STATIONARY MEASURES AND INVARIANT SUBSETS OF HOMOGENEOUS SPACES (III)

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Abstract. Let $G$ be a real Lie group, $A$ be a lattice in $G$ and $\Gamma$ be a compactly generated closed subgroup of $G$. If the Zariski closure of the group $\text{Ad}(\Gamma)$ is semisimple with no compact factor, we prove that every $\Gamma$-orbit closure in $G/A$ is a finite volume homogeneous space. We also establish related equidistribution properties.

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1. Introduction

1.1. Orbits closures, the real case. Let $G$ be a real Lie group with Lie algebra $\mathfrak{g}$. We let $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ denote the adjoint representation of $G$. In this article, by using the results of [4], we shall prove the following

**Theorem 1.1.** Let $G$ be a real Lie group, $\Lambda$ be a lattice in $G$ and $\Gamma$ be a compactly generated closed sub-semigroup of $G$. We assume that the Zariski closure of the semigroup $\text{Ad}(\Gamma) \subset \text{GL}(\mathfrak{g})$ is semisimple with no compact factor. Then, for every $x$ in $G/\Lambda$, there exists a closed subgroup $H$ of $G$ with $\Gamma \subset H$ such that $\Gamma x = Hx$ and $Hx$ carries a $H$-invariant probability measure $\nu_x$.

In all this article, by a semisimple algebraic group, we mean a Zariski connected semisimple algebraic group.

This result on orbits closures answers a question by Shah [26] and Margulis [17]. In case $\text{Ad}\Gamma$ itself is a semisimple subgroup of $\text{GL}(\mathfrak{g})$ with no compact factor, it follows from Ratner’s Theorem [24]. Under the stronger assumption that $G$ is simple and $\Gamma$ is Zariski dense in $G$, Theorem 1.1 is the main result of [2].

Theorem 1.1 is already new in the following “concrete” cases:
- when $G = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ and $\Gamma$ is Zariski dense in $G$.
- when $G = \text{SL}(3, \mathbb{R})$ and $\Gamma$ is Zariski dense in the subgroup $\text{SO}(2, 1)$.

Let $G$, $\Lambda$ and $\Gamma$ be as above and set $X = G/\Lambda$. For any $x$ in $X$, we let $\nu_x$ be, as in Theorem 1.1, the unique probability measure on $\Gamma x$ which is invariant under the stabilizer of this set in $G$. We shall say that a sequence of Borel probability measures $(\nu_n)$ on $X$ converges toward a Borel probability measure $\nu$ if, for any continuous compactly supported function $\varphi$ on $X$, $\int_X \varphi \, d\nu_n$ converges toward $\int_X \varphi \, d\nu$. Theorem 1.1 will follow from the following equidistribution result.

**Theorem 1.2.** Let $G$, $\Lambda$ and $\Gamma$ be as above and let $\mu$ be a compactly supported Borel probability measure on $\Gamma$ whose support spans a dense sub-semigroup of $\Gamma$. Then, for every $x$ in $G/\Lambda$, one has

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu^k \ast \delta_x = \nu_x.$$
Theorem 1.3. Let $G$, $\Lambda$, $\Gamma$ and $\mu$ be as above. Let $g_1, \ldots, g_n, \ldots$ be a sequence of independent identically distributed random elements of $\Gamma$ with law $\mu$. Then, for every $x$ in $G/\Lambda$, almost surely,

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} \xrightarrow{n \to \infty} \nu_x.$$ 

In Theorem 1.3 “almost surely” means for $\mu^\otimes \mathbb{N}^*$-almost every choice of the sequence $g_1, \ldots, g_n, \ldots$

This result may be seen as a random analogue of the equidistribution properties of unipotent flows on homogeneous spaces, due to Ratner [24] and Dani-Margulis [10].

1.2. Orbits closures, the S-adic case. Let $p$ be a prime number. As in [25], a $p$-adic Lie group $G$ is said to be weakly regular if any two of its one-parameter subgroups $\varphi_1, \varphi_2 : \mathbb{Q}_p \to G$ are equal as soon as their derivatives at $e$ are equal. Any real Lie group is said to be weakly regular.

Fix a finite set $S$ whose elements are prime numbers and $\infty$. In this paper, as in [4], by a weakly regular $S$-adic Lie group, we shall mean a topological group that is isomorphic to a closed subgroup of a product of weakly regular $p$-adic Lie groups, $p \in S$.

Let $G$ be a weakly regular $S$-adic Lie group with Lie algebra $\mathfrak{g} = \bigoplus_{p \in S} \mathfrak{g}_p$ and $\Gamma$ be a sub-semigroup of $G$. We let $\overline{\text{Ad} \Gamma^Z}$ denote the Zariski closure of the image of $\Gamma$ under the adjoint representation of $G$, that is the product $\prod_{p \in S} \overline{\text{Ad} \mathfrak{g}_p} \Gamma^Z$ of the Zariski closures of the images of $\Gamma$ in $\text{GL}(\mathfrak{g}_p)$, $p \in S$. We also let $\overline{\text{Ad} \Gamma^Z,nc}$ denote the smallest normal Zariski closed subgroup of $\overline{\text{Ad} \Gamma^Z}$ such that the image of $\Gamma$ in $\overline{\text{Ad} \Gamma^Z} / \overline{\text{Ad} \Gamma^Z,nc}$ is bounded.

We get the following $S$-adic extension of Theorems 1.1, 1.2 and 1.3:

Theorem 1.4. Let $G$ be a weakly regular $S$-adic Lie group, $\Lambda$ be a lattice in $G$ and $\Gamma$ be a closed compactly generated sub-semigroup of $G$ such that $\overline{\text{Ad} \Gamma^Z}$ is semisimple and equal to $\overline{\text{Ad} \Gamma^Z,nc}$.

a) For every $x$ in $G/\Lambda$, there exists a closed subgroup $H \supset \Gamma$ of $G$ such that $\Gamma x = H x$ and $H x$ carries a $H$-invariant probability measure $\nu_x$.

b) If $\mu$ is a compactly supported Borel probability measure on $\Gamma$ whose support spans a dense sub-semigroup of $\Gamma$, then

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^k * \delta_x \xrightarrow{n \to \infty} \nu_x.$$
c) More precisely, if \( g_1, \ldots, g_n, \ldots \) is a sequence of independent identically distributed random elements of \( \Gamma \) with law \( \mu \), then, almost surely,

\[
\frac{1}{n} \sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} \xrightarrow{n \to \infty} \nu_x.
\]

Note that Theorem 1.4 is already new in the following “concrete” cases:

- when \( G = \text{SL}(2, \mathbb{Q}_p) \) and \( \Gamma \) is Zariski dense and unbounded in \( G \).
- when \( G = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{Q}_p) \) and the projection of \( \Gamma \) on each factor is Zariski dense and unbounded.

1.3. **Equidistribution of invariant subsets.** Our methods also allow us to get some properties of the set of invariant homogeneous subsets.

Let \( G \) be a locally compact topological group, \( \Lambda \) be a discrete subgroup of \( G \) and \( X = G/\Lambda \). We shall say that a closed subset \( Y \) of \( X \) is a **finite volume homogeneous subspace** if the stabilizer \( G_Y \) of \( Y \) in \( G \) acts transitively on \( Y \) and preserves a Borel probability measure \( \nu_Y \) on \( Y \). If \( \Gamma \) is a sub-semigroup of \( G_Y \), we shall say that \( Y \) is \( \Gamma \)-**ergodic** if \( \Gamma \) acts ergodically on \( (Y, \nu_Y) \).

Let \( \Gamma \) be a sub-semigroup of \( G \). We set

\[
S_X(\Gamma) := \{ \text{\Gamma-invariant and \Gamma-ergodic finite volume homogeneous subspaces } Y \text{ of } X \}.
\]

When \( G, \Lambda, X, \Gamma \) are as in Theorem 1.4, this set plays a key role in the proofs, since, according to the main result of [4], every \( \Gamma \)-invariant \( \Gamma \)-ergodic probability measure on \( X \) is equal to \( \nu_Y \) for some \( Y \) in \( S_X(\Gamma) \).

We may identify \( S_X(\Gamma) \) with a set of Borel probability measures on \( X \) through the map \( Y \mapsto \nu_Y \). In particular, we endow \( S_X(\Gamma) \) with the topology of weak convergence, so that a sequence \((Y_n)\) in \( S_X(\Gamma) \) converges toward \( Y_\infty \in S_X(\Gamma) \) if and only if \( \nu_{Y_n} \) converges toward \( \nu_{Y_\infty} \).

For every compact subset \( K \subset X \), we set

\[
S_K(\Gamma) := \{ Y \in S_X(\Gamma) \mid Y \cap K \neq \emptyset \}.
\]

As we will see in the Corollaries, the following Theorem 1.5 is very efficient to compute the limit of a sequence in \( S_X(\Gamma) \).

**Theorem 1.5.** Let \( G, \Lambda, \Gamma \) be as in Theorem 1.4 and let \( L \) be the centralizer of \( \Gamma \) in \( G \).

a) For every compact subset \( K \) of \( X \), the set \( S_K(\Gamma) \) is compact.

b) If \((Y_n) \subset S_X(\Gamma)\) converges to \( Y_\infty \in S_X(\Gamma) \), then there exists a sequence \((\ell_n) \subset L\) converging to \( e \) such that, for \( n \) large, \( Y_n \subset \ell_n Y_\infty \).
In particular, when $\Lambda$ is cocompact, the set $S_X(\Gamma)$ itself is compact.

We denote by $\overline{X} := X \cup \{\infty\}$ the one point compactification of $X$ and by $\delta_\infty$ the Dirac mass at $\infty$. The set $S_X(\Gamma) \cup \{\delta_\infty\}$ can be seen as a set of Borel probability measures on $\overline{X}$.

**Corollary 1.6.** Let $G$, $\Lambda$ and $\Gamma$ be as in Theorem 1.4. Then the set $S_X(\Gamma) \cup \{\delta_\infty\}$ is compact.

Corollary 1.6 is an analogue of the main theorem of Mozes and Shah in [20] (see also [12]) which asserts, in case $G$ is a real Lie group, if $E$ is the space of finite volume homogeneous subsets of $X$ which are invariant and ergodic under some Ad-unipotent one-parameter subgroup of $G$, then the set $E \cup \{\delta_\infty\}$ is compact.

When $\Gamma$ has discrete centralizer, the statement of Theorem 1.5 becomes simpler. A subset $F$ of $X = G/\Lambda$ is said to be $\Gamma$-invariant if $gF \subset F$ for all $g$ in $\Gamma$.

