ON AN ARTICLE BY W. MAGNUS ON THE FRICKE CHARACTERS OF FREE GROUPS

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Abstract. Another way of deriving the structure of the ring of Fricke characters of free groups and a classification of unitary and associative algebras over an arbitrary field with elements having a minimal polynomial of degree at most 2 are given.

I. Introduction.

Identities linking traces are not new. Those of proposition 1 appear in Fricke and Klein [3] and had already been stated by Vogt, as mentioned in [5]. The fact that the trace of a product, the factors of which belong to a finite set $A_1, A_2, \ldots A_n$ of elements of $SL(2,\mathbb{C})$, can be expressed as a polynomial in the traces of the $2^n - 1$ products $A_{j_1}A_{j_2}\cdots, A_{j_k}$, where $1 \le j_1 < j_2 < \cdots < j_k \le n$, conjectured by Fricke and proved by Horowitz [4], has been rediscovered several times, mainly on the occasion of studies in quasi-crystals (see [6] and [7] for a tentative compilation of the corresponding literature). The structure of the ideal of polynomials in $2^n - 1$ variables which vanish on any system of traces $\{\operatorname{tr} A_{j_1}A_{j_2}\cdots A_{j_k}\}$ has been elucidated by Whittemore [11] in the case $n = 4$ and by Magnus [5] in general. This structure also has been rediscovered in the context of quasi-crystals [1,2,10].

In this work, a new method of obtaining these results in given. It relies upon explicit calculations made possible by the existence of computer algebra softwares.

Rather than placing ourselves at once in $M_2(K)$, the algebra of 2×2-matrices on a field K, we prefer to consider first an algebra in which any element has a minimal polynomial of degree at most 2. Indeed, it is well known that polynomial identities between traces of matrices come from the Cayley-Hamilton identity [8,9]. Despite the fact that establishing identities is somewhat harder in this context, this has the advantage of yielding the structure of these algebras (theorems 1 and 3) and a new characterization of $M_2(K)$.

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Theorem 2 is a slight improvement on theorem 2.2 of [5]. Also, identity (6) is already in [5].

II. Algebras with elements having a minimal polynomial of degree at most 2.

Let $\mathcal{A}$ be an associative and unitary algebra on a field K the elements of which have a minimal polynomial of degree at most 2.

In this section, the characteristic of K is supposed to be different from 2, except in lemmas 1 and 2 and in proposition 1.

Lemma 1. *There exists a linear form, $a \mapsto \overline{a}$, and a quadratic form, $a \mapsto \widetilde{a}$, on $\mathcal{A}$, assuming values 2 and 1 respectively at the unit, and such that, for any element a in $\mathcal{A}$, one has $a^2 - (\overline{a})\, a + \widetilde{a} = 0$.*

Moreover, if a and b are in $\mathcal{A}$, one has $\overline{ab} = \overline{ba}$ and $\widetilde{ab} = \widetilde{a}\widetilde{b}$, as well as the following relations:

$$ab + ba = \overline{ab} - \overline{a}\,\overline{b} + \overline{b}\,a + \overline{a}\,b \tag{1}$$

$$aba = \overline{ab}\,a + \widetilde{a}\,(b - \overline{b}) \tag{2}$$

$\overline{a}$ et $\widetilde{a}$ will be called the trace and the determinant of a.

Proof. If $a \in \mathcal{A}$ and if a is not proportionnal to the unit, its minimal polynomial writes $a^2 - (\overline{a})\, a + \widetilde{a}$. Obviously, one has $\overline{t\,a} = t\,\overline{a}$ and $\widetilde{t\,a} = t^2\widetilde{a}$ for any scalar t.

Let a and b be two elements of $\mathcal{A}$ such that 1, a and b be independent. One has

$$\begin{aligned}(a+b)^2 &= \overline{a+b}\,(a+b) - \widetilde{a+b}\\ &= \overline{a}\,a - \widetilde{a} + \overline{b}\,b - \widetilde{b} + ab + ba,\end{aligned}$$

from which it follows $ab + ba = \left(\overline{a+b} - \overline{a}\right) a + \left(\overline{a+b} - \overline{b}\right) b + \widetilde{a} + \widetilde{b} - \widetilde{a+b}$, which will be provisionnally written $ab + ba = \alpha\, a + \beta\, b + \gamma$.

Let t be a scalar. By expanding $(a+tb)^2$ one gets

$$\overline{a+tb}\,(a+tb) - \widetilde{a+tb} = (a+tb)^2 = (\overline{a} + t\alpha)\, a + t\,(t\,\overline{b} + \beta)\, b + t\gamma - \widetilde{a} - \widetilde{b}t^2.$$

It follows that, for all $t \neq 0$, one has $\overline{a+tb} = \overline{a} + t\alpha = t\overline{b} + \beta$, whence $\overline{a+b} = \overline{a} + \overline{b}$ (in the case $K = \mathbb{Z}/2\mathbb{Z}$, one first considers an extension of K).

Now, let a be an element of $\mathcal{A}$ and t a scalar. One has $(a+t)^2 = a^2 + 2t\,a + t^2 = (\overline{a} + 2t)(a+t) - \widetilde{a} - t\,a - t^2$, whence $\overline{a+t} = \overline{a} + 2t$. When 1, a and b are dependent, it is easy to check the formula $\overline{a+b} = \overline{a} + \overline{b}$.

One has

$$\begin{aligned}aba &= -a^2 b + \overline{b}\,a^2 + \overline{a}\,ab + \left(\widetilde{a} + \widetilde{b} - \widetilde{a+b}\right) a\\ &= \widetilde{a}\,b + \overline{b}\,a^2 - \overline{a}\,ab + \left(\widetilde{a} + \widetilde{b} - \widetilde{a+b}\right) a\\ &= \widetilde{a}\,b + \left(\overline{a}\,\overline{b} + \widetilde{a} + \widetilde{b} - \widetilde{a+b}\right) a - \widetilde{a}\,\overline{b},\end{aligned}$$

whence

$$(ab)^2 = \left(\overline{a}\,\overline{b} + \widetilde{a} + \widetilde{b} - \widetilde{a+b}\right) ab - \widetilde{a}\,\widetilde{b}.$$

From this one deduces $\overline{ab} = \overline{ba} = \overline{a}\,\overline{b} + \widetilde{a} + \widetilde{b} - \widetilde{a+b}$, $\widetilde{ab} = \widetilde{a}\,\widetilde{b}$, and formulas (1) and (2).

