(A)}{vol(\mathbb{S}^n)} = \int_{\mathcal{MC}} \mu(A)\, d\Pi(\mu).$$

Proof.

The identity to be proved holds for finite partitions. According to Lemma 3.5.2, the function $\mu \mapsto \mu(A)$ is continuous on $\mathcal{MC}^+$. Therefore the identity still holds for vague limits of finite partitions. This completes the proof of Lemma 3.5.9.

3.5.1 Choice of a center map

In the previous sections, we didn't make any particular assumption about the center map. In fact the only property of this map which was used was continuity. In this section we construct a family of center maps which will lead us to the proof of the waist theorem.

Definition 3.5.3 (approximate centers of convexely derived measures) *Let $\mu \in \mathcal{MC}$, let $r > 0$. Consider the function $\mathbb{S}^n \to \mathbb{R}$, $x \mapsto v_{r,\mu}(x) = \mu(B(x,r))$. Let $M_r(\mu)$ be the set of points where $v_{r,\mu}$ achieves its maximum on* support(μ).

If the support of μ is ℓ-dimensional, $\ell < n$, we denote by $M_0(\mu)$ the unique point where the density of μ achieves its maximum.

The next Lemma states a semi-continuity property of M_r.

Notation 3.5.10 *When A_i, $i \in \mathbb{N}$, are subsets of a topological space, we shall denote by*

$$\lim_{i\to\infty} A_i = \bigcap_i \overline{\bigcup_{j\geq i} A_j}.$$

the set of all possible limits of subsequences $x_{i(j)} \in A_{i(j)}$.

Lemma 3.5.11 *Let μ_i be convexely derived measures which converge to $\mu \in \mathcal{MC}^+$. Then, for all $r > 0$,*

$$\lim_{i\to\infty} M_r(\mu_i) \subset M_r(\mu).$$

If follows that

$$\lim_{i\to\infty} \text{conv. hull}(M_r(\mu_i)) \subset \text{conv. hull}(M_r(\mu)).$$

Proof.

Let $x \in \lim_{i\to\infty} M_r(\mu_i)$, i.e. $x = \lim_{i\to\infty} x_i$ for some $x_i \in M_r(\mu_i)$. Pick $y \in$ support(μ). Pick a sequence $y_i \in$ support(μ_i) converging to y. According to Lemma 3.5.7,

$$v_{r,\mu}(x) = \lim_{i\to\infty} v_{r,\mu_i}(x_i), \quad v_{r,\mu}(y) = \lim_{i\to\infty} v_{r,\mu_i}(y_i).$$

Since $v_{r,\mu_i}(x_i) \geq v_{r,\mu_i}(y_i)$, we get $v_{r,\mu}(x) \geq v_{r,\mu}(y)$, showing that $x \in M_r(\mu)$.

We claim that for arbitrary compact sets $A_i \in \mathbb{S}^n$,

$$\lim_{i\to\infty} \text{conv. hull}(A_i) \subset \text{conv. hull}(\lim_{i\to\infty} A_i).$$

Indeed, taking cones, it suffices to check this in Euclidean space. If $x \in \lim_{i\to\infty}$ conv. hull(A_i), $x = \lim x_i$ with $x_i \in$ conv. hull(A_i), then there exist $n+1$ numbers $t_{i,j} \in [0,1]$ and points $a_{i,j} \in A_i$ such that $\sum_j t_{i,j} = 1$, $x_i = \sum_j t_{i,j} a_{i,j}$. One can assume that all sequences $i \mapsto t_{i,j}$, $a_{i,j}$ converge to t_j, a_j. Then $t_j \in [0,1]$, $\sum_j t_j = 1$, $a_j \in A = \lim_{i\to\infty} A_i$ and $x = \sum_j t_j a_j \in$ conv. hull(A). This completes the proof of Lemma 3.5.11.

The above semi-continuity property is sufficient to apply Ernest Michael's theory of continuous selections, [20].

Theorem 22 (Michael continuous selection Theorem) *Let X be paracompact, Y be a Banach Space, and $\mathfrak{S}$ the space of closed convex non-empty subsets of Y. Then every lower semi-continuous map $\phi : X \to \mathfrak{S}$ admits a continuous selection.*

Let $N_r(S)$ denotes the convex hull of $M_r(S)$ in $\mathbb{R}^{n+1}$. By definition of the convex sets in $\mathbb{S}^n$, $N_r(S)$ is a closed convex set which does not contain the origin of $\mathbb{R}^{n+1}$. We apply the Theorem 22 to the map $S \to N_r(S)$. We obtain a continuous map $S \to D_r(S)$ which never takes the value 0. We pose $C_r(S) = D_r(S)/\|D_r(S)\|$ and we obtain the continuous selection on $\mathbb{S}^n$. Hence the following

Definition 3.5.4 (centers of open convex sets) *Let $r > 0$. According to Theorem 22, we can choose a continuous map $C_r : \mathcal{MC}^n \to \mathbb{S}^n$, such that for every $S \in \mathcal{MC}^n$, $C_r(S)$ belongs to* $\mathrm{conv.\,hull}(M_r(S))$.

3.5.2 Construction of partitions adapted to a continuous map

Definition 3.5.5 (partitions adapted to a continuous map) *Let $f : \mathbb{S}^n \to \mathbb{R}^k$ be a continuous map. Let $r \geq 0$. Say a convex partition $\Pi \in \mathcal{CP}$ is r-*adapted to *f if there exists $z \in \mathbb{R}^k$ such that $f^{-1}(z)$ intersects the convex hull of $M_r(\mu)$ for all measures μ in the support of Π. Let*

$$\mathcal{F}_r = \{\Pi \in \mathcal{CP} \mid \bigcap_{\mu \in \mathrm{support}(\Pi)} f(\mathrm{conv.\,hull}(M_r(\mu))) \neq \emptyset\}$$

denote the set of partitions which are r-adapted to f.

Corollary 3.5.12 *For all $r > 0$, $\mathcal{F}_r$ is closed in $\mathcal{CP}$.*

Proof.

If $\lim_{i\to\infty} \Pi_i = \Pi$, $\mathrm{support}(\Pi) \subset \lim_{i\to\infty} \mathrm{support}(\Pi_i)$, i.e. every piece μ of Π is the limit of a sequence of pieces μ_i of Π_i. By assumption, there is a $z_i \in \mathbb{R}^k$ which belongs to all $f(\mathrm{conv.\,hull}(M_r(\mu)))$, $\mu \in \mathrm{support}(\Pi_i)$. One can assume z_i converges to z. Then z belongs to all $f(\mathrm{conv.\,hull}(M_r(\mu)))$, $\mu \in \mathrm{support}(\Pi)$. Indeed, in general, if g is a continuous map and A_i are subsets of a compact space, $g(\lim_{i\to\infty} A_i) = \lim_{i\to\infty} g(A_i)$. So if $\mu = \lim \mu_i$, $\mu_i \in \mathrm{support}(\Pi_i)$,

$$\begin{aligned} z = \lim_{i\to\infty} z_i &\in \lim_{i\to\infty} f(\mathrm{conv.\,hull}(M_r(\mu_i))) \\ &\subset f\left(\lim_{i\to\infty} \mathrm{conv.\,hull}(M_r(\mu_i))\right) \\ &\subset f(\mathrm{conv.\,hull}(M_r(\mu))), \end{aligned}$$

thanks to Lemma 3.5.11.

Remark 3.5.13 *Theorem 3 states that for every $r > 0$, $\mathcal{F}_r$ contains uniform atomic measures with arbitrarily many pieces. Theorem 4 produces elements of $\mathcal{F}_r$ whose support is contained in arbitrary thin neighborhoods of the compact subset $\mathcal{MC}^{\leq k}$. With Corollary 3.5.12, this gives elements in $\mathcal{F}_r \cap \mathcal{CP}^{\leq k}$.*

3.5.3 Convergence of $M_r(\mu)$ as r tends to 0

Lemma 3.5.14 *Let $\ell < n$. For every ℓ-dimensional convexely derived measure μ,*

$$\lim_{r \to 0} d_H(M_r(\mu), M_0(\mu)) = 0.$$

Proof.

We prove the Lemma by contradiction. Otherwise, we get a $\delta > 0$ and a sequence of radii r_i tending to 0 such that $d_H(M_{r_i}(\mu), M_0(\mu)) \geq \delta$. Pick a point $x_i \in S$ where $v_{r_i,\mu}$ achieves its maximum and such that $d(x_i, M_0(\mu)) \geq \delta$. Up to extracting a subsequence, we can assume that x_i converges to $x \in S$. Then $v_{r_i,\mu}(x_i)/\alpha_k r_i^k$ converges to $\phi_\mu(x)$. For every $y \in S$, $v_{r_i,\mu}(y) \leq v_{r_i,\mu}(x)$ and $v_{r_i,\mu}(y)/\alpha_k r_i^k$ converges to $\phi_\mu(y)$. Therefore $\phi_\mu(y) \leq \phi_\mu(x)$. This shows that $\{x\} = M_0(\mu)$, contradiction.

A stronger statement will be given after the following technical lemmas.

Lemma 3.5.15 *Let μ be a convexely derived measure on $\mathbb{S}^n$ whose support is a k-dimensional convex set S. Write $d\mu = \phi\, dvol_k$. Then*

$$\max_S \phi \leq \frac{2^{n+1}}{vol_k(S)}.$$

Proof.

Replace S with $C = co(S) \subset \mathbb{R}^{n+1}$, and ϕ by its $n-k$-homogeneous extension. Then $\phi^{1/(n-k)}$ is concave. Assume ϕ achieves its maximum at $x \in C$. Translate C so that $x = 0$. On $\frac{1}{2}C$, $\phi^{1/(n-k)} \geq \frac{1}{2}\phi^{1/(n-k)}(x)$, thus

$$\begin{aligned} 1 = \mu(S) &\geq \int_{\frac{1}{2}C} \phi\, dvol_{k+1} \\ &\geq \frac{1}{2^{n-k}}\phi(x) vol_{k+1}(\frac{1}{2}C) \\ &= \frac{1}{2^{n+1}}\phi(x) vol_{k+1}(C) \\ &= \frac{1}{2^{n+1}}\phi(x) vol_k(S). \end{aligned}$$

Lemma 3.5.16 *Let S, S_i be full compact convex subsets of $\mathbb{R}^n$ such that S_i Hausdorff-converges to S. Let $\phi : S_i \to [0,1]$ be concave functions. Then there exists a concave function $\phi : S \to [0,1]$ and a subsequence with the following properties.*

- *On every compact subset of the interior of S, ϕ_i converges uniformly to ϕ.*
- *For all $x \in \partial S$ and all sequences $x_i \in S_i$ converging to x,*

$$\limsup_{i\to\infty} \phi_i(x_i) \leq \phi(x).$$

Proof.

In general, bounded concave functions f on compact convex sets Σ are locally Lipschitz,

$$\textit{for } x \in \Sigma \textit{ with } d(x, \partial\Sigma) = r, \textit{ and all } y \in \Sigma,\ |f(x) - f(y)| \leq \frac{1}{r} d(x,y).$$

Indeed, let $[x', y']$ be the intersection of Σ with the line through x and y, with x', x, y' and y' sitting along the line in this order. Let ℓ be the affine function on $[x', y']$ such that $\ell(x') = f(x')$ and $\ell(x) = f(x)$. Then $f(y) \leq \ell(y)$, thus $f(y) - f(x) \leq \frac{1}{d(x',x)}|f(x) - f(x')|d(x,y) \leq \frac{1}{r}d(x,y)$. Also, let ℓ' be the affine function on $[x', y']$ such that $\ell'(x) = f(x)$ and $\ell'(y') = f(y')$. Then $f(y) \geq \ell'(y)$, thus $f(y) - f(x) \geq -\frac{1}{d(x,y')}|f(x) - f(y')|d(x,y) \geq -\frac{1}{r}d(x,y)$.

This shows that on every compact subset of the interior of S, the sequence f_j is equicontinuous, so a subsequence can be found which converges uniformly on all such compact sets to a continuous function ϕ. Of course, ϕ is concave and bounded, so it extends continuously to ∂S. Let $x \in \partial S$ and $x_i \in S_i$ converge to x. Pick an interior point x_0 of S and a second interior point $x' \neq x_0$ such that x_0 lies on the segment $[x', x]$. Pick x_i' on the line passing through x_0 and x_i and converging to x'. The Lipschitz estimate for ϕ_i reads

$$\phi_i(x_i) - \phi_i(x_0) \leq \frac{d(x_0, x_i)}{d(x_0, x_i')}|\phi_i(x_i') - \phi_i(x_0)|.$$

Letting i tend to infinity yields

$$\limsup \phi_i(x_i) \leq \phi(x_0) + \frac{d(x_0, x)}{d(x_0, x')}|\phi(x') - \phi(x_0)|.$$

Letting x_0 and x' tend to x (while keeping x', x_0 and x aligned and $\frac{d(x_0,x)}{d(x_0,x')}$ bounded) gives $\limsup \phi_i(x_i) \leq \phi(x)$.

Lemma 3.5.17 *For each $k < n$, the restriction of $(\mu, r) \mapsto d_H(M_r(\mu), M_0(\mu))$ to $\mathbb{R}_+ \times \mathcal{MC}^k$ tends to 0 along $\{0\} \times \mathcal{MC}^k$, i.e. for all $\mu \in \mathcal{MC}^k$,*

$$\lim_{r \to 0,\, \mu' \to \mu,\, \mu' \in \mathcal{MC}^k} d_H(M_r(\mu), M_0(\mu)) = 0.$$

Proof.

Let $\mu \in \mathcal{MC}^k$. Let μ_i be a sequence of k-dimensional convexely derived measures which converges to μ and r_i be positive numbers tending to 0. Let $g_i \in O(n+1)$ be a rotation mapping the support of μ_i into the k-sphere which contains the support of μ. One can assume that g_i converges to identity, and then change μ_i to $(g_i)_*\mu_i$, since this does not change the convergence of centers $C_{r_i}(\mu_i)$. In other words, one can assume that all μ_i have support S_i in the same k-sphere. Of course, S_i Hausdorff-converges to the support S of μ. Let ϕ_i denote the density of μ_i with respect to k-dimensional volume. Since $vol_k(S_i)$ does not tend to 0, ϕ_i are uniformly bounded, by Lemma 3.5.15. Furthermore, on any compact convex subset K of the relative interior of S, the ϕ_i are equicontinuous (this follows by the cone construction from Lemma 3.5.16). Therefore one can assume that ϕ_i converge uniformly on compact subsets of the relative interior of S. Since for all $r' > 0$, v_{r',μ_i} converges to $v_{r',\mu}$, the limit must be equal to the density ϕ of μ. From Lemma 3.5.16, one can assert that at boundary points $x \in \partial S$, for every sequence $x_i \in S_i$ converging to x, $\limsup \phi_i(x_i) \leq \phi(x)$.