**Theorem 1.7.** Let $G$, $\Lambda$ and $\Gamma$ be as in Theorem 1.4. Assume the centralizer $L$ of $\Gamma$ in $G$ is discrete.

a) The set $S_X(\Gamma)$ is compact.
b) If $(Y_n) \subset S_X(\Gamma)$ converges to $Y_\infty \in S_X(\Gamma)$, then, for $n$ large, one has $Y_n \subset Y_\infty$.
c) Every closed $\Gamma$-invariant subset of $X$ is a finite union of elements of $S_X(\Gamma)$.

In particular, if $(Y_n)$ is a sequence in $S_X(\Gamma)$ such that, for any $Y \in S_X(\Gamma)$ with $Y \neq X$, for all but finitely many $n$, one has $Y_n \not\subset Y$, then $\nu_{Y_n} \rightarrow \nu_X$, that is the orbits $Y_n$ become equidistributed in $X$ when $n$ is large.

Let us state a particular case of this result:

**Corollary 1.8.** Let $G$ be a connected semisimple real Lie group with no compact factor, $\Lambda$ be an irreducible lattice in $G$ and $\Gamma$ be a Zariski dense subgroup of $G$.

Every infinite $\Gamma$-invariant subset of $X$ is dense in $X$; any sequence $(Y_n)$ of distinct finite $\Gamma$-orbits in $X$ satisfies $\nu_{Y_n} \rightarrow \nu_X$.

Under the stronger assumption that $G$ is simple, Corollary 1.8 is the main result of [2]. It also extends previous results by Clozel, Oh and Ullmo in [9] about equidistribution of Hecke orbits (see also [13]).

1.4. **Actions on tori and nilmanifolds.** We now specialize our results to automorphisms of tori and other nilmanifolds.

Let $N$ be a connected simply connected nilpotent real Lie group, $\Lambda$ be a lattice in $N$ and $X$ be the compact nilmanifold $X = N/\Lambda$. As in
[4, Sect. 1.2], we say a closed subset of $X$ is an \textit{affine submanifold} if it is a translate of the image in $X$ of some closed connected subgroup of $N$. By Mal’cev’s rigidity Theorem (see [23, II.2.11]), we may consider the group $\text{Aut}(\Lambda)$ of automorphisms of $\Lambda$ as a subgroup of the group $\text{Aut}(N)$ of automorphisms of $N$. In particular, we may see $X$ as a quotient of the group $\text{Aut}(\Lambda) \ltimes N$ by its lattice $\text{Aut}(\Lambda) \ltimes \Lambda$. Then, the action of $\text{Aut}(\Lambda)$ on $X$ reads as its natural action by automorphisms. Any homogeneous subspace of $X$, viewed as a homogeneous space of $\text{Aut}(\Lambda)$, is a finite union of affine submanifolds.

\textbf{Example 1.9.} If $N = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$, we retrieve the standard action of $\text{GL}(d, \mathbb{Z})$ on the torus $T^d$. Any homogeneous subspace is a finite union of (parallel) subtori.

Theorems [1.4] and [1.5] and their corollaries now give the following partial answers to [17, Prob. 3]:

\textbf{Corollary 1.10.} Let $X = N/\Lambda$ be a compact nilmanifold and $\Gamma \subset \text{Aut}(\Lambda)$ be a finitely generated sub-semigroup whose Zariski closure in $\text{Aut}(N)$ is a Zariski connected semisimple subgroup with no compact factor. Let $L \subset N$ be the subgroup of $\Gamma$-invariant elements in $N$.

a) Every $\Gamma$-orbit closure is a finite homogeneous union of affine submanifolds.

b) Let $\mu$ be a finitely supported Borel probability measure on $\Gamma$ whose support spans $\Gamma$ and $g_1, \ldots, g_n, \ldots$ be a sequence of random independent identically distributed elements of $\text{Aut}(\Lambda)$ with law $\mu$. Then, almost surely, as $n$ goes to $\infty$, $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x}$ converges to the homogeneous probability measure of $\overline{\Gamma x}$.

c) The set $S_X(\Gamma)$ is compact. If $(Y_n) \subset S_X(\Gamma)$ converges to $Y_\infty \in S_X(\Gamma)$, then there exists a sequence $(\ell_n) \subset L$ converging to $e$ such that, for $n$ large, $Y_n \subset \ell_n Y_\infty$.

\textbf{Corollary 1.11.} Assume moreover the centralizer $L$ of $\Gamma$ in $N$ is trivial. Then every closed $\Gamma$-invariant subset $F$ in $X$ is a finite union of affine submanifolds.

\textbf{Corollary 1.12.} Assume $X$ is a torus $T^d$ and $\Gamma$ acts strongly irreducibly on $\mathbb{Q}^d$. Then every infinite $\Gamma$-invariant subset of $X$ is dense in $X$. Any sequence of distinct finite $\Gamma$-orbits in $X$ equidistributes toward the Haar probability of $T^d$.

The statement about invariant closed subsets in Corollary [1.12] is due to Guivarc’h and Starkov [14] and Muchnik [21]. In case $\Gamma$ is proximal and acts strongly irreducibly on $\mathbb{R}^d$, the one about equidistribution of finite orbits follows from the results of Bourgain, Furman, Lindenstrauss and Mozes [7].
To conclude, we point out that even the following special case of Corollary 1.10(b) seems to be new:

**Corollary 1.13.** Let $g_1, \ldots, g_n, \ldots$ be a sequence of independent identically distributed random elements of $\text{SL}(2, \mathbb{Z})$ whose law $\mu$ has finite support and generates a non-solvable group. Then, starting from any irrational point $x$ in the 2-torus $\mathbb{T}^2$, almost surely, the trajectory $g_n \cdots g_1 x$ equidistributes toward the Haar probability on $\mathbb{T}^2$.

1.5. **Structure of the article.** The remainder of this paper is devoted to the proof of Theorems 1.4, 1.5 and 1.7. After replacing $G$ by a compactly generated open subgroup which contains $\Gamma$ and acts transitively on $X$, we may assume $G$ is second countable.

In Chapter 2, we prove homogeneity of the orbit closures and convergence in law in Theorem 1.4. Knowing the recurrence results from [11] and [5] and the classification of stationary probability measures from [4], the main problem is to check the centralizer $L$ of $\Gamma$ has only countably many orbits in $S_X(\Gamma)$. We give the proof of this property for real Lie groups and postpone the general case to the appendix.

In Chapter 3, we establish preliminary results on Markov chains, mainly in order to prove the almost sure equidistribution statement in Theorem 1.4. Our starting point is Breiman’s law of large numbers for Markov chains with a unique invariant measure [8]. In our case, as there may be several stationary measures, we have to prove that almost surely, starting from a point which does not belong to a given $Y$ in $S_X(\Gamma)$, the trajectory will spend most of the time far away from $Y$. This is done by exhibiting an exponentially recurrent subset in the complement of $Y$ and by establishing large deviation properties for the return times in this subset. These techniques strengthen certain ideas from [18] and [4, Sect. 6].

In Chapter 4, we apply the results of Chapter 3 to the proofs of the almost sure equidistribution statement in Theorem 1.4 and of Theorem 1.5. We then deduce Theorem 1.7 from these.

In the appendix, we establish the countability of $L$-orbits in $S_X(\Gamma)$ in the $S$-adic case. In Appendix A, we develop some tools to overcome the difficulty due to the fact that lattices in semiconnected $S$-adic Lie groups are not necessarily finitely generated, whereas Appendix B is devoted to the proof itself, whose structure mimics the real case.

2. **Equidistribution in law**

In this section, we establish the homogeneity of the orbit closures and the convergence in law in Theorem 1.4. We will first prove a countability statement for the set $S_X(\Gamma)$ (Proposition 2.1). The claims
will then follow from the recurrence result from [5] and the classification of stationary probability measures from [4].

2.1. Countability of ergodic invariant homogeneous subsets.

Note that, if $\Gamma^\pm$ denotes the closed subgroup spanned by $\Gamma$, we have $S_X(\Gamma) = S_X(\Gamma^\pm)$, so that we will assume in the next two sections $\Gamma$ is a subgroup of $G$.

**Proposition 2.1.** Let $G$ be a second countable weakly regular $S$-adic Lie group, $\Lambda$ be a discrete subgroup of $G$, $X = G/\Lambda$, $\Gamma$ be a compactly generated subgroup of $G$ such that $\text{Ad}\Gamma^Z$ is semisimple and equal to $\text{Ad}\Gamma^Z,nc$, and $L$ be the centralizer of $\Gamma$ in $G$. Then, there exists a countable set $\mathcal{Y} \subset S_X(\Gamma)$ such that

$$S_X(\Gamma) = \{ \ell Y \mid \ell \in L, Y \in \mathcal{Y} \}.$$

In particular, when the centralizer of $\Gamma$ in $g$ is null, the set $S_X(\Gamma)$ is countable.

As the proof of Proposition 2.1 in the general $S$-adic case is highly technical, we will first give it in the real case. The general case is dealt with in the appendix. First, let us prove two elementary facts whose proofs are the same both in the real and $S$-adic cases and which will be of use below.

One is the following Lemma, which, in case $H$ is discrete, follows from [4, Lem. 5.16].

**Lemma 2.2.** Let $G$ be a second countable $S$-adic Lie group, $H$ be a closed subgroup of $G$, $\Gamma$ be a closed subgroup of $G$ such that $\text{Ad}\Gamma^Z$ is semisimple and $L$ be the centralizer of $\Gamma$ in $G$. When $S \neq \{\infty\}$, we assume that the group $\Gamma$ is compactly generated. Then the set of fixed points of $\Gamma$ in $G/H$ is a countable union of $L$-orbits.

**Proof.** Set $X = G/H$ and let $X^\Gamma$ be the set of fixed points of $\Gamma$ in $X$. We shall prove that the orbits of $L$ in $X^\Gamma$ are open, that is for any $x$ in $X^\Gamma$, $Lx$ contains a neighborhood of $x$ in $X^\Gamma$. We may assume $x$ is the base point of $X = G/H$. In particular $\Gamma$ is contained in $H$.