When the characteristic of K is different from 2, one has $\widetilde{a} = \left(\overline{a}^2 - \overline{a^2}\right)/2$.

In the sequel it will be convenient to denote by $\mathcal{A}_0$ the kernel of the trace.

Proposition 1 (Fricke formulas). *If a, b, and c are elements of $\mathcal{A}$ one has*

$$\overline{abc} + \overline{acb} = \overline{a}\,\overline{bc} + \overline{b}\,\overline{ca} + \overline{c}\,\overline{ab} - \overline{a}\,\overline{b}\,\overline{c}$$

and

$$\begin{aligned}\overline{abc}\,\overline{acb} = {} & \overline{a}^2\widetilde{b}\,\widetilde{c} + \overline{b}^2\widetilde{c}\,\widetilde{a} + \overline{c}^2\widetilde{a}\,\widetilde{b} + \overline{bc}^2\widetilde{a} + \overline{ca}^2\widetilde{b} + \overline{ab}^2\widetilde{c} \\ & -\overline{a}\,\overline{b}\,\widetilde{c}\,\overline{ab} - \overline{b}\,\overline{c}\,\widetilde{a}\,\overline{bc} - \overline{c}\,\overline{a}\,\widetilde{b}\,\overline{ca} + \overline{bc}\,\overline{ca}\,\overline{ab} - 4\,\widetilde{a}\,\widetilde{b}\,\widetilde{c}.\end{aligned}$$

Proof. One has $a(bc+cb) = (\overline{bc} - \overline{b}\,\overline{c})a + \overline{b}(\overline{a}\,a - \widetilde{a}) + \overline{a}\,ab$, which gives the first formula.

Let us compute the product $abcacb$ in two ways: write $(abca)cb = ab(cac)b$ and use formula (2). One has

$$\begin{aligned}\big(\overline{abc}\,a + \widetilde{a}(bc - \overline{bc})\big)cb &= ab\big(\overline{ca}\,c + \widetilde{c}(a - \overline{a})\big)b \\ &= \overline{ca}\,a\big(\overline{bc}\,b + \widetilde{b}(c - \overline{c})\big) + \widetilde{c}(\overline{ab}\,ab - \widetilde{ab}) - \overline{a}\,\widetilde{c}\,a(\overline{b}\,b - \widetilde{b}),\end{aligned}$$

from which the second formula follows by taking the traces.

Lemma 2. *If a, b, and c are three elements of $\mathcal{A}$ such that $\overline{abc} \neq \overline{acb}$, one has*

$$\begin{aligned}\overline{b}\,abc = {} & -\overline{a}\,\widetilde{b}\,\overline{c} + \widetilde{b}\,\overline{ca} + \overline{b}\,\overline{abc} - \overline{ab}\,\overline{bc} \\ & + \widetilde{b}\,\overline{c}\,a - \overline{abc}\,b + \overline{a}\,\widetilde{b}\,c \\ & + \overline{bc}\,ab + \overline{ab}\,bc - 2\widetilde{b}\,ca.\end{aligned}$$

Proof. First, a direct computation shows that the conclusion of this lemma holds when $c = a$ and when $c = b$.

One evaluates the product $cabcbc$ in two ways by writing $[c(ab)c]bc = ca(bc)^2$ and using formula (2) and the Cayley-Hamilton theorem. One has

$$[c(ab)c]bc = \big(\overline{abc}\,c + \widetilde{c}(ab - \overline{ab})\big)bc = \overline{abc}(\overline{bc}\,c + \widetilde{c}\,b - \widetilde{c}\,\overline{b}) + \widetilde{c}(\overline{b}\,abc - \widetilde{b}\,ac) - \widetilde{c}\,\overline{ab}\,bc$$

and

$$ca(bc)^2 = \overline{bc}\,cabc - \widetilde{bc}\,ca = \overline{bc}(\overline{abc}\,c + \widetilde{c}\,ab - \widetilde{c}\,\overline{ab}) - \widetilde{bc}\,ca.$$

By subtracting, one gets the relation

$$\widetilde{c}\big(-\overline{b}\,abc - \overline{a}\,\widetilde{b}\,\overline{c} + \widetilde{b}\,\overline{ca} + \overline{b}\,\overline{abc} - \overline{ab}\,\overline{bc} + \widetilde{b}\,\overline{c}\,a - \overline{abc}\,b + \overline{a}\,\widetilde{b}\,c + \overline{bc}\,ab + \overline{ab}\,bc - 2\widetilde{b}\,ca\big) = 0.$$

If $\widetilde{c} \neq 0$, we are done.

If $\widetilde{c} = 0$, as the relation to be proven is linear in c, it suffices to prove it when c is replaced by $c + \alpha a + \beta b$. As $(c + \alpha a + \beta b)\widetilde{\ } = \alpha^2\widetilde{a} + \beta^2\widetilde{b} + \widetilde{c} - \alpha(\overline{ca} - \overline{c}\,\overline{a}) - \beta(\overline{bc} - \overline{b}\,\overline{c}) - \alpha\beta(\overline{ab} - \overline{a}\,\overline{b})$, one can choose such a linear combination the determinant of which does not vanish, unless all the terms be zero, which, due to proposition 1, would imply $\overline{abc} = \overline{acb} = \overline{a}\,\overline{b}\,\overline{c}$.

From now on the characteristic of K is supposed to be different from 2.