We repeat the argument of Lemma 3.5.14. If $M_{r_i}(\mu_i)$ does not converge to $M_0(\mu)$, some sequence $x_i \in M_{r_i}(\mu_i)$ satisfies $d(x_i, M_0(\mu)) \geq \delta$ for some $\delta > 0$. Up to extracting a subsequence, we can assume that x_i converges to $x \in S$. If $x \notin \partial S$, then $v_{r_i,\mu}(x_i)/\alpha_k r_i^k$ converges to $\phi(x)$. If $x \in \partial S$, $\limsup v_{r_i,\mu}(x_i)/\alpha_k r_i^k \leq \phi(x)$. For every $y \in S \setminus \partial S$, $v_{r_i,\mu}(y) \leq v_{r_i,\mu}(x)$ and $v_{r_i,\mu}(y)/\alpha_k r_i^k$ converges to $\phi(y)$. Therefore $\phi(y) \leq \phi(x)$. Since $S \setminus \partial S$ is dense in S, this holds for all $y \in S$, thus ϕ achieves its maximum at x, i.e. $\{x\} = M_0(\mu)$, contradiction.

Corollary 3.5.18 *On any compact subset of $\mathcal{MC}^k$, the functions*

$$\mu \mapsto d_H(M_r(\mu), M_0(\mu))$$

converge uniformly to 0 as r tends to 0.

Proposition 23 *Assume $f : \mathbb{S}^n \to \mathbb{R}^k$ is a generic smooth map. Let r_i tend to 0 and let $\Pi_i \in \mathcal{CP}^{\leq k} \cap \mathcal{F}_{r_i}$ be convex partitions of dimension $\leq k$, r_i-adapted to f. Then, for all $\varepsilon > 0$,*

$$\max_{z \in \mathbb{R}^k} \frac{vol(f^{-1}(z) + \varepsilon)}{vol(\mathbb{S}^n)} \geq \frac{vol(\mathbb{S}^{n-k} + \varepsilon)}{vol(\mathbb{S}^n)} \limsup_{i \to \infty} \Pi_i(\mathcal{MC}^k).$$

Proof.

By assumption, for each i, there exists $z_i \in \mathbb{R}^k$ such that for all $\mu \in \text{support}(\Pi_i)$, there exists $x_{i,\mu} \in \text{conv. hull}(M_{r_i}(\mu))$ such that $f(x_{i,\mu}) = z_i$. Let $\mathcal{K} \subset \mathcal{MC}^k$ be a compact set. According to Corollary 3.5.18 and Lemma 3.5.7, for all $\varepsilon > 0$,

$$\delta_i := \sup_{\mu \in \mathcal{K}} |\mu(B(x_{i,\mu}, \varepsilon)) - \mu(B(M_0(\mu), \varepsilon))|$$

tends to 0. Considerations in section 5 show that for every k-dimensional convexely derived measure μ,

$$\mu(B(M_0(\mu), \varepsilon)) \geq \frac{vol(\mathbb{S}^{n-k} + \varepsilon)}{vol(\mathbb{S}^n)}.$$

For a generic smooth map f, the intersection of $f^{-1}(z_i) + \varepsilon$ with k-dimensional convex sets has vanishing k-dimensional volume, so the desintegration formula applies, and

$$\begin{aligned} \frac{vol(f^{-1}(z_i) + \varepsilon)}{vol(\mathbb{S}^n)} &= \int_{\mathcal{MC}} \mu(f^{-1}(z_i) + \varepsilon)\, d\Pi_i(\mu) \\ &\geq \int_{\mathcal{K}} \mu(B(x_{i,\mu}, \varepsilon))\, d\Pi_i(\mu) \\ &\geq \Pi_i(\mathcal{K}) \frac{vol(\mathbb{S}^{n-k} + \varepsilon)}{vol(\mathbb{S}^n)} - \delta_i. \end{aligned}$$

Taking the supremum over all compact subsets of $\mathcal{MC}^k$ and then a limit as i tends to infinity yields the announced inequality.

3.5.4 End of the proof of Gromov's theorem

There remains to show that convex partitions in $\mathcal{CP}^{\leq k} \cap \mathcal{F}_r$, r small, put most of their weight on k-dimensional pieces. This will be proven indirectly. Pieces of dimension $< k$ may exist, but they provide a lower bound on $vol(f^{-1}(z) + r)$ which is so large, that they must have small weight. We shall need a weak concavity property of $v_{\mu,r}$, which in turn relies on the corresponding Euclidean statement.

Lemma 3.5.19 *Let $S \subset \mathbb{R}^n$ be an open convex set, ϕ an m-concave function defined on S. Let $\mu = \phi dvol_n$. Then the map $x \mapsto \mu(B(x, r) \cap S)$ is $m + n$-concave on S.*

Proof.

We use the following estimate (Generalized Prekopa-Leindler inequality), which can be found in [15]. For $\alpha \in [-\infty, +\infty]$ and $\theta \in [0,1]$, the α-mean of two nonnegative numbers a and b with weight θ is

$$M_\alpha^{(\theta)}(a,b) = (\theta a^\alpha + (1-\theta)b^\alpha)^{1/\alpha}.$$

Let $-\frac{1}{n} \leq \alpha \leq +\infty$, $\theta \in [0,1]$, u, v, w nonnegative measurable functions on $\mathbb{R}^n$ such that for all x, $y \in \mathbb{R}^n$,

$$w(\theta x + (1-\theta)y) \geq M_\alpha^{(\theta)}(u(x), v(y)).$$

Let $\beta = \frac{\alpha}{1+\alpha n}$. Then

$$\int w \geq M_\beta^{(\theta)}(\int u, \int v).$$

We apply this to restrictions of ϕ to balls, $u = 1_{B(x,r)}\phi$, $v = 1_{B(y,r)}\phi$, $w = 1_{B(\theta x+(1-\theta)y,r)}\phi$. By m-convexity of ϕ, the assumptions of the generalized Prekopa-Leindler inequality are satisfied with $\alpha = 1/m$. Then for $\beta = \frac{1}{m+n}$,

$$\mu(B(\theta x + (1-\theta)y), r)) \geq M_\beta^{(\theta)}(\mu(B(x,r)), \mu(B(y,r))),$$

which means

$$\mu(B(\theta x + (1-\theta)y), r))^{\frac{1}{m+n}} \geq \theta\mu(B(x,r))^{\frac{1}{m+n}} + (1-\theta)\mu(B(y,r))^{\frac{1}{m+n}}.$$

Lemma 3.5.20 *The functions $v_{\mu,r}$ on $\mathbb{S}^n$ are weakly concave. In other words, there exists a constant $c = c(n) > 0$ such that for every convexely derived measure μ and every sufficiently small $r > 0$, if $K \subset \mathrm{support}(\mu)$, then*

$$\min_{\mathrm{conv}(K)} v_{\mu,\frac{r}{c}} \geq c \min_K v_{\mu,r}.$$

Proof.

Since a half-sphere is projectively equivalent with Euclidean space, it suffices to prove weak concavity when K consists of 2 points.

Let μ be a k-dimensional convexely derived measure on $\mathbb{S}^n$. Denote its density by ϕ, a $\sin^{n-k}$-concave function on the support S of μ. Let Φ denote the $(n-k)$-homogeneous extension of ϕ to the cone on S. This is $(n-k)$-concave. Fix a point $x_0 \in \mathbb{S}^n$, let $\mathbb{R}^n$ denote the tangent space of $\mathbb{S}^n$ at x_0. Denote by ϕ' the restriction of Φ to $\mathbb{R}^n$, and μ' the measure with density ϕ'. Lemma 3.5.19 implies that $x' \mapsto \mu(B(x',r))$ is $(2n-k)$-concave.

This implies that for every x', $y' \in \mathbb{R}^n$ and z' belonging to the middle third of the line segment $[x', y']$,

$$\mu'(B(z', r)) \geq \frac{1}{3^{2n-k}} \max\{\mu'(B(x', r)), \mu'(B(y', r))\}.$$

The radial projection from a neighborhood $V \subset \mathbb{S}^n$ of x_0 to $\mathbb{R}^n$ is nearly isometric and nearly maps ϕ' to ϕ. Thus there exists a constant $c_1 > 0$ such that if x, $y \in V$ and z belongs to the middle third of the geodesic segment $[x, y]$,

$$\mu(B(z, \frac{r}{c_1})) \geq c_1 \max\{\mu(B(x, r)), \mu(B(y, r))\}.$$

Covering long segments $[x, y]$ with N neighborhoods like V (N can be bounded independantly of n) provides a constant $c > 0$ such that for all $z \in [x, y]$ which is not too close to the endpoints,

$$\mu(B(z, \frac{r}{c_1^N})) \geq c_1^N \max\{\mu(B(x, r)), \mu(B(y, r))\}.$$

In particular, for $c = c_1^N$,

$$\mu(B(z, \frac{r}{c})) \geq c \min\{\mu(B(x, r)), \mu(B(y, r))\}.$$

Proposition 24 *There exists a constant $c = c(n) > 0$ such that if Π belongs to $\mathcal{F}_r \cap \mathcal{CP}^{\leq k}$ for some small enough $r > 0$, then for all $\ell \leq k$,*

$$\max_{z \in \mathbb{R}^k} vol(f^{-1}(z) + \frac{r}{c}) \geq c \sum_{\ell=0}^{k} vol(\mathbb{S}^{n-\ell} + r)\Pi(\mathcal{MC}^\ell).$$

Proof.

By assumption, there exists $z \in \mathbb{R}^k$ such that for every measure μ in the support of Π, there exists $x \in$ conv. hull$(M_r(\mu))$ such that $f(x) = z$. If the support of μ is ℓ-dimensional, Lemmata 3.5.11 and 3.4.9 give

$$\begin{aligned}
\mu(f^{-1}(z) + \frac{r}{c}) &\geq \mu(B(x, \frac{r}{c})) \\
&= v_{\mu, \frac{r}{c}}(x) \\
&\geq c \min_{M_r(\mu)} v_{\mu,r} \\
&= c \max_{\text{support}(\mu)} v_{\mu,r} \\
&\geq c\, v_{\mu,r}(M_0(\mu)) \\
&= c\, \mu(B(M_0(\mu), r)) \\
&\geq c \frac{vol(\mathbb{S}^{n-\ell} + \rho)}{vol(\mathbb{S}^n)}.
\end{aligned}$$

Integrating this with respect to Π yields

$$\begin{aligned}\frac{vol(f^{-1}(z)+r)}{vol(\mathbb{S}^n)} &= \int_{\mathcal{MC}} \mu(f^{-1}(z)+r)\, d\Pi(\mu) \\ &\geq c\sum_{\ell=0}^{k} \frac{vol(\mathbb{S}^{n-\ell}+r)}{vol(\mathbb{S}^n)}\Pi(\mathcal{MC}^\ell).\end{aligned}$$

Proof of Gromov's theorem.

At last, we prove Theorem 3 : *Let $\varepsilon > 0$. Let $f : \mathbb{S}^n \to \mathbb{R}^k$ be a continuous map. Then*

$$\max_{z\in\mathbb{R}^k} vol(f^{-1}(z)+\varepsilon) \geq vol(\mathbb{S}^{n-k}+\varepsilon).$$

Assume first that f is smooth and generic. Then there exists a constant W such that for all sufficiently small r,

$$\max_{z\in\mathbb{R}^k} vol(f^{-1}(z)+r) \leq Wr^k.$$

For every $r > 0$, there exists a convex partition $\Pi_r \in \mathcal{CP}^{\leq k} \cap \mathcal{F}_r$ which is r-adapted to f (Corollary 3.5.13). Proposition 24 yields

$$\sum_{\ell=0}^{k} vol(\mathbb{S}^{n-\ell}+r)\Pi_r(\mathcal{MC}^\ell) \leq \frac{1}{c}\max_{z\in\mathbb{R}^k} vol(f^{-1}(z)+\frac{r}{c}) \leq \frac{W}{c}(\frac{r}{c})^k.$$

As r tends to 0, this implies that for all $\ell < k$, $\Pi_r(\mathcal{MC}^\ell)$ tends to 0, and thus $\Pi_r(\mathcal{MC}^k)$ tends to 1. Letting r tend to 0 in Proposition 23 then shows that

$$\max_{z\in\mathbb{R}^k} \frac{vol(f^{-1}(z)+\varepsilon)}{vol(\mathbb{S}^n)} \geq \frac{vol(\mathbb{S}^{n-k}+\varepsilon)}{vol(\mathbb{S}^n)}.$$

Every continuous map $f : \mathbb{S}^n \to \mathbb{R}^k$ is a uniform limit of smooth generic maps. Hausdorff semi-continuity of $X \mapsto vol(X+\varepsilon)$ then extends the result to all continuous maps. Indeed, let the continuous map $f : \mathbb{S}^n \to \mathbb{R}^k$ of the waist theorem be fixed. Let $g_j : \mathbb{S}^n \to \mathbb{R}^k$ be a sequence of C^∞ maps such that $\delta_j = \|g_j - f\|_{C^0}$ tends to 0. For every j, there exists a $z_j \in \mathbb{R}^k$ such that $vol(g_j^{-1}(z_j)+\varepsilon) \geq w(\varepsilon) := vol(\mathbb{S}^{n-k}+\varepsilon)$. We know that for every j, $g_j^{-1}(z_j) \subseteq f^{-1}(B(z_j,\delta_j))$. Then

$$vol_n(f^{-1}(B(z_j,\delta_j))+\varepsilon) \geq vol_n(g_j^{-1}(z_j)+\varepsilon) \geq w(\varepsilon).$$

Up to extracting a subsequence, we can assume that $\{z_j\}$ converges to a point z. There exists a decreasing sequence $\varepsilon_j \to 0$ such that for every j, $|z - z_j| \leq \varepsilon_j$. Then

$$f^{-1}(B(z_j, \delta_j)) + \varepsilon \subseteq f^{-1}(B(z, \delta_j + \varepsilon_j)) + \varepsilon,$$

thus for all j

$$vol_n(f^{-1}(B(z, \delta_j + \varepsilon_j) + \varepsilon) \geq w(\varepsilon),$$

and by Fatou Lemma

$$vol_n(\bigcap_j f^{-1}(B(z, \delta_j + \varepsilon_j)) + \varepsilon) \geq w(\varepsilon).$$

If for all j, $x \in f^{-1}(B(z, \delta_j + \varepsilon_j)) + \varepsilon$, then there exists y_j such that $d(x, y_j) \leq \varepsilon$ and $f(y_j) \in B(z, \delta_j + \varepsilon_j)$. We choose a subsequence y_k which converges to y. By construction, $d(x, y) \leq \varepsilon$, $f(y) = z$ thus $x \in f^{-1}(z) + \varepsilon$. Hence

$$\bigcap_j f^{-1}(B(z, \delta_j + \varepsilon_j)) + \varepsilon) \subset f^{-1}(z) + \varepsilon,$$

and

$$vol_n(f^{-1}(z) + \varepsilon) \geq w(\varepsilon).$$

Chapitre 4

Waist of the unit sphere of uniformly convex normed spaces

4.1 Introduction

The classical isoperimetric inequality for a metric space relates the measure of compact sets to the measure of their boundaries. These inequalities are codimension 1 isoperimeric inequalities (simply because the difference of the dimension of a compact set and the dimension of its boundary is equal to 1).