Let $\mathfrak{l}$ be the Lie algebra of $L$. If $S = \{\infty\}$, $\mathfrak{l}$ is necessarily the centralizer of $\Gamma$ in $\mathfrak{g}$. If not, this is still the case, since we then assumed $\Gamma$ to be compactly generated. As the linear span of $\text{Ad}\Gamma$ in the space of endomorphisms of $\mathfrak{g}$ is finite dimensional, there exists $g_1, \ldots, g_r$ in $\Gamma$ such that

$$\mathfrak{l} = \{ v \in \mathfrak{g} \mid \forall 1 \leq i \leq r, g_i v = v \}.$$

Let $\mathfrak{h}$ be the Lie algebra of $H$. As $\text{Ad}\Gamma^Z$ is semisimple, $\mathfrak{h}$ admits a $\Gamma$-invariant complementary subspace $\mathfrak{v}$. Now, there exists a standard open subset $\Omega$ of $G$ (see [4, Sect. 5] or Section A.2 below),
with exponential map $\exp_\Omega : O \to \Omega$, such that the map $(O \cap \mathfrak{v}) \to X; v \mapsto \exp_\Omega(v) x$ is a diffeomorphism onto its image. We can assume $\exp_\Omega(O \cap \mathfrak{v}) \subset L$ and, by [4, Lem. 5.2], for any $v$ in $O$ and $1 \leq i \leq r$, if $g_i v \in O$, then $\exp_\Omega(g_i v) = g_i \exp_\Omega(v) g_i^{-1}$.

Set $U = O \cap \bigcap_{i=1}^r g_i^{-1} O \cap \mathfrak{v}$. Then $\exp_\Omega(U)x$ is a neighborhood of $x$ and we shall prove that $\exp_\Omega(U)x \cap X_\Gamma \subset Lx$, which finishes the proof. Indeed, for $y = \exp_\Omega(v)x$ in $X_\Gamma$ with $v$ in $U$, we have, for any $1 \leq i \leq r$,

$$\exp_\Omega(g_i v)x = g_i \exp_\Omega(v) g_i^{-1} x = g_i \exp_\Omega(v) x = g_i y = y = \exp_\Omega(v)x,$$

hence, as both $v$ and $g_i v$ belong to $O \cap \mathfrak{v}$, $g_i v = v$. This gives $v \in \Gamma$ and $y = \exp_\Omega(v)x \in Lx$, what should be proved. □

We let $Q_S$ denote the locally compact algebra $\prod_{p \in S} Q_p$. By definition, a finite dimensional $Q_S$-module is a product $V = \prod_{p \in S} V_p$, where, for any $p$ in $S$, $V_p$ is a finite dimensional $Q_p$-vector space. We then let $GL(V) = \prod_{p \in S} GL(V_p)$ be the linear group of $V$ and $Gr(V) = \prod_{p \in S} Gr(V_p)$ be its Grassmann variety.

One ingredient of the proof of Proposition 2.1 is the following more or less classical

**Lemma 2.3.** Let $V$ be a finitely generated $Q_S$-module and $\Gamma$ be a subgroup of $GL(V)$. Assume that the Zariski closure $\overline{\Gamma}^Z$ is semisimple and equal to $\overline{\Gamma}^{Z,nc}$. Then every $\Gamma$-invariant probability measure $\eta$ on $Gr(V)$ is concentrated on the set of fixed points of $\Gamma$ in $Gr(V)$.

**Proof.** By taking projections and replacing subspaces by exterior powers, it suffices to prove Lemma 2.3 when $V = V_p$ for some $p$ in $S$, $\Gamma$ acts irreducibly on $V$ and $\eta$ is a $\Gamma$-invariant probability measure on $\mathbb{P}(V)$. We will prove that the action of $\Gamma$ on $V$ is then trivial.

We first check that for any subspace $W \subset V$, one has $\eta(\mathbb{P}(W)) = 0$. Indeed, let $W$ be the set of subspaces $W$ of $V$ such that $\eta(\mathbb{P}(W)) > 0$ and the dimension of $W$ is minimal among the subspaces satisfying this property. For any $W \neq W'$ in $W$, we have

$$\eta(\mathbb{P}(W) \cup \mathbb{P}(W')) = \eta(\mathbb{P}(W)) + \eta(\mathbb{P}(W')).$$

Hence, $W$ contains only finitely many elements $W_1, \ldots, W_r$ such that

$$\eta(\mathbb{P}(W_1)) = \cdots = \eta(\mathbb{P}(W_r)) = \max_{W \in W} \eta(\mathbb{P}(W)).$$

Now, as $\eta$ is $\Gamma$-invariant, the set $\{W_1, \ldots, W_r\}$ is $\Gamma$-invariant. As $\Gamma$ is Zariski connected and acts irreducibly on $V$, we get $W_1 = \cdots = W_r = V$ and $W = \{V\}$ as required.
Assume $\Gamma$ acts non trivially on $V$. By assumption, $\Gamma$ is a non relatively compact subgroup of $\text{SL}(V)$, hence the closure of $\mathbb{Q}_p \Gamma$ in the space of endomorphisms of $V$ contains a non-zero singular map $f$. We have just shown $\eta(\ker f) = 0$. As $\eta$ is $\Gamma$-invariant, we get $\eta(\text{im } f) = 1$, but we have just shown $\eta(\text{im } f) = 0$, whence a contradiction. $\square$

2.2. Proof of countability in the real case. Let $G$ be a real Lie group and let $\Delta \subset \Sigma$ be discrete subgroups of $G$.

**Definition 2.4.** We let $\mathcal{T}(G, \Delta, \Sigma)$ denote the set of closed subgroups $H$ of $G$ such that

(i) $\Sigma$ is contained in $H$ and $\Sigma$ is a lattice in $H$.
(ii) one has $\Delta = \Sigma \cap H^\circ$, where $H^\circ$ is the connected component of $H$.
(iii) there exists a subgroup $\Gamma$ of $H$ such that $\text{Ad}^\Gamma Z$ is semisimple with no compact factor and $\Gamma$ acts ergodically on the $H$-invariant measure of $H/\Sigma$.

The core of the proof of Proposition 2.1 in the real case is the following

**Lemma 2.5.** Let $G$ be a second countable real Lie group and $\Delta \subset \Sigma$ be finitely generated discrete subgroups of $G$. The set $\mathcal{T}(G, \Delta, \Sigma)$ is countable.

Note that the ergodicity condition (iii) is crucial to get this countability statement. Indeed, when $G = \text{SL}(2, \mathbb{R})$, $G$ contains uncountably many compact subgroups.

We shall need several preparatory lemmas. The semisimplicity assumption plays an essential role in the following

**Lemma 2.6.** Let $G$ be a real Lie group and $\Delta \subset \Sigma$ be discrete subgroups of $G$. Let $H_1$ and $H_2$ belong to $\mathcal{T}(G, \Delta, \Sigma)$. Then $H_1$ normalizes $H_2^\circ$.

A similar phenomenon appears in the proof of [25, Prop. 1.7].

**Proof.** It suffices to prove that $H_1$ normalizes the Lie algebra $\mathfrak{h}_2$ of $H_2$. Now, as $\Sigma$ is contained in $H_2$, the map

$$H_1 \rightarrow \text{Gr}(\mathfrak{g}); h_1 \mapsto \text{Ad}h_1(\mathfrak{h}_2)$$

factors as a map $H_1/\Sigma \rightarrow \text{Gr}(\mathfrak{g})$. Let $\eta$ be the image of the $H_1$-invariant measure of $H_1/\Sigma$ under this map. Let $\Gamma$ be a subgroup of $H_1$ such that $\text{Ad}^\Gamma Z$ is semisimple with no compact factor and that $\Gamma$ acts ergodically on $H_1/\Sigma$. As $\eta$ is $\Gamma$-ergodic, by Lemma 2.3, $\nu$ is a Dirac mass on a fixed point of $\Gamma$, that is $\mathfrak{h}_2$ is normalized by $H_1$, what should be proved. $\square$

We shall also need the two following elementary facts.
Lemma 2.7. Let $G$ be a second countable real Lie group. Then the set of normal compact subgroups of $G$ is countable.

Proof. Let $K$ be a normal compact subgroup of $G$. The Lie algebra $\mathfrak{k}$ of $K$ may be decomposed in a unique way as a direct sum $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{a}$ where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is abelian. As $\mathfrak{k}$ is a $G$-invariant ideal of $\mathfrak{g}$, so are $\mathfrak{s}$ and $\mathfrak{a}$. As the Lie algebra $\mathfrak{g}$ of $G$ contains only finitely many semisimple ideals, we may assume that $\mathfrak{s}$ is fixed. As the connected analytic subgroup $S$ of $G$ with Lie algebra $\mathfrak{s}$ is compact and normal in $G$, after replacing $G$ by $G/S$, we may assume $\mathfrak{s} = \{0\}$.

In other terms, we have to prove that $G$ contains countably many compact normal subgroups $K$ whose Lie algebra $\mathfrak{k}$ is abelian. Since such a $K$ is compact, $\mathfrak{k}$ admits a $K$-invariant complementary subspace $v$ in $\mathfrak{g}$. As $v$ is $K$-invariant and $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, $\mathfrak{k}$ is central in $\mathfrak{g}$, hence, $K^\circ$ is a central subgroup of $G^\circ$. Now, the connected component of the center of $G^\circ$ is isomorphic to a product $\mathbb{R}^p \times \mathbb{T}^q$ and $\mathbb{T}^q$ admits countably many closed subgroups. Hence we may assume that $K^\circ$ is fixed. Thus, after replacing $G$ by the group $G/K^\circ$, we may assume that $K$ is finite. Since $K$ is normal, it then centralizes $G^\circ$ and Lemma 2.7 follows from Lemma 2.8 below. □

Lemma 2.8. Let $G$ be a second countable real Lie group. Then the set of compact subgroups of $G$ which centralize $G^\circ$ is countable.

Proof. By replacing $G$ by the centralizer of $G^\circ$, we may assume $G^\circ$ is central in $G$ and we have to prove that $G$ contains countably many compact subgroups. Now, as above, $G^\circ$ being an abelian connected group, it admits countably many compact subgroups, hence we may fix the intersection of our compact subgroups with $G^\circ$. As this intersection is central, up to replacing $G$ by a quotient, we may assume it is trivial, and we therefore only have to prove that $G$ contains countably many finite subgroups $F$ such that $F \cap G^\circ = \{e\}$. For such a subgroup $F$, the group $FG^\circ$ is isomorphic to the product $F \times G^\circ$ and contains therefore only finitely many finite subgroups $F'$ such that $F' \cap G^\circ = \{e\}$ and $F'G^\circ = FG^\circ$. As $G/G^\circ$ is countable, the result follows. □

We can now give the

Proof of Lemma 2.5. We can assume the set $\bigcup_{H \in \mathcal{T}(G,\Delta,\Sigma)} H$ spans a dense subgroup of $G$. Set $L = \bigcap_{H \in \mathcal{T}(G,\Delta,\Sigma)} H^\circ$. By Lemma 2.6 $L$ is a normal subgroup of $G$. Since $L \cap \Sigma = \Delta$ is a lattice in $L$, the image of $\Sigma$ in $G/L$ is still discrete, so that, after replacing $G$ by $G/L$, we may assume $L = \{e\}$. In particular, this gives $\Delta = \{e\}$ and thus, for any $H$ in $\mathcal{T}(G,\Delta,\Sigma)$, $H^\circ$ is compact. As, still by Lemma 2.6 $H^\circ$ is normal.
in $G$, and, by Lemma 2.7, the set of normal compact subgroups of $G$ is countable, we can suppose $H^\circ$ is fixed, thus after replacing $G$ by $G/H^\circ$, we can assume $H$ is discrete.