Corollary. *If a, b, and c are three elements of $\mathcal{A}_0$ such that $\overline{abc} \neq 0$, one has*

$$bc - \frac{1}{2}\overline{bc} = -\frac{1}{2\,\overline{abc}}\left[\left(4\,\widetilde{b}\,\widetilde{c} - \overline{bc}^2\right) a + \left(\overline{bc}\,\overline{ca} + 2\,\overline{ab}\,\widetilde{c}\right) b + \left(\overline{ab}\,\overline{bc} + 2\,\overline{ca}\,\widetilde{b}\right) c\right]. \tag{3}$$

Proof. It follows from proposition 1 that $-\overline{abc}^2 = \overline{ab}^2\widetilde{c} + \overline{bc}^2\widetilde{a} + \overline{ca}^2\widetilde{b} + \overline{ab}\,\overline{bc}\,\overline{ca} - 4\widetilde{a}\,\widetilde{b}\,\widetilde{c}$. Lemma 2 gives

$$\overline{bc}\left(ab - \frac{1}{2}\overline{ab}\right) + \overline{ab}\left(bc - \frac{1}{2}\overline{bc}\right) - 2\,\widetilde{b}\left(ca - \frac{1}{2}\overline{ca}\right) = \overline{abc}\,b$$

and two other relations obtained by circular permutation. One so gets a linear system, with determinant $-2\,\overline{abc}^2$, in the unknowns $bc - \frac{1}{2}\overline{bc}$, $ca - \frac{1}{2}\overline{ca}$, and $ab - \frac{1}{2}\overline{ab}$. By solving this system, one gets the claimed formula as well as two other ones obtained by circular permutation.

Proposition 2. *If there exist three elements a, b, and c of $\mathcal{A}_0$ such that $\overline{abc} \neq 0$, then, for all $d \in \mathcal{A}_0$, one has*

$$\begin{aligned}
&\Big[\left(-4\,\widetilde{b}\,\widetilde{c} + \overline{bc}^2\right)\overline{ad} + \left(-2\,\widetilde{c}\,\overline{ab} - \overline{ac}\,\overline{bc}\right)\overline{bd} + \left(-2\,\widetilde{b}\,\overline{ac} - \overline{ab}\,\overline{bc}\right)\overline{cd}\Big]\, a + \\
&\Big[\left(-2\,\widetilde{c}\,\overline{ab} - \overline{ac}\,\overline{bc}\right)\overline{ad} + \left(-4\,\widetilde{a}\,\widetilde{c} + \overline{ac}^2\right)\overline{bd} + \left(-2\,\widetilde{a}\,\overline{bc} - \overline{ab}\,\overline{ac}\right)\overline{cd}\Big]\, b + \\
&\Big[\left(-2\,\widetilde{b}\,\overline{ac} - \overline{ab}\,\overline{bc}\right)\overline{ad} + \left(-2\,\widetilde{a}\,\overline{bc} - \overline{ab}\,\overline{ac}\right)\overline{bd} + \left(-4\,\widetilde{a}\,\widetilde{b} + \overline{ab}^2\right)\overline{cd}\Big]\, c + \\
&\Big[-8\,\widetilde{c}\,\widetilde{b}\,\widetilde{a} + 2\,\widetilde{c}\,\overline{ab}^2 + 2\,\widetilde{b}\,\overline{ac}^2 + 2\,\widetilde{a}\,\overline{bc}^2 + 2\,\overline{bc}\,\overline{ac}\,\overline{ab}\Big]\, d = 0
\end{aligned} \tag{4}$$

Proof. By multiplying to the right both sides of (3) by d and taking the traces one gets

$$\overline{bcd} = -\frac{1}{2\,\overline{abc}}\left[\left(4\,\widetilde{b}\,\widetilde{c} - \overline{bc}^2\right)\overline{ad} + \left(\overline{bc}\,\overline{ca} + 2\,\overline{ab}\,\widetilde{c}\right)\overline{bd} + \left(\overline{ab}\,\overline{bc} + 2\,\overline{ca}\,\widetilde{b}\right)\overline{cd}\right]. \tag{5}$$

Now, let us prove formula (4). Due to proposition 1, the coefficient of d in (4) equals $-2\,\overline{abc}^2$.

First, suppose that $\overline{bc}^2 - 4\,\widetilde{b}\,\widetilde{c}$ and $\overline{bcd}$ do not vanish. By writing formula (3) with a, b, and c, and with d, b, and c and by taking (5) into account, one gets two expressions of bc. After cancelling the factor $\overline{bc}^2 - 4\,\widetilde{b}\,\widetilde{c}$, one gets (4).

If $\overline{bc}^2 - 4\,\widetilde{b}\,\widetilde{c} \neq 0$ and $\overline{bcd} = 0$, then $\overline{bc(a+d)} \neq 0$ and one can apply the above computation to a, b, c, and $a + d$. One gets relation (4) again.

It remains to get rid of the extra assumption $\overline{bc}^2 - 4\,\widetilde{b}\,\widetilde{c} \neq 0$. If one of the quantities $\overline{ab}^2 - 4\,\widetilde{a}\,\widetilde{b}$ or $\overline{ac}^2 - 4\,\widetilde{a}\,\widetilde{c}$ is nonzero, it suffices to perform a permutation of variables. Therefore we suppose that $\overline{ab}^2 - 4\,\widetilde{a}\,\widetilde{b} = \overline{ac}^2 - 4\,\widetilde{a}\,\widetilde{c} = \overline{bc}^2 - 4\,\widetilde{b}\,\widetilde{c} = 0$. One has $\overline{b(a+c)}^2 - 4\,\widetilde{b}\,\widetilde{(a+c)} = (\overline{ab} + \overline{bc})^2 - 4\,\widetilde{b}\,(\widetilde{a} + \widetilde{c} - \overline{ac}) = 2\left(\overline{ab}\,\overline{bc} + 2\,\widetilde{b}\,\overline{ac}\right)$. If this last expression vanished, due to the three preceding relations and to the expression of $\overline{abc}^2$ recalled above, one would have $\overline{abc} = 0$, contrary to the hypothesis. Therefore, one can apply what precedes to a, b, $a + c$, and d. This gives the same formula again.