During his research on a Morse theory for the space of cycles of a manifold, F. Almgren gave a sharp lower bound for the volume of a minimal k-cycle in the sphere $\mathbb{S}^n$ for every k (see [22],[7]). This is an instance of an higher codimensional isoperimetric type inequality.

Another important example of higher codimensional isoperimetric inequality, which in fact is a generalisation of the Almgren isoperimetric inequality on the sphere, is the waist of the sphere theorem of Gromov presented in [8].

In this paper we prove a higher codimensional isoperimetric inequality for the unit sphere of a uniformly convex normed space. The idea follows [8].

In [9], M. Gromov and V. Milman give an isoperimetric inequality for the unit sphere of a uniformly convex normed space by using the localization technique (a nice exposition of this can be found in [1]). The main result of this chapter, Theorem 5, generalizes the isoperimetric inequality of Gromov-Milman. Let us reproduce it here.

Theorem 25 *Let X be a uniformly convex normed space of finite dimension $n+1$. Let $S(X)$ be the unit sphere of X, for which the distance is induced from the norm of X. The measure defined on $S(X)$ is the conical probability measure. Then a lower bound for the*

waist of $S(X)$ relative to $\mathbb{R}^k$ is given by

$$w(\varepsilon) = \frac{1}{1 + (1 - 2\delta(\frac{\varepsilon}{2}))^{n-k}(k+1)^{k+1}\frac{F(k,\frac{\varepsilon}{2})}{G(k,\frac{\varepsilon}{2})}}$$

where $\delta(\varepsilon)$ is the modulus of convexity,

$$F(k,\varepsilon) = \int_{\psi_2(\varepsilon)}^{\frac{\pi}{2}} \sin(x)^{k-1}\, dx.$$

and

$$G(k,\varepsilon) = \int_0^{\psi_1(\varepsilon)} \sin(x)^{k-1}\, dx.$$

And where

$$\psi_1(\varepsilon) = 2\arcsin(\frac{\varepsilon}{4\sqrt{k+1}})$$

and

$$\psi_2(\varepsilon) = 2\arcsin(\frac{\varepsilon}{2\sqrt{k+1}})$$

The next section will be concerned with preliminaries and tools which we need to prove this theorem. In the last section we will discuss the relation of our result with Gromov-Milman's isoperimetric inequality and some applications of our theorem.

We thank S. Alesker whose exposition [1] has helped us a lot.

4.2 Preliminaries

Let us consider a uniformly convex normed space of dimension $(n+1)$, $X = (\mathbb{R}^{n+1}, \|\quad\|)$ which we fix once for all.

Definition 4.2.1 (Modulus of convexity) *The space X has modulus of convexity δ if for all $\varepsilon > 0$, for all vectors x, $y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ we have*

$$\frac{\|x+y\|}{2} \leq 1 - \delta(\varepsilon).$$

Example 4.2.1 *Let E be a Euclidean space. In this case, the modulus of convexity is easily determined from the parallegram identity. And we have*

$$\delta_E(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Remark δ is a monotone increasing function. We use this remark later on to prove the Lemma 5.2.

We denote by $B(X) := \{x \in X| \quad \|x\| \leq 1\}$ the unit ball of X and $\partial B(X) = S(X) := \{x \in X| \quad \|x\| = 1\}$ the unit sphere of X.

We define a probability measure μ on $S(X)$ and we call it the conical measure,

Definition 4.2.2 (conical probability measure) *For any Borel set $A \subset S(X)$ we define*

$$\mu(A) := \frac{m_{n+1}\{\bigcup tA|0 \leq t \leq 1\}}{m_{n+1}(B(X))}$$

where m_n is the n-dimensional Lebesgue measure on X.

We can check that the measure μ is a probability measure on $S(X)$, indeed

$$\mu(S(X)) = \frac{m_{n+1}\{tS(X), 0 \leq t \leq 1\}}{m_{n+1}B(X)} = 1.$$

Remark : For the Euclidean norm on $\mathbb{R}^{n+1}$, where the distance between two points is the Euclidean distance and where the unit sphere is the canonical n-dimensional sphere $\mathbb{S}^n$, the conical measure is the canonical Riemannian probability measure on $\mathbb{S}^n$.

The mm-space on which we are going to work is $(S(X), \mu, d)$ with μ the conical probability measure and d the distance induced on $S(X)$ from the norm defined on X (i.e. for all $x, y \in S(X)$, $d(x, y) = \|x - y\|$).

4.3 Scheme of proof of Theorem 5

We fix a continuous map $f : S(X) \to \mathbb{R}^k$. The proof of theorem 1 goes as follows,

- Use a generalisation of the Borsuk-Ulam theorem giving rise to a finite convex partition of the sphere and a fiber of f (i.e $f^{-1}(z)$ for some $z \in \mathbb{R}^k$) passing through the centers of all the pieces of the partition (the center of a convex set has to be defined).
- Narrow the pieces of the partition (by increasing their number) so that almost all of them are Hausdorff close to a k-dimensional convex set. Pass to a limit infinite partition of the sphere by convex subsets of dimension less than or equal to k.
- On each piece of the partition, there exists a probability measure, convexely derived from the conical measure. This brings the n-dimensional volume estimate of the waist down to a k-dimensional measure estimate on each convex set of the partition. This method is called the localization technique. But usually, the localization or

the needle decomposition, brings the n-dimensional measure estimate down to a 1-dimensional problem. The use of a multi-dimensional localization technique first appears in [8].

- On each piece of the partition, Lemma 4.4.6 gives an estimate of the measure of an ε-ball centered at a point where the measure of the convex set is mostly concentrated. By integrating this estimate over the space of pieces of the partition, we obtain the result of Theorem 5.
  There are some difficulties due to the l-dimensional convex sets of the infinite partition for all $l < k$. We prove that these "bad" convex sets do not affect the estimate on waist. Or better say, the measure of these convex sets in the space of pieces of the partition is equal to zero.

4.4 Convexely derived measures on convex sets of $S(X)$

The topics studied in this section follows the ideas used in [1] and [9]. For every subset $S \in S(X)$ we define the subset $co(S) \in B(X)$ as

$$co(S) := \{\bigcup tS | 0 \leq t \leq 1\}.$$

Hence $co(S)$ is the cone centered at the origin of the ball over S.

Suppose we have a sequence of open convex sets $\{S_i\}$ of $S(X)$ which Hausdorff converges to a convex set $S' \in S(X)$ where we suppose that the dimension of S' is equal to k with $k < n$. It is clear that the sequence $\{co(S_i)\}$ Hausdorff converges to the set $co(S')$ where $\dim co(S') = k+1$. We define a probability measure μ' on $co(S')$ as follows. For every $i \in \mathbb{N}$, we define the measure $\mu'_i = \frac{m_{n+1}|S_i}{m_{n+1}(S_i)}$. A subsequence of this sequence of measures vaguely converges to a probability measure μ on $co(S')$. We call this measure a convexely derived measure. We recall that the support of the measure μ is automatically equal to $co(S')$ as the sequence converges to this set. By Brunn's Theorem, the measure μ is $(n+1-(k+1))$-concave so by Borell's Theorem, μ admits a density function f with respect to the $(k+1)$-dimensional Lebesgue measure defined on A. The function f is $(n-k)$-concave. Hence

$$\mu = f dm_{k+1}$$

where m_{k+1} is the $(k+1)$-dimensional Lebesgue measure. Morever we have :

Lemma 4.4.1 *The measure μ is $(n+1)$-homogeneous and the function f is $(n-k)$-homogeneous.*

This means $\mu(tA) = t^{n+1}\mu(A)$ for $0 \leq t \leq 1$ and $f(tx) = t^{n-k}f(x)$ for all $x \in co(S')$.

Proof of the Lemma

The measure μ is convexely derived from the normalized $(n+1)$-dimensional Lebesgue measure. As the $(n+1)$-dimensional Lebesgue measure is $(n+1)$-homogeneous then μ is $(n+1)$-homogeneous. From the equality $\mu = f dm_{k+1}$, and the fact that μ is $(n+1)$-homogeneous and m_{k+1} is $(k+1)$-homogeneous, then clearly f is $(n-k)$-homogeneous and the proof of the Lemma follows.

The convexely derived measure μ' defined on $co(S')$ defines a probability measure μ on S' convexely derived from the conical measure of $S(X)$ and obtained from the sequence $\{S_i\}$, where for every $X \subset S'$ we have

$$\mu(X) = \mu'(co(X)).$$

And on the other hand, there exists another probability measure defined on S' which is the canonical k-dimensional conical measure conically induced by m_{k+1}, we denote this measure by ν. For every Borel subset U of S'

$$\nu(U) = \frac{m_{k+1}(co(U))}{m_{k+1}(co(S'))}.$$

S' is a subset of the unit sphere of $\mathbb{R}^{k+1}$ equipped with a norm satisfying the same modulus of convexity.

Then we have

$$\mu(U) = \mu'(co(U)) = \int_{co(U)} f dm_{k+1} = \int_U f d\nu.$$

Hence in conclusion we have

$$d\mu = f d\nu$$

where we take the restriction of f on the set U.

The function f is $(n-k)$-concave on $co(A)$ but the restriction of this function on the spherical part of the border of $co(A)$ is not anymore $(n-k)$-concave.

However the restriction function still has nice concavity properties as we will explain now.

Definition 4.4.1 *An* arc $\sigma \subset S(X)$ *is subarc of the intersection of a 2-plane passing through the origin of the ball with* $S(X)$.

We know that $\forall x, y \in S_\pi$,

$$f^{1/(n-k)}(\frac{x+y}{2}) \geq \frac{f^{1/(n-k)}(x) + f^{1/(n-k)}(y)}{2}.$$

But the point $\frac{x+y}{2}$ is no more on $S(X)$, so we set $z = \frac{x+y}{2}/\|\frac{x+y}{2}\| \in S(X)$.
By the definition of the modulus of convexity we have

$$\|\frac{x+y}{2}\| \leq 1 - \delta(\|x-y\|) \tag{4.1}$$

So we can conclude the following Lemma.

Lemma 4.4.2 *Let f denote the density of a convexely derived measure on $S(X)$. Let x, $y \in S_\pi$, let $z = \frac{x+y}{2}/\|\frac{x+y}{2}\| \in S_\pi$. Then*

$$\frac{f^{1/(n-k)}(x) + f^{1/(n-k)}(y)}{2} \leq (1 - \delta(\|x-y\|)f^{1/(n-k)}(z).$$

Proof of the Lemma
As $\frac{x+y}{2} = \|\frac{x+y}{2}\|z$ and as the function f is $(n-k)$-homogeneous

$$f^{1/(n-k)}(\frac{x+y}{2}) = \|\frac{x+y}{2}\|f^{1/(n-k)}(z)$$

and by equation 4.1 the proof of the Lemma follows.

Definition 4.4.2 *Let f be a function defined on an arc of $S(X)$. Say f is* weakly $(n-k)$-concave *if $\forall x$, $y \in \sigma$, $z = \frac{x+y}{2}/\|\frac{x+y}{2}\|$,*

$$\frac{f^{1/(n-k)}(x) + f^{1/(n-k)}(y)}{2} \leq (1 - \delta(\|x-y\|)f^{1/(n-k)}(z).$$

Lemma 4.4.3 *A nonzero weakly $(n-k)$-concave function defined on an arc of $S(X)$ has at most one maximum point and has no local minima.*

Proof of the Lemma
If there were two distinct maxima x and y and we would get $f^{1/(n-k)}(x) \leq (1-\delta(\|x-y\|)f^{1/(n-k)}(x)$, contradiction. Suppose f has a local minimum at point m. Take nearby points x' and y' such that $m = \frac{x'+y'}{2}$. Then $x = \frac{x'}{\|x'\|}$ and $y = \frac{y'}{\|y'\|}$ belong to the arc, and $m = \frac{x+y}{2}/\|\frac{x+y}{2}\| = m$. This leads again to a contradiction. The proof of the Lemma follows.

Let f be the density of a convexely derived measure on supported on a k-dimensional convex subset S of $S(X)$. By Lemma 4.4.3 we can conclude that there exists at most one point $z \in S$ at which f achieves its maximum. Indeed suppose the f achieves its maximum in at least two points x_1 and x_2. Since there exists an arc passing through x_1 and x_2 and contained in S, this would contradict Lemma 4.4.3.

Let z be the point of S where f achieves its maximum. We want to give a (uniform) lower bound for $\mu(B(z,\varepsilon))$ where $B(z,\varepsilon)$ is the k-dimensional ball in S of norm-radius ε,

$$B(z,\varepsilon) := \{x \in S_\pi | \quad \|x-z\| \leq \varepsilon\}.$$

Therefore, from now on, the mm-space we are working on is $(S, \mu, \| \quad \|)$.

We define two subsets on S : $A := B(z,\varepsilon)$, $B := S \setminus B(z,2\varepsilon) = B(z,2\varepsilon)^c$ and we are interested in estimating the ratio

$$\frac{\mu_\pi(B)}{\mu_\pi(A)}.$$

We need the following lemma.

Lemma 4.4.4 *Let f be the density of a convexely derived measure supported on a k-dimensional convex subset S of $S(X)$. Assume f achieves its maximum at z. Let $x \in B(z,2\varepsilon)^c = S \setminus B(z,2\varepsilon)$ and consider the arc $\sigma = [z,x]$ in $S(X)$. Then*

$$f(x) \leq (1-2\delta(\varepsilon))^{n-k} Min_{\sigma \cap B(z,\varepsilon)} f.$$

Proof of the Lemma

(Compare [1]). Pick $y \in [x,z] \cap B(z,\varepsilon)$. By weak concavity, we know that f is monotone nondecreasing along $[x,z]$, so

$$f(x) \leq f(y) \leq f(z).$$

So the maximum of f on the subarc $[x,y]\sigma$ is achieved at y. By Lemma 4.4.2,

$$\frac{f^{1/(n-k)}(x) + f^{\frac{1}{n-k}}(y)}{2} \leq (1-\delta(\|x-y\|) Max_{w \in [x,y]} f^{1/(n-k)}(w),$$

which implies

$$f(x) \leq (1-2(\delta(\|x-y\|))^{n-k} f(y).$$

By the triangle inequality, $\|x-y\| \geq \varepsilon$ and we remember that the modulus of convexity is nondecreasing, so

$$\delta(\|x-y\|) \geq \delta(\varepsilon).$$

Hence

$$(1-2\delta(\|x-y\|))^{n-k} \leq (1-2\delta(\varepsilon))^{n-k}.$$

And at last we have

$$f(x) \leq (1-2\delta(\|x-y\|))^{n-k} f(y) \leq (1-2\delta(\varepsilon))^{n-k} f(y).$$

And the proof of the Lemma follows.