In other terms, we only have to prove that, setting $V(G, \Sigma)$ to be the set of discrete subgroups $H$ of $G$ which contain $\Sigma$ as a finite index subgroup and which admit a subgroup $\Gamma$ such that $\Ad\Gamma^Z$ is semisimple with no compact factor and $H = \Gamma\Sigma$, then $V(G, \Sigma)$ is countable. Now, if $H$ belongs to $V(G, \Sigma)$, $H$ normalizes a finite index subgroup $\Theta$ of $\Sigma$. As $\Sigma$ is finitely generated, the set of finite index subgroups of $\Sigma$ is countable, and we can assume $\Theta$ is fixed. We set $G'$ to be the closure of the subgroup of $G$ spanned by the $H$ in $V(G, \Sigma)$ which normalize $\Theta$ and replace $G$ by $G'/\Theta$. Now, we just have to prove that, if $\Sigma$ is finite, $V(G, \Sigma)$ is countable. In this case, if $H$ belongs to $V(G, \Sigma)$, then $H$ is finite and, if $\Gamma$ is a subgroup of $H$ such that $\Ad\Gamma^Z$ is semisimple with no compact factor and $H = \Gamma\Sigma$, then $\Gamma$ is finite. Therefore $\Ad\Gamma$ is finite, hence trivial. In other terms, $\Gamma$ is a finite subgroup of $G$ which centralizes $G^\circ$. By Lemma 2.8 the set of such subgroups is countable and we are done. 

We can now conclude the

Proof of Proposition 2.1 in the real case. Let $Y$ be in $S_X(\Gamma)$ and recall that $G_Y$ denotes the stabilizer of $Y$ in $G$. We pick $g$ in $G$ such that $g\Lambda$ belongs to $Y$ and we set $H = g^{-1}(\Gamma G^\circ_Y)g$. As $H$ and $H^\circ$ are open in $g^{-1}G_Y g$ and $g\Lambda g^{-1} \cap G_Y$ is a lattice in $G_Y$, $\Lambda \cap H$ is a lattice in $H$ and $\Lambda \cap H^\circ$ is lattice in $H^\circ$. In particular, by [23, 6.18], $\Lambda \cap H^\circ$ is finitely generated. Since $\Gamma$ is compactly generated, the real Lie group $H$ is also compactly generated and its lattice $\Lambda \cap H$ is also finitely generated. In particular, as $\Lambda$ admits countably many finitely generated subgroups, we can assume the groups $$\Delta := \Lambda \cap H^\circ \quad \text{and} \quad \Sigma := \Lambda \cap H$$ are fixed. Now, the group $H$ belongs to $T(G, \Delta, \Sigma)$ so that, by Lemma 2.5, we can also assume it to be fixed. The point $gH \in G/H$ is $\Gamma$-invariant, hence, by Lemma 2.2, we can assume that the $L$-orbit $LgH$ is fixed.

Now, if $g_1$ is an element of $G$ such that $Lg_1H = LgH$, one can write $g_1 = \ell gh$ with $\ell$ in $L$ and $h$ in $H$. Hence one gets $g_1H\Lambda = \ell Y$ and the result follows.

2.3. Proof of equidistribution in law. We will need the following elementary Lemma which asserts that any two distinct elements in $S_X(\Gamma)$ which are open in $X$ are disjoint.
Lemma 2.9. Let $G$ be a second countable locally compact topological group, $\Lambda$ be a discrete subgroup of $G$, $X = G/\Lambda$ and $\Gamma$ be a closed subsemigroup of $G$. Any two distinct elements $Y \neq Y'$ of $\mathcal{S}_X(\Gamma)$ which are open in $X$ are disjoint.

In particular the set $\mathcal{S}_{op}(\Gamma) := \{ Y \in \mathcal{S}_X(\Gamma) \, | \, Y \text{ open in } X \}$ is countable.

Proof. Assume the intersection $Y'' = Y \cap Y'$ is not empty. Then, as $Y''$ is open in $Y$, one has $\nu_Y(Y'') > 0$ and, as $Y''$ is $\Gamma$-invariant and $Y$ is $\Gamma$-ergodic, one has $\nu_Y(Y'') = 1$. Since $Y''$ is closed, we get $Y = Y''$. In the same way, $Y' = Y''$. □

Using Proposition 2.1 and the results of [4] and [5], we get the

Proof of Theorem 1.4.a) and 1.4.b). We proceed by induction on the dimension of $G$. If this dimension equals 0, then the space $G/\Lambda$ is finite and the result is evident.

Assume $G$ has positive dimension and fix $x$ in $X$. If there exists a non-open $Y$ in $\mathcal{S}_X(\Gamma)$ such that $x$ belongs to $Y$, we get the result by induction. Thus, we can assume this is not the case, and we will prove that there exists a unique $Y_x$ in $\mathcal{S}_{op}(\Gamma)$ containing $x$ and that the sequence of probability measures $\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \mu^* \delta_x$ on $X$ converges toward $\nu_{Y_x}$. Now, let $\nu$ be a limit point of $\nu_n$ as $n$ goes to $\infty$, so that $\nu$ is $\mu$-stationary. By [5, Th. 7.2], $\nu$ is a probability measure.

By [4, Th. 2.5], $\nu$ being $\mu$-stationary, it is $\Gamma$-invariant and every $\mu$-ergodic component of $\nu$ is equal to some $\nu_Y$ for $Y$ in $\mathcal{S}_X(\Gamma)$. Let $L$ be the centralizer of $\Gamma$ in $G$. By [4, Prop. 6.24 and Cor. 6.25], for every non-open $Y$ in $\mathcal{S}_X(\Gamma)$ and every compact subset $K_L$ of $L$, one has $\nu(K_L Y) = 0$, hence $\nu(L Y) = 0$. Since, by Proposition 2.1, $\mathcal{S}_X(\Gamma)$ is a countable union of $L$-orbits, almost every ergodic component of $\nu$ is equal to some $\nu_Y$ with $Y$ in $\mathcal{S}_{op}(\Gamma)$.

Since by Lemma 2.9, $\mathcal{S}_{op}(\Gamma)$ is countable, there exists $Y_x$ in $\mathcal{S}_{op}(\Gamma)$ such that $\nu(Y_x) > 0$. But then, the point $x$ belongs to $Y_x$, so that, by Lemma 2.9, $Y_x$ is the unique element of $\mathcal{S}_{op}(\Gamma)$ containing $x$. By construction, $\nu$ does not give mass to any other element of $\mathcal{S}_{op}(\Gamma)$. Therefore, $\nu = \nu_{Y_x}$ and we are done.

□

3. Markov Operators

We will now develop abstract probabilistic tools for proving the almost sure statement in Theorem 1.4. An important ingredient in our method comes from the proof of Breiman’s law of large numbers for Markov chains [8], which, in the context of group actions, states as follows:
Let $G$ be a locally compact group, $X$ a compact metrizable $G$-space, and $\mu$ a Borel probability measure on $G$ such that there exists a unique $\mu$-stationary Borel probability measure $\nu$ on $X$. Let $g_1, \ldots, g_n, \ldots$ be a sequence of random elements of $G$ which are independent and identically distributed with law $\mu$. Then, for any $x$ in $X$, almost surely, one has

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} \underset{n \to \infty}{\longrightarrow} \nu.$$ 

In our situation, we have to update the strategy of [8] since $X$ is not compact and there may be several $\mu$-stationary probability measures on $X$. To do this, we shall need abstract general informations on Markov chains.

3.1. Markov measures and the law of large numbers. Let $(X, \mathcal{X})$ be a standard Borel space. By a Markov operator on $X$, we mean a Borel map $x \mapsto P_x$ from $X$ to the space of Borel probability measures on $X$. Given such an operator, for any bounded Borel function $\varphi$ on $X$ and any $x$ in $X$, we set

$$P \varphi(x) = \int_X \varphi \, dP_x.$$ 

Let us recall the construction of the Markov measures associated to $P$ on the space of trajectories. We set $W = X^\mathbb{N}$ and we equip it with the product $\sigma$-algebra $\mathcal{X}^\otimes \mathbb{N}$. An element $w$ in $W$ will be written as a sequence $w = (w_0, w_1, w_2, \ldots)$. For any $x$ in $X$, there exists a unique Borel probability measure $\omega_x$ on $W$ such that, for any bounded Borel functions $\varphi_0, \ldots, \varphi_n$ on $X$, one has

$$\int_W \varphi_0(w_0) \cdots \varphi_n(w_n) \, d\omega_x(w) = (\varphi_0 P(\ldots (\varphi_{n-1} P(\varphi_n)) \ldots))(x).$$

In other terms, $\omega_x$ is implicitly defined by $\omega_x = \delta_x \otimes (\int_X \omega_y \, dP_x(y))$. We say $\omega_x$ is the Markov measure associated to $P$ and $x$.

Example 3.1. Let a locally compact topological group $G$ act measurably on $X$. Fix a Borel probability measure $\mu$ on $G$ and set, for any $x$ in $X$, $P_x = \mu * \delta_x$. This defines a Markov operator $P$ on $X$ which represents formally the notion of a random walk on $X$ with law $\mu$. For any $x$ in $X$, the associated Markov measure $\omega_x$ on $W$ is the image of the measure $\mu^{\otimes \mathbb{N}}$ on $G^\mathbb{N}$ under the map $(g_k)_{k \in \mathbb{N}} \mapsto (g_{k-1} \cdots g_0 x)_{k \in \mathbb{N}}$.

Lemma 3.2. (Breiman [3]) Let $(X, \mathcal{X})$ be a standard Borel space, $P$ be a Markov operator and $\varphi$ be a bounded Borel function on $X$. For
every } x \text{ in } X, \text{ for } \omega_x\text{-almost every } w \text{ in } W, \text{ one has }
\frac{1}{n} \sum_{k=0}^{n-1} \varphi(w_k) - \frac{1}{n} \sum_{k=0}^{n-1} P\varphi(w_k) \longrightarrow 0.