Proposition 3. *If a, b, c, and d are four elements of $\mathcal{A}_0$ such that $\overline{abc} \neq 0$, one has*

$$\begin{aligned}
&+16\,\widetilde{d}\,\widetilde{c}\,\widetilde{b}\,\widetilde{a} \\
&-4\,(\widetilde{d}\,\widetilde{c}\,\overline{ab}^2 + \widetilde{d}\,\widetilde{b}\,\overline{ac}^2 + \widetilde{d}\,\widetilde{a}\,\overline{bc}^2 + \widetilde{c}\,\widetilde{b}\,\overline{ad}^2 + \widetilde{c}\,\widetilde{a}\,\overline{bd}^2 + \widetilde{b}\,\widetilde{a}\,\overline{cd}^2) \\
&-4\,(\widetilde{d}\,\overline{bc}\,\overline{ac}\,\overline{ab} + \widetilde{c}\,\overline{bd}\,\overline{ad}\,\overline{ab} + \widetilde{b}\,\overline{cd}\,\overline{ad}\,\overline{ac} + \widetilde{a}\,\overline{cd}\,\overline{bd}\,\overline{bc}) \\
&-2\,(\overline{cd}\,\overline{bd}\,\overline{ac}\,\overline{ab} + \overline{cd}\,\overline{bc}\,\overline{ad}\,\overline{ab} + \overline{bd}\,\overline{bc}\,\overline{ad}\,\overline{ac}) \\
&+\,(\overline{cd}^2\,\overline{ab}^2 + \overline{bd}^2\,\overline{ac}^2 + \overline{bc}^2\,\overline{ad}^2) = 0
\end{aligned} \tag{6}$$

Proof. Proposition 1 gives expressions of the squares of $\overline{abc}$ and $\overline{abd}$ as a polynomial in the determinants of a, b, c, and d and in the traces of their products two by two. By taking into account formula (5), one gets a polynomial relation with integer coefficients between the determinants of a, b, c, and d and the traces of their products. This polynomial is the product of $\overline{bc}^2 - 4\,\widetilde{b}\,\widetilde{c}$ and of the left hand side of (6). Therefore, if $\overline{bc}^2 - 4\,\widetilde{b}\,\widetilde{c} \neq 0$, the proposition is proven. One gets rid of this extra assumption in the same way as in the proof of proposition 2.

Proposition 4. *If one can extract square roots in the field K and if there exists in $\mathcal{A}_0$ three elements a, b, and c such that $\overline{abc} \neq 0$, the algebra $\mathcal{A}$ is isomorphic to $M_2(K)$, the algebra of 2×2-matrices over K.*

Proof. Given a, b, and c, one can find three 2×2-matrices A, B, and C with zero traces and such that $\det A = \widetilde{a}$, $\det B = \widetilde{b}$, $\det C = \widetilde{c}$, $\operatorname{tr} AB = \overline{ab}$, $\operatorname{tr} BC = \overline{bc}$, $\operatorname{tr} CA = \overline{ca}$, and $\operatorname{tr} ABC = \overline{abc}$ (apply proposition 7, the proof of which uses propositions 2 and 3 only, and proposition 1). The linear map from $\mathcal{A}$ into $M_2(K)$ which sends 1, a, b, and c on Id, A, B, and C is an homomorphism of unitary algebras due to formula (4) and to formula (3) and to the ones obtained by circular permutations. The fact that this is an isomorphism comes from proposition 6, the proof of which uses proposition 2 only.

Remarks. It is necessary to assume that is is possible to extract square roots in K. Indeed, let us consider the $\mathbb{R}$-algebra $\mathcal{A}$ generated, as a vector space, by 1, a, b, and c such that $a^2 = b^2 = c^2 = -1$, $ab = -ba = c$, $ca = -ac = b$, and $bc = -cb = a$ (observe that $abc = -1$). This algebra cannot be isomorphic to $M_2(\mathbb{R})$ because there exists no real 2×2-matrices A and B such that $A^2 = B^2 = -Id$ and $\operatorname{tr} AB = 0$ (lemma 3).

Even when K is a field where square roots exist, $\mathcal{A}$ is not necessarily isomorphic to $M_2(K)$: take $\mathcal{A} = K \oplus V$ and declare that the product of two elements of V is always 0. The following theorem gives the structure of algebras $\mathcal{A}$ when $\overline{abc} = 0$ for all a, b, and c with zero traces.

Theorem 1.

(1) *If there exist in $\mathcal{A}_0$ three elements a, b, and c such that $\overline{abc} \neq 0$, the algebra $\mathcal{A}$ has dimension 4 and any element d such that $\overline{d} = 0$ decomposes on the base $\{a, b, c\}$ according to formula (4).*

(2) *If, for any elements a, b, and c of $\mathcal{A}_0$, one has $\overline{abc} = 0$, then the set $\mathcal{N}$ consisting in the elements of $\mathcal{A}$ which are nilpotent of order 2 is a two sided ideal, any product of three elements of $\mathcal{N}$ equals 0, and the quotient $\mathcal{A}/\mathcal{N}$ has dimension 1 or 2.*

Proof. The first assertion comes from the preceding proposition by extending K, if necessary.

Let a, b, and c be three elements of $\mathcal{A}_0$. By using repeatedly formula (1) one gets

$$abc = -cab + \overline{ab}\, c = (ac - \overline{ac})\, b + \overline{ab}\, c = a\, (\overline{bc} - bc) - \overline{ac}\, b + \overline{ab}\, c,$$

whence

$$2\, abc = a\, \overline{bc} - b\, \overline{ac} + c\, \overline{ab} \tag{7}$$

(this last identity appears already in [1]).

By hypothesis, if a and b are two elements of $\mathcal{A}_0$, one has $\overline{ab\left(ab - \frac{1}{2}\overline{ab}\right)} = 0$, that is $\overline{ab}^2 = 4\widetilde{a}\widetilde{b}$.

