On the discrete Poincaré–Friedrichs inequalities for nonconforming approximations of the Sobolev space $H^1$

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Abstract

We present a direct proof of the discrete Poincaré–Friedrichs inequalities for a class of nonconforming approximations of the Sobolev space $H^1(\Omega)$, indicate optimal values of the constants in these inequalities, and extend the discrete Friedrichs inequality onto domains only bounded in some direction. We consider a polygonal domain $\Omega$ in two or three space dimensions and its shape regular simplicial triangulation. The nonconforming approximations of $H^1(\Omega)$ consist of functions from $H^1$ on each element such that the mean values of their traces on inter-element boundaries coincide. The key idea is to extend the proof of the discrete Poincaré–Friedrichs inequalities for piecewise constant functions used in the finite volume method.

Résumé

Nous présentons une démonstration des inégalités de Poincaré–Friedrichs discrètes pour une classe d’approximations non-conformes de l’espace de Sobolev $H^1(\Omega)$, indiquons les valeurs optimales des constantes dans ces inégalités et montrons l’inégalité de Friedrichs discrète pour des domaines bornés dans une direction uniquement. Nous considérons un domaine polygonal $\Omega$ en dimension deux ou trois d’espace et sa triangulation régulière par des triangles ou tétraèdres. Les approximations non-conformes de $H^1(\Omega)$ sont données par des fonctions de $H^1$ sur chaque élément de maillage telles que les moyennes de ces traces sur les frontières entre les éléments coïncident. L’idée essentielle est d’étendre la démonstration des inégalités de Poincaré–Friedrichs discrètes connues pour des fonctions constantes par morceaux dans le cadre de la méthode des volumes finis.

Key words: Poincaré–Friedrichs inequalities, Sobolev space $H^1$, nonconforming approximation

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1 Introduction

The Friedrichs (also called Poincaré) inequality
\[ \int_{\Omega} g^2(x) \, dx \leq c_F \int_{\Omega} |\nabla g(x)|^2 \, dx \quad \forall g \in H^1_0(\Omega) \] (1)
and the Poincaré (also called mean Poincaré) inequality
\[ \int_{\Omega} g^2(x) \, dx \leq c_P \int_{\Omega} |\nabla g(x)|^2 \, dx + \tilde{c}_P \left( \int_{\Omega} g(x) \, dx \right)^2 \quad \forall g \in H^1(\Omega) \] (2)
(cf. [8]) play an important role in the theory of partial differential equations. We consider here a bounded polygonal domain (open and connected set) \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), \( H^1(\Omega) \) is the Sobolev space of \( L^2(\Omega) \) functions with square-integrable generalized derivatives, and \( H^1_0(\Omega) \) is the subspace of \( H^1(\Omega) \) of functions with zero trace on the boundary \( \partial\Omega \) of \( \Omega \). We refer for instance to [1] for details on the spaces \( H^1(\Omega) \), \( H^1_0(\Omega) \).

Let \( \{T_h\}_h \) be a family of simplicial triangulations of \( \Omega \) (consisting of triangles in space dimension two and of tetrahedra in space dimension three). Let the spaces \( W(T_h) \) be formed by functions locally in \( H^1(K) \) on each \( K \in T_h \) such that the mean values of their traces on interior sides coincide. Finally, let \( W_0(T_h) \subset W(T_h) \) be such that the mean values of the traces on exterior sides of functions from \( W_0(T_h) \) are equal to zero (precise definitions of these spaces are given in the next section). We investigate in this paper the discrete equivalents of (1) and (2) in the forms
\[ \int_{\Omega} g^2(x) \, dx \leq C_F \sum_{K \in T_h} \int_K |\nabla g(x)|^2 \, dx \quad \forall g \in W_0(T_h), \forall h > 0, \] (3)
\[ \int_{\Omega} g^2(x) \, dx \leq C_P \sum_{K \in T_h} \int_K |\nabla g(x)|^2 \, dx + \tilde{c}_P \left( \int_{\Omega} g(x) \, dx \right)^2 \quad \forall g \in W(T_h), \forall h > 0. \] (4)

The validity of (3) for \( W_0(T_h) \) consisting of piecewise linear functions (used e.g. in the Crouzeix–Raviart finite element method) has been established in [10, Proposition 4.13] provided that \( \Omega \) is convex and in [4] for a generally nonconvex \( \Omega \) but with triangulations which are not locally refined. These results have been later extended in [7] onto \( T_h \) only satisfying the shape regularity (minimal angle) assumption and onto spaces that include \( W_0(T_h) \). Another proof of this last result is presented in [2]. This paper also shows how to extend the discrete Friedrichs and Poincaré inequalities onto general polygonal (nonmatching) partitions of \( \Omega \) and on functions that do not satisfy the equality of the means of traces on interior sides, provided that (3), (4) are satisfied.

It was shown in [7, 2] that the constants \( C_F, C_P \) only depend on the domain \( \Omega \) and on the shape regularity of the meshes. We establish in this paper the exact dependence of \( C_F, C_P \) on these parameters. We show that in space dimension two \( C_F \) only depends on the area of \( \Omega \) and that in space dimension two or three \( C_F \) only depends on the square of the diameter of \( \Omega \) in one chosen direction. For convex domains, \( C_P \) only depends on the square of the diameter of \( \Omega \) and on the ratio between the area of the circumscribed ball and the area of \( \Omega \). For nonconvex domains, our results involve a more complicated dependence of \( C_P \) on \( \Omega \). The dependencies which we mentioned above are optimal in the sense that they coincide with the dependencies of \( c_F, c_P \) on \( \Omega \) in the continuous case. The dependence of \( C_F \) on \( \Omega \) also allows for the extension of the discrete Friedrichs inequality to domains which are only
bounded in some direction. We finally show that $C_F$ depends, in space dimension two and provided that it is expressed using the area of $\Omega$, on the square of a parameter describing the shape regularity of the meshes given in the next section. This dependence still holds true for $C_F$ in space dimension two or three and expressed using the square of the diameter of $\Omega$ in one chosen direction and also for $C_P$, provided that the mesh is not locally refined. We present an example showing that this dependence is optimal. For locally refined meshes, our results involve a more complicated dependence on the shape regularity parameter.

Our proof of the discrete Friedrichs and Poincaré inequalities on the spaces $W_0(T_h)$, $W(T_h)$ respectively is more direct than those presented in [7] and in [2]; in particular, all the necessary intermediate results are proved here. In [7] the author uses a Clément type interpolation operator (cf. [3]) mapping the space $W_0(T_h)$ to $H^1_0(\Omega)$. In [2] the key idea is to construct nonconforming $P_1$ interpolants of functions from $W(T_h)$ and to connect the nonconforming $P_1$ finite elements and conforming $P_2$ finite elements (in space dimension two) or conforming $P_3$ finite elements (in space dimension three). In both cases one finally makes use of the continuous inequalities (1), (2). Our main idea is to construct a piecewise constant interpolant and to extend the discrete Friedrichs–Poincaré inequalities for piecewise constant functions known from finite volume methods, see [5], [6]. In particular, we do not make use of the continuous inequalities; since $H^1_0(\Omega) \subset W_0(T_h)$ and $H^1(\Omega) \subset W(T_h)$, we rather prove them.

The rest of the paper is organized as follows. In Section 2 we describe the assumptions on $T_h$, define a dual mesh $D_h$ where the dual elements are associated with the sides of $T_h$, define the function spaces used in the sequel, and introduce the interpolation operator. In Section 3 we give the discrete Friedrichs inequality for piecewise constant functions on $D_h$. In Section 4 we prove some interpolation estimates on functions from $H^1(K)$, where $K$ is a simplex in two or three space dimensions. In Section 5 we prove the discrete Friedrichs inequality for functions from $W_0(T_h)$, using their interpolation by piecewise constant functions on $D_h$. In Section 6 we show how this proof simplifies for Crouzeix–Raviart finite elements in two space dimensions. Finally, Section 7 is devoted to the proof of the discrete Poincaré inequality for piecewise constant functions on $D_h$ and Section 8 to the extension of this result to functions from $W(T_h)$.

2 Notation and assumptions

Throughout the paper, we shall consider by ‘segment’ a segment of a straight line. Let us consider a domain $K \subset \mathbb{R}^d$, $d = 2, 3$. We denote by $\| \cdot \|_{0,k}$ the norm on $L^2(K)$, $\|g\|_{0,K} = \left( \int_K g^2(x) \, dx \right)^{1/2}$, by $|K|$ is the $d$-dimensional Lebesgue measure of $K$, by $|\sigma|$ the $(d - 1)$-dimensional Lebesgue measure of $\sigma$, a part of a hyperplane in $\mathbb{R}^d$, and by $|s|$ the length of a segment $s$. Let $b$ be a vector. We shall mean by the diameter of $K$ in the direction of $b$, denoted by $\text{diam}_b(K)$, the supremum of the lengths of segments $s$ with the direction vector $b$ such that $s \subset K$. The diameter of $K$ is the supremum of the lengths of all the segments $s$ such that $s \subset K$.