We are ready now to integrate both sides of the inequality of Lemma 4.4.4 and give an upper bound for $\frac{\mu(B)}{\mu(A)}$.

Lemma 4.4.5 *Let $\varepsilon > 0$ be given. Let $S \subset S(X)$ be a k-dimensional convex set. Let a convexely derived measure μ be defined on S. Let z be the maximum point for the density function of the measure μ. Let $A := B(z, \varepsilon)$, $B := S \setminus B(z, 2\varepsilon)$. Then*

$$\frac{\mu(B)}{\mu(A)} \leq (1 - 2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{F(k,\varepsilon)}{G(k,\varepsilon)}.$$

$$F(k,\varepsilon) = \int_{\psi_2(\varepsilon)}^{\frac{\pi}{2}} \sin(x)^{k-1}\, dx.$$

and

$$G(k,\varepsilon) = \int_0^{\psi_1(\varepsilon)} \sin(x)^{k-1}\, dx.$$

And where

$$\psi_1(\varepsilon) = 2\arcsin(\frac{\varepsilon}{4\sqrt{k+1}})$$

and

$$\psi_2(\varepsilon) = 2\arcsin(\frac{\varepsilon}{2\sqrt{k+1}})$$

Proof of the Lemma

Let σ be an arc of $S(X)$ emanating from z. Denote by

$$m = Min \underset{\sigma \cap B(z,\varepsilon)}{f}.$$

Then

$$x \in \sigma \cap B(z, 2\varepsilon)^c \Rightarrow f(x) \leq (1 - 2\delta(\varepsilon))^{n-k} m,$$

and

$$y \in \sigma \cap B(z, \varepsilon) \Rightarrow f(y) \geq m.$$

Assume first that the norm $\|\cdot\|$ is Euclidean. We need to convert Euclidean distances into Riemannian distances along the unit sphere, i.e. angles. If x and y are unit vectors making an angle ϕ, then $|x - y| = 2\sin(\phi/2)$. Therefore $|x - y| = \epsilon$ corresponds to an angle ϕ_1 and $|x - y| = 2\epsilon$ corresponds to an angle ϕ_2. Therefore, for a fixed θ, $t \leq \phi_1 \Rightarrow f(t,\theta) \geq m(\theta)$ and $t \geq \phi_2 \Rightarrow f(t,\theta) \leq (1 - 2\delta(\varepsilon))^{n-k} m(\theta)$. Using polar coordinates (t, θ) on the unit sphere, we compute

$$\begin{aligned}\frac{\mu(B)}{\mu(A)} &\leq \frac{\int_{\phi_2}^{\pi}\int_{\mathbb{S}^{k-1}} f(t,\theta)\, \sin(t)^{k-1}\, dt\, d\theta}{\int_0^{\phi_1}\int_{\mathbb{S}^{k-1}} f(t,\theta)\, \sin(t)^{k-1}\, dt\, d\theta} \\ &\leq \max_{\theta \in \mathbb{S}^{k-1}} \frac{\int_{\phi_2}^{\pi} f(t,\theta)\, \sin(t)^{k-1}\, dt}{\int_0^{\phi_1} f(t,\theta)\, \sin(t)^{k-1}\, dt}.\end{aligned}$$

For each θ,

$$\begin{aligned}\frac{\int_{\phi_2}^{\frac{\pi}{2}} f(t,\theta)\,\sin(t)^{k-1}\,dt}{\int_0^{\phi_1} f(t,\theta)\,\sin(t)^{k-1}\,dt} &\leq \frac{\int_{\phi_2}^{\pi}(1-2\delta(\varepsilon))^{n-k}m(\theta)\sin(t)^{k-1}\,dt}{\int_0^{\phi_1} m(\theta)\sin(t)^{k-1}\,dt}\\ &= \frac{\int_{\phi_2}^{\pi}\sin(t)^{k-1}\,dt}{\int_0^{\phi_1}\sin(t)^{k-1}\,dt}(1-2\delta(\varepsilon))^{n-k}.\end{aligned}$$

To handle general norms, we use the fact that the Banach-Mazur distance between any $k+1$-dimensional normed space and Euclidean space is at most $\sqrt{k+1}$. On the affine extension of $co(S)$ there exists a Euclidean structure $|\cdot|$ such that for every $x \in Aff(co(S))$ we have

$$\frac{1}{\sqrt{k+1}}|x| \leq \|x\| \leq |x|.$$

Or equivalently we have

$$B \subset K \subset \sqrt{k+1}B,$$

where B is the Euclidean ball of dimension $k+1$ and K is the uniformly convex ball defined by $S(X)$.

We denote by pr the radial projection of the uniformly convex sphere ∂K to the Euclidean sphere ∂B. Recall that ν is the conical measure on ∂K and we denote by dv_k the conical measure on ∂B, i.e. the Riemannian probability measure. Then the density $h = \dfrac{pr_* d\nu}{dv_k}$ satisfies

$$\frac{1}{\sqrt{k+1}^{k+1}} \leq h \leq \sqrt{k+1}^{k+1}.$$

Let x, $y \in \partial K$, $x' = pr(x)$, $y' = pr(y)$. Since radial projection to the sphere decreases Euclidean distance outside the Euclidean ball,

$$|x'-y'| \leq |x-y| \leq \sqrt{k+1}\|x-y\|.$$

For a general norm, radial projection to the unit sphere is 2-Lipschitz. Indeed, let x'', y'' be points such that $1 \leq \|x''\| \leq \|y''\|$. Rescaling both by $\|x''\|$ decreases $\|x''-y''\|$, so we can assume that $\|x''\| = 1$. Then $\|y''\| \leq 1 + \|x''-y''\|$ and

$$\begin{aligned}\|x'' - \frac{y''}{\|y''\|}\| &= \|\frac{x''}{\|y''\|} - \frac{y''}{\|y''\|} + x(1-\frac{1}{\|y''\|})\|\\ &\leq \|x''-y''\| + \|y''\| - 1 \leq 2\|x''-y''\|.\end{aligned}$$

If $x'' = \sqrt{k+1}x'$ and $y'' = \sqrt{k+1}y'$, then

$$\|x-y\| \leq 2\|x''-y''\| = 2\sqrt{k+1}\|x'-y'\| \leq 2\sqrt{k+1}|x'-y'|.$$

We radially project the set S to a set S' on the sphere. S' is k dimensional and is a convex set as radial projection preserves convexity. We denote the projection of the point z on the sphere by $z' = pr(z)$. In polar coordinates (t,θ) centered at z', fix θ. Let $\psi_1(\theta)$ (resp. $\psi_2(\theta)$) denote the angle t such that $y = pr^{-1}(t,\theta) \in \partial K$ satisfies $\|y-z\| = \varepsilon$ (resp. $= 2\varepsilon$). The above distance estimates yield

$$2\sin\frac{\psi_1(\theta)}{2} \geq \frac{\varepsilon}{2\sqrt{k+1}}$$

and

$$2\sin\frac{\psi_2(\theta)}{2} \geq \frac{\varepsilon}{\sqrt{k+1}}.$$

Then

$$\begin{aligned}\frac{\mu(B)}{\mu(A)} &\leq \frac{\int_{\mathbb{S}^{k-1}}\int_{\psi_2(\theta)}^{\pi} h(t,\theta)f(t,\theta)\,\sin(t)^{k-1}\,dt\,d\theta}{\int_{\mathbb{S}^{k-1}}\int_0^{\psi_1(\theta)} h(t,\theta)f(t,\theta)\,\sin(t)^{k-1}\,dt\,d\theta}\\ &\leq \max_{\theta\in\mathbb{S}^{k-1}} \frac{\int_{\psi_2(\theta)}^{\pi} h(t,\theta)f(t,\theta)\,\sin(t)^{k-1}\,dt}{\int_0^{\psi_1(\theta)} h(t,\theta)f(t,\theta)\,\sin(t)^{k-1}\,dt}.\end{aligned}$$

For each θ,

$$\begin{aligned}\frac{\int_{\psi_2(\theta)}^{\pi} h(t,\theta)f(t,\theta)\,\sin(t)^{k-1}\,dt}{\int_0^{\psi_1(\theta)} h(t,\theta)f(t,\theta)\,\sin(t)^{k-1}\,dt} &\leq \frac{\int_{\psi_2}^{\pi}(1-2\delta(\varepsilon))^{n-k}m(\theta)h(t,\theta)\sin(t)^{k-1}\,dt}{\int_0^{\psi_1} m(\theta)h(t,\theta)\sin(t)^{k-1}\,dt}\\ &= (1-2\delta(\varepsilon))^{n-k}\frac{\int_{\psi_2}^{\pi} h(t,\theta)\sin(t)^{k-1}\,dt}{\int_0^{\psi_1} h(t,\theta)\sin(t)^{k-1}\,dt}\\ &\leq (1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{\int_{\psi_2}^{\pi}\sin(t)^{k-1}\,dt}{\int_0^{\psi_1}\sin(t)^{k-1}\,dt}.\end{aligned}$$

Replacing ψ_1 and ψ_2 with the above lower bounds yields

$$\begin{aligned}\frac{\mu(B)}{\mu(A)} &\leq (1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{\int_{\psi_2}^{\pi}\sin(t)^{k-1}\,dt}{\int_0^{\psi_1}\sin(t)^{k-1}\,dt}\\ &\leq (1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{F(k,\varepsilon)}{G(k,\varepsilon)}.\end{aligned}$$

And the proof of the Lemma follows.

Lemma 4.4.6 *Let S be a convex set of dimension k in $S(x)$. Let a convexely derived measure μ be defined on S. Let z be the maximum point of the density of the measure μ. For every $\varepsilon > 0$ we have the following estimation*

$$\mu(B(z,\varepsilon) \geq \frac{1}{1+(1-2\delta(\frac{\varepsilon}{2}))^{n-k}(k+1)^{k+1}\frac{F(k,\frac{\varepsilon}{2})}{G(k,\frac{\varepsilon}{2})}}$$

Where the functions F and G are as defined before.

Proof of the Lemma
We use the result of the previous Lemma which tells

$$\frac{\mu(B)}{\mu(A)} \leq (1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{F(k,\varepsilon)}{G(k,\varepsilon)}.$$

We remind that μ is a probability measure and we have

$$\frac{\mu(B(z,2\varepsilon))}{\mu(B(z,2\varepsilon))^c} \geq \frac{\mu(B(z,\varepsilon))}{\mu(B(z,2\varepsilon))^c} \geq \frac{1}{(1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{F(k,\varepsilon)}{G(k,\varepsilon)}}.$$

Hence

$$\mu(B(z,2\varepsilon)) = \frac{\mu(B(z,2\varepsilon))}{\mu(B(z,2\varepsilon))+\mu(B(z,2\varepsilon))^c} \geq \frac{1}{1+(1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{F(k,\varepsilon)}{G(k,\varepsilon)}}$$

And the proof of the Lemma follows.

4.5 Partition of $S(X)$ following Gromov

In this section we follow the ideas used in [8] and [19]. Let $f : S(X) \to \mathbb{R}^k$ be as theorem 1. We want to partition the sphere $S(X)$ by at most k-dimensional convex sets. The continuous map f defines a continuous map $Pr(f)$ on the sphere $\mathbb{S}^n$ which is the radial projection of f on $\mathbb{S}^n$. We would like to use the following slight variant of a theorem announced by Gromov in [8]. The author remarks that the following theorem is not entirely proved in [8] and unfortunately we are not able to give a proof of it. However, if we believe Gromov, then the proof of our Theorem 5 becomes much easier. On the other hand, we will give another method, which will be independent of the following Theorem to finalize the results of this paper.

Theorem 26 (Gromov) *Let $f : S(X) \to \mathbb{R}^k$ be a continuous map. There exists an infinite partition of the sphere by at most k-dimensional convex sets, denoted by Π_∞ and a point $z \in \mathbb{R}^k$ such that for every piece Π_∞, $f^{-1}(z)$ passes through the maximum point of the density of the convexely derived measure defined on that piece.*

We are ready to give a conditional proof of Theorem 5.

Theorem 26 provides an infinite partition of $S(X)$ by at most k-dimensional convex sets and a fiber $f^{-1}(z)$ passing through all the maximum points of the densities of the convexely derived mesure defined on all pieces of the partition. Hence on every piece S, we have

$$\mu_\pi((f^{-1}(z)+\varepsilon)\cap S) \geq \mu_\pi(B(x_\pi,\varepsilon)) \geq w(\varepsilon).$$

And at the end

$$\begin{aligned}\mu(f^{-1}(z)+\varepsilon) &= \int_{\Pi_\infty} \mu_S((f^{-1}(z)+\varepsilon)\cap S)d\pi(S) \\ &= \int_{dimS=k} \mu_S((f^{-1}(z)+\varepsilon)\cap S)d\pi(S) + \int_{dimS<k} \mu_S((f^{-1}(z)+\varepsilon)\cap S)d\pi(S).\end{aligned}$$

The measure of the measurable partition is equal to one. In Proposition 24, we prove that the measure of the set of pieces of partition which has dimension $< k$ on the sphere is equal to zero, radially projecting this on $S(X)$ implies that the measure of the set of pieces of partition of $S(X)$ which has dimension $< k$ is also equal to zero, hence we have

$$\mu(f^{-1}(z)+\varepsilon) \geq w(\varepsilon).$$

Hence the proof of the theorem follows.

4.6 Partition of $S(X)$ following section 3.5

4.6.1 Approximation of General Norms By Smooth Norms

For technical reason imposed by Lemma 3.5.2, we need to approximate general norms by smooth norms. Indeed as we will see in the next subsection, we can not allow the convexely derived measures charging any mass for the boundary of balls. In this subsection we show by approximation that we can in fact exclude this technical problem.

Lemma 4.6.1 *Let X denote a finite dimensional space equipped with a C^2-smooth norm. Let $S(X)$ denote its unit sphere. Fix an auxiliary Euclidean structure. There exists K such that for every 2-plane Π passing through the origin, $S(X)\cap P$ is a disjoint union of curves whose curvatures κ satisfy $|\kappa| \leq K$ at all points.*

Proof.