It follows that, for any $a \in \mathcal{N}$ and for any $b \in \mathcal{A}$, one has $\overline{ab} = 0$. In particular, $aba = \overline{ab}\, a + \widetilde{a}\, (b - \overline{b}) = 0$. Therefore ab and ba are in $\mathcal{N}$.

Let a and b be two elements of $\mathcal{N}$. One has $(a + b)^2 = ab + ba = \overline{ab} = 0$. This means that $\mathcal{N}$ is a two sided ideal. The fact that any product of three elements of $\mathcal{N}$ is 0 comes from formula (7).

Let a and b be two non-zero elements of $\mathcal{A}_0$. As, for any $t \in K$, one has $(a+tb)^2 = -\widetilde{a}+t\,\overline{ab}-t^2\widetilde{b} = -\widetilde{b}\,\left(t - \frac{\overline{ab}}{2\,\widetilde{b}}\right)^2$, there exists a $t \in K$ such that $a+tb \in \mathcal{N}_0$. This proves that the dimension of $\mathcal{A}/\mathcal{N}$ is 1 or 2.

Proposition 5. *Let a, b, and c be three elements of $\mathcal{A}_0$ such that $\overline{abc} \neq 0$. If x, y, z, and t are four elements of K satisfying the following relation*

$$\begin{aligned}
&+ 16\, t\, \widetilde{c}\, \widetilde{b}\, \widetilde{a} \\
&- 4\, (t\, \widetilde{c}\, \overline{ab}^2 + t\, \widetilde{b}\, \overline{ac}^2 + t\, \widetilde{a}\, \overline{bc}^2 + \widetilde{c}\, \widetilde{b}\, x^2 + \widetilde{c}\, \widetilde{a}\, y^2 + \widetilde{b}\, \widetilde{a}\, z^2) \\
&- 4\, (t\, \overline{bc}\, \overline{ac}\, \overline{ab} + \widetilde{c}\, y\, x\, \overline{ab} + \widetilde{b}\, z\, x\, \overline{ac} + \widetilde{a}\, z\, y\, \overline{bc}) \\
&- 2\, (z\, y\, \overline{ac}\, \overline{ab} + z\, \overline{bc}\, x\, \overline{ab} + y\, \overline{bc}\, x\, \overline{ac}) \\
&+ \ (z^2\, \overline{ab}^2 + y^2\, \overline{ac}^2 + \overline{bc}^2\, x^2) = 0,
\end{aligned}$$

there exists a unique $d \in \mathcal{A}$ such that $d = 0$, $\overline{ad} = x$, $\overline{bd} = y$, $\overline{cd} = z$, and $\widetilde{d} = t$.

Proof. Uniqueness comes from theorem 1. Set, as formula (4) suggests it,

$$\begin{aligned}
d = \frac{1}{2\,\overline{abc}^2}\big\{ &\big[\big(-4\, \widetilde{b}\, \widetilde{c} + \overline{bc}^2 \big)\, x + \big(-2\, \widetilde{c}\, \overline{ab} - \overline{ac}\, \overline{bc} \big)\, y + \big(-2\, \widetilde{b}\, \overline{ac} - \overline{ab}\, \overline{bc} \big)\, z \,\big]\, a + \\
&\big[\big(-2\, \widetilde{c}\, \overline{ab} - \overline{ac}\, \overline{bc} \big)\, x + \big(-4\, \widetilde{a}\, \widetilde{c} + \overline{ac}^2 \big)\, y + \big(-2\, \widetilde{a}\, \overline{bc} - \overline{ab}\, \overline{ac} \big)\, z \,\big]\, b + \\
&\big[\big(-2\, \widetilde{b}\, \overline{ac} - \overline{ab}\, \overline{bc} \big)\, x + \big(-2\, \widetilde{a}\, \overline{bc} - \overline{ab}\, \overline{ac} \big)\, y + \big(-4\, \widetilde{a}\, \widetilde{b} + \overline{ab}^2 \big)\, z \,\big]\, c \,\big\}.
\end{aligned}$$

Then, a direct computation proves the proposition.

This is a generalization of the decomposition of a 2×2-matrix on the Pauli matrices.

III. Application to 2×2-matrices.

The algebra $\mathcal{A}$ is the algebra $M_2(K)$ of 2×2-matrices with entries in K. Let $M_2^0(K)$ stand for the set of matrices with zero traces. In this section, the characteristic of K is supposed to be different from 2.

Proposition 6. *Three matrices a, b, and c in $M_2^0(K)$ are independent if and only if the trace of abc is not zero. If it is so, any matrix d in $M_2^0(K)$ decomposes on the base (a,b,c) according to formula* (4).

Proof. If $K=\mathbb{C}$, by continuity, formula (4) holds for any four matrices with zero traces. As this formula is a polynomial identity with integer coefficients linking the entries of matrices a, b, c, and d, its holds for any field K (even of characteristic 2).

The same argument also shows that relation (6) holds for any four matrices.

If $\overline{abc}\neq 0$, formula (4) gives an expression of any matrix with zero trace as a linear combination of a, b, and c.

As a matter of fact, as F. Choucroun pointed it out to the author, $\overline{abc}$ is the determinant of (a,b,c) with respect to the base $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

Corollary 1. *Given three matrices a, b, and c, the system $(1,a,b,c)$ is a base of $M_2(K)$ if and only if $\Delta_{a,b,c}=\left(\overline{abc}-\overline{acb}\right)^2\neq 0$. If it is so, any matrix d can be written as $d=\alpha\, a+\beta\, b+\gamma\, c+\delta$, where coefficients α, β, γ, and δ are linear combinations of $\overline{d}$, $\overline{ad}$, $\overline{bd}$, and $\overline{cd}$ the coefficients of which are rational fractions (with integer coefficients) in the variables $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{ab}$, $\overline{ac}$, $\overline{bc}$, and $\overline{abc}$, the denominators of which are powers of $\Delta_{a,b,c}$ times a power of 2.*

Proof. Apply the preceding theorem to matrices $a-\overline{a}/2$, $b-\overline{b}/2$, $c-\overline{c}/2$, and $d-\overline{d}/2$.