2.1 Triangulation

We suppose that $T_h$ for all $h > 0$ consists of closed triangles ($d=2$) or tetrahedra ($d=3$) such that $\overline{T} = \bigcup_{K \in T_h} K$ and such that if $K, L \in T_h$, $K \neq L$, then $K \cap L$ is either an empty set or a common face, edge, or vertex of $K$ and $L$. The parameter $h$ is defined by
\( h = \max_{K \in \mathcal{T}_h} \text{diam}(K) \). We denote by \( \mathcal{E}_h \) the set of all sides (edges when \( d = 2 \), faces when \( d = 3 \)), by \( \mathcal{E}_{h}^{\text{int}} \) the set of all interior sides, by \( \mathcal{E}_{h}^{\text{ext}} \) the set of all exterior sides, and by \( \mathcal{E}_K \) the set of all the sides of an element \( K \in \mathcal{T}_h \). We make the following shape regularity assumption on \( \{\mathcal{T}_h\}_h \):

**Assumption (A) (Shape regularity assumption)**

There exists a constant \( \kappa_T > 0 \) such that

\[
\min_{K \in \mathcal{T}_h} \frac{|K|}{\text{diam}(K)^d} \geq \kappa_T \quad \forall h > 0.
\]

Assumption (A) is equivalent to the existence of a constant \( \theta_T > 0 \) such that

\[
\max_{K \in \mathcal{T}_h} \frac{\text{diam}(K)}{\rho_K} \leq \theta_T \quad \forall h > 0,
\]  

where \( \rho_K \) is the diameter of the largest ball inscribed in the simplex \( K \). Finally, Assumption (A) is equivalent to the existence of a constant \( \phi_T > 0 \) such that

\[
\min_{K \in \mathcal{T}_h} \phi_K \geq \phi_T \quad \forall h > 0.
\]

Here, \( \phi_K \) is the smallest angle of the simplex \( K \) (plain angle in radians for \( d = 2 \) and spheric angle in steradians for \( d = 3 \)).

In the sequel we shall consider apart triangulations which may not be locally refined, i.e. the case where the following assumption holds:

**Assumption (B) (Inverse assumption)**

There exists a constant \( \zeta_T > 0 \) such that

\[
\max_{K \in \mathcal{T}_h} \frac{h}{\text{diam}(K)} \leq \zeta_T \quad \forall h > 0.
\]

Assumptions (A) and (B) imply

\[
\min_{K \in \mathcal{T}_h} \frac{|K|}{h^d} \geq \kappa_T \quad \forall h > 0,
\]

where \( \kappa_T = \frac{\kappa_T}{\zeta_T} \).

### 2.2 Dual mesh

In the sequel we will use a dual mesh \( \mathcal{D}_h \) to \( \mathcal{T}_h \) such that \( \overline{\Omega} = \bigcup_{D \in \mathcal{D}_h} D \). There is one dual element \( D \) associated with each side \( \sigma_D \in \mathcal{E}_h \). We construct it by connecting the barycentres of every \( K \in \mathcal{T}_h \) that contains \( \sigma_D \) through the vertices of \( \sigma_D \). For \( \sigma_D \in \mathcal{E}_{h}^{\text{ext}} \), the contour of \( D \) is completed by the side \( \sigma_D \). We refer to Fig. 1 for the two-dimensional case. We denote by \( \mathcal{D}_{h}^{\text{int}} \) the set of all interior and by \( \mathcal{D}_h^{\text{ext}} \) the set of all boundary dual elements. As for the primal mesh, we set \( \mathcal{F}_h, \mathcal{F}_{h}^{\text{int}}, \mathcal{F}_h^{\text{ext}} \), and \( \mathcal{F}_D \) for the dual mesh sides. We denote by \( Q_D \) the barycentre of a side \( \sigma_D \) and for two adjacent elements \( D, E \in \mathcal{D}_h \), we set \( \sigma_{D,E} = \partial D \cap \partial E, \ d_{D,E} = |Q_E - Q_D| \), and \( K_{D,E} \) the element of \( \mathcal{T}_h \) such that \( \sigma_{D,E} \subset K_{D,E} \). We remark that

\[
|K \cap D| = \frac{|K|}{d+1}
\]  

(8)
for each $K \in T_h$ and $D \in D_h$ such that $\sigma_D \in \mathcal{E}_h$. Let us now consider $\sigma_{D,E} \in \mathcal{F}_h^{\text{int}}$, $\sigma_{D,E} = \partial D \cap \partial E$ in the two-dimensional case. Let $K_{D,E} \cap D$ be in the clockwise direction from $K_{D,E} \cap E$. We then define $\nu_{D,E}$ as the height of the triangle $|K_{D,E} \cap D|$ with respect to its base $\sigma_{D,E}$ and have (see Fig. 1)

$$|K_{D,E} \cap D| = \frac{|\sigma_{D,E}|\nu_{D,E}}{2}. \quad (9)$$

### 2.3 Function spaces

We define the space $W(T_h)$ by

$$W(T_h) = \left\{ g \in L^2(\Omega) : g|_K \in H^1(K) \quad \forall K \in T_h, \right. \int_{\sigma_{K,L}} g|_K(x) \, d\gamma(x) = \int_{\sigma_{K,L}} g|_L(x) \, d\gamma(x) \quad \forall \sigma_{K,L} \in \mathcal{E}_h^{\text{int}}, \sigma_{K,L} = \partial K \cap \partial L \} . \quad (10)$$

We keep the same notation for the function $g$ and its trace and denote $d\gamma(x)$ the integration symbol for the Lebesgue measure on a hyperplane of $\Omega$. The space $W_0(T_h)$ is defined by

$$W_0(T_h) = \left\{ g \in W(T_h) : \int_{\sigma} g(x) \, d\gamma(x) = 0 \quad \forall \sigma \in \mathcal{E}_h^{\text{ext}} \right\} . \quad (11)$$

We finally define

$$|g|_{1,T} = \left( \sum_{K \in T_h} \int_K |\nabla g(x)|^2 \, dx \right)^{\frac{1}{2}},$$

which is a seminorm on $W(T_h)$ and a norm on $W_0(T_h)$. The spaces $X(T_h) \subset W(T_h)$ and $X_0(T_h) \subset W_0(T_h)$ are defined by piecewise linear functions on $T_h$. Note that the functions from $X(T_h)$ are continuous in barycentres of interior sides and that the functions from $X_0(T_h)$ are moreover equal to zero in barycentres of exterior sides.

The space $Y(D_h)$ is the space of piecewise constant functions on $D_h$,

$$Y(D_h) = \left\{ c \in L^2(\Omega) : c|_D \text{ is constant } \forall D \in D_h \right\} ,$$
and $Y_0(D_h)$ is its subspace of functions equal to zero on all $D \in D_h^{ext}$,

$$Y_0(D_h) = \left\{ c \in Y(D_h) : c|_D = 0 \quad \forall D \in D_h^{ext} \right\}.$$ 

For $c \in Y(D_h)$ given by the values $c_D$ on $D \in D_h$, we define

$$|c|_{1,T,*} = \left( \sum_{\sigma_D \in F_h^{int}} \frac{|\sigma_D,E|}{v_D,E} (c_E - c_D)^2 \right)^{\frac{1}{2}},$$

$$|c|_{1,T,\dagger} = \left( \sum_{\sigma_D \in F_h^{int}} \frac{|\sigma_D,E|}{\text{diam}(K_D,E)} (c_E - c_D)^2 \right)^{\frac{1}{2}},$$

$$|c|_{1,T,\ddagger} = \left( \sum_{\sigma_D \in F_h^{int}} \frac{|\sigma_D,E|}{d_D,E} (c_E - c_D)^2 \right)^{\frac{1}{2}};$$

$|\cdot|_{1,T,*}$, $|\cdot|_{1,T,\dagger}$, and $|\cdot|_{1,T,\ddagger}$ are seminorms on $Y(D_h)$ and norms on $Y_0(D_h)$.

### 2.4 Interpolation operator

The interpolation operator $I$ associates to a function $g \in W(T_h)$ a function $I(g) \in Y(D_h)$ such that

$$I(g)|_D = g_D = \frac{1}{|\sigma_D|} \int_{\sigma_D} g|_K(x) \, d\gamma(x) \quad \forall D \in D_h,$$

where $K \in T_h$ is such that $\sigma_D \in \mathcal{E}_K$. Note that by (10), if $\sigma_D \in \mathcal{E}_K$ and $\sigma_D \in \mathcal{E}_L$, $K \neq L$, the choice between $K$ and $L$ does not matter. We recall that $\sigma_D \in \mathcal{E}_h$ is the side associated with the dual element $D \in D_h$. Note that for $g \in W_0(T_h)$, $I(g) \in Y_0(D_h)$.

### 3 Discrete Friedrichs inequality for piecewise constant functions

In finite volume methods (cf. [5]) one can prove the discrete Friedrichs inequality for piecewise constant functions for meshes that satisfy the following orthogonality property: there exists a point associated with each element of the mesh such that the straight line connecting these points for two neighboring elements is orthogonal to the common side of these two elements. The proofs in [5, 6] rely on this property of the meshes. We present in this section analogies of Lemma 9.5 and consequent Remark 9.13 and of Lemma 9.1 of [5] for the mesh $D_h$, where the orthogonality property is not necessarily satisfied.