Since the norm is homogeneous of degree 1, its derivative along a line passing through the origin does not vanish. It follows that at every point $x \in S(X)$, the restriction of the differential to P does not vanish identically, i.e. P is transverse to the tangent hyperplane $T_x S(X)$. This shows that $S(X) \cap P$ is a C^2-smooth 1-dimensional submanifold, i.e. a finite disjoint union of curves. Furthermore, the curvature $\kappa(x, P)$ of $S(X) \cap P$ at x is a continuous function of $(x, P) \in I = \{(x, P) \,|\, x \in \partial B(0,1),\, x \in P\}$. Since I is compact, κ is bounded.

Notation 4.6.2 *The* Hessian *of a C^2-smooth function $f : \mathbb{R}^d \to \mathbb{R}$ at x is the quadratic form*

$$Hess_x(v) = \frac{\partial^2}{\partial t^2} f(x + tv)_{|t=0}.$$

Say a C^2-smooth norm on a finite dimensional vectorspace is strongly convex *if at every nonzero point, the Hessian of $x \mapsto \| x \|^2$ is positive definite.*

Proposition 27 *Let X denote a finite dimensional space equipped with a C^2-smooth strongly convex norm. Let $S(X)$ denote its unit sphere. There exists $r_0 > 0$ such that, for every $r < r_0$, for every 2-plane P passing through the origin, for every $x \in S(X)$, $S(X) \cap P \cap \partial B(x, r)$ is a finite set.*

Proof.

The map $x \mapsto Hess_x \| \cdot \|^2$ is homogeneous of degree 0. Fix an auxiliary Euclidean inner product on X. By compactness of the unit sphere, there exists a positive constant c such that for all $x \neq 0$ and all v,

$$(Hess_x \| \cdot \|^2)(v, v) \geq c\, v \cdot v. \tag{4.2}$$

Also, the differential $x \mapsto D_x \| \cdot \|^2$ is homogeneous of degree 1. Therefore there exists a positive constant C such that for all $x \neq 0$ and all v,

$$|(D_x \| \cdot \|^2)(v)| \leq C \, \| x \| \sqrt{v \cdot v}. \tag{4.3}$$

Fix $x \in X$. Let P be a 2-plane. Let f denote the restriction of $z \mapsto \| z - x \|^2$ to P. It satisfies the previous two inequalities. Let $s \mapsto \gamma(s)$ be a C^2-smooth curve in P parametrized by arclength, $z = \gamma(0)$, $\tau = \gamma'(0)$. Then

$$\gamma(s) = z + s\tau + \frac{s^2}{2}\gamma''(0) + o(s^2).$$

Since, for all small v,

$$f(z+v) = f(z) + D_z f(v) + \frac{1}{2} Hess_x f(v,v) + o(v \cdot v),$$

$$f(\gamma(s)) = f(z) + D_z f(s\tau + \frac{s^2}{2}\gamma''(0)) + \frac{1}{2} Hess_z f(\tau,\tau) + o(s^2).$$

Now assume that $f(\gamma(s_j)) = f(z)$ for a sequence s_j that tends to 0. Then, comparing asymptotic expansions gives

$$D_z f(\tau) = 0, \quad D_z f(\gamma''(0)) + Hess_z f(\tau,\tau).$$

Since $\tau \cdot \tau = 1$, inequalities (4.2) and (4.3) give

$$c \leq -D_z f(\gamma''(0)) \leq C \parallel z - x \parallel \sqrt{\gamma''(0) \cdot \gamma''(0)}.$$

This shows that the curvature κ of the plane curve at γ at z satisfies

$$\kappa(z) \geq \frac{c}{C \parallel z - x \parallel}.$$

Therefore, if z is an accumulation point of $\gamma \cap P \cap \partial B(x,r)$, the curvature of γ ay z is $\geq \frac{c}{Cr}$. With Lemma 4.6.1, we conclude that if $r < r_0 := c/CK$, for all P, $S(X) \cap P \cap \partial B(x,r)$ has only isolated points, thus is finite.

Lemma 4.6.3 *Let X_1 be a finite dimensional normed space. Let $S(X_1)$ denote its unit sphere. For every $\lambda > 1$, there exists a C^2-smooth strongly convex norm on X_1, with unit sphere $S(X_2)$, such that the radial projection $S(X_1) \to S(X_2)$ is λ-biLipschitz.*

Proof.

Fix an auxiliary Euclidean inner product on X_1. Fix a smooth compactly supported nonnegative function $\psi : X \to \mathbb{R}_+$ such that $\int \psi = 1$. The convolution

$$f(x) = \int_{X_1} \parallel y \parallel_1 \psi(x-y)\, dy = \int_{X_1} \parallel x - y \parallel_1 \psi(y)\, dy$$

is smooth and convex. For all $x \in X_1$,

$$|f(x) - \parallel x \parallel_1 | \leq \int_{X_1} \parallel y \parallel_1 \psi(y)\, dy$$

is uniformly bounded. Therefore, when one restricts f to a large Euclidean sphere and extends it to become positively homogeneous of degree 1, one gets a smooth norm $\| \cdot \|'$ uniformly close to $\| \cdot \|_1$. By convexity, the Hessian of $\| \cdot \|'^2$ is nonnegative. For $\delta > 0$, let

$$\| v \|_\delta = \sqrt{\| v \|'^2 + \delta\, v \cdot v}.$$

This is a smooth norm, and $Hess(\| v \|_\delta^2) \geq \delta\, v \cdot v$ is positive definite. For δ small enough, this norm is close to $\| \cdot \|_1$, therefore radial projection between unit spheres is λ-biLipschitz.

Lemma 4.6.3 allows to reduce the proof of Theorem 5 to the special case of C^2-smooth strongly convex norms, for which we know, from Proposition 27, that convexely derived measures do not give any mass to small enough spheres. Until the end of section 4.6.2, we suppose the norm of class C^2 and strongly convex.

4.6.2 Infinite Partitions

This section follows closely the *Infinite partitions* section of chapter 3. We invite the reader to consult that chapter for the necessary definitions and notations. Basically, nearly all the Lemmas and Propositions can be applied here by just radially projecting the round sphere $\mathbb{S}^n$ to $S(X)$. However, there will be some minor changes which we will attempt to explain here. We keep the same notation as there, knowing that the space of convexely derived measures are defined on $S(X)$ in the same manner than it is defined on $\mathbb{S}^n$.

For every $\nu \in \mathcal{MC}$, there exists a lower bound and an upper bound for $\nu(B(x,r)$, depending on r, which tends to zero. This is obtained from Corollary 3.5.3 and Lemma 3.5.6 by radial projection.

The next Lemma is proved independently from the previous chapter of this Thesis. One should note the importance of the proposition 27 for the proof of the next Lemma.

Lemma 4.6.4 *Let $\rho > 0$. Let $\mathcal{K}$ be a compact set of probability measures on $S(X)$ with the following property : for every $\nu \in \mathcal{K}$, all x and all $r < \rho$, $\nu(\partial B(x,r)) = 0$. Then the function $(\nu, x, r) \mapsto \nu(B(x,r))$ is uniformly continuous on $\mathcal{K} \times S(X) \times (0,\rho)$. It follows that it is continuous on $\mathcal{MC}^+ \times S(X) \times [0,\rho)$.*

Proof.

Let $(\nu_i, x_i, r_i) \to (\nu, x, r)$. Let $\{x_i'\}$ (resp x') be the sequence of points (resp the point) on $\mathbb{S}^n$ image of radial projection of the sequence $\{x_i\}$ (resp x). Let $\phi_i \in Iso(\mathbb{R}^{n+1})$ be such that $\lim_{i\to\infty} \phi_i = Id$ and for every i, $\phi_i(x_i') = x'$. Such a sequence of isometry acts on

$S(X)$ by taking the action on $\mathbb{S}^n$ and projecting to $S(X)$. For every $\delta > 0$, for big enough i we have

$$B(x, r-\delta) \subset \phi_i(B(x_i, r_i)) \subset B(x, r+\delta).$$

This implies

$$\nu_i(\phi_i^{-1}(B(x, r-\delta))) < \nu_i(B(x_i, r_i)) < \nu_i(\phi_i^{-1}(B(x, r+\delta))).$$

Hence

$$\limsup \nu_i(B(x_i, r_i)) < \lim_{i\to\infty} \phi_{i*}\nu_i(B(x, r+\delta)) = \nu(B(x, r+\delta))$$
$$\liminf \nu_i(B(x_i, r_i)) > \lim_{i\to\infty} \phi_{i*}\nu_i(B(x, r-\delta)) = \nu(B(x, r-\delta)).$$

Let $\delta \to 0$. As we supposed the norm being smooth, we know that the $\nu(\partial B(x, r)) = 0$. We can apply the Lemma 3.5.2 and deduce that $\lim_{\delta\to 0} \nu(B(x, r+\delta)) = \nu(B(x, r))$. We can apply the Lemma 3.5.6 and the continuity on $\mathcal{MC}^+ \times S(X) \times [0, \rho)$ is deduced.

By the same arguments as in the previous chapter, more specifically

- Theorem 21 of chapter 3,
- semi-continuity of $\nu \to M_r(\nu)$,
- compactness of $\mathcal{CP}^{\leq k}$,

we show that for all $r > 0$, there exists a $z \in \mathbb{R}^k$ and an infinite partition of $S(X)$ into convex pieces ν of at most dimension k such that for every piece ν of the partition, conv. hull$(M_r(\nu))$ intersects $f^{-1}(z)$.

By following again the same arguments as in chapter 3 for the estimation on the density functions of convexely derived measures this time,

- a bound on the density functions when the measure of the support is not too small.
- a bound on the Lipschitz constant of the density which is deduced from some concavity properties deduced in Lemma 3.5.16,

we obtain (by taking some sub-sequence) the convergence of the density functions, hence the convergence of the sets $M_r(\nu)$ when r tends to zero and when ν converges to a measure of the same dimension.

Proposition 28 *Assume $f : S(X) \to \mathbb{R}^k$ is a generic smooth map. Let r_i tend to 0 and let $\Pi_i \in \mathcal{CP}^{\leq k} \cap \mathcal{F}_{r_i}$ be convex partitions of dimension $\leq k$, r_i-adapted to f. Then, for all $\varepsilon > 0$,*

$$\max_{z\in\mathbb{R}^k} \mu(f^{-1}(z) + \varepsilon) \geq w(\varepsilon) \limsup_{i\to\infty} \Pi_i(\mathcal{MC}^k).$$

Where

$$w(\varepsilon) = \frac{1}{1 + (1 - 2\delta(\frac{\varepsilon}{2}))^{n-k}(k+1)^{k+1}\frac{F(k,\frac{\varepsilon}{2})}{G(k,\frac{\varepsilon}{2})}}.$$

And where the functions $F(.,.)$ and $G(.,.)$ were defined previously.

The proof of this proposition is exactly the same as in the case of the round sphere.

Proposition 29 *There exists a constant $c = c(n) > 0$ such that if $f : S(X) \to \mathbb{R}^k$ is smooth and generic and Π belongs to $\mathcal{F}_r \cap \mathcal{CP}^{\leq k}$ for some small enough $r > 0$, then,*

$$\max_{z \in \mathbb{R}^k} \mu(f^{-1}(z) + \frac{r}{c}) \geq c \sum_{\ell=0}^{k} w_l(r)\Pi(\mathcal{MC}^\ell).$$

Where $w_l(r)$ is equal to $w(r)$ in codimension l.

The proof is (again) the same as in the case of the round sphere.
The next Lemma and Proposition 29 prove the main theorem 5.

Lemma 4.6.5 *For every $l < k$, we have*

$$\lim_{r \to 0} w_l(r)/w_k(r) \backsim \lim_{r \to 0} r^{l-k} = \infty$$

Proof.
For every $m \in \mathbb{N}$, $\lim_{r \to 0} G(m,r) \to 0$, and $\lim_{r \to 0} F(m,r) = 1$. Furthermore

$$\frac{w_l(r)}{w_k(r)} = \frac{1 + (1 - 2\delta(r/2))^{n-k}\frac{F(k,r/2)}{G(k,r/2)}(k+1)^{k+1}}{1 + (1 - 2\delta(r/2))^{n-l}\frac{F(l,r/2)}{G(l,r/2)}(l+1)^{l+1}} \backsim_{r \to 0} C\frac{G(l,r)}{G(k,r)}.$$

And by the well known asymptotic behavior of the function $G(m,r)$ we have

$$\frac{G(l,r)}{G(k,r)} \backsim_{r \to 0} r^{l-k}.$$

Hence the proof of the Lemma follows.

4.7 Why all these complications ?

Remember the waist of the sphere theorem of Gromov which we proved in the previous chapter. Several times during the last sections, we used the radial projection between the canonical sphere and the unit sphere $S(X)$. One could ask why bothering with all we did and not just radially projecting the result of theorem 18 on $S(X)$. Indeed, this gives another lower bound for the waist of $S(X)$ as we will show in the next

Proposition 30 *Let X be a uniformly convex normed space of finite dimension $n+1$. Let $S(X)$ be the unit sphere of X, for which the distance is induced from the norm of X. The measure defined on $S(X)$ is the conical probability measure. So a lower bound for the waist of $S(X)$ relative to $\mathbb{R}^k$ is given by*

$$w_2(\varepsilon) = (n+1)^{-n-1}\frac{vol(\mathbb{S}^{n-k} + \frac{\varepsilon}{n+1})}{vol(\mathbb{S}^n)}.$$

Proof of the Proposition

Let pr be the radial projection of $\mathbb{S}^n$ to $S(X)$. We apply theorem 18 to the map $g = pr^{-1} \circ f$. Hence there exists a fiber X such that for every $\varepsilon > 0$

$$vol(X+\varepsilon) \geq vol(\mathbb{S}^{n-k} + \varepsilon)$$

We radially project $X+\varepsilon$ to $S(X)$. We have

$$pr(X+\varepsilon) \subset pr(X) + (n+1)\varepsilon$$

Hence

$$\begin{aligned}
\mu(pr(X)+\varepsilon) &\geq \mu(pr(X + \frac{\varepsilon}{n+1})) \\
&\geq (n+1)^{-n-1}\frac{vol(X+\frac{\varepsilon}{n+1})}{vol(\mathbb{S}^n)} \\
&\geq (n+1)^{-n-1}\frac{vol(\mathbb{S}^{n-k}+\frac{\varepsilon}{n+1})}{vol(\mathbb{S}^n)}.
\end{aligned}$$

And the Proposition is proved.

We see that a brutal application of Gromov's theorem gives a lower bound for the waist of the unit sphere of a uniformly convex normed space, $S(X)$. But comparing $w_1(\varepsilon)$ and $w_2(\varepsilon)$, we can see that the lower bound $w_1(\varepsilon)$ has a much better dependence on the variable n, even if the dependence on the variable k is very bad.