\text{Proof.} \text{ We first recall the following version of the classical law of large numbers:}
\text{Let } (Y, \mathcal{Y}, \eta) \text{ be a probability space and } (\zeta_i) \text{ be a bounded sequence of elements of } L^2(Y, \mathcal{Y}, \eta) \text{ with, for any } n, \mathbb{E}(\zeta_i | \zeta_{i-1}, \ldots, \zeta_1) = 0, \text{ then }
\frac{1}{n} \sum_{k=1}^n \zeta_k \longrightarrow 0 \text{ almost everywhere.}

\text{For any integer } n \geq 1 \text{ set, for } w \text{ in } W,
\zeta_n(w) = \varphi(w_n) - P\varphi(w_{n-1}).

\text{This sequence of functions on } W \text{ is bounded by } 2 \sup_X |\varphi| \text{ and, as } \zeta_n \text{ only depends on } w_n, \ldots, w_0, \text{ one has, for any } n \geq 1,
\mathbb{E}_{\omega_x}(\zeta_n | \zeta_{n-1}, \ldots, \zeta_1) = \mathbb{E}_{\omega_x}(\mathbb{E}_{\omega_x}(\zeta_n | w_{n-1}, \ldots, w_0) | \zeta_{n-1}, \ldots, \zeta_1).

\text{By construction, one has }
\mathbb{E}_{\omega_x}(\zeta_n | w_{n-1}, \ldots, w_0) = 0,
\text{ hence }
\mathbb{E}_{\omega_x}(\zeta_n | \zeta_{n-1}, \ldots, \zeta_1) = 0.

\text{Therefore, we have, } \omega_x\text{-almost everywhere, } \frac{1}{n} \sum_{k=1}^n \zeta_k \longrightarrow 0. \text{ The result follows, since } \varphi \text{ is bounded.} \square

\text{We say a Borel measure } \nu \text{ on } X \text{ is } P\text{-invariant if, for any nonnegative Borel function } \varphi \text{ on } X, \text{ one has } \int_X P\varphi \, d\nu = \int_X \varphi \, d\nu.

\text{Recall, if } X \text{ is a compact space, a Markov-Feller operator on } X \text{ is a nonnegative operator } P \text{ on the space of continuous functions on } X \text{ such that } P1 = 1. \text{ In other terms, a Markov-Feller operator is a Markov operator on } X \text{ such that the map } x \mapsto P_x \text{ is continuous, when the space of Borel probability measures of } X \text{ is equipped with the weak-* topology.}

\text{¿From Lemma 3.2, we get}

\textbf{Corollary 3.3. Let } X \text{ be a compact metrizable topological space and } P \text{ be a Markov-Feller operator on } X. \text{ Then, for any } x \text{ in } X, \text{ for } \omega_x\text{-almost any } w \text{ in } W, \text{ any weak-* limit of } \frac{1}{n} \sum_{k=0}^{n-1} \delta_{w_k} \text{ is } P\text{-invariant.}

\text{In particular, using the weak-*-compactness of the space of probability measures on } X, \text{ we retrieve Breiman’s law of large numbers in } [8]:
If moreover there exists a unique $P$-invariant probability measure $\nu$ on $X$, then for any $x$ in $X$, for $\omega_x$-almost any $w$ in $W$, one has
\[ \frac{1}{n} \sum_{k=0}^{n-1} \delta_{w_k} \xrightarrow{n \to \infty} \nu. \]

3.2. Recurrent subsets. We need to understand weak limits as in Corollary 3.3 when the space $X$ is not compact and in particular to get a handleable criterion for them to have total mass 1. To this aim, we study recurrent subsets.

If $Y \subset X$ is a Borel subset, we say $Y$ is $P$-recurrent if, for any $x$ in $Y$, one has $\omega_x(\{w \in W \mid \exists k \geq 1 \ w_k \in Y\}) = 1$, that is if, almost surely, the trajectories issued from $Y$ turn back to $Y$. For any $w$ in $W$, we let
\[ \tau_Y(w) = \min\{k \geq 1 \mid w_k \in Y\} \in [1, \infty] \]
denote the first return time in $Y$. In the same way, for any $w$ in $W$ with $\#\{k \in \mathbb{N} \mid w_k \in Y\} = \infty$, we set $\tau_Y^1(w) = \tau_Y(w)$ and, for any $p \geq 2$,
\[ \tau_Y^p(w) = \min\{k > \tau_Y^{p-1}(w) \mid w_k \in Y\}. \]

If $w_0 \in Y$, we also write $\tau_Y^0(w) = 0$. These are the successive return times of $w$ in $Y$.

Assume $Y$ is $P$-recurrent and let, for any $x$ in $Y$, $Q_y$ denote the probability measure on $Y$ that is the image of $\omega_x$ under the map $w \mapsto w_{\tau_Y(w)}$, which is defined $\omega_x$-almost everywhere on $W$. We say $Q$ is the Markov operator induced by $P$ on $Y$. One easily checks that, for any $x$ in $Y$, the Markov measure associated to $Q$ and $x$ is the image of $\omega_x$ under the map
\[ W \to Y^\mathbb{N}, \quad w \mapsto (w_{\tau_Y^p(w)})_{p \in \mathbb{N}}. \]

We say a $P$-invariant Borel measure $\nu$ on $X$ is ergodic if, for any Borel function $\varphi$ on $X$ with $P\varphi = \varphi$, $\varphi$ is constant $\nu$-almost everywhere.

Lemma 3.4. Let $(X, X)$ be a standard Borel space, $P$ be a Markov operator on $X$, $Y$ be a $P$-recurrent Borel subset of $X$ and $Q$ be the Markov operator induced by $P$ on $Y$. Let $\nu$ be a $P$-invariant Borel measure on $X$. Then $\nu|_Y$ is $Q$-invariant. Moreover, if $\nu$ is $P$-ergodic, then $\nu|_Y$ is $Q$-ergodic.

Proof. Let $\varphi$ be a nonnegative Borel function on $X$ and let us prove $\int_Y Q\varphi \ d\nu = \int_Y \varphi \ d\nu$. We introduce the function $\psi : X \to [0, \infty)$ given by $\psi(x) = \varphi(x)$ if $x \in Y$ and $\psi(x) = \int_W 1_{\{\tau_Y(w) < \infty\}} \varphi(w_{\tau_Y(w)}) \ d\omega_x(w)$ else. By construction, one has
\[ \psi = \varphi 1_Y + \psi 1_{Y^c} \]
\[ P\psi = (Q\varphi) 1_Y + \psi 1_{Y^c} \]
(3.1)
hence
\[ \int_X \psi \, d\nu = \int_Y \varphi \, d\nu + \int_{Y^c} \psi \, d\nu \]
\[ \int_X P\psi \, d\nu = \int_Y Q\varphi \, d\nu + \int_{Y^c} \psi \, d\nu. \]
Since \( \nu \) is \( P \)-invariant, this gives
\[ \int_Y Q\varphi \, d\nu = \int_Y \varphi \, d\nu. \]
Now, assume \( \nu \) is \( P \)-ergodic and \( Q\varphi = \varphi \) and let still \( \psi \) be as above. From (3.1), we get \( P\psi = \psi \), hence \( \psi \) is constant \( \nu \)-almost everywhere and so is \( \varphi \).

3.3. **Exponentially recurrent subsets.** Assuming the return times enjoy strong uniform moment properties, we will now get almost sure estimates on the asymptotic behaviour as \( p \) goes to \( \infty \) of the \( p \)-th return time of a given trajectory.

Let still \( Y \) be \( P \)-recurrent. As in [4, Sect. 6], we say \( Y \) is **exponentially \( P \)-recurrent** if there exists \( 0 < a < 1 \) with
\[ \sup_{x \in Y} \int_W e^{a \tau_Y} \, d\omega_x < \infty. \]
The following Lemma asserts that return times in exponentially recurrent subsets satisfy a large deviation principle.

**Lemma 3.5.** Let \((X, \mathcal{X})\) be a standard Borel space, \( P \) be a Markov operator on \( X \) and \( Y \) be an exponentially recurrent subset of \( X \). We set
\[ \theta := \sup_{x \in Y} \int_W \tau_Y \, d\omega_x < \infty. \]
Then, for any \( \varepsilon > 0 \), there exists \( \alpha > 0 \) such that, for any \( x \) in \( Y \) and \( p \) in \( \mathbb{N} \), we have
\[ \omega_x \{ w \in W \mid \tau_Y^p(w) \geq p(\theta + \varepsilon) \} \leq e^{-p\alpha}. \]
In particular, for \( \omega_x \)-almost any \( w \) in \( W \), one has
\[ \limsup_{p \to \infty} \frac{1}{p} \tau_Y^p(w) \leq \theta. \]

**Proof.** Let \( \alpha_0 > 0 \) be such that
\[ \sup_{x \in Y} \int_W e^{\alpha_0 \tau_Y} \, d\omega_x < \infty. \]
For any \( 0 < \alpha \leq \alpha_0 \), for any \( x \) in \( Y \), we get
\[ \omega_x \{ w \in W \mid \tau_Y^p(w) \geq p(\theta + \varepsilon) \} \leq e^{-p\alpha(\theta + \varepsilon)} \int_W e^{\alpha \tau_Y^p(w)} \, d\omega_x(w). \]
Now, by the Markov property for the operator induced by \( P \) on \( Y \), we have
\[ \int_W e^{\alpha \tau_Y^p} \, d\omega_x \leq \left( \sup_{y \in Y} \int_W e^{\alpha \tau_Y} \, d\omega_y \right)^p. \]
Since for every $t \geq 0$ one has $e^t \leq 1 + t + t^2e^t$, there exists $C > 0$ such that, for $0 < \alpha \leq \alpha_0/2$, one has

$$\sup_{y \in Y} \int_W e^{\alpha \tau_Y} d\omega_y \leq 1 + \alpha \theta + C\alpha^2.$$ 

Thus, if $\alpha$ is small enough, we get

$$e^{-\alpha(\theta+\varepsilon)} \sup_{y \in Y} \int_W e^{\alpha \tau_Y} d\omega_y < 1$$

and the first part of the Lemma is proved. The second follows, by Borel-Cantelli Lemma.

Lemma 3.5 yields the following Corollary, which we shall not use but which is of independent interest:

**Corollary 3.6.** Let $(X, \mathcal{X})$ be a standard Borel space, $Y \supset Z$ be Borel subsets of $X$ and $P$ be a Markov operator on $X$. Assume $Y$ is $P$-recurrent and let $Q$ be the Markov operator induced by $P$ on $Y$. If $Y$ is exponentially $P$-recurrent and $Z$ is exponentially $Q$-recurrent, then $Z$ is exponentially $P$-recurrent.