Notice that, due to proposition 1, $\Delta_{a,b,c}$ is a polynomial with integer coefficients in the variables $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{ab}$, $\overline{ac}$, $\overline{bc}$.

Corollary 2. *Two matrices a and b span $M_2(K)$ as an algebra if and only if $D_{a,b}=\overline{a}^2\,\widetilde{b}+\overline{b}^2\,\widetilde{a}+\overline{ab}^2-\overline{a}\,\overline{b}\,\overline{ab}-4\,\widetilde{a}\,\widetilde{b}\neq 0$.*

Proof. It suffices to express that $a-\overline{a}/2$, $b-\overline{b}/2$, and $ab-\overline{ab}/2$ are independent.

Lemma 3. *If a and b are two elements of $M_2^0(K)$ there exist α and β in K such that $\overline{ab}^2-4\,\widetilde{a}\,\widetilde{b}=4\alpha^2\widetilde{a}+\beta^2$.*

Proof. If $\widetilde{a}\neq 0$, up to conjugation, one may assume that $a=\begin{pmatrix}0&\widetilde{a}\\-1&0\end{pmatrix}$. Then $b=\begin{pmatrix}\alpha&u\\v&-\alpha\end{pmatrix}$, with $\alpha^2+uv=-\widetilde{b}$ and $v\,\widetilde{a}-u=\overline{ab}$. One obtains the following relation $v^2\widetilde{a}-\overline{ab}\,v+\alpha^2+\widetilde{b}=0$, from which one can deduce the result.

Lemma 4. *If d_a, d_b, and y_{ab} are three elements in K, there exist two matrices a and b in $M_2^0(K)$ such that $\widetilde{a}=d_a$, $\widetilde{b}=d_b$, and $\overline{ab}=y_{ab}$ if and only if there exist two elements α and β in K such that $y_{ab}^2-4\,d_a d_b=\alpha^2 d_a+\beta^2$.*

Proof. If $d_a\neq 0$, take

$$a=\begin{pmatrix}0&d_a\\-1&0\end{pmatrix}\quad\text{et}\quad b=\frac{1}{2}\begin{pmatrix}\alpha&\beta-y_{ab}\\(\beta+y_{ab})/d_a&-\alpha\end{pmatrix}.$$

If $d_a=0$ and $y_{ab}\neq 0$, take $a=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $b=\begin{pmatrix}0&y_{ab}\\-d_b/y_{ab}&0\end{pmatrix}$.

If $d_a=y_{ab}=0$, take $a=0$ and $b=\begin{pmatrix}0&d_b\\-1&0\end{pmatrix}$.

Proposition 7. *If one can extract square roots in K, given seven elements in K, y_{ab}, y_{bc}, y_{ca}, d_a, d_b, d_c, and z such that*

$$z^2 + y_{ab}^2 d_c + y_{bc}^2 d_a + y_{ca}^2 d_b + y_{ab} y_{bc} y_{ca} - 4\, d_a\, d_b\, d_c = 0, \tag{8}$$

there exist three matrices a, b, and c in $M_2^0(K)$ such that $\overline{ab} = y_{ab}$, $\overline{bc} = y_{bc}$, $\overline{ca} = y_{ca}$, $\widetilde{a} = d_a$, $\widetilde{b} = d_b$, $\widetilde{c} = d_c$, and $\overline{abc} = z$.

Proof. If $y_{ab}^2 - 4\, d_a d_b \neq 0$, one chooses (lemma 4) two matrices a and b such that $\widetilde{a} = d_a$, $\widetilde{b} = d_b$, and $\overline{ab} = y_{ab}$. One has $\overline{ab(ab - \overline{ab}/2)} = (\overline{ab}^2 - 4\,\widetilde{a}\,\widetilde{b})/2 \neq 0$. Proposition 5, applied to a, b, and $ab - \overline{ab}/2$, gives a matrix c such that $\overline{c} = 0$, $\overline{ac} = y_{ac}$, $\overline{bc} = y_{bc}$, and $\overline{(ab - \overline{ab}/2)c} = z$. In this case, once matrices a and b chosen, the matrix c is unique.

If $y_{ab}^2 - 4\, d_a d_b = y_{bc}^2 - 4\, d_b d_c = y_{ca}^2 - 4\, d_c d_a = 0$, there are three alternatives.

If $z \neq 0$, set $y'_{ab} = y_{ab} + y_{bc}$, $y'_{bc} = y_{bc}$, $y'_{ca} = y_{ca} - 2\, d_c$, $d'_a = d_a + d_c - y_{ca}$, $d'_b = d_b$, $d'_c = d_c$ et $z' = z$. The dashed letters also fulfill relation (8). Moreover, one has ${y'_{ab}}^2 - 4\, d'_a d'_b \neq 0$. Then one constructs, as previously, three matrices a', b', and c'. Matrices $a = a' - c'$, $b = b'$, and $c = c'$ answer the question.

If $z = 0$ and $d_a d_b d_c \neq 0$, one can take for a, b, and c matrices of the form $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$.

If $z = 0$ and if, for instance, $d_a = 0$, one can take for a, b, and c the matrices 0, $\begin{pmatrix} 0 & -d_b \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} \sqrt{-d_c} & y_{bc} \\ 0 & -\sqrt{-d_c} \end{pmatrix}$.

Let consider the following polynomial

$$\begin{aligned}\Lambda(x_1, x_2, x_3, y_{12}, y_{13}, y_{23}, z) = \\ z^2 - (x_1 y_{23} + x_2 y_{13} + x_3 y_{12} - x_1 x_2 x_3)\, z + x_1^2 + x_2^2 + x_3^2 + y_{23}^2 \\ + y_{13}^2 + y_{12}^2 - x_1 x_2 y_{12} - x_2 x_3 y_{23} - x_3 x_1 y_{13} + y_{23} y_{13} y_{12} - 4\end{aligned}$$

and its discriminant $\Delta(x_1, x_2, x_3, y_{12}, y_{13}, y_{23})$ with respect to z.