For example, if k is fixed and n tends to infinity, $w_2(\varepsilon)$ tends (exponentially fast) to 0 while for this case, the lower bound $w_1(\varepsilon)$ tends to 1. One can hope to have a better dependence on the variable k by knowing the best degree of dilation of the radially projection of $\mathbb{S}^n \to S(X)$. Here we gave a trivial bound for the degree of dilation, not taking into account uniform convexity.

4.7.1 Comparison with Gromov-Milman's isoperimetric inequality

We need to compare the result of theorem 5 for $k = 1$ with Gromov-Milman's isoperimetric inequality which we remind here. This Theorem was proved first by Gromov-Millman in [9] where the proof was completed later on by S. Alesker in [1] (S. Sodin had the kindness of referring Alesker's paper to the author). There is a very short and easy proof of this Theorem given by J. Arias-de-Reyna, K. Ball and R. Villa in [6].

Theorem 31 *Let $S(X)$ be a uniformly convex unit sphere with modulus δ. For every Borel set $A \subset S(X)$ such that $\mu(A) \geq \frac{1}{2}$ and for every $\varepsilon > 0$ we have*

$$\mu(A + \varepsilon) \geq 1 - e^{-a(\varepsilon)n},$$

where $a(\varepsilon) = \delta(\frac{\varepsilon}{8} - \theta_n)$ and where $\theta_n = 1 - (\frac{1}{2})^{1/(n-1)}$.

Our theorem 5, in case $k = 1$, implies a similar isoperimetric inequality. This requires Proposition 3 of chapter 2, which is not optimal for small ε and fixed n, so comparison with Theorem 31 is hopeless.

On the other hand, let ε be fixed and let $n \to \infty$. In this regime, our main theorem 5 combined with Proposition 3 yields

$$\max\{\mu(A + \varepsilon), \mu(A^c + \varepsilon)\} \geq 1 - e^{-b(\varepsilon)n - c(\varepsilon)},$$

where $b(\varepsilon) = 2\delta(\frac{\varepsilon}{2})$ and $c(\varepsilon)$ has an ugly expression. Since, $b > a$, our theorem 5 gives a better estimate.

4.8 Waist of L^p Unit Spheres

The best examples covered by Theorem 5 are provided by the waist of unit spheres of L^p spaces where $1 < p < \infty$. As in these cases it is well known that L^p spaces are uniformly convex Banach spaces. For the main Theorem of this chapter be useful, it is

not hurtful to indicate some words about the waist of the unit spheres of the L^p spaces. First we remind that the modulus of convexity of the L^p spaces were found by O.Hanner in [11]. We remind the following

Theorem 32 (O.Hanner) *Let x and y be two elements of L^p or l^p where $\|x\| = 1$, $\|y\| = 1$ and $\|x-y\| \geq \varepsilon$ and $0 < \varepsilon < 2$. Then $\|\frac{x+y}{2}\| \leq 1-\delta(\varepsilon)$, where $\delta(\varepsilon)$ is determined in the following way :*

- *when $1 < p < 2$*

$$(1-\delta+\frac{\varepsilon}{2})^p + |1-\delta-\frac{\varepsilon}{2}|^p = 2,$$

- *when $p \geq 2$*

$$\delta = 1-(1-(\frac{\varepsilon}{2})^p)^{\frac{1}{p}}.$$

For every ε, we can chose x and y such that $\|\frac{x+y}{2}\| = 1-\delta(\varepsilon)$.

Here for the sake of simpliness, we suppose $p \geq 2$. Then we can immediately give a lower bound for the waist as it is shown in the following

Theorem 33 *A lower bound for the waist of the unit spheres of the L^p spaces where $p \geq 2$ and where $0 < \varepsilon < 4$ is given by*

$$w(\varepsilon) = \frac{1}{1+(1-2(1-(1-(\frac{\varepsilon}{4})^p)^{\frac{1}{p}})^{n-k}(k+1)^{k+1}\frac{F(k,\frac{\varepsilon}{2})}{G(k,\frac{\varepsilon}{2})}}.$$

4.9 Trivial Lower Bound for the Waist of the Cubes and L^∞ Unit Balls

In this section, we use the invariant property of the waist to give some trivial bounds for the waist of the Euclidean cube and the L^∞ unit spheres. We use the trivial proposition 4 of the introduction.

Definition 4.9.1 (The moment map) *Let $f : \mathbb{R}^n \to]0,1[^n$ be the diffeomorphism such that $df_(x_1, \cdots, x_n) = [e^{-\pi x_1^2}, \cdots, e^{-\pi x_n^2}]$. Then f is called the moment map and it sends n-dimensional Gaussian space to the open Euclidean cube. One can trivially see that f is 1-Lipschitz.*

Theorem 34 *Let I be the (open) n-dimensional Euclidean cube. Then for every $k \leq n$ we have*

$$wst(I \to \mathbb{R}^k, \varepsilon) \geq wst(Ga^n \to \mathbb{R}^k, \varepsilon) = \int_0^{\varepsilon} (x^{k-1}e^{-x^2/2})dx / \int_0^{\infty} (x^{k-1}e^{-x^2/2})dx.$$

Proof

From proposition 4 and the waist of Gaussian spaces (Theorem 4), the proof of the Theorem is straightforward.

4.9.1 Asymptotic Behavior of Theorem 34

As one can remark, the result for the lower bound on the k-waist of the n-dimensional cube only depends on k. This can give a good asymptotic result for the k-waist of the infinite dimensional cube.

Theorem 35 *Let I^∞ be the infinite dimensional cube. Then*

$$wst(I^\infty \to \mathbb{R}^k, \varepsilon) \geq \int_0^{\varepsilon} (x^{k-1}e^{-x^2/2})dx / \int_0^{\infty} (x^{k-1}e^{-x^2/2})dx.$$

Proof

By Theorem 34 and by tending $n \to \infty$ we get the required result.

4.9.2 Waist of L^∞ Balls

Let $(\mathbb{R}^n, \|.\|_\infty, \mu_\infty^n)$ be the n-dimensional L^∞ Banach space. The measure μ_∞^n is the Haar measure defined such that

$$\mu_\infty^n(B_\infty(0,1)) = Vol_n(B_2(0,1)),$$

where $B_2(0,1)$ is the Euclidean unit ball.

Let $B_\infty = (B_\infty(0,1), \|, \|, \frac{1}{Vol_n(B_2(0,1))}\mu_\infty^n)$ be the pm-space of the unit ball of the n-dimensional L^∞ Banach space. The goal of this section is to give a lower bound for the waist of B_∞.

There exists an isomorphism between B^∞ and the n-dimensional Euclidean cube. We denote such isomorphism by $i : I^n \to B^\infty$. In the other hand we know that for every $x \in B^\infty$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty.$$

This shows that the isomorphism i is a $\frac{1}{\sqrt{n}}$-Lipschitz mapping. We may apply the results of the previous section and prove the following

Theorem 36 *A lower bound for the waist of the unit ball of the L^∞ Banach space is given by*

$$\begin{aligned} wst(B^\infty \to \mathbb{R}^k, \varepsilon) &\geq wst(I^n \to \mathbb{R}^k, \frac{1}{\sqrt{n}}\varepsilon) \\ &\geq \int_0^{\frac{1}{\sqrt{n}}\varepsilon} (x^{k-1}e^{-x^2/2})dx / \int_0^\infty (x^{k-1}e^{-x^2/2})dx. \end{aligned}$$

Chapitre 5

Measure of topological complexity of minimal graphs

5.1 Minimal graphs

Definition 5.1.1 *Let M be a Riemannian manifold and let A be a finite subset of M. A minimal graph with attaching points A is a finite embedded graph G in M such that the following conditions are satisfied :*

I. Each edge of the graph is a geodesic segment

II. Every $a \in A$ is a vertex of degree 1.

III. The sum of unit vectors of edges outcoming from each vertex of degree greater than 1 is equal to zero.

Minimal graphs are critical points for the length functional on the space of embedded graphs with fixed attaching points. In this paper we are not interested in the length of graphs. The purpose of our study here is the combinatorial structure of minimal graphs.

There exist minimal graphs with empty attaching set. For example on the canonical two sphere we can have a 3-regular minimal graph with only two vertices. This cannot exist in the plane. For more details about minimal graphs see [17].

In this paper, we will consider only the classes of 3 or 4-regular minimal graphs. We are concerned with estimates on the number of vertices of such graphs with n attaching points for a given number n. Using the term "regular" graph is abusive as all our graphs will have some vertices of degree 1 (attaching points), so in our use, a regular graph is everywhere regular except at the attaching points.

Notation 5.1.1 *Let $n \geq 2$, we denote by $f_3(n)$ (resp. $f_4(n)$) the supremum of the numbers of vertices of 3 (resp. 4)-regular minimal graphs on the plane attached to n points.*

Here are the two main theorems of this paper.

Theorem 37 *The maximal number of vertices of a 3-regular minimal graph on the plane with n attaching points, $f_3(n)$ satisfies the following equalities*

- *if $n = 6k$,*
$$f_3(n) = 6k^2 + 6k. \tag{5.1}$$
- *if $n = 6k + 1$,*
$$f_3(n) = 6k^2 + 8k. \tag{5.2}$$
- *if $n = 6k + 2$,*
$$f_3(n) = 6k^2 + 10k + 2. \tag{5.3}$$
- *if $n = 6k + 3$,*
$$f_3(n) = 6k^2 + 12k + 4. \tag{5.4}$$
- *if $n = 6k + 4$,*
$$f_3(n) = 6k^2 + 14k + 6. \tag{5.5}$$
- *if $n = 6k + 5$,*
$$f_3(n) = 6k^2 + 16k + 8. \tag{5.6}$$

Theorem 37 is sharp. Furthermore, the combinatorial structure of minimal graphs which maximize the number of vertices is unique and will be described in Definition 5.3.3.

Theorem 38 *Let G be a 4-regular minimal graph on the plane with n attaching points. Then G has at most $\binom{n/2}{2} + n$ vertices. In other words*

$$f_4(n) = \binom{n/2}{2} + n \qquad \text{if } n \text{ is even}$$

$$f_4(n) = 0 \qquad \text{otherwise.}$$

This is sharp. For each n, there is a minimal 4-regular graph which achieves this bound.

Theorem 37 (resp. 38) is proven in sections 2 and 3 (resp. 4). Both proofs are elementary.

As paradoxal as it can be, for a variant of this problem, where graphs are allowed to bear densities, there is an opposite result : no bound on the number of vertices. We will give more details in Section 5, together with some open questions.

5.2 3-regular minimal graphs

In this section we prove Theorem 37, i.e. an upper bound on the number of vertices of 3-regular minimal graphs with n attaching points.

5.2.1 Preliminaries

Let us give a simple example which illustrates definitions.

Example 5.2.1 *The minimum number of attaching points of such a graph is* 2 *and for this case we have* $f_3(2) = 2$ *(the graph consists of only one edge connecting the two attaching points). Things get more interesting for* $n = 3$ *because in this case we know that there exists some configuration of* 3 *points on the plane and a* 3*-regular minimal tree with only one vertex of degree* 3 *attached to these points. And so* $f_3(3) \geq 4$*, but can we actually find a minimal graph attached to* 3 *points with more than* 4 *vertices ?*

The answer of this question turns out to be negative but we will have to wait a little before proving this statement.

Let G be a 3 regular minimal graph with n attaching points. By the definition 1.1 we know that the angle between the edges directed from every vertex is equal to 120 degree. This is actually the only geometric restriction put on the graph which will make the estimation of the function $f_3(n)$ easy.

Notation 5.2.1 *For every graph* G *we denote by* $\sharp G$ *the number of vertices of* G*.*

Lemma 5.2.2 $f_3(n)$ *is a increasing function of* n*.*

Proof of Lemma 5.2.2.
Let G be a 3-regular graph attached to n points. We choose one attaching point x. As we supposed that all attaching points have degree equal to 1 then there is only one edge e directed from x. We add two edges directed from x in a way that the angles between each of them and e are equal to 120 degrees. The new graph G' is 3-regular and minimal with $n+1$ attaching points, and $\sharp G' = \sharp G + 1$. This completes the proof of the lemma.

Definition 5.2.1 (Cycle, Interior) *A* cycle C *in* G *is a* 2*-regular subgraph of* G*. As a cycle is a simple closed curve in the plane, it separates the plane into two component. The bounded one is called the* interior *of* C*, and denoted by* $\mathrm{Int}(C)$*.*

Definition 5.2.2 (Ingoing and Outgoing vertices and edges) *Let* C *be a cycle in* G*.*

- *A vertex v of C is called* ingoing *if the one edge e at v which does not belong to C is contained in the interior of C. e is called an* ingoing *edge.*
- *Otherwise we call the vertex v and the edge e* outgoing.

We denote by V_{out}^C the number of outgoing vertices of the cycle C, and by V_{in}^C the number of ingoing vertices of the cycle C.

We define a partial order on the set of cycles.

Definition 5.2.3 (Maximal cycles) *Let C and C' be two cycles. We define $C \leq C'$ if* $\mathrm{Int}(C) \subseteq \mathrm{Int}(C')$. *We call a cycle C of G* maximal *if C is maximal for this partial order.*

Lemma 5.2.3 *Let C be a maximal cycle. For every outgoing vertex v of C the outgoing edge attached to v does not belong to any cycle.*

Proof of Lemma 5.2.3.
By contradiction. Assume there is an outgoing edge e of C which belongs to a cycle C'. The cycles C and C' will have some edges in common. The outer boundary component of $\mathrm{Int}(C) \cup \mathrm{Int}(C')$ is a cycle. This new cycle contradicts the maximality of C.

Lemma 5.2.4 *If a graph does not have any maximal cycle then this graph is a tree.*

Proof of the Lemma.
If the graph does not have any cycle then there is nothing to prove, so we assume that the graph has some cycles. In this case the set of cycles is non empty so it most have a maximal element for the partial order of definition 5.2.3. And the proof follows.

Lemma 5.2.5

$$V_{out}^C - V_{in}^C = 6. \tag{5.7}$$

Proof of Lemma 5.7.

This is a simple consequence of the Gauss-Bonnet theorem. Let us walk along C, keeping the interior of C on our left hand side. The cycle C consist of finitely many line segments with exterior angles equal to $+60$ degrees at outgoing vertices and -60 degrees at ingoing vertices. So

$$60V_{in}^C - 60V_{out}^C = 360.$$

Corollary 5.2.6 *Let G be a minimal graph which is not a tree. Then G has at least 6 attaching points.*

Proof of the Corollary.
By assumption, G contains a maximal cycle C. Let $G' = G$ with the edges of C removed. Then no two outgoing vertices of C can be connected in G'. Otherwise, let v_1, v_2 be outgoing vertices of C connected in G' by an arc σ of minimal length (over all paths and all pairs v_1, v_2). Then $\sigma \cap C = \{v_1, v_2\}$. Let δ be one of the arcs of C from v_1 to v_2. Then $\sigma \cup \delta$ is a cycle, contradicting maximality of C.
Each connected component of G' sitting outside C is minimal with attaching points consisting of a subset of attaching points of G and exactly one outgoing vertex of C. It must have at least 2 attaching points. Therefore

$$\sharp att(G) \geq V_{out}^C = 6 + V_{in}^C \geq 6.$$

where $\sharp att(G)$ denotes the number of attaching points of G.