**Proof.** By Lemma 3.5, there exists $\gamma \geq 1$ and $\alpha > 0$ such that, for any $x$ in $Y$ and $p$ in $\mathbb{N}$, one has

$$\omega_x(\{w \in W \mid \tau_Y^p(w) > \gamma p\}) \leq e^{-\alpha p}.$$ 

As $Z$ is exponentially $Q$-recurrent, there exists $\beta > 0$ such that, for $p$ large enough, for any $x$ in $Z$, one has

$$\omega_x(\{w \in W \mid \tau_Z(w) > \tau_Y^p(w)\}) \leq e^{-\beta p}.$$ 

We get

$$\omega_x(\{w \in W \mid \tau_Z(w) > \gamma p\}) \leq e^{-\alpha p} + e^{-\beta p}$$

and the result follows. \qed

We now aim at proving that, on a given trajectory which starts from an exponentially recurrent subset $Y$, most of the time is spent at a close temporal distance from $Y$.

To be more precise, we introduce some notations. Let still $Y \subset X$ be a Borel subset. If $w \in W$ is such that $w_0 \in Y$ and $\sharp\{k \in \mathbb{N} \mid w_k \in Y\} = \infty$, we set, for any natural integers $p, T$,

$$\sigma_Y^p(w) := \tau_Y^{p+1}(w) - \tau_Y^p(w),$$

which are the successive excursion times of $w$ out of $Y$, and

$$\sigma_Y^{p,T}(w) := \sigma_Y^p(w)1_{\{\sigma_Y^p(w) \geq T\}} \quad \text{and} \quad \tau_Y^{p,T}(w) := \sum_{0 \leq q < p} \sigma_Y^{p,T}(w),$$
which is the total duration among the \( p \) first excursions outside \( Y \) of those of length \( \geq T \).

**Lemma 3.7.** Let \((X, \mathcal{X})\) be a standard Borel space, \( P \) be a Markov operator on \( X \) and \( Y \) be an exponentially \( P \)-recurrent subset of \( X \). For any \( \varepsilon > 0 \), there exists \( T \) in \( \mathbb{N} \) such that, for any \( x \) in \( Y \), for \( \omega_x \)-almost any \( w \) in \( W \), one has

\[
\limsup_{p \to \infty} \frac{1}{p} \mathbb{E}^{\omega_x}_{Y,T}(w) \leq \varepsilon
\]

**Proof.** Since \( Y \) is exponentially \( P \)-recurrent, there exists \( \alpha_0 > 0 \) such that \( \sup_{x \in Y} \int_{W} e^{\alpha_0 \tau_{Y,T}} d\omega_x < \infty \). Hence, for \( T \) large enough, we have

\[
\sup_{x \in Y} \int_{W} \tau_{Y,T}^{1} d\omega_x \leq \varepsilon.
\]

We can now conclude as in the proof of Lemma 3.5, using the fact that \( \sup_{x \in Y} \int_{W} e^{\alpha_0 \tau_{Y,T}} d\omega_x < \infty \).

3.4. **Excursions of trajectories.** Let \( u : X \to [0, \infty] \) be a Borel function such that there exist \( 0 \leq a < 1 \) and \( C \geq 0 \) with \( Pu \leq au + C \). In [4] Prop. 6.3], we proved, if \( M > 0 \) is large enough, the set \( X_M = u^{-1}([0, M]) \) is exponentially recurrent. In Proposition 3.9 below, we will see that, on the set \( \{u < \infty\} \), almost surely, the trajectories spend most of the time in subsets of the form \( X_M \) with \( M \) large. This is a key step for proving Theorem 1.4.

We start with a much weaker result, whose conclusion serves as a motivation for the exact formulation of Proposition 3.9 and which will also be of use in the proof of Theorem 1.5.

**Lemma 3.8.** Let \((X, \mathcal{X})\) be a standard Borel space, \( P \) be a Markov operator on \( X \) and \( u : X \to [0, \infty] \) be a Borel function such that there exist \( 0 \leq a < 1 \) and \( C \geq 0 \) with \( Pu \leq au + C \). Let \( \nu \) be a \( P \)-invariant \( P \)-ergodic Borel probability measure on \( X \) such that \( \nu(\{u < \infty\}) > 0 \). Then one has \( \int_X u d\nu \leq \frac{C}{1-a} \).

**Proof.** According to Chacon-Ornstein ergodic Theorem, for \( \nu \)-almost every \( x \) in \( X \), one has

\[
\frac{1}{n} \sum_{k=0}^{n-1} P^k u(x) \xrightarrow{n \to \infty} \int_X u d\nu.
\]

Since \( \nu(\{u < \infty\}) > 0 \), we can choose such a \( x \) with \( u(x) < \infty \). Now, by assumption, for every \( k \geq 0 \), one has

\[
P^k u(x) \leq a^k u(x) + \frac{C}{1-a}.
\]
hence
\[
\frac{1}{n} \sum_{k=0}^{n-1} P^k u(x) \leq \frac{u(x)}{(1-a)n} + \frac{C}{1-a}.
\]
Letting \( n \) go to infinity, we get
\[
\int_X u \, d\nu \leq \frac{C}{1-a}.
\]

The main result of this chapter now states

**Proposition 3.9.** Let \((X, \mathcal{X})\) be a standard Borel space, \(P\) be a Markov operator on \(X\) and \(u : X \to [0, \infty]\) be a Borel function such that there exist \(0 \leq a < 1\) and \(C \geq 0\) with \(Pu \leq au + C\). Then, for any \(x\) in \(X\) with \(u(x) < \infty\), for \(\omega_x\)-almost any \(w\) in \(W\), for any \(M > 0\), one has
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{0 \leq k < n} u(w_k) \leq C(1-a)M.
\]

Note, if \(\nu\) is a \(P\)-invariant Borel probability measure, by Birkhoff Theorem and Lemma 3.8 above, the conclusion of Proposition 3.9 holds for \(\nu\)-almost any \(x\) with \(u(x) < \infty\).

As a first step toward the proof, we establish a weaker result that would be sufficient for our purpose.

**Lemma 3.10.** Let \((X, \mathcal{X})\) be a standard Borel space, \(P\) be a Markov operator on \(X\) and \(u : X \to [0, \infty]\) be a Borel function such that there exist \(0 \leq a < 1\) and \(C \geq 0\) with \(Pu \leq au + C\). Then, for any \(x\) in \(X\) with \(u(x) < \infty\) and \(\varepsilon > 0\), there exists \(M > 0\) such that, for \(\omega_x\)-almost any \(w\) with \(u(x) < \infty\),
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{0 \leq k < n} u(w_k) \leq \varepsilon.
\]

**Proof.** Choose \(M_0 > 0\) and set \(Y = u^{-1}([0, M_0])\). By [4, Prop. 6.3], if \(M_0\) is large enough, the set \(Y\) is exponentially \(P\)-recurrent. More precisely, pick \(a_0\) in \((a, 1]\) and assume \(a_0 - a - C/M_0 > 0\). Reusing ideas of the proof of [4, Prop. 6.3], let us prove the following uniform bound for weighted Birkhoff sums of \(u\) up to the return time in \(Y\): for any \(x\) in \(X\),
\[
\int_W \sum_{k=1}^{\tau_Y(w)} a_0^{-k} u(w_k) \, d\omega_x(w) \leq \frac{au(x) + C}{a_0 - a - C/M_0}.
\]

In order to prove this bound, we set, for \(x\) in \(X\) and \(n \geq 1\):
\[
U_n(x) := \int_W \sum_{k=1}^{\min(\tau_Y(w), n)} a_0^{-k} u(w_k) \, d\omega_x(w),
\]
which can be rewritten as

\[ U_n(x) = \sum_{k=1}^{n} a_0^{-k} \int_{\{\tau_Y(w) \geq k\}} u(w_k) \, d\omega_x(w). \]

In particular, one has \( U_n(x) \leq \sum_{k=1}^{n} a_0^{-k} P^k u(x) < \infty \). Besides, the function \( 1_{\{\tau_Y \geq k\}} \) being a function of \( w_1, \ldots, w_{k-1} \), by the Markov property, one gets

\[
U_n(x) = \sum_{k=1}^{n} a_0^{-k} \int_{\{\tau_Y(w) \geq k\}} (Pu)(w_{k-1}) \, d\omega_x(w) \\
\leq \sum_{k=1}^{n} a_0^{-k} \int_{\{\tau_Y(w) \geq k\}} (a u(w_{k-1}) + C) \, d\omega_x(w) \\
\leq \frac{au(x) + C}{a_0} + \sum_{k=2}^{n} a_0^{-k} \int_{\{\tau_Y(w) \geq k\}} (a + C/M_0) u(w_{k-1}) \, d\omega_x(w) \\
\leq \frac{au(x) + C}{a_0} + \frac{a + C/M_0}{a_0} U_n(x),
\]

that is, since \( a_0 - a - C/M_0 > 0 \),

\[ U_n(x) \leq \frac{au(x) + C}{a_0 - a - C/M_0}. \]

Letting \( n \) go to \( \infty \), we get (3.2).

For any \( x \) in \( Y \) and \( p \) in \( \mathbb{N} \), we set, for \( \omega_x \)-almost any \( w \) in \( W \),

\[ v_p(w) = \max_{\tau_Y^p(w) \leq k < \tau_Y^{p+1}(w)} \log_+ u(w_k). \]

According to (3.2) with \( a_0 = 1 \), one has

\[
\sup_{x \in Y} \int_{W} e^{v_0(w)} \, d\omega_x < \infty.
\]

Set \( \rho = \sup_{x \in Y} \int_{W} v_0(w) \, d\omega_x \). Reasoning again as in the proof of Lemma 3.5 yields, for any \( x \) in \( Y \), for \( \omega_x \)-almost any \( w \) in \( W \),

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{p=0}^{n-1} v_p(w) \leq \rho.
\]

Now, by Lemma 3.7, there exists \( T \) in \( \mathbb{N} \) such that, for any \( x \) in \( Y \), for \( \omega_x \)-almost any \( w \) in \( W \), one has

\[
\limsup_{n \to \infty} \frac{1}{n} \tau_{Y,T}^n(w) \leq \frac{\varepsilon}{2}.
\]
Since, for any \( n \in \mathbb{N} \), \( \tau^n_x \geq n \), we get, for any \( M > 1 \),
\[
\#\{0 \leq k < n \mid u(w_k) > M\} \leq \sum_{p=0}^{n-1} \sigma_p^u(w) 1_{\{v_p(w) > \log M\}} \\
\leq \tau^n_{Y,T}(w) + T\#\{0 \leq p < n \mid v_p(w) > \log M\} \\
\leq \tau^n_{Y,T}(w) + \frac{T}{\log M} \sum_{p=0}^{n-1} v_p(w),
\]
thus, for \( \omega \)-almost any \( w \) in \( W \),
\[
\limsup_{n \to \infty} \frac{1}{n}\#\{0 \leq k < n \mid u(w_k) > M\} \leq \varepsilon + \frac{T\rho}{\log M}.
\]
The result follows, since \( M \) is arbitrarily large. \( \square \)