Let $\mathcal{K}$ denote the quotient field of $\mathbb{Z}[x_1, x_2, x_3, y_{12}, y_{13}, y_{23}, z]/\Lambda$.

If $n \geq 3$, a polynomial $P \in \mathcal{K}[x_j,\ 4 \leq j \leq n;\ y_{jk},\ 1 \leq j \leq 3,\ 4 \leq k \leq n]$ will be also denoted by $P(x_j,\ 1 \leq j \leq n;\ y_{jk},\ 1 \leq j \leq 3,\ j < k \leq n;\ z)$.

Given four unimodular 2×2-matrices A_1, A_2, A_3, and B, the left hand side, multiplied by a suitable power of 2, of the identity obtained by replacing in (6) a, b, c, and d respectively by $A_1 - \overline{A_1}/2$, $A_2 - \overline{A_2}/2$, $A_3 - \overline{A_3}/2$, and $B - \overline{B}/2$ is a polynomial ξ with integer coefficients in the variables $x_1 = \overline{A_1}$, $x_2 = \overline{A_2}$, $x_3 = \overline{A_3}$, $u = \overline{B}$, $y_{12} = \overline{A_1 A_2}$, $y_{13} = \overline{A_1 A_3}$, $y_{23} = \overline{A_2 A_3}$, $v_1 = \overline{A_1 B}$, $v_2 = \overline{A_2 B}$, and $v_3 = \overline{A_3 B}$. Let ξ be viewed as an element of $\mathcal{K}[u, v_1, v_2, v_3]$ and let $\mathcal{J}$ denote the ideal of $\mathcal{K}[x_j,\ 4 \leq j \leq n;\ y_{jk},\ 1 \leq j \leq 3,\ 4 \leq k \leq n]$ generated by the $n-3$ polynomials $\xi(x_j, y_{1j}, y_{2j}, y_{3j})$ $(3 < j \leq n)$.

Theorem 2. *Given an element w of the free group $\Gamma_{\langle a_1,a_2,\cdots,a_n\rangle}$, there exists a polynomial $P_w \in \mathcal{K}[x_j,\, 4 \le j \le n;\, y_{jk},\, 1 \le j \le 3,\, 4 \le k \le n]$, which is unique modulo $\mathcal{J}$ if K is infinite, and such that, for any representation φ of $\Gamma_{\langle a_1,a_2,\cdots,a_n\rangle}$ in $SL(2,K)$ such that $\Delta(\overline{\varphi(a_1)},\overline{\varphi(a_2)},\overline{\varphi(a_3)},\overline{\varphi(a_1a_2)},\overline{\varphi(a_1a_3)},\overline{\varphi(a_2a_3)}) \neq 0$, one has*

$$\operatorname{tr}\varphi(w) = P_w\big(\overline{\varphi(a_j)},\, 1 \le j \le n;\, \overline{\varphi(a_ja_k)},\, 1 \le j \le 3,\, j<k \le n;\, \overline{\varphi(a_1a_2a_3)}\big).$$

This statement improves on theorem 2.2 in [5]. Indeed, there is no need for the hypothesis $\overline{\varphi(a_1)}^2 + \overline{\varphi(a_2)}^2 + \overline{\varphi(a_1a_2)}^2 - \overline{\varphi(a_1)}\,\overline{\varphi(a_2)}\,\overline{\varphi(a_1a_2)} - 4 \neq 0$ which is the reformulation in our setting of the hypothesis $D(\varphi(a_1),\varphi(a_2)) \neq 2$ assumed in [5].

Proof. Let us first prove the existence of P_w. It results from theorem 1 that, for $j>3$, the matrix $\varphi(a_j)$ can be written as $\alpha + \beta_1\varphi(a_1) + \beta_2\varphi(a_2) + \beta_3\varphi(a_3)$, where the coefficients α, β_1, β_2, and β_3 are linear combinations of $\overline{\varphi(a_j)}$, $\overline{\varphi(a_1a_j)}$, $\overline{\varphi(a_2a_j)}$, and $\overline{\varphi(a_3a_j)}$ the coefficients of which are rational fractions in the variables $\overline{\varphi(a_1)}$, $\overline{\varphi(a_2)}$, $\overline{\varphi(a_3)}$, $\overline{\varphi(a_1a_2)}$, $\overline{\varphi(a_1a_3)}$, $\overline{\varphi(a_2a_3)}$, and $\overline{\varphi(a_1a_2a_3)}$. The same fact holds for $\varphi(a_j^{-1})$, because $\varphi(a_j^{-1}) = \overline{\varphi(a_j)} - \varphi(a_j)$. One ends by using the corollary to lemma 2.

Uniqueness results from propositions 1, 3, 7, and 5.

IV. The case of characteristic 2.

Now, the characteristic of K equals 2 and $\mathcal{A}$ is an associative and unitary algebra with elements having a minimal polynomial of degree at most 2. As previously, $\mathcal{A}_0$ stands for the kernel of the trace. This time, the unit belongs to $\mathcal{A}_0$.

Lemma 5. *For any a, b, and c in $\mathcal{A}$, one has*

$$\overline{bc}\,a + \overline{ca}\,b + \overline{ab}\,c + \overline{a}\,bc + \overline{b}\,ca + \overline{c}\,ab + \overline{abc} = 0. \tag{9}$$

Proof. It goes as the proof of formula (7).

Theorem 3.