Example 5.2.2 *In example 5.2.1, we showed that $f_3(3) \geq 4$. It is only here, at this stage, that we can give the exact value of $f_3(3) = 4$.*

Indeed, we saw in the previous two lemmas that a graph with 3 attaching points must be a tree, and then it will have 4 vertices.

Beside this example, for minimal graphs having 4 and 5 attaching points, the optimal minimal graphs maximising the number of vertices have to be trees, hence $f_3(4) = 6$ and $f_3(5) = 8$. Morever, the isomorphism classes of such trees are unique. Hence for $n \leq 5$ we have a classification of 3-regular minimal graphs maximising the number of vertices.

Lemma 5.2.7 *Let F be a disjoint union of k 3-regular trees attached at a total of n points (i.e. F has n vertices of degree 1, all others have degree 3). Then $\sharp F = 2n - 2k$.*

Proof.
For each component T of F with n_T attaching points, $\sharp T = 2n_T - 2$. Summing over all components yields $\sharp F = 2n - 2k$.

5.2.2 Three types of graphs

Now, let us come back to the estimation of the function f_3. There are 3 possibilities for a graph G :

I. G is a tree.

II. G has one and only one maximal cycle.

III. G has more than one maximal cycle.

Accordingly the proof of Theorem 37 splits into 3 cases, covered in the following 3 lemmas.

Lemma 5.2.8 *Let G be a minimal graph attached on n points. If G is a tree, then $\sharp G \leq 2n-2$.*

Proof.
The number of vertices of the binary tree attached to n points is equal to $2n-2$ (n attaching points plus $n-2$ vertices of degree equal 3). Therefore $\sharp G \leq 2n-2$.

Lemma 5.2.9 *Let G be a minimal graph attached on n points. If G has only one maximal cycle then $\sharp G \leq f_3(n-6)+2n$. If equality holds, then either $n=6$ and G is the union of a 6-cycle and its 6 outgoing edges, or $n>6$ and G is obtained from a minimal graph G'' attached at $n-6$ points by completing its $n-6$ attaching points into a $2n-6$-cycle and adding the n outgoing edges of this cycle.*

Proof.
G has exactly 1 maximal cycle C. Let m' denote the number of outgoing vertices of the cycle. From Lemma 5.7, the number of ingoing vertices is equal to $m'-6$. Let n' be the number of attaching points outside the cycle. Of course the number of attaching points inside the cycle is equal to $n-n'$.
If outside the cycle C, there exists some (non maximal) cycle then the set of cycles outside C must have a maximal element, hence a maximal cycle which is disjoint from the cycle C. This contradicts the assumption. Hence outside C the graph is a forest (a disjoint union of trees).
Now let us remove all the edges of C, and consider vertices of C as attaching points for the remaining graph G'. G' consists of a graph G'' whose edges were inside C and of a collection F of trees whose edges were outside C. F has $m'+n'$ attaching points.
Each outgoing vertex of C is the root of one of the trees of the forest F. Thus the number of components of F is m'. From Lemma 5.2.7, $\sharp F = 2(m'+n') - 2m' = 2n'$.
Each of these trees has at leat one attaching point, apart from its root, thus $n' \geq m'$.
G'' has at most $n-n'+m'-6$ attaching points. By the definition of the function $f_3(n)$ we know that the number of the vertices of G'' is at most $f_3(n+m'-n'-6)$. Then

$$\sharp G \leq f_3(n+m'-n'-6)+2n'.$$

Thus there exists a $k \leq n-6$ such that

$$\sharp G \leq f_3(k)+2n'.$$

On the other hand, we showed that $f_3(n)$ is nondecreasing. We conclude that

$$\sharp G \leq f_3(n-6)+2n. \tag{5.8}$$

Equality implies that $n = n' = m'$ and that F is a disjoint union of n edges. Thus F consists of the outgoing edges of C. If G'' is nonempty, the attaching points of G'' are the ingoing vertices of C. Otherwise, C has no ingoing verices, this means that C is a 6-cycle. This completes the proof of Lemma 5.2.9.

Lemma 5.2.10 *Let G be a minimal graph attached on n points. If G has more than one maximal cycle, then $\exists n'$ $6 \leq n' \leq n-4$ such that*

$$\sharp G \leq f_3(n') + f_3(n - n' + 2).$$

Proof. We shall use the following terminology.

Definition 5.2.4 *Let C and D be two maximal cycles in a minimal 3-regular graph. A connecting set for C and D is a triple (v_C, v_D, σ) such that v_C (resp. v_D) is an outgoing vertex of C (resp. D) and σ a path which joins these vertices outside* $\mathrm{C} \cup \mathrm{D}$.

Let C and D be two maximal cycles. Let σ be a path connecting C to D with a minimum number of edges. Let e be the first edge traversed by σ. Then e disconnects C from D. Indeed, otherwise, there would exist a path connecting C to D away from e. Let γ be the shortest path in $G \setminus \{e\}$ joining the endpoints of e. Then $\gamma \cup e$ is a cycle touching C and thus contradicting maximality of C. So cutting the edge e will disconnect C from D.
Let $e' \subset e$ be a proper interval. Let G'(resp. G'') be the connected component of $G \setminus \{e\}$ containing C (resp. D). Let $n' = \sharp att(G')$. Then G'' has $n - n' + 2$ attaching points (the cut through e' produces two extra attaching points). By Lemma 5.7, we conclude that $n' \geq 6$ and $n - n' + 2 \geq 6$.
And so in final for an n' such that $6 \leq n' \leq n-4$, we have

$$\sharp G \leq f_3(n') + f_3(n - n' + 2). \tag{5.9}$$

This completes the proof of Lemma 5.2.10.

5.2.3 Combining three recursion inequations

To sum up, for every minimal graph G with n attaching points, the number of vertices of G satisfies

$$\begin{array}{ll} \text{either} & \sharp G \leq 2n-2, \\ & \sharp G \leq f_3(n-6) + 2n, \\ \text{or} & (\exists\, 6 \leq k \leq n-4) \quad \sharp G \leq f_3(k) + f_3(n-k+2). \end{array}$$

So f_3 satisfies at least one of three recursion inequations.

Lemma 5.2.11 *Let f be a function on integers. Assume that $f(2) = 2$ and, for every $n \geq 3$,*

$$f(n) \leq \max \begin{cases} 2n-2, & \\ f(n-6)+2n & \text{if } n \geq 6, \\ \max\{f(k)+f(n-k+2)\,;\, 6 \leq k \leq n-4\} & \text{if } n \geq 10. \end{cases}$$

Then $f_3(n) \leq \frac{1}{6}n^2 + n$. If equality holds for some n, then $f(n) = f(n-6) + 2n$.

Proof :
We prove this lemma by induction on n.
For $n = 2$ the inequality is verified as we assumed $f(2) = 2$.
Let's suppose that the inequality is verified for every $k \leq n-1$ and we want to prove it for $k = n$.
If $f(n)$ satisfies the first inequation :

$$f(n) \leq 2n - 2 < \frac{1}{6}n^2 + n.$$

If $f(n)$ satisfies the second inequation :

$$\begin{aligned} f(n) &\leq f(n-6) + 2n \\ &= \frac{1}{6}(n-6)^2 + (n-6) + 2n \\ &= \frac{1}{6}n^2 + n. \end{aligned}$$

If $f(n)$ satisfies the last inequation, there exists $6 \leq k \leq n-4$ such that

$$\begin{aligned} f(n) &\leq f(k) + f_3(n-k+2) \leq \frac{1}{6}k^2 + \frac{1}{6}(n-k+2)^2 + n + 2 \\ &= (\frac{1}{6}n^2 + n) + \frac{1}{3}(k^2 - kn + 2n - 2k + 8). \end{aligned}$$

But as $n \geq 10$ and $6 \leq k \leq (n-4)$, we have

$$\frac{1}{3}((k-n)(k-2)+8) \leq \frac{1}{3}(8-n) < 0.$$

And so

$$f(n) \leq \frac{1}{6}n^2 + n.$$

Note that equality can hold only in the second case. This completes the proof of Lemma 5.2.11, and the

Corollary 5.2.12

$$f_3(n) \leq \frac{1}{6}n^2 + n$$

And of Theorem 37 except for the equality case which will be discussed in the next section.

5.3 3-regular minimal graphs which maximise the number of vertices

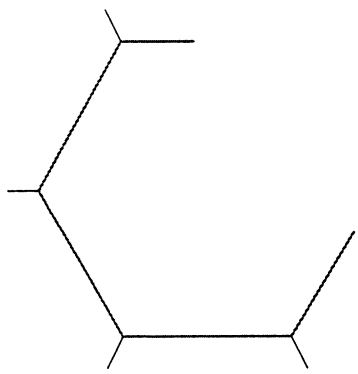
Theorem 1.1 gives an upper bound for the maximal number of vertices of 3-regular minimal graphs. A natural question to ask is how good is this estimate.

In this section we will study the equality case of the inequations of the last section. For each n, we will actually find the combinatorial class of graphs which maximise the number of vertices.

Before characterising these graphs, we need a construction which applies to a class of minimal graphs.

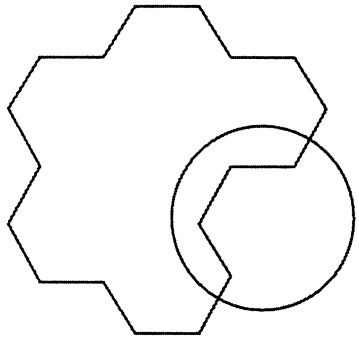
Definition 5.3.1 (Simple minimal graphs) *Let G be a minimal embedded graph in the plane. Say that G is* simple *if*

- *Either G is a tree with the following property : it does not contain paths consisting of 5 edges and turning on the same side (like 5 consecutive edges of a convex hexagon).*



Forbidden configuration in a simple minimal tree.

- *Or G has a unique maximal cycle which surrounds all vertices except attaching points. Furthermore, no two consecutive vertices in the maximal cycle are both ingoing vertices.*



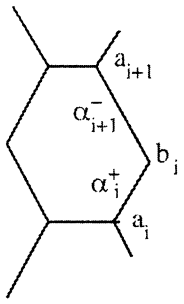
Forbidden configuration in the maximal cycle of a simple minimal graph

The pictures of section 5.6 all feature simple minimal graphs. Note that, on the set of attaching points of a simple minimal graph, there is a natural circular order.

Definition 5.3.2 (Padding of a simple minimal graph) *Let G be a simple minimal embedded graph in the plane. The* padding *of G, denoted by $P(G)$ is the minimal graph obtained as follows.*

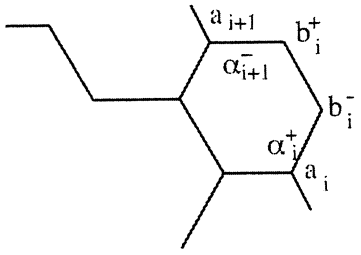
We number the attaching points of G in circular order $a_1, \cdots, a_n$ where $a_{n+1} = a_1$. From each attaching point a_i we draw two half-lines, α_i^+ and α_i^- obtained by turning the edge which connects a_i to G by respectively 120 *and* 240 *degrees. Next, one considers the portion of the cycle between a_i and a_{i+1}, and completes this set of edges into an hexagon having two edges carried by α_i^+ and α_{i+1}^-. For this, one is led to place between 1 and 4 new vertices, depending on the configuration.*

- *If the angle $\angle(\alpha_i^+, \alpha_{i+1}^-) = \frac{-2\pi}{3}$, we cut the half-lines α_i^+ and α_{i+1}^- on their intersection point b_i. We obtain two edges making an angle equal to* 120 *degrees at b_i.*



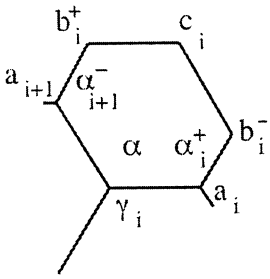
Adding one vertex

- *If the angle $\angle(\alpha_i^+, \alpha_{i+1}^-) = \frac{-\pi}{3}$, we place on α_i^+ (resp α_{i+1}^-), two points b_i^- (resp b_i^+) such that the vector $\overrightarrow{b_i^+ b_i^-}$ makes an angle equal to $\frac{\pi}{3}$ with α_i^+ (i.e $\angle(\alpha_i^+, b_i^+ b_i^-) = \frac{\pi}{3}$ and $\angle(b_i^+ b_i^-, \alpha_{i+1}^-) = \frac{\pi}{3}$).*



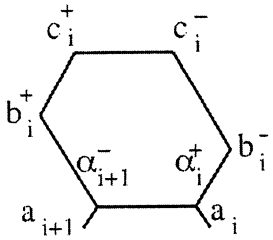
Adding two vertices

- *If the angle* $\angle(\alpha_i^+, \alpha_{i+1}^-) = 0$, *we add a point* b_i^- *on* α_i^+, *a point* b_i^+ *on* α_{i+1}^- *and a point* $_i$ *such that* $a_i b_i^- c_i b_i^+ a_{i+1} \gamma_i$ *is a hexagon with interior angles all equal to* 120 *degree and where* γ_i *is the vertex connected to* a_i *and* a_{i+1} *by two edges of* G.



Adding three vertices

- *If the angle* $\angle(\alpha_i^+, \alpha_{i+1}^-) = \pi/3$ *(this happens only if* G *is the one-edge graph), we place a point* b_i^- *on* α_i^+, *a point* b_i^+ *on* α_{i+1}^- *and points* c_i^-, c_i^+ *such that* $a_i b_i^- c_i^- c_i^+ b_i^+ a_{i+1}$ *is a hexagon with interior angles all equal to* 120 *degree.*



Adding four vertices

The edges $\cdots \alpha_i^+ (b_i^+, \gamma_i, b_i^-) \alpha_{i+1}^- \cdots$ *form a cycle for which the vertices* a_i *are ingoing vertices and the* b_i,b_i^+,b_i^-,γ_i *are outgoing vertices. We add to* G *these edges with segments (and vertices) attached to each outgoing vertices which will form the attaching points of the new minimal graph* $P(G)$.

The padded graph $P(G)$ is a simple minimal graph. Indeed, by construction, it has a cycle which surrounds all vertices except attaching points. In this cycle, the a_i's are ingoing vertices, and between two consecutive a_i's, outgoing vertices ($b_i^{\pm}$'s and $c_i^{\pm}$'s) are inserted. Therefore, the padding operation can be iterated.