**Proof of Proposition 3.9.** For any \( M > 0 \), set \( X = u^{-1}([0,M]) \).

Fix \( \varepsilon > 0 \) and \( M > 0 \). By Lemma 3.10 there exists \( M' > 0 \) such that, for \( \omega \)-almost any \( w \) in \( W \), for \( n \) large enough, one has
\[
\frac{1}{n} \sum_{k=0}^{n-1} 1_{X_{M'}^\varepsilon}(w_k) \leq \varepsilon.
\]

Pick \( m \geq 1 \) such that \( a^m M' \leq \varepsilon \). By Lemma 3.2 for \( \omega \)-almost any \( w \) in \( W \), for \( n \) large enough, we have
\[
\frac{1}{n} \sum_{k=0}^{n-1} 1_{X_{M'}^\varepsilon}(w_k) \leq \frac{1}{n} \sum_{k=0}^{n-1} P^m 1_{X_{M'}^\varepsilon}(w_k) + \varepsilon \\
\leq \frac{1}{n} \sum_{k=0}^{n-1} 1_{X_{M'}^\varepsilon}(w_k) P^m 1_{X_{M'}^\varepsilon}(w_k) + 2\varepsilon \leq \frac{C}{(1-a)M} + 3\varepsilon,
\]
since, for any \( y \) in \( X_{M'} \), one has
\[
P^m 1_{X_{M'}^\varepsilon}(y) \leq \frac{1}{M} P^m u(y) \leq \frac{1}{M} \left(a^m u(y) + \frac{C}{1-a}\right) \leq \varepsilon + \frac{C}{(1-a)M}.
\]
The Proposition follows. \( \square \)

### 4. Equidistribution through Markov chains

In this chapter, we apply the results on Markov chains established in Chapter 3 to the proof of Theorems 1.4.c), 1.5 and 1.7.

The key geometric input we will need from [4] and [5] is the following
Lemma 4.1. Let $G$, $\Lambda$, $X$, $\Gamma$ and $\mu$ be as in Theorem 1.4 and $L$ be the centralizer of $\Gamma$ in $G$. We denote by $P$ the Markov operator on $X$ with transition probabilities $P_x = \mu * \delta_x$, $x \in X$.

Let $K$ be a compact subset of $X$. There exist a lower semicontinuous function $u : X \to [0, \infty]$ which is bounded on $K$ and $0 \leq a < 1$ and $C > 0$ with $Pu \leq au + C$, such that, for any $M > 0$, the set $X_M = u^{-1}([0, M])$ is compact.

If $X_M$ is $P$-recurrent, we let $P_M$ denote the Markov operator induced by $P$ on $X_M$.

Let $Y$ be a closed $\Gamma$-invariant homogeneous subspace of $X$ and $K_L$ be a compact subset of $L$. If $M$ is large enough, the set $X_M$ is $P$-recurrent and there exist a lower semicontinuous function $v_M : X_M \to [0, \infty]$, a compact neighborhood $K_L'$ of $K_L$ in $L$ and $0 \leq a_M < 1$ and $C_M > 0$ with

$$P_Mv_M \leq a_Mv_M + C_M$$

$$v_M < \infty \quad \text{on} \quad (K_L'Y)^c \cap X_M$$

$$v_M = \infty \quad \text{on} \quad K_LY \cap X_M.$$

Proof. The existence of $u$ follows from [5, Prop. 7.4]; the one of $v_M$ from the proof of [4, Prop. 6.24]. Note that, by Proposition 3.9, the set $X_M$ is $P$-recurrent as soon as $C/M < 1$. □

4.1. Almost sure equidistribution of trajectories.

Proof of Theorem 1.4(c). We set $X = G/\Lambda$ and $P$ for the Markov operator on $X$ with transition probabilities $P_x = \mu * \delta_x$, $x \in X$. Fix $x$ in $X$. As in the first part of the proof of Theorem 1.4 we can assume $x$ does not belong to any non-open $Y$ in $\mathcal{S}_X(\Gamma)$.

By Lemma 4.1 there exist a lower semicontinuous function $u : X \to [0, \infty]$ and $0 \leq a < 1$ and $C > 0$ with $Pu \leq au + C$, $u(x) < \infty$ and such that, for any $M > 0$, the set $X_M = u^{-1}([0, M])$ is compact. Therefore, by Proposition 3.9 for $\mu$-almost any $u$ in $W$, any weak limit point $\nu$ of $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{w_k}$ is a probability measure. By Lemma 3.2, such a limit point is $\mu$-stationary. By [4, Th. 2.5], $\nu$ is an average of probability measures $\nu_Y$ with $Y$ in $\mathcal{S}_X(\Gamma)$. By Proposition 2.1 and Lemma 2.9 it suffices to prove that, for any non-open $Y$ in $\mathcal{S}_X(\Gamma)$ and any compact subset $K_L$ of $L$, one has $\nu(K_LY) = 0$ (where $L$ is the centralizer of $\Gamma$ in $G$).

Fix $0 < \varepsilon < 1$ and $M > 0$ large enough, so that $\frac{C}{(1-a)M} \leq \varepsilon$. By Proposition 3.9 the set $X_M = u^{-1}([0, M])$ is $P$-recurrent. We let $P_M$ be the Markov operator induced by $P$ on $X_M$ and $\tau^p_M$, $p \in \mathbb{N}$, be the successive return times in $X_M$. By Lemma 4.1 if $M$ is large enough,
there exist a lower semicontinuous function \( v_M : X_M \to [0, \infty] \) and \( 0 \leq a_M < 1 \) and \( C_M > 0 \) with \( P_M v_M \leq a_M v_M + C_M \), \( v_M(x) < \infty \) and such that \( v_M = \infty \) on \( K_L Y \cap X_M \). Therefore, by Proposition 3.9, applied to the Markov operator \( P_M \), for \( \omega \)-almost any \( w \) in \( W \), any weak limit point of the sequence of probability measures \( \frac{1}{n} \sum_{p=0}^{n-1} \delta_{w, \omega} \) gives mass 0 to \( K_L Y \cap X_M \). Hence, applying Proposition 3.9 to the operator \( P \), we get \( \nu(K_L Y \cap X_M) \leq \varepsilon \). The result follows, since \( \varepsilon \) is arbitrarily small and \( M \) arbitrarily large. \( \square \)

4.2. Equidistribution of invariant homogeneous subsets. The proof of Theorem 1.5 essentially relies on the following

**Lemma 4.2.** Let \( G, \Lambda, X \) and \( \Gamma \) be as in Theorem 1.4 and \( L \) be the centralizer of \( \Gamma \) in \( G \). Fix a compact subset \( K \) of \( X \) and let \( (Y_n) \subset S_K(\Gamma) \) and \( \nu_\infty \) be a limit point of \( \nu_{Y_n} \) as \( n \to \infty \).

a) The measure \( \nu_\infty \) has total mass 1.
b) If \( Y \in S_X(\Gamma) \) is such that, for any compact subset \( K_L \) of \( L \), for all but finitely many \( n \), one has \( Y_n \not\subset K_L Y \), then one has \( \nu_\infty(LY) = 0 \).

**Proof.** The proof follows the same lines as the one of Theorem 1.4 (c). For any \( n \), we set \( \nu_n = \nu_{Y_n} \). By replacing \( K \) by a compact neighborhood, we may assume one has \( \nu_n(K) > 0 \).

Let \( \mu \) be a compactly supported Borel probability measure on \( \Gamma \) whose supports spans \( \Gamma \) and let \( P \) be the Markov operator on \( X \) with transition probabilities \( P_x = \mu \ast \delta_x, \ x \in X \). By Lemma 4.1, there exist a lower semicontinuous function \( u : X \to [0, \infty] \) which is bounded on \( K \) and \( 0 \leq a < 1 \) and \( C > 0 \) with \( Pu \leq au + C \), such that, for any \( M > 0 \), the set \( X_M = u^{-1}([0, M]) \) is compact. Since for any \( n \), \( \nu_n \) is \( \Gamma \)-invariant, it is \( P \)-invariant. Hence, as \( \nu_n(K) > 0 \), by Lemma 3.8, one has \( \int_X u d\nu_n \leq \frac{C}{1-a} \) and therefore

\[
\nu_n(X_M^c) \leq \frac{C}{(1-a)M}.
\]

We get \( \nu_\infty(X_M^c) \leq \frac{C}{(1-a)M} \) and \( \nu_\infty \) is a probability measure, that is a) is proved.

Let \( Y \) be as in b) and let us prove \( \nu_\infty(LY) = 0 \). Fix \( 0 < \varepsilon < 1 \) and \( M > 0 \) large enough, so that \( \frac{C}{(1-a)M} \leq \varepsilon \). By Proposition 3.9, the set \( X_M \) is \( P \)-recurrent. We let \( P_M \) be the induced Markov operator on \( X_M \). Now, if \( M \) is large enough, the second part of Lemma 4.1 holds: there exist a lower semicontinuous function \( v_M : X_M \to [0, \infty] \), a compact neighborhood \( K'_L \) of \( K_L \) in \( L \) and \( 0 \leq a_M < 1 \) and \( C_M > 0 \) with \( P_M v_M \leq a_M v_M + C_M \), such that \( v_M < \infty \) on \( (K'_L Y)^c \cap X_M \) and \( v_M = \infty \) on \( K_L Y \cap X_M \).
For any $M' > 0$, we set $X_{M,M'} = v_M^{-1}([0, M'])$. Let us dominate
$\nu_n(X_M \setminus X_{M,M'})$. By Lemma 3.4 for all $n$, the restriction of $\nu_n$ to $X_M$
and $P_M$-invariant and $P_M$-ergodic. By assumption, for all but finitely
many $n$, we have $\nu_n(K_LY) = 0$. Hence, by Lemma 3.8 we get
$$\int_{X_M} v_M \, d\nu_n \leq \nu_n(X_M) \frac{C_M}{(1 - a_M)} \leq \frac{C_M}{(1 - a_M)},$$
and $\nu_n(X_M \setminus X_{M,M'}) \leq \frac{C_M}{(1 - a_M)M'}$.

Since $X_{M,M'}$ is compact, using (4.1) and letting $n$ go to $\infty$, this gives
$\nu_\infty(X_{M,M'}) \geq 1 - \varepsilon - \frac{C_M}{(1 - a_M)M'}$, thus, since $M'$ is arbitrary, $\nu_\infty(K_LY \cap
X_M) \leq \varepsilon$. Since $\varepsilon$ is arbitrarily small and $M$ arbitrarily large, this
proves $\nu_\infty(LY) = 0$ as required. □

Proof of Theorem 1.5. Since $b)$ directly follows from Lemma 4.2 we
only have to prove $a)$. Note $b)$ implies $S_K(\Gamma)$ is closed.