(1) *If there exists a and b in $\mathcal{A}_0$ such that $\overline{ab} \neq 0$, then $\dim\mathcal{A} = 4$ and $\mathcal{A}$ is generated, as a unitary algebra, by a and b. Moreover, if K is algebraically closed, $\mathcal{A}$ is isomorphic to $M_2(K)$.*

(2) *If, for any a and b in $\mathcal{A}_0$ one has $\overline{ab} = 0$, then $\mathcal{A}_0$ is a commutative algebra. Moreover, if there exists f in $\mathcal{A} \setminus \mathcal{A}_0$, the set $\mathcal{J} = \{a \in \mathcal{A}_0 \mid \overline{af} = 0\}$ is an ideal. If $\dim\mathcal{J} \ge 2$, then $ab = 0$ for any a and b in $\mathcal{J}$. If $\dim\mathcal{J} = 1$, the algebra $\mathcal{A}$ is generated by elements 1, f, and a subject to relations $f^2 = f$ and $af = fa = a^2 = 0$ or $f^2 = f + \lambda + \lambda^2$, $af = fa = \lambda a$, and $a^2 = 0$, where λ is a parameter in K.*

Proof. Let us prove assertion (1). Due to formula (9), for any $c \in \mathcal{A}_0$, one has $c = \dfrac{\overline{abc} + \overline{bc}\,a + \overline{ca}\,b}{\overline{ab}}$, which proves that $\{1, a, b, ab\}$ is a base for $\mathcal{A}$.

If one can find a' and b' in $\mathcal{A}_0$ such that ${a'}^2 = {b'}^2 = 0$ and $(a'b')^2 = a'b'$, then the map $a \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $b \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ extends as an isomorphism of unitary algebras from $\mathcal{A}$ to $M_2(K)$. If $\widetilde{a} \neq 0$ or $\widetilde{b} \neq 0$, this is possible if the equation $\widetilde{a}\, t^2 + \overline{ab}\, tu + \widetilde{b}\, u^2 = 0$ has two distinct projective roots.

Now suppose that $\overline{ab} = 0$ for all a and b in $\mathcal{A}_0$. It results from formula (1) that $ab = ba$ for all a and b in $\mathcal{A}_0$.

If $\mathcal{A} = \mathcal{A}_0$, there is not much to say. So, from now on, we suppose that there exists $f \in \mathcal{A}$ such that $\overline{f} \neq 0$. We may assume that $\overline{f} = 1$.

Let $\mathcal{J}$ stand for the set $\{a \in \mathcal{A}_0 \mid \overline{af} = 0\}$. If a and b are elements of $\mathcal{J}$, one has $ab = \overline{abf}$ (due to formula (9)), therefore $ab \in \mathcal{J}$; moreover, one has $\widetilde{a}\, b = a^2 b = a(ab) = \overline{abf}\, a$, which means that, if $\dim \mathcal{J} > 1$, the product of any two elements of $\mathcal{J}$ is 0. If $a \in \mathcal{J}$, one has $fa = af + a$, $(af)f = af^2 = af + \widetilde{f}\, a$ and, by formula (2), $f(af) = \widetilde{f}\,(a + \overline{a})$. This proves that $\mathcal{J}$ is an ideal.

If $\dim \mathcal{J} = 1$, the algebra $\mathcal{A}$ is generated by 1, f, and a third element a such that $af = fa = \lambda a$ and $a^2 = \widetilde{a}$. By looking for associativity, one sees there are only the two alternatives as stated above.

For the sake of completeness, we give the analogues of formulas (4) and (6) in characteristic 2.

Proposition 8. *Let a_1, a_2, a_3, and a_4 be four elements in $M_2(K)$. Then*

$$\begin{aligned}
&\left(\overline{a_2}\,\overline{a_3}\,\overline{a_4} + \overline{a_2}\,\overline{a_3 a_4} + \overline{a_3}\,\overline{a_2 a_4} + \overline{a_4}\,\overline{a_2 a_3}\right) a_1 + \textit{three similar terms} \\
&+ \overline{a_1}\,\overline{a_2}\,\overline{a_3 a_4} + \textit{five similar terms} \\
&+ \overline{a_1 a_2}\,\overline{a_3 a_4} + \textit{two similar terms} \\
&+ \overline{a_1}\,\overline{a_2}\,\overline{a_3}\,\overline{a_4} = 0
\end{aligned}$$

and

$$\begin{aligned}
&\left(\overline{a_2}\,\overline{a_3}\,\overline{a_4} + \overline{a_2}\,\overline{a_3 a_4} + \overline{a_3}\,\overline{a_2 a_4} + \overline{a_4}\,\overline{a_2 a_3}\right)^2 \widetilde{a_1} + \textit{three similar terms} \\
&+ \overline{a_1}\,\overline{a_2}\,\overline{a_3}\,\overline{a_4}\left(\overline{a_1}\,\overline{a_2}\,\overline{a_3 a_4} + \textit{five similar terms}\right) \\
&+ \overline{a_1}^2\,\overline{a_2}\,\overline{a_3}\,\overline{a_2 a_4}\,\overline{a_3 a_4} + \textit{eleven similar terms} \\
&+ \overline{a_1}\,\overline{a_2}\,\overline{a_3 a_4}\left(\overline{a_1 a_2}\,\overline{a_3 a_4} + \overline{a_1 a_3}\,\overline{a_2 a_4} + \overline{a_1 a_4}\,\overline{a_2 a_3}\right) + \textit{five similar terms} \\
&+ \overline{a_1}^2\,\overline{a_2 a_3}\,\overline{a_2 a_4}\,\overline{a_3 a_4} + \textit{three similar terms} \\
&+ \left(\overline{a_1 a_2}\,\overline{a_3 a_4} + \overline{a_1 a_3}\,\overline{a_2 a_4} + \overline{a_1 a_4}\,\overline{a_2 a_3} + \overline{a_1}\,\overline{a_2}\,\overline{a_3}\,\overline{a_4}\right)^2 = 0
\end{aligned}$$

Proof. Assume for a moment that $K = \mathbb{C}$, consider four matrices a_1, a_2, a_3, and a_4, replace a by $a_1 - \overline{a_1}$, b by $a_2 - \overline{a_2}$, c by $a_3 - \overline{a_3}$, and d by $a_4 - \overline{a_4}$ in formulas (4) and (6), and multiply each of the obtained identities by a suitable power of 2 so as to have no denominators. Then by taking these identities modulo 2, one gets the above formulas.

Of course, one can state an analogue of theorem 2 for representations of $\Gamma_{\langle a_1, a_2, \cdots, a_n \rangle}$ in $SL(2, K)$ when the characteristic of K equals 2.

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