**Theorem 3.1 (Discrete Friedrichs inequality for piecewise constant functions in 2-D)** Let $d = 2$. Then for all $c \in Y_0(D_h)$,

$$\|c\|_{0,\Omega}^2 \leq \frac{1}{2} |\cdot|_{1,T,*} c_{1,T,*}.$$

**Proof:**

Let $b_1 = (1,0)$ and $b_2 = (0,1)$ be two fixed unit vectors in the axis directions. For all $x \in \Omega$, let $B_x^1$ and $B_x^2$ be the straight lines going through $x$ and defined by the vectors $b_1$, $b_2$ respectively. Let the functions $\chi^{(i)}(x)$, $i = 1, 2$ for each $\sigma \in F_h^{int}$ be defined by

$$\chi^{(i)}(x) = \begin{cases} 1 & \text{if } \sigma \cap B_x^i \neq \emptyset \\ 0 & \text{if } \sigma \cap B_x^i = \emptyset \end{cases}.$$
Let finally \( D \in D_h^{\text{int}} \) be fixed. Then for a.e. \( x \in D, B^i_x, i = 1, 2 \) do not contain any vertex of the dual mesh and \( B^i_x \cap \sigma, i = 1, 2 \) contain at most one point of all \( \sigma \in F_h \). This implies that for a.e. \( x \in D, B^i_x, i = 1, 2 \) always have to intersect the interior of some \( E \in D_h^{\text{ext}} \) before ‘leaving’ or after ‘entering’ \( \Omega \) (we recall that \( \Omega \) may be nonconvex). Using this, the fact that \( c_E = 0 \) for all \( E \in D_h^{\text{ext}} \), and the triangle inequality, we have

\[
2|c_D| \leq \sum_{\sigma, F, G \in F_h^{\text{int}}} |c_G - c_F| \chi_{\sigma, F, G}^{(i)}(x) \quad \text{for a.e. } x \in D, i = 1, 2.
\]

This gives

\[
|c_D|^2 \leq \frac{1}{4} \sum_{\sigma, F, G \in F_h^{\text{int}}} |c_G - c_F| \chi_{\sigma, F, G}^{(1)}(x) \sum_{\sigma, F, G \in F_h^{\text{int}}} |c_G - c_F| \chi_{\sigma, F, G}^{(2)}(x) \quad \text{for a.e. } x \in D,
\]

which is obviously valid also for \( D \in D_h^{\text{ext}} \), considering that \( c_D = 0 \) on \( D \in D_h^{\text{ext}} \). Integrating the above inequality over \( D \) and summing over \( D \in D_h \) yields

\[
\sum_{D \in D_h} c_D^2(D) \leq \frac{1}{4} \int_\Omega \left( \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| \chi_{\sigma, D, E}^{(1)}(x) \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| \chi_{\sigma, D, E}^{(2)}(x) \right) dx.
\]

Let \( \alpha = \inf \{x_1; (x_1, x_2) \in \Omega \} \) and \( \beta = \sup \{x_1; (x_1, x_2) \in \Omega \} \). For each \( x_1 \in (\alpha, \beta) \), we denote by \( J(x_1) \) the set of \( x_2 \) such that \( x = (x_1, x_2) \in \Omega \). We now notice that \( \chi_{\sigma}^{(1)}(x) \) only depends on \( x_2 \) and that \( \chi_{\sigma}^{(2)}(x) \) only depends on \( x_1 \). Thus

\[
\int_\alpha^\beta \int_{J(x_1)} \left( \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| \chi_{\sigma, D, E}^{(1)}(x_2) \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| \chi_{\sigma, D, E}^{(2)}(x_1) \right) dx_2 dx_1 =
\]

\[
= \int_\alpha^\beta \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| \chi_{\sigma, D, E}^{(2)}(x_1) \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| \int_{J(x_1)} \chi_{\sigma, D, E}^{(1)}(x_2) dx_2 dx_1 \leq
\]

\[
\leq \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| |\sigma, D, E| \int_\alpha^\beta \sum_{\sigma, D, E \in F_h^{\text{int}}} |c_E - c_D| \chi_{\sigma, D, E}^{(2)}(x_1) dx_1,
\]

where we have used \( \int_{J(x_1)} \chi_{\sigma, D, E}^{(1)}(x_2) dx_2 \leq |\sigma, D, E| \). Using analogously \( \int_\alpha^\beta \chi_{\sigma, D, E}^{(2)}(x_1) dx_1 \leq |\sigma, D, E| \), we come to

\[
\sum_{D \in D_h} c_D^2(D) \leq \frac{1}{4} \left( \sum_{\sigma, D, E \in F_h^{\text{int}}} |\sigma, D, E| |c_E - c_D| \right)^2.
\]

Finally, using the Cauchy–Schwarz inequality, we have

\[
\sum_{D \in D_h} c_D^2(D) \leq \frac{1}{4} \sum_{\sigma, D, E \in F_h^{\text{int}}} |\sigma, D, E| v_{D, E} \sum_{\sigma, D, E \in F_h^{\text{int}}} |\sigma, D, E| v_{D, E} (c_E - c_D)^2.
\]

The equality \( \sum_{\sigma, D, E \in F_h^{\text{int}}} |\sigma, D, E| v_{D, E} = 2 |\Omega| \) which follows from (9) permits to conclude the proof. \( \square \)
Remark 3.2 (Discrete Friedrichs inequality for piecewise constant functions on orthogonal meshes) Let $b$ be a fixed unit vector and let $T_h$ consist of equilateral simplices. Then for all $c \in Y_0(D_h)$,

$$\|c\|_{0,\Omega}^2 \leq \text{diam}_b(\Omega) + 2h^2|c|^2_{1,T_h}.$$ 

This follows from [5, Lemma 9.1] (cf. alternatively [6, Lemma 1]), since the dual mesh $D_h$ satisfies the orthogonality property for $T_h$ consisting of equilateral simplices.

Lemma 3.3 Let Assumption (B) be satisfied and let $b \subset \Omega$ be a segment which does not contain any vertex of the dual mesh $D_h$. Then

$$A = \sum_{\sigma_{D,E} \in F_h^{int}, \sigma_{D,E} \cap b \neq \emptyset} \text{diam}(K_{D,E}) \leq C_{d,T} \text{diam}_b(\Omega),$$

where

$$C_{d,T} = \frac{2^d(d-1)}{\kappa_T} (1 + 2\theta_T). \quad (12)$$

Proof:

The number of nonzero terms of $A$ is equal to the number of interior dual sides intersected by $b$. In view of the fact that $b$ does not contain any vertex of the dual mesh, this number is bounded by $2(d-1)$-times the number of simplices $K \in T_h$ whose interior is intersected by $b$. All intersected simplices have to be entirely in the rectangle/rectangular parallelepiped constructed around $b$, with the distance between $b$ and its boundary equal to $h$. Considering the consequence (7) of Assumptions (A) and (B), we can estimate the number of intersected elements by

$$\frac{(2h)^{d-1}(|b| + 2h)}{\kappa_T h^d}.$$ 

Using in addition $\text{diam}(K_{D,E}) \leq h$ and $|b| \leq \text{diam}_b(\Omega)$, we have

$$A \leq \frac{2^d(d-1)}{\kappa_T} (\text{diam}_b(\Omega) + 2h).$$

Noticing that

$$h \leq \theta_T \text{diam}_b(\Omega) \quad (13)$$

by the consequence (5) of Assumption (A) concludes the proof. □

Lemma 3.4 Let $b \subset \Omega$ be a segment which does not contain any vertex of the dual mesh $D_h$. Then

$$A = \sum_{\sigma_{D,E} \in F_h^{int}, \sigma_{D,E} \cap b \neq \emptyset} \text{diam}(K_{D,E}) \leq C_{d,T} \text{diam}_b(\Omega), \quad (14)$$

where

$$C_{d,T} = 4N(d-1)\theta_T^{2N}, \quad N = \frac{2^{d-1}\pi}{\phi_T}. \quad (15)$$
Proof:

The number of nonzero terms of \( A \) is equal to the number of interior dual sides intersected by \( b \). In view of the fact that \( b \) does not contain any vertex of the dual mesh, this number is bounded by \( 2(d-1) \) times the number of simplices \( K \in T_h \) whose interior is intersected by \( b \). We consider two different cases.

If the segment \( b \) intersects at most \( N \) simplices, where \( N \) is given by (15), we use the estimate

\[
\text{diam}(K) \leq \theta_T \rho_K \leq \theta_T \text{diam}_b(\Omega) \quad \forall K \in T_h,
\]

which follows from the consequence (5) of Assumption (A) to see that

\[
A \leq 2(d-1)N \theta_T \text{diam}_b(\Omega).
\]

This in view of \( N \geq 1 \) and \( \theta_T > 1 \) implies (14) with \( C_{d,T} \) given by (15).

We next consider the case where the segment \( b \) intersects at least \( N + 1 \) simplices. We divide it into a system of non overlapping segments \( \{b_k\}_{k=1}^M \) such that \( b = \bigcup_{k=1}^M b_k \). We further require that each \( b_k \) intersects at least \( N \) and at most \( 2N \) simplices and that no simplex has an intersection with a positive 1-dimensional Lebesgue measure with two different segments.