Let $(Y_n)$ be a sequence in $S_K(\Gamma)$ and let us construct a converging
subsequence. First, we can assume the sequence $(\nu_{v_n})$ converges to a
measure $\nu_\infty$. By Lemma 4.2 $a)$, $\nu_\infty$ is a probability measure. Since
$\nu_\infty$ is $\Gamma$-invariant, by [4, Th. 2.5], every $\Gamma$-ergodic component of $\nu_\infty$
is equal to $\nu_Y$ for some $Y$ in $S_X(\Gamma)$, hence, as by Proposition 2.1 $S_X(\Gamma)$
is a countable union of $L$-orbits, there exists $Y_\infty$ in $S_X(\Gamma)$ such that
$\nu_\infty(LY_\infty) > 0$. By Lemma 4.2 $b)$, there exists a compact subset $K_L$ of
$L$ such that, for large $n$, one has $Y_n \subset K_LY_\infty$. Assume the dimension of
$Y_\infty$ is minimal and let us prove $\nu_\infty = \nu_{Y_\infty}$ for some $\ell$ in $K_L$.

Indeed, for $n$ large, since $Y_n$ is $\Gamma$-ergodic, there exists $\ell_n$ in $K_L$
such that $Y_n \subset \ell_n Y_\infty$. After again extracting and replacing $Y_\infty$ by a trans-
late, we can assume $\ell_n \xrightarrow{n \to \infty} e$. Since $\nu_{\ell_n} - \nu_{\ell_{n+1}} \xrightarrow{n \to \infty} 0$, we can
assume, for all $n$, one has $\ell_n = e$, that is $Y_n \subset Y_\infty$. Now, still by [4,
Th. 2.5], every $\Gamma$-ergodic component of $\nu_\infty$ is equal to $\nu_Y$ for some
$Y$ in $S_X(\Gamma)$, $Y \subset Y_\infty$. But, by assumption and by Lemma 4.2 $b)$, if
$Y \subsetneq Y_\infty$, one has $\nu_\infty(LY) = 0$. Hence, again by Proposition 2.1 we have
$\nu_\infty = \nu_{Y_\infty}$, what should be proved. □

4.3. Closed invariant subsets.

Proof of Theorem 1.5 $a)$ and $b)$. Note $b)$ follows directly from Theo-
rem 1.5 $b)$ and the fact that $L$ is now assumed to be discrete.

Let us prove $a)$, that is the set $S_X(\Gamma)$ is compact. By Theorem 1.5 $a)$,
it suffices to construct a compact subset $K$ of $X$ such that, for every
$x$ in $X$, one has $\Gamma x \cap K \neq \emptyset$. Now, fix a compactly supported Borel
probability measure $\mu$ on $\Gamma$ whose support spans $\Gamma$. By [5, Th. 7.2.b]
(see also [11] and [5, Th. 1.4] in the real case), there exists a compact
subset $K$ of $X$ such that, for any $x$ in $X$, for $n$ large enough, one has $\mu^n \ast \delta_x(K) \geq 1/2$ and we are done. \hfill \Box

To prove Theorem 1.7(c), we will need the following complement to Lemma 2.9.

**Lemma 4.3.** Let $G$, $\Lambda$, $X$ and $\Gamma$ be as in Theorem 1.4. If the centralizer of $\Gamma$ in $G$ is discrete, then the set $S_{op}(\Gamma)$ is a finite cover of $X$.

**Proof.** We have to check the set $S_{op}(\Gamma)$ is finite and one has $X = \bigcup_{Y \in S_{op}(\Gamma)} Y$. Assume there exists a sequence $(Y_n)$ of distinct elements of $S_{op}(\Gamma)$. Since, by Theorem 1.7(a), $S_X(\Gamma)$ is compact, we can assume $(Y_n)$ converges to some $Y$ in $S_X(\Gamma)$. Now, by Theorem 1.7(b), we get, for large $n$, $Y_n \subset Y$, hence, since $Y_n$ is open and $Y$ is $\Gamma$-ergodic, $Y_n = Y$, which is a contradiction. Therefore, $S_{op}(\Gamma)$ is finite. In particular, the set $X \setminus \bigcup_{Y \in S_{op}(\Gamma)} Y$ is open in $X$. Besides, we have $X = \bigcup_{Y \in S_X(\Gamma)} Y$ by Theorem 1.4 and $\nu_X(\bigcup_{Y \not\in S_{op}(\Gamma)} Y) = 0$ since, by Proposition 2.1, $S(\Gamma)$ is countable. Thus, we get $X \setminus \bigcup_{Y \in S_{op}(\Gamma)} Y = \emptyset$. \hfill \Box

**Proof of Theorem 1.7(c).** Let $F$ be a closed $\Gamma$-invariant subset of $X$, and let us prove $F$ is a finite union of elements of $S_X(\Gamma)$. Using Lemma 4.3, we can assume $X$ is $\Gamma$-ergodic. We proceed by induction on the dimension of $X$. If it is zero, there is nothing to prove. Assume it is positive.

If there exists $Y_1, \ldots, Y_r$ in $S_X(\Gamma) \setminus \{X\}$ such that $F \subset Y_1 \cup \cdots \cup Y_r$, then we are done by induction, since, for $1 \leq i \leq r$, $F \cap Y_i$ is a closed $\Gamma$-invariant subset of $Y_i$.

Assume this is not the case and let us prove $F = X$. Let $(Y_n)_{n \geq 1}$ be the elements of $S_X(\Gamma) \setminus \{X\}$, which is countable by Proposition 2.1. For any $n \geq 1$, pick $x_n \in F \setminus (Y_1 \cup \cdots \cup Y_n)$ and set $Z_n = \Gamma x_n$. By Theorem 1.4, we have $Z_n \in S_X(\Gamma)$. Since, by $a)$, $S_X(\Gamma)$ is compact, $(Z_n)$ admits a limit point $Z_{\infty}$ in $S_X(\Gamma)$. Now, by construction, for any $Y \neq X$ in $S_X(\Gamma)$, for all but finitely many $n$, one has $Z_n \not\subset Y$, hence, by $b)$, $Z_{\infty} \not\subset Y$. We get $Z_{\infty} = X$, hence $F = X$, what should be proved. \hfill \Box

**Example 4.4.** If $L$ is not discrete, there may exist closed $\Gamma$-invariant subsets in $X$ which are not finite unions of sets of the form $KL Y$, where $K_L$ is a compact subset of $L$ and $Y$ is in $S(\Gamma)$.

Here are two examples. We set $G = SL(3, \mathbb{R})$, $\Lambda = SL(3, \mathbb{Z})$ and $\Gamma \simeq GL(2, \mathbb{Z})$ to be the stabilizer of $\mathbb{Z}^2 \times \{0\}$ in $\Lambda$. Let $x_0 = \mathbb{Z}^3$ be the base point of $X$, $H \simeq SL(2, \mathbb{R})$ be the semisimple part of the stabilizer of $\mathbb{R}^2 \times \{0\}$ in $G$, $L = \{\ell_t = diag(t, t, t^{-1}) \mid t \neq 0\}$.

We set $Y_\infty = Hx_0$, $(Y_n)$ to be the sequence of all finite $\Gamma$-orbits in $Y_\infty$ and $(t_n) \subset (1, \infty)$ a sequence
converging toward $t_\infty = 1$. Then $F_1 = LY_\infty$ and $F_2 = Y_\infty \cup \bigcup_n \ell_{t_n}Y_n$ are closed $\Gamma$-invariant subsets.

Appendix A. Lattices in $S$-adic Lie groups

We now aim at proving Proposition 2.1 for general $S$-adic Lie groups. In order to adapt the strategy we followed in the real case, we will use the notions introduced in [25] and [4, Sect. 5].

We will first study lattices in $S$-adic Lie groups and in particular give conditions for them to be finitely generated. The main results of this appendix are Proposition A.1 and Lemma A.9.

A.1. Finite generation of lattices. One of the main difficulties for extending the proof of Proposition 2.1 is the fact that some of the lattices we may encounter are not finitely generated: for example, if $p$ is a prime number, the group $\mathbb{Z}[\frac{1}{p}]$ is not finitely generated, although it embeds as a lattice in $\mathbb{R} \times \mathbb{Q}_p$.

In this section, we precisely describe which $S$-adic Lie groups have finitely generated lattices:

Proposition A.1. Let $G$ be a compactly generated $S$-adic Lie group. Any lattice in $G$ is finitely generated.

In case $G$ is $S$-algebraic and the lattice is arithmetic, this result is due to Kneser [15]. We follow the same strategy.

Note that, if a locally compact group admits a finitely generated lattice, it is compactly generated. Conversely, any cocompact lattice in a compactly generated locally compact group is finitely generated (see [16, IX.3] or [4, Prop. 5.22]), but this is not true for non-cocompact lattices: indeed, for any prime number power $q$, the group $\Lambda = \text{SL}(2, \mathbb{F}_q[T^{-1}])$ embeds as a lattice in the compactly generated locally compact group $G = \text{SL}(2, \mathbb{F}_q((T)))$, but $\Lambda$ is not finitely generated.

Proof. We will show how to deduce this statement from the well-known case where $G$ is a real Lie group. Let $G^\circ$ be the connected component of $G$. The closure $\Omega$ of the image of $\Lambda$ in $G/G^\circ$ has finite covolume. Since $G/G^\circ$ is a compactly generated non-archimedean $S$-adic Lie group, by [6, Lem. 5.2], $\Omega$ is cocompact in $G/G^\circ$, hence it is compactly generated (see for instance [4, Prop. 5.22]). We may thus assume that the group $\Lambda G^\circ$ is dense in $G$. We conclude thanks to Lemma A.2 below. □

Lemma A.2. Let $G$ be a compactly generated locally compact group, $G^\circ$ its connected component and $\Lambda$ be a lattice in $G$ such that the group $\Lambda G^\circ$ is dense in $G$. Then the group $\Lambda$ is finitely generated.
Proof. By Montgommery-Zippin theorem in [19], there exists an open subgroup $H$ of $G$ and a compact normal subgroup $K$ of $H$ such that $H/K$ is a connected real Lie group (note that if $G$ is an $S$-adic Lie group, the existence of $H$ and $K$ does not rely on [19]). The group $\Lambda \cap H/\Lambda \cap K$ is a lattice in $H/K$. Hence, by [16, IX.3], $\Lambda \cap H$ is finitely generated.

Recall, if $G'$ is a topological group, $U'$ an open subset of $G'$ which spans $G'$ and $\Lambda'