5.3.1 The Graphs H_n

For each n, we define a graph H_n which is a subgraph of the standard tiling of the plane by regular hexagons. This definition is by induction on n. We begin by defining these graphs for $2 \leq n \leq 7$. As we saw in the previous section, for $n \leq 5$, the minimal graphs are trees and it is not hard to know that for each $n \leq 5$, there exists only one class of isomorphism of a tree which maximises the number of vertices. We denote an element of this class which is a subset of the standard hexagonal tilling by H_n. H_2 consists of two vertices joined by a single edge.

By convention, H_6 is a minimal graph consisting of a maximal cycle of length 6 (a hexagon) and the outgoing edges and vertices attached to the hexagon. H_7 is the minimal graph which consists of adding a minimal tree of length 3 to a vertex of the hexagon of H_6 (see the first 7 pictures of section 5.6).

We are now ready to define the family of graphs H_n.

Definition 5.3.3 *For each $n \geq 8$, define H_n inductively as follows. H_n is the graph obtained by padding H_{n-6}, i.e*

$$H_n = P(H_{n-6}).$$

Section 5.6 shows the first 19 H_n.

Remark For every $n \geq 6$, H_n has n attaching points and only one maximal cycle of length $2n-6$. The n attaching points correspond to the outgoing vertices of the cycle.

Next we enumerate for each n, the number of vertices of H_n.

Lemma 5.3.1 *Denote by N_n the number of vertices of H_n. If $n = 6k + i$, $0 \leq i \leq 5$ and $n \geq 2$, $N_n = \frac{1}{6}(6k)^2 + 6k + 2ik + \epsilon(i)$ where*

$$\epsilon(0) = 0, \quad \epsilon(1) = 0, \quad \epsilon(2) = 2, \quad \epsilon(3) = 4, \quad \epsilon(4) = 6, \quad \epsilon(5) = 8.$$

Proof of the Lemma.
By the recursive definition of H_n, we can easily conclude that

$$N(n) = N(n-6) + 2n.$$

This gives the fact that ϵ is periodic.

We will show now that the graphs H_n maximize the number of vertices.

Lemma 5.3.2 *For every n, H_n has the maximum number of vertices among all the 3-regular minimal graphs attached to n points.*

Proof of the Lemma.
For $n = 6k$, $6k+2$, $6k+3$, $6k+4$, by the obvious following calculation

$$(\frac{1}{6}(6k+i)^2 + (6k+i)) - (\frac{1}{6}(6k)^2 + (6k)) - 2ik = i + \frac{i^2}{6},$$

we deduce $\frac{1}{6}n^2 + n - N(n) < 1$ hence $f_3(n) \leq \lfloor \frac{1}{6}n^2 + n \rfloor \leq N(n)$. Then the Lemma follows from Corollary 2.1.
The difficulty is for the two cases $n = 6k+1$ and $n = 6k+5$, when the above calculation shows the possibility of existence of a minimal graph having one vertex more than H_n (see Lemma 5.3.1). We prove the Lemma for $n = 6k+1$, for the other case, the argument is the same.
Let suppose that there exists a graph G, attached to $n = 6k+1$ vertices, and having $(6k^2 + 6k) + (2k+1)$ vertices. It is obvious that G is not a tree. If G has at least two maximal cycles then by Lemma 5.2.11 we know that there exists k with $6 \leq k \leq n-4$ such that $f(n) \leq f(k) + f(n-k+2)$. Following the calculation in the proof of Lemma 5.2.11 we have

$$\begin{aligned} f(n) - (\frac{1}{6}n^2 + n) &\leq \frac{1}{3}((k-n)(k-2)+8) \\ &\leq \frac{4}{3}(4-k) \\ &\leq \frac{-8}{3} \\ &\leq -2 \end{aligned}$$

and hence if G has at least two maximal cycles

$$f(n) \leq \frac{1}{6}n^2 + n - 2.$$

with $n = 6k+1$ and $f(n) = 6k^2 + 8k + 1$ we find a contradiction.
Then G has one maximal cycle. To every outgoing vertex of the maximal cycle, there is a tree which is attached. If the tree is not a segment, then we can eliminate two neighbouring edges of the tree and obtain a new graph G' with $6k$ attaching points. The number of vertices of G' is equal to $6k^2 + 8k - 1$. We know that the maximum number of vertices of a minimal graph attached to $6k$ points is equal to $6k^2 + 6k$. but

$$6k^2 + 8k - 1 \geq 6k^2 + 6k$$

and this is not possible. So we deduce that to every outgoing vertex of the maximal cycle of the graph G only a segment can be attached. This is important for us because we can now use induction. Eliminating the outgoing vertices of the maximal cycle of G with the attaching segments (and points) attached to them, we obtain a new graph G' with $6(k-1)+1$ attaching points (by Lemma 5.7) and $f(n)-12k-2$ vertices. As

$$f(n)-12k-2 = 6k^2+8k+1-12k-2 = 6k^2-4k-1 = \frac{1}{6}(6(k-1)^2)+6(k-1)+2(k-1)+1$$

we can apply the same operation as we did on the graph G to G'. Carrying the induction we arrive to the case where $k=1$, which means a minimal graph attached to 7 points which has 15 vertices. The next Lemma provides the desired contradiction.

Lemma 5.3.3 *The maximum number of vertices of a 3-regular minimal graph H_7 attached to 7 points is equal to 14. Furthermore, every 3-regular minimal graph with 7 attaching points and 14 vertices is isomorphic to H_7.*

Proof of the Lemma
H_7 can be presented in the tiled plane by a hexagon with 5 segments attached to 5 vertices of the hexagon and a tree with 4 vertices having one attaching point in the 6th vertex of the hexagon.

We need to prove that this graph has the maximum number of vertices among all 3-regular minimal graphs with 7 attaching points.

Let suppose that there exists a graph H having more vertices than H_7. By Corollary 2.5, H has only one maximal cycle and the length of the maximal cycle is equal to 6. Then each vertex of the maximal cycle can be considered as an attaching point for a minimal tree (attached to the vertex). If there exist two vertices of the maximal cycle such that the two minimal trees attached to them are not segments, then the number of attaching points of H will be more than 7 and this is not possible. So to 5 vertices of the maximal cycle are attached 5 segments, and the proof of the Lemma follows.

By Lemma 5.3.3 we can conclude that a 3-regular minimal graph attached to $n=6k+1$ points can't have $f(n)=6k^2+8k+1$ vertices and the proof of the lemma follows.

The conclusion of Lemma 5.3.2 is that for every n, H_n maximises the number of vertices for n attaching points. In fact, we can prove more.

Lemma 5.3.4 *For every n, a graph G which maximises the number of vertices among 3-regular minimal graphs attached to n points is combinatorially isomorphic to H_n.*

Proof of the Lemma
As seen before, Lemma 3.5 holds for all $n \leq 7$. For $n \geq 8$ we prove it by induction on

n (the proof repeats some arguments in Lemma 3.3). Let $n \geq 8$. Let G be a 3-regular minimal graph attached to n points with $N(n)$ vertices. Copying the proof of Lemma 5.3.2 we know that G has one maximal cycle. We also know that a segment is attached to every outgoing vertex of the maximal cycle (otherwise by eliminating one edge from a (non trivial) tree attached to an outgoing vertex of the maximal cycle we obtain a graph G' with $n-1$ attaching points having more than $\frac{1}{6}(n-1)^2 + (n-1)$ vertices, which is impossible). Eliminating the outgoing vertices of the maximal cycle and the segments attached to them, we obtain a graph G' with $n-6$ attaching points. The number of vertices of G' is equal $N(n-6)$. Then G' is a minimal graph attached to $n-6$ points and maximising the number of vertices among all 3-regular minimal graphs attached to $n-6$ points. By induction hypothesis, G' is isomorphic to H_{n-6}. To reconstruct G from G', one must first glue in a cycle whose ingoing vertices are the attaching points of G'. This operation is exactly the padding operation of Definition 3.1, hence $P(G')$ is isomorphic to $P(H_{n-6})$ and the proof of the Lemma follows.

5.4 4-regular minimal graphs

Here we prove Theorem 38.

Let G be a 4-regular minimal graph with n attaching points. Then G is made up of line segments intersecting each other in the way that when any two segments intersect at a point (vertex of G) there are no other segments passing through the intersecting point. Thus every intersection points will be a vertex of G and the minimality condition is verified. Every line segment joins two of the attaching points. Then the problem of estimating $f_4(n)$ is equivalent to finding the maximum number of intersection points of $n/2$ line segments in the plane such that only two lines pass through the intersecting points.

For n odd it is impossible to attach a minimal graph of degree 4 to n points, and $f_4(n) = 0$ in this case. For n even, the number of intersecting points will not exceed $\binom{n/2}{2}$.

To complete the proof of the theorem, we show that, for every even n, there exists a collection of $n/2$ line segments intersecting at exactly $\binom{n/2}{2}$ points. We prove the existence of such a collection by induction on n. For $n = 1$ there is nothing to prove and i guess for $n = 2$ my non-born baby could find two lines which intersect at one point in the plane. We suppose that such a collection is constructed for n and we need to add a single line L to this collection such that L does not pass throw the intersection points and such that L intersects all the lines of the collection. As the number of lines and their intersection is finite, it is always possible to add such a line L with required property. From the existence of such a collection the prove of Theorem 38 follows.

5.5 Remarks and open questions

The problem of estimating the maximum number of vertices of a minimal graph attached to some points in the plane for the case of 3 and 4-regular graphs turned out to be very elementary. We saw that without any difficulties we could even classify maximizing graphs. But the same question for a non necessarily regular graph is violently more complicated. Indeed for vertices of degree 3, the angle between any two outgoing edges is equal to 120 degree, this simple fact let us have a Gauss-Bonnet type lemma and made our estimates possible. But for degrees greater than 4, there exist infinitely many possible geometric configurations of outgoing edges.

Notation 5.5.1 *Let δ be a natural number, we denote by $g_\delta(n)$ the supremum of the number of vertices of minimal graphs with n attaching points and degree bounded by δ.*

The general questions are
- for $\delta = 4$, find a sharper upper bound for $g_4(n)$.
- for $\delta > 4$, is $g_\delta(n)$ finite?

A first guess is $g_\delta(n) = f_3(n)$ and that the class of 3-regular graphs have the largest $g_\delta(n)$ for all the value of n and among all the minimal graphs with bounded degrees. Indeed, one can imagine that locally every minimal graph with degree greater than 3 can be mapped by a homotopy to a 3-regular minimal graph such that the number of vertices of the image by the homotopy increase. The non-obvious part is that these local homotopies will move the position of the attaching points and that we can't glue back the local part of the graph correctly together and get a new 3-regular minimal graph. Thus the initial guess may be misleading.

Let us modify the problem by introducing *weights*. We consider finite planar graphs equipped with a positive weight for each edge. We replace the total length functional by the weighted length, i.e. the sum of lengths of edges multiplied with weights. This changes the minimality condition slightly. Allard and Almgren gave an example of a family of 3-regular weighted minimal graphs with 16 attaching points and with arbitrarily large numbers of vertices. These examples are known as the spider web-like varifolds, see [2], [3] and [5]. They motivate the following conjecture.

Conjecture 5.5.1 *There exist minimal graphs attached to some fixed finite set of points in the plane with arbitrary large number of vertices.*

If this conjecture is true, it will be interesting to study infinite minimal graphs. This can motivate also the study of Morse theory in the space of 1-cycles with infinitely many edges.

However (paradoxically), the author conjectures

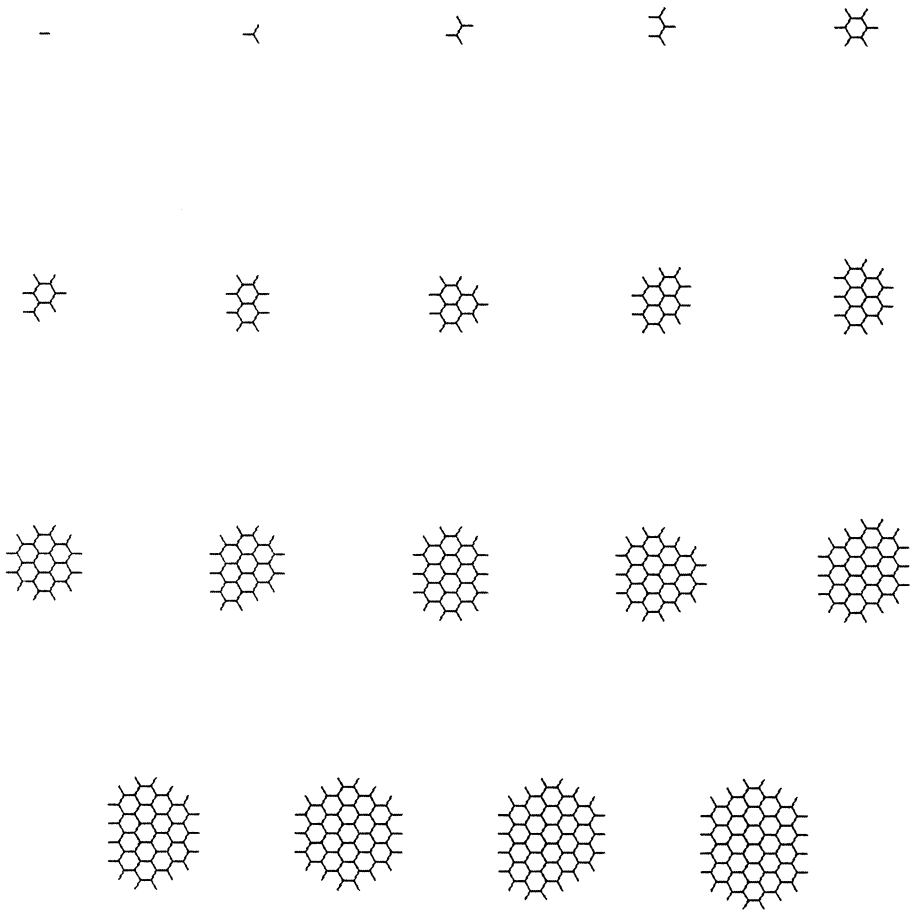
Conjecture 5.5.2 *For all $d \geq 3$, the number of vertices of a d-regular minimal graph attached to n points is $\leq f_3(n)$.*

One can begin with the case where all the angles of the d-regular minimal graphs are equal to $\frac{2\pi}{d}$ and try to obtain an intermediate result like Lemma 2.4.

The problem of estimating the maximum number of vertices in a minimal graph with some fixed conditions can also be asked in a more general context. One can ask the same questions about the graphs embedded in compact Riemannian manifold such as spheres, tori, etc.

The known results concern mostly the two sphere with a non necessarily canonical metric (see [12] and [13]).

5.6 Figures



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