We then have

\[
A \leq 2(d-1) \sum_{k=1}^M \sum_{K \in T_h: K \cap b_k \neq \emptyset} \text{diam}(K).
\]

Next it follows from the consequence (5) of Assumption (A) that \( \rho_K \leq \theta_T \rho_L \) if \( K, L \in T_h \) are neighboring elements. Recall that \( \rho_K \) is the diameter of the largest ball inscribed in the simplex \( K \). Thus we come to

\[
\frac{\max_{K \in T_h: K \cap b_k \neq \emptyset} \rho_K}{\min_{K \in T_h: K \cap b_k \neq \emptyset} \rho_L} \leq \theta_T^{2N-1} \quad \forall k = 1, \ldots, M.
\]

We further claim that

\[
\min_{K \in T_h: K \cap b_k \neq \emptyset} \rho_K \leq |b_k| \quad \forall k = 1, \ldots, M,
\]

i.e. if we take \( N \) simplices intersected by a straight line, where \( N \) is given by (15), then the length of the intersection is at least equal to the smallest diameter of the inscribed balls of the simplices. We show this by contradiction. Let us suppose that \( N \) simplices are intersected by a segment \( l \) and that the length of the intersection is smaller or equal to the smallest diameter of the inscribed balls of the simplices. By contradiction, the centers of all the inscribed balls have to lie outside of \( l \). Now with each simplex intersected by \( l \), we add an angle greater or equal to \( \phi_T \) by the consequence (6) of Assumption (A). Since we have \( N \) simplices, their angles fill the whole circle \((2\pi, d = 2)\) or sphere \((4\pi, d = 3)\), which yields the contradiction.

Using the last two estimates, the fact that each \( b_k \) intersects at most \( 2N \) simplices, and once more the consequence (5) of Assumption (A), we have

\[
A \leq 2(d-1) \sum_{k=1}^M 2N \theta_T^{2N} |b_k| \leq 4N(d-1) \theta_T^{2N} |b| \leq 4N(d-1) \theta_T^{2N} \text{diam}_b(\Omega).
\]

This proves (14) with \( C_{d,T} \) given by (15) for the second case and consequently the whole lemma. \( \square \)
Theorem 3.5 (Discrete Friedrichs inequality for piecewise constant functions) Let $b$ be a fixed unit vector. Then for all $c \in Y_0(D_h)$,

$$\|c\|_{0,\Omega}^2 \leq C_{d,T}[\text{diam}_b(\Omega)]^2|c|_{1,T,1}^2,$$

where $C_{d,T}$ is given by (12) when Assumption (B) is satisfied and by (15) in the general case.

Proof:

For all $x \in \Omega$, we denote by $B_x$ the straight semi-line defined by the origin $x$ and the vector $b$. Let $y(x) \in \partial\Omega \cap B_x$ be the point where $B_x$ intersects $\partial\Omega$ for the first time. Then $[x,y(x)] \subset \overline{\Omega}$. We finally define a function $\chi_\sigma(x)$ for each $\sigma \in \mathcal{F}_h^{int}$ by

$$\chi_\sigma(x) = \begin{cases} 
1 & \text{if } \sigma \cap [x,y(x)] \neq \emptyset, \\
0 & \text{if } \sigma \cap [x,y(x)] = \emptyset.
\end{cases}$$

Let $D \in \mathcal{D}_h^{int}$ be fixed. Then for a.e. $x \in D$, $B_x$ does not contain any vertex of the dual mesh and $B_x \cap \sigma$ contains at most one point of all $\sigma \in \mathcal{F}_h$. This implies that for a.e. $x \in D$, $B_x$ always has to intersect the interior of some $E \in \mathcal{D}_h^{ext}$ before "leaving" $\Omega$. Using this, the fact that $c_E = 0$ for all $E \in \mathcal{D}_h^{ext}$, and the triangle inequality, we have

$$|c_D| \leq \sum_{\sigma_F,G \in \mathcal{F}_h^{int}} |c_G - c_F|\chi_{\sigma_F,G}(x) \quad \text{for a.e. } x \in D.$$  

The Cauchy–Schwarz inequality yields

$$|c_D|^2 \leq \sum_{\sigma_F,G \in \mathcal{F}_h^{int}} \chi_{\sigma_F,G}(x) \text{ diam}(K_{F,G}) \sum_{\sigma_F,G \in \mathcal{F}_h^{int}} \frac{(c_G - c_F)^2}{\text{diam}(K_{F,G})} \chi_{\sigma_F,G}(x) \quad \text{for a.e. } x \in D,$$

which is obviously valid also for $D \in \mathcal{D}_h^{ext}$, considering that $c_D = 0$ on $D \in \mathcal{D}_h^{ext}$. Integrating the above inequality over $D$, summing over $D \in \mathcal{D}_h$, and using Lemma 3.3 when Assumption (B) is satisfied and Lemma 3.4 in the general case yields

$$\sum_{D \in \mathcal{D}_h} |c_D|^2|D| \leq C_{d,T}[\text{diam}_b(\Omega)] \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \frac{(c_E - c_D)^2}{\text{diam}(K_{D,E})} \int_{\Omega} \chi_{\sigma_{D,E}}(x) \, dx.$$

Now the value $\int_{\Omega} \chi_{\sigma_{D,E}}(x) \, dx$ is the measure of the set of points of $\Omega$ which are located inside a cylinder whose basis is $\sigma_{D,E}$ and generator vector is $-b$. Thus

$$\int_{\Omega} \chi_{\sigma_{D,E}}(x) \, dx \leq |\sigma_{D,E}| \text{ diam}_b(\Omega),$$

which leads to the assertion of the lemma. \(\square\)

4 Interpolation estimates on functions from $H^1(K)$

Lemma 4.1 Let $K$ be a simplex, $\sigma$ its side, and $g \in H^1(K)$. We set

$$g_K = \frac{1}{|K|} \int_K g(x) \, dx,$$

$$g_\sigma = \frac{1}{|\sigma|} \int_\sigma g(x) \, d\gamma(x).$$
Then

\[(g_K - g_\sigma)^2 \leq c_d \frac{\text{diam}(K)^2}{|K|} \int_K |\nabla g(x)|^2 \, dx, \tag{20}\]

\[
\int_K [g(x) - g_\sigma]^2 \, dx \leq c_d \text{diam}(K)^2 \int_K |\nabla g(x)|^2 \, dx, \tag{21}\]

where

\[c_d = 6 \text{ for } d = 2, \quad c_d = 9 \text{ for } d = 3. \tag{22}\]

**Proof:**

The inequality (20) is proved as a part of [5, Lemma 9.4] or [6, Lemma 2] for \(d = 2\). In these references a general convex polygonal element \(K\) is considered; the fact that \(c_d = 6\) follows by considering a triangular element. The inequality (21) also follows from these proofs, using the Cauchy–Schwarz inequality. We now give the proof for the three-dimensional case, following the ideas of the proof for \(d = 2\).

Let us consider a tetrahedron \(K\) and its face \(\sigma\). Let us denote the space coordinates by \(x_1, x_2, x_3\). We assume, without loss of generality, that \(\sigma \subset \{0\} \times \mathbb{R} \times \mathbb{R}^+\), that one vertex of \(\sigma\) lies in the origin, that the longest edge of \(\sigma\) lies on \(x_2^+\), and that \(K \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}\). Let \(a = (\alpha, \beta, \gamma)\) be the vertex which does not lie on \(\sigma\). For all \(x_1 \in [0, \alpha]\), we set \(J(x_1) = \{x_2 \in \mathbb{R} \mid (x_1, x_2, x_3) \in K\}\). For all \(x_2 \in J(x_1)\) with \(x_1 \in [0, \alpha]\) given, we set \(J(x_1, x_2) = \{x_3 \in \mathbb{R} \mid (x_1, x_2, x_3) \in K\}\). For a.e. \(x = (x_1, x_2, x_3) \in K\) and a.e. \(y = (y_1, y_2, y_3) \in \sigma\), we set \(z(x, y) = ta + (1-t)y\) with \(t = \frac{x_1}{\alpha}\). Since \(K\) is convex, \(z(x, y) \in K\) and we have \(z(x_1, y) = (x_1, z_2(x_1, y_2), z_3(x_1, y_3))\) with \(z_2(x_1, y_2) = \frac{x_1}{\alpha} \beta + (1 - \frac{x_1}{\alpha}) y_2\) and \(z_3(x_1, y_3) = \frac{x_1}{\alpha} \gamma + (1 - \frac{x_1}{\alpha}) y_3\).

Using the Cauchy–Schwarz inequality, we have

\[
\int_K [g(x) - g_\sigma]^2 \, dx = \int_K \left[ \frac{1}{|\sigma|} \int_\sigma g(x) \, d\gamma(y) - \frac{1}{|\sigma|} \int_\sigma g(y) \, d\gamma(y) \pm
\right.
\]

\[
\pm \frac{1}{|\sigma|} \int_\sigma g(z(x, y)) \, d\gamma(y) \right]^2 \, dx \leq \frac{2}{|\sigma|^2} \int_K \left[ \int_\sigma \left( g(x) - g(z(x, y)) \right) \, d\gamma(y) \right]^2 \, dx +
\]

\[
+ \frac{2}{|\sigma|^2} \int_K \left[ \int_\sigma \left( g(z(x, y)) - g(y) \right) \, d\gamma(y) \right]^2 \, dx \leq \frac{2}{|\sigma|}(A + B),
\]

where

\[A = \int_K \int_\sigma \left( g(x) - g(z(x, y)) \right)^2 \, d\gamma(y) \, dx, \]

\[B = \int_K \int_\sigma \left( g(z(x, y)) - g(y) \right)^2 \, d\gamma(y) \, dx.\]

Similarly,

\[(g_K - g_\sigma)^2 \leq \frac{2}{|K||\sigma|}(A + B).\]

We denote by \(D_i g\) the partial derivative of \(g\) with respect to \(x_i, i \in \{1, 2, 3\}\) and estimate \(A\) and \(B\) separately. For this purpose, we suppose that \(g \in C^1(K)\) and use the density of \(C^1(K)\) in \(H^1(K)\) to extend the estimates to \(g \in H^1(K)\).
We first estimate \( A \). We have

\[
A = \int_0^a \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0,y_2)} \int_{J(0,y_1)} \left( g(x_1, x_2, x_3) - g(x_1, z_2(x_1, y_2), z_3(x_1, y_3)) \right)^2 dy_3 dy_2 dx_3 dx_2 dx_1.
\]

Let us suppose that \( x_3 \geq z_3 \). This implies that \([x_1, x_2, z_3(x_1, y_3)] \in K\), since the cross-section of \( K \) and the plane \( x_1 = \text{const} \) is a triangle whose bottom edge is horizontal and the longest of its three edges. We deduce the inequality

\[
\left( g(x_1, x_2, x_3) - g(x_1, z_2(x_1, y_2), z_3(x_1, y_3)) \right)^2 = \left( g(x_1, x_2, x_3) - g(x_1, x_2, z_3(x_1, y_3)) + g(x_1, x_2, z_3(x_1, y_3)) - g(x_1, z_2(x_1, y_2), z_3(x_1, y_3)) \right)^2 = \left( \int_{x_3}^{x_2} D_3 g(x_1, x_2, s) ds + \int_{z_2(x_1,y_2)}^{x_2} D_2 g(x_1, s, z_3(x_1, y_3)) ds \right)^2 \leq 2\text{diam}(K) \int_{J(x_1,x_2)} \left[ D_3 g(x_1, x_2, s) \right]^2 ds + 2\text{diam}(K)(1 - \frac{x_1}{a}) \int_{z_2(x_1,y_2)}^{x_2} \left[ D_2 g(x_1, s, z_3(x_1, y_3)) \right]^2 ds,
\]

where we have used the Newton integration formula and the Cauchy–Schwarz inequality. Defining \( D_i g, i \in \{1, 2, 3\} \) by 0 outside of \( K \) and considering also \( x_3 < z_3 \), we come to

\[
A \leq 2\text{diam}(K)(A_1 + A_2 + A_3 + A_4)
\]

with

\[
A_1 = \int_0^a \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0,y_2)} \int_{J(0,y_1)} \left[ D_3 g(x_1, x_2, s) \right]^2 ds dy_3 dy_2 dx_3 dx_2 dx_1,
\]

\[
A_2 = \int_0^a \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0,y_2)} \int_{J(0,y_1)} \left( 1 - \frac{x_1}{a} \right) \int_{z_2(x_1,y_2)} \left[ D_2 g(x_1, s, z_3(x_1, y_3)) \right]^2 ds dy_3 dy_2 dx_3 dx_2 dx_1,
\]

\[
A_3 = \int_0^a \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0,y_2)} \int_{J(0,y_1)} \int_{z_2(x_1,y_2)}^{x_2} \left[ D_2 g(x_1, s, x_3) \right]^2 ds dy_3 dy_2 dx_3 dx_2 dx_1,
\]

\[
A_4 = \int_0^a \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0,y_2)} \int_{J(0,y_1)} \left( 1 - \frac{x_1}{a} \right) \int_{z_3(x_1,y_3)}^{x_3} \left[ D_3 g(x_1, z_2(x_1, y_2), s) \right]^2 ds dy_3 dy_2 dx_3 dx_2 dx_1.
\]

We easily see that

\[
A_1 \leq \text{diam}(K)|\sigma| \int_K \left[ D_3 g(x) \right]^2 dx.
\]

Next, we estimate \( A_2 \). Using the Fubini theorem and the change of variables \( z_3 = z_3(x_1, y_3) \), we have

\[
\int_{J(0,y_2)} \left( 1 - \frac{x_1}{a} \right) \int_{z_2(x_1,y_2)}^{x_2} \left[ D_2 g(x_1, s, z_3(x_1, y_3)) \right]^2 ds dy_3 = \int_{z_2(x_1,y_2)}^{x_2} \int_{J(x_1,z_2(x_1,y_2))} \left[ D_2 g(x_1, s, z_3) \right]^2 dz_3 ds \leq \int_{J(x_1)} \int_{J(x_1,s)} \left[ D_2 g(x_1, s, z_3) \right]^2 dz_3 ds,
\]

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where the estimate follows by extending the integration region. Hence
\[ A_2 \leq \text{diam}(K)|\sigma| \int_K [D_2g(x)]^2 \, dx. \]

Using the Fubini theorem, we similarly estimate \( A_3 \) and \( A_4 \),
\[ A_3 \leq \text{diam}(K)|\sigma| \int_K [D_2g(x)]^2 \, dx, \]
\[ A_4 \leq \text{diam}(K)|\sigma| \int_K [D_3g(x)]^2 \, dx, \]
which finally yields
\[ A \leq 4\text{diam}(K)^2|\sigma| \int_K |\nabla g(x)|^2 \, dx. \quad (23) \]

We now turn to the study of \( B \). We write it as
\[
B = \int_0^\alpha \int_{J(x_1)} \int_{J(x_1, x_2)} J(0) \int_{J(0, y_2)} (g(x_1, z_2(x_1, y_2), z_3(x_1, y_3)) - \\
g(0, y_2, y_3))^2 \, dy_2 \, dx_3 \, dx_2 \, dx_1.
\]

Using the Newton integration formula and the Cauchy–Schwarz and Hölder inequalities, we have
\[
\left( g(x_1, z_2(x_1, y_2), z_3(x_1, y_3)) - g(0, y_2, y_3) \right)^2 = \left( \int_0^{x_1} \left[ D_1g(s, z_2(s, y_2), z_3(s, y_3)) + \\
D_2g(s, z_2(s, y_2), z_3(s, y_3)) \frac{\beta - y_2}{\alpha} + D_3g(s, z_2(s, y_2), z_3(s, y_3)) \frac{\gamma - y_3}{\alpha} \right] ds \right)^2 \leq \\
\alpha \left( 1 + \left( \frac{\beta - y_2}{\alpha} \right)^2 + \left( \frac{\gamma - y_3}{\alpha} \right)^2 \right) \int_0^{x_1} \sum_{i=1}^{3} [D_i g(s, z_2(s, y_2), z_3(s, y_3))]^2 \, ds.
\]

Hence
\[
B \leq \alpha \left( 1 + \left( \frac{\beta - y_2}{\alpha} \right)^2 + \left( \frac{\gamma - y_3}{\alpha} \right)^2 \right) \sum_{i=1}^{3} B_i
\]
with
\[
B_i = \int_0^\alpha \int_{J(x_1)} \int_{J(x_1, x_2)} J(0) \int_{J(0, y_2)} \int_0^{x_1} [D_i g(s, z_2(s, y_2), z_3(s, y_3))]^2 \, ds \, dy_2 \, dx_3 \, dx_2 \, dx_1,
\]
i \in \{1, 2, 3\}. Using the Fubini theorem, we have
\[
B_i = \int_{J(0)} \int_{J(0, y_2)} \int_0^\alpha [D_i g(s, z_2(s, y_2), z_3(s, y_3))]^2 \int_s^\alpha \int_{J(x_1)} \int_{J(x_1, x_2)} \, dx_3 \, dx_2 \, dx_1 \, ds \, dy_3 \, dy_2.
\]
Hence
\[
B_i \leq \frac{|\sigma|}{2\alpha} \int_0^\alpha \int_{J(0, y_2)} [D_i g(s, z_2(s, y_2), z_3(s, y_3))]^2 (\alpha - s)^2 \, dy_3 \, dy_2 \, ds,
\]

where we have used the estimate \( \int_{J(x_1)} \int_{J(x_1,x_2)} dx_3 \, dx_2 \leq |\sigma|(1 - \frac{x_1}{\alpha}) \) on the area of the cross-section of \( K \) and the plane \( x_1 = \text{const} \). Now using the change of variables \( z_3 = z_3(s,y_3) \) and \( z_2 = z_2(s,y_2) \) gives

\[
\int_{J(0)} \int_{J(0,y_2)} [D_i g(s,z_2(s,y_2), z_3(s,y_3))]^2 (\alpha - s)^2 \, dy_3 \, dy_2 = \\
= \alpha^2 \int_{J(s)} \int_{J(s,z_2)} [D_i g(s,z_2,z_3)]^2 \, dz_3 \, dz_2
\]

and thus

\[
B_i \leq \frac{|\sigma|\alpha}{2} \int_K [D_i g(x)]^2 \, dx,
\]

which finally gives, noticing that \( \alpha^2 + (\beta - y_2)^2 + (\gamma - y_3)^2 = |\mathbf{a} - \mathbf{y}|^2 \leq \text{diam}(K)^2 \),

\[
B \leq \frac{|\sigma|}{2} \text{diam}(K)^2 \int_K |\nabla g(x)|^2 \, dx. \tag{24}
\]

Now combining (23) and (24) leads to the assertion of the lemma for \( d = 3 \). \( \Box \)

## 5 Discrete Friedrichs inequality

**Lemma 5.1** Let \( d = 2 \). Then

\[
|I(g)|^2_{1,T,*} \leq \frac{C_d}{k_h} |g|^2_{1,T} \quad \forall g \in W(T_h),
\]

where \( C_d \) is given by (26) below.

**Proof:**

Let \( K \in T_h \) and \( \sigma_D, \sigma_E \in \mathcal{E}_K \). We define \( g_K \) by (18) and deduce from the inequality \((a - b)^2 \leq 2a^2 + 2b^2\) and from (20) that

\[
(g_E - g_D)^2 \leq 2(g_E - g_K)^2 + 2(g_D - g_K)^2 \leq 4c_d \frac{\text{diam}(K)^2}{|K|} \int_K |\nabla g(x)|^2 \, dx. \tag{25}
\]

Using this, the definition of \( |\cdot|_{1,T,*}, |\sigma_{D,E}| \leq \frac{2}{3} \text{diam}(K) \), (9) and (8), the fact that each \( K \in T_h \) contains exactly three dual edges, and Assumption (A), we have

\[
|I(g)|^2_{1,T,*} = \sum_{\sigma_{D,E} \in \mathcal{F}_h} \frac{|\sigma_{D,E}|}{v_{D,E}} (g_E - g_D)^2 \leq \\
\leq 4c_d \sum_{K \in T_h} \sum_{\sigma_{D,E} \in \mathcal{F}_h, \sigma_{D,E} \in K} \frac{|\sigma_{D,E}|^2}{v_{D,E}|\sigma_{D,E}|} \frac{\text{diam}(K)^2}{|K|} \int_K |\nabla g(x)|^2 \, dx \leq \\
\leq 8c_d \sum_{K \in T_h} \left[ \frac{\text{diam}(K)^2}{|K|} \right]^2 \int_K |\nabla g(x)|^2 \, dx \leq \frac{8c_d}{k_h^2} \sum_{K \in T_h} \int_K |\nabla g(x)|^2 \, dx. \quad \square
\]
Lemma 5.2 There holds

\[ |I(g)|_{1,T}^2 \leq \frac{C_d}{\kappa_T} |g|_{1,T}^2 \quad \forall g \in W(T_h), \]

where

\[ C_d = 8c_d \text{ for } d = 2, \quad C_d = \frac{27}{4} c_d \text{ for } d = 3, \tag{26} \]

and \( c_d \) is given by (22).

Proof:
Using the definition of \( \cdot \|_1,T,\cdot \), (25), \( |\sigma_{D,E}| \leq C_d \text{diam}(K_{D,E})^{d-1} \) with \( C_d = \frac{2}{3} \) when \( d = 2 \) and \( C_d = \frac{9}{32} \) when \( d = 3 \), the fact that each \( K \in T_h \) contains \( \left( \frac{d+1}{2} \right) = \frac{(d+1)d}{2} \) dual sides, and Assumption (A), we have

\[ |I(g)|_{1,T}^2 = \sum_{\sigma_{D,E} \in F_{h}^{int}} \frac{|\sigma_{D,E}|}{\text{diam}(K_{D,E})} (g_E - g_D)^2 \leq 4c_d \sum_{K \in T_h} \sum_{\sigma_{D,E} \in F_{h}^{int}, \sigma_{D,E} \subset K} \frac{|\sigma_{D,E}|}{|K|} \int_K |\nabla g(x)|^2 \, dx \leq 2c_d(d+1) d C_d \sum_{K \in T_h} \frac{\text{diam}(K)^d}{|K|} \int_K |\nabla g(x)|^2 \, dx \leq \frac{C_d}{\kappa_T} \sum_{K \in T_h} \int_K |\nabla g(x)|^2 \, dx. \quad \square \]

Lemma 5.3 (Interpolation estimate) There holds

\[ ||g - I(g)||_{0,\Omega}^2 \leq c_d h^2 |g|_{1,T}^2 \quad \forall g \in W(T_h). \]

Proof:
We have

\[ ||g - I(g)||_{0,\Omega}^2 = \sum_{K \in T_h} \sum_{\sigma_{D,E} \in E_K} \int_{K \cap D} [g(x) - g_D]^2 \, dx \leq c_d \sum_{K \in T_h} \sum_{\sigma_{D,E} \in E_K} (\text{diam}(K \cap D))^2 \int_{K \cap D} |\nabla g(x)|^2 \, dx \leq c_d h^2 \sum_{K \in T_h} \int_K |\nabla g(x)|^2 \, dx, \]

using the estimate (21) for the simplex \( K \cap D \) and \( \text{diam}(K \cap D) \leq h. \quad \square \]

We state below the first of the two main results of this paper.

Theorem 5.4 (Discrete Friedrichs inequality) Let \( \mathbf{b} \) be a fixed unit vector. Then

\[ ||g||_{0,\Omega}^2 \leq C_F |g|_{1,T}^2 \quad \forall g \in W_0(T_h), \forall h > 0 \]

with

\[ C_F = \frac{C_d}{\kappa_T} |\Omega| + 2c_d h^2 \text{ for } d = 2, \quad C_F = 2C_d \frac{C_d}{\kappa_T} |\text{diam}_n(\Omega)|^2 + 2c_d h^2 \text{ for } d = 2, 3, \]

where \( C_d, T \) is given by (12) when Assumption (B) is satisfied and by (15) in the general case, \( c_d \) is given by (22), and \( C_d \) is given by (26).
Proof:

One has

$$\|g\|_{0,\Omega}^2 \leq 2\|g - I(g)\|_{0,\Omega}^2 + 2\|I(g)\|_{0,\Omega}^2.$$ 

The error $$\|g - I(g)\|_{0,\Omega}^2$$ of the approximation follows from Lemma 5.3. Note that $$I(g) \in Y_0(D_h)$$ and hence the discrete Friedrichs inequality for piecewise constant functions given by Theorem 3.1 and Lemma 5.1 yield

$$\|I(g)\|_{0,\Omega}^2 \leq \frac{C_d}{2\kappa_T} |\Omega| \|g\|_{1,T}^2$$

for the case where $$d = 2$$. Similarly, using discrete Friedrichs inequality for piecewise constant functions given by Theorem 3.5 and Lemma 5.2, one has

$$\|I(g)\|_{0,\Omega}^2 \leq C_d \frac{\kappa_T}{\kappa_T} [\text{diam}_b(\Omega)]^2 \|g\|_{1,\mathcal{T}}^2$$

for the case where $$d = 2, 3$$. 

Remark 5.5 (Dependence of $$C_F$$ on $$\Omega$$) We have $$h^2 \leq \frac{|\Omega|}{\kappa_T}$$ by Assumption (A) and $$h \leq \theta_T \text{diam}_b(\Omega)$$ by the consequence (5) of Assumption (A). Hence the constant in the discrete Friedrichs inequality only depends on the area of $$\Omega$$ when $$d = 2$$ and on the square of the diameter of $$\Omega$$ in one chosen direction when $$d = 2, 3$$. This dependence is optimal: [8, Theorem 1.1] gives the same dependence for the Friedrichs inequality and $$H_0^1(\Omega) \subset W_0(\mathcal{T}_h)$$. 

Remark 5.6 (Dependence of $$C_F$$ on the shape regularity parameter) One can see that $$C_F$$ depends on $$\frac{1}{\kappa_T^2}$$ when $$d = 2$$ and when it is expressed using $$|\Omega|$$. We are able to establish the same result also when $$C_F$$ is expressed using $$\text{diam}_b(\Omega)$$, but only when the meshes are not locally refined (when Assumption (B) is satisfied). Indeed, $$C_F$$ in this case depends on $$\frac{C_d}{\kappa_T}$$ and the constant $$C_d$$ given by (12) is of the form $$\frac{2^d(d-1)C_d^2(2C+1)}{\kappa_T}$$; this follows by replacing the inequality (13) by $$h \leq C\text{diam}_b(\Omega)$$ for some suitable constant $$C$$. Example 6.3 shows that this dependence is optimal. However, in the case where the meshes are only shape regular, we were only able to establish (15). 

Remark 5.7 (Discrete Friedrichs inequality for domains only bounded in some direction) We see that the constant $$C_F$$ only depends on the diameter of $$\Omega$$ in one chosen direction. Thus the discrete Friedrichs inequality may be extended onto domains only bounded in one direction, as it is the case for the Friedrichs inequality (cf. [8, Remark 1.1]). 

Remark 5.8 (Discrete Friedrichs inequality for functions only fixed to zero on a part of the boundary) The discrete Friedrichs inequality can be extended to functions only fixed to zero on a part of the boundary, see Lemma 7.2 and Remark 7.3 below. Then, for convex domains, $$C_F$$ depends on the square of the diameter of $$\Omega$$; for nonconvex domains, the dependence of $$C_F$$ on $$\Omega$$ is more complicated. Our results indicate that the dependence of $$C_F$$ on the shape regularity parameter is in this case given by $$\frac{C_d}{\kappa_T}$$, cf. Remark 5.6.
6 Discrete Friedrichs inequality for Crouzeix–Raviart finite elements in space dimension two

We show in this section how the proofs from the previous sections simplify for the case of Crouzeix–Raviart finite elements in two space dimensions. Let us consider the space $X(T_h)$ introduced in Section 2. The basis of this space is spanned by the shape functions $\varphi_D$, $D \in \mathcal{D}_h$ such that $\varphi_D(Q_E) = \delta_{DE}$, $E \in \mathcal{D}_h$, $\delta$ being the Kronecker delta.

**Lemma 6.1** Let $d = 2$. Then for all $c \in X(T_h)$,

$$\|c\|_{0,\Omega} = \|I(c)\|_{0,\Omega}.$$

**Proof:**

Let us write $c = \sum_{D \in \mathcal{D}_h} c_D \varphi_D$. We have

$$\int_{\Omega} c^2(x) \, dx = \sum_{K \in \mathcal{T}_h} \int_K c^2(x) \, dx = \sum_{K \in \mathcal{T}_h} \frac{1}{3} |K| \sum_{\sigma_D \in \mathcal{E}_K} c^2(Q_D) = \sum_{D \in \mathcal{D}_h} c^2_D |D|,$$

where we have used the fact that the quadrature formula $\int_K \psi \, dS \approx \frac{1}{3} |K| \sum_{\sigma_D \in \mathcal{E}_K} \psi(Q_D)$ is exact for quadratic functions on triangles and (8).

**Lemma 6.2 (Discrete Friedrichs inequality for Crouzeix–Raviart finite elements in 2-D)** Let $d = 2$ and let $b$ be a fixed unit vector. Then

$$\|c\|_{0,\Omega}^2 \leq C_F |c|_{1,T}^2 \quad \forall c \in X_0(T_h), \forall h > 0,$$

where $C_F = \frac{1}{4\kappa_T^2} |\Omega|$ or $C_F = \frac{C_{d,T}}{2\kappa_T} \|\text{diam}_B(\Omega)\|^2$.

**Proof:**

Let $c \in X_0(T_h)$, $c = \sum_{D \in \mathcal{D}_h} c_D \varphi_D$. Note that by the definition of $X_0(T_h)$, $c_D = 0$ for all $D \in \mathcal{D}_h$. Using respectively Lemma 6.1 and Theorem 3.1 or Theorem 3.5, we get

$$\|c\|_{0,\Omega}^2 \leq \frac{|\Omega|}{2} |I(c)|_{1,T}^2, \quad \|c\|_{0,\Omega}^2 \leq C_{d,T} \|\text{diam}_B(\Omega)\|^2 |I(c)|_{1,T}^2.$$

Finally, we deduce that

$$|I(c)|_{1,T}^2 = \sum_{\sigma_D \in F_{int}^h} \frac{|\sigma_{D,E}|^2}{c_{D,E}^2 |\sigma_{D,E}|} \left|\nabla c|_{K_{D,E}} \cdot (Q_E - Q_D)\right|^2 \leq \frac{2}{3} \sum_{\sigma_D \in F_{int}^h} \frac{\text{diam}(K_{D,E})^2}{|K_{D,E}|} \left|\nabla c|_{K_{D,E}} \right|^2 \frac{d_{D,E}^2}{|K_{D,E}|} \leq \frac{1}{2\kappa_T^2} \sum_{K \in \mathcal{T}_h} \left|\nabla c|_K \right|^2 |K| = \frac{1}{2\kappa_T^2} |c|_{1,T}^2,$$

using (9) and (8), $|\sigma_{D,E}| \leq \frac{2}{3} \text{diam}(K_{D,E})$, the fact that the gradient of $c$ is elementwise constant and that each $K \in \mathcal{T}_h$ contains exactly three dual edges, $d_{D,E} \leq \frac{\text{diam}(K_{D,E})}{2}$, and Assumption (A). Similarly, $|I(c)|_{1,T}^2 \leq \frac{1}{2\kappa_T^2} |c|_{1,T}^2$. □
Figure 2: Domain $\Omega$, triangulation $T_h$, dual mesh $D_h$, and values of a function $c \in X_0(T_h)$ for the optimality example

Example 6.3 (Optimality of the dependence of $C_F$ on the shape regularity parameter) Let us consider a domain $\Omega = (-\infty, +\infty) \times (0, v)$, its triangulation $T_h$ consisting of isosceles triangles with basis $h$ and height $v$, and a function $c \in X_0(T_h)$ given by the values 0, 1, -1 as depicted in Figure 2. Using Lemma 6.1, we immediately have

$$k_c k_2^0 = \sum_{K \in T_h} \frac{1}{3} |K| (0 + 1 + 1) = \frac{2}{3} |\Omega|.$$ We have $\nabla c|_K = \frac{4}{h}$ on each $K \in T_h$, hence

$$|c|_0^2 = \frac{16}{h^2} |\Omega|.$$ The term occurring on the right hand side of the discrete Friedrichs inequality for Crouzeix–Raviart finite elements in 2-D given by Lemma 6.2, independent of the shape regularity parameter, is $\frac{1}{2} |\text{diam}_b(\Omega)|^2 |c|_1^2 = \frac{8v^2}{h^2} |\Omega|$. This term can be arbitrarily smaller than $\|c\|_0^2$, letting $h \to +\infty$. Next, $\kappa_T = \frac{v}{2h}$. Note that $T_h$ satisfies Assumption (B) and hence $C_{d,T} \approx \frac{1}{\kappa_T}$. In fact, by a simple estimation of the term $A$ from Lemma 3.3, one has

$$C_{d,T} = \frac{1}{\kappa_T}$$ and thus $\frac{C_{d,T}}{\kappa_T} = \frac{1}{\kappa_T^2} = \frac{4h^2}{v^2}$. One immediately sees that the multiplication by this term is necessary.

Corollary 6.4 (Discrete Friedrichs inequality for Crouzeix–Raviart finite elements on equilateral triangles) Let $d = 2$, let $b$ be a fixed unit vector, and let $T_h$ consist of equilateral triangles. Then

$$\|c\|_0^2 \leq C_F |c|_{1,T}^2 \quad \forall c \in X_0(T_h), \forall h > 0,$$

where $C_F = \frac{|\Omega|}{2}$ or $C_F = [\text{diam}_b(\Omega) + 2h]^2$.

Proof:

Let $c$ be as in the previous lemma. For equilateral triangles, one has $d_{D,E} = v_{D,E}$ and thus the norms $| \cdot |_{1,T,*}$ and $| \cdot |_{1,T,3}$ coincide. By (9) and (8), $|\sigma_{D,E}|v_{D,E} = \frac{2}{3} |K|$, $\cos^2(\alpha) + \cos^2(\alpha + \frac{\pi}{3}) + \cos^2(\alpha + \frac{2\pi}{3}) = \frac{3}{2}$, so that

$$\sum_{K \in T_h} \sum_{\sigma_{D,E} \in F_h^{int}, \sigma_{D,E} \in K} \frac{|\sigma_{D,E}|}{d_{D,E}} |\nabla c|_K^2 d_{D,E}^2 \cos^2(\nabla c|_K, Q_E - Q_D) = \sum_{K \in T_h} |\nabla c|_K^2 |K|.$$ Using respectively Lemma 6.1, Theorem 3.1 or Remark 3.2, and the above equality yields the assertion. $\square$
Remark 6.5 \((C_F\) for Crouzeix–Raviart finite elements on equilateral triangles)\)
Let \(d = 2\) and let \(b\) be a fixed unit vector. Then the constant in the Friedrichs inequality may be expressed as \(c_F = \frac{\Omega}{2}\) or \(c_F = |\text{diam}_b(\Omega)|^2\), cf. [8, Theorem 1.1]. Corollary 6.4 shows that for Crouzeix–Raviart finite elements and equilateral triangles, we are able to achieve the same result (up to \(h\)) also for the constant \(C_F\) from the discrete Friedrichs inequality. We however remark that there exist sharper estimates on the constant \(c_F\) in the Friedrichs inequality, see e.g. [9].

7 Discrete Poincaré inequality for piecewise constant functions

As in the case of the discrete Friedrichs inequality, we start with the discrete Poincaré inequality for piecewise constant functions. [5, Lemma 10.2] states the discrete Poincaré inequality for piecewise constant functions on meshes satisfying the orthogonality property. We present in this section an analogy of this lemma for the mesh \(D_h\), where the orthogonality property is not necessarily satisfied.

Lemma 7.1 Let \(\omega\) be an open convex subset of \(\Omega\), \(\omega \neq 0\), and let \(m_\omega(c) = \frac{1}{|\omega|} \int_\omega c(x) \, dx\). Then for all \(c \in Y(D_h)\),
\[
\|c - m_\omega(c)\|_{0, \omega}^2 \leq \frac{|B_\Omega|}{|\omega|} C_{d,T} [\text{diam}(\Omega)]^2 |c|_{1,T,\Omega}^2,
\]
where \(B_\Omega\) is the ball of \(\mathbb{R}^d\) with center \(0\) and radius \(\text{diam}(\Omega)\) and \(C_{d,T}\) is given by (12) when Assumption (B) is satisfied and by (15) in the general case.

The proof of this lemma follows the proof of the first step of [5, Lemma 10.2], using the techniques introduced in Section 3 for meshes where the orthogonality property is not satisfied.

Lemma 7.2 Let \(\omega\) be a polygonal open convex subset of \(\Omega\) and let \(I \subset \partial \omega\) with \(|I| > 0\). Let \(E \in D_h\) be such that \(I \cap E\) is not an empty set and not just a point (such dual element always exists). Then for all \(c \in Y(D_h)\) with \(c_E = 0\),
\[
\|c\|_{0, \omega}^2 \leq C_{d,T} [\text{diam}(\Omega)]^2 |c|_{1,T,\Omega}^2,
\]
where \(C_{d,T}\) is given by (12) when Assumption (B) is satisfied and by (15) in the general case.

Proof:
The proof is similar to that of Theorem 3.5. There exist a set of vectors \(b_i\) and of nonempty non overlapping subsets \(\omega_i\) of \(\omega\), \(i = 1, \ldots, M\) (\(M\) may be equal to \(+\infty\)) with the following properties: (i) \(b_i\) is such that \(C_{b_i} \cap \omega \neq \emptyset\), where \(C_{b_i}\) is the cylinder whose basis is \(I \cap E\) and generator vector is \(-b_i\); (ii) \(\omega_i = C_{b_i} \cap \omega \setminus \bigcup_{j=1}^{i-1} \omega_j\); (iii) \(\bigcup_{i=1}^{M} \omega_i = \omega\). Note that the fact that \(\omega\) is convex is important. For all \(x \in \omega_i\), we set \(E^x_\omega\) as the straight semi-line defined by the origin \(x\) and the vector \(b_i\). Let \(y(x) = I \cap E \cap E^x_\omega\). Let the function \(\chi_{\sigma}(x)\) be given by (16) for each \(\sigma \in F^h_{\text{int}}\). Let \(D \in D_h\), \(D \cap \omega_i \neq \emptyset\) be fixed. We then have (17) for a.e. \(x \in D \cap \omega_i\), as in Theorem 3.5. Integrating (17) over \(D \cap \omega_i\), summing over all \(D \in D_h\) such
that $D \cap \omega_i \neq \emptyset$, and using Lemma 3.3 when Assumption (B) is satisfied and Lemma 3.4 in the general case yields

$$\sum_{D \in \mathcal{D}_h} |c_D|^2 |D \cap \omega_i| \leq C_{d,T} \text{diam}(\Omega) \sum_{\sigma_{D,E} \in \mathcal{F}_h} \frac{(c_E - c_D)^2}{\text{diam}(K_{D,E})} \int_{\omega_i} \chi_{\sigma_{D,E}}(x) \, dx.$$ 

Using the inequality

$$\int_{\omega_i} \chi_{\sigma_{D,E}}(x) \, dx \leq |\sigma_{D,E} \cap \omega_i| \text{diam}(\omega),$$

diam($\omega$) $\leq$ diam($\Omega$), and summing over all $i$ concludes the proof. $\square$

**Remark 7.3** Let $\Omega$ be convex. Then we can take $\omega = \Omega$ in Lemma 7.2 and have an extension of Theorem 3.5 onto functions from $Y(D_h)$ which vanish on only one boundary dual element.

**Remark 7.4** Lemma 7.2 is an alternative to the second step of the proof of [5, Lemma 10.2].

**Theorem 7.5** (Discrete Poincaré inequality for piecewise constant functions) Let $m_\Omega(c) = \frac{1}{|\Omega|} \int_{\Omega} c(x) \, dx$. Then for all $c \in Y(D_h)$,

$$||c - m_\Omega(c)||_{0,\Omega}^2 \leq C_\Omega C_{d,T} [\text{diam}(\Omega)]^2 |c|_{1,T,\Omega}^2,$$

where

$$C_\Omega = \frac{|B_\Omega|}{|\Omega|}$$

when $\Omega$ is convex and

$$C_\Omega = 2 \sum_{i=1}^n \frac{|B_{\Omega_i}|}{|\Omega_i|} + 16(n - 1)^2 \frac{|\Omega|}{|\Omega_i|_{\text{min}}} \left( \frac{|B_\Omega|}{|\Omega_i|_{\text{min}}} + 1 \right)$$

when $\Omega$ is not convex but there exists a finite number of disjoint open convex polygonal sets $\Omega_i$ such that $\Omega = \bigcup_{i=1}^n \Omega_i$. Here, $|\Omega_i|_{\text{min}} = \min_{i=1, \ldots, n} \{|\Omega_i|\}$, $B_\Omega$ is the ball of $\mathbb{R}^d$ with center 0 and radius diam($\Omega$), and $C_{d,T}$ is given by (12) when Assumption (B) is satisfied and by (15) in the general case.

**Proof:**

When $\Omega$ is convex, the assertion of this Theorem coincides with that of Lemma 7.1 for $\omega = \Omega$. When $\Omega$ is not convex, we have Lemmas 7.1 and 7.2 for each $\Omega_i$. Then the third step of the proof of [5, Lemma 10.2] yields the assertion of the theorem. $\square$

**Remark 7.6** One has

$$||c||_{0,\Omega}^2 \leq 2||c - m_\Omega(c)||_{0,\Omega}^2 + 2||m_\Omega(c)||_{0,\Omega}^2.$$

Hence Theorem 7.5 implies the discrete Poincaré inequality for piecewise constant functions in the form

$$||c||_{0,\Omega}^2 \leq 2C_\Omega C_{d,T} [\text{diam}(\Omega)]^2 |c|_{1,T,\Omega}^2 + \frac{2}{|\Omega|} \left( \int_{\Omega} c(x) \, dx \right)^2 \quad \forall c \in Y(D_h), \forall h > 0.$$
8 Discrete Poincaré inequality

We state below the second of the two main results of this paper.

**Theorem 8.1 (Discrete Poincaré inequality)** There holds

\[ \|g\|_{0, \Omega}^2 \leq C_P |g|_{1,T}^2 + \frac{4}{|\Omega|} \left( \int_{\Omega} g(x) \, dx \right)^2 \quad \forall g \in W(T_h), \forall h > 0 \]

with

\[ C_P = 4C_dC_{\Omega}C_{d,T} [\text{diam}(\Omega)]^2 + 8c_dh^2, \]

where \( C_\Omega \) is given by (27) when \( \Omega \) is convex and by (28) otherwise, \( C_{d,T} \) is given by (12) when Assumption (B) is satisfied and by (15) in the general case, \( c_d \) is given by (22), and \( C_d \) is given by (26).

**Proof:**

One has

\[ \|g\|_{0, \Omega}^2 \leq 4\|g - I(g)\|_{0, \Omega}^2 + 4\|I(g) - m_\Omega[I(g)]\|_{0, \Omega}^2 + 4\|m_\Omega[I(g)] - m_\Omega(g)\|_{0, \Omega}^2 + 4\|m_\Omega(g)\|_{0, \Omega}^2, \]

where \( m_\Omega(f) = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx \). The discrete Poincaré inequality for piecewise constant functions given by Theorem 7.5 and Lemma 5.2 imply

\[ \|I(g) - m_\Omega[I(g)]\|_{0, \Omega}^2 \leq C_dC_{\Omega}C_{d,T} [\text{diam}(\Omega)]^2 |g|_{1,T}^2. \]

We have

\[ \|m_\Omega[I(g)] - m_\Omega(g)\|_{0, \Omega}^2 \leq \|g - I(g)\|_{0, \Omega}^2 \]

by the Cauchy–Schwarz inequality. Finally, the error \( \|g - I(g)\|_{0, \Omega}^2 \) of the approximation follows from Lemma 5.3.

**Remark 8.2 (Dependence of \( C_P \) on \( \Omega \))** Let \( \Omega \) be a cube. We then have \( h \leq \text{diam}(\Omega) \)

and \( C_\Omega \leq \frac{\sqrt{3}}{2} \) and hence the constant in the discrete Poincaré inequality in this case only depends on the square of the diameter of \( \Omega \). This dependence is optimal: [8, Theorem 1.3] gives the same dependence for the Poincaré inequality and \( H^1(\Omega) \subset W(T_h) \).

**Remark 8.3 (Dependence of \( C_P \) on the shape regularity parameter)** Our results indicate that the dependence of \( C_P \) on the shape regularity parameter is given by \( \frac{C_{d,T}}{\kappa_T} \), cf. Remark 5.6.

**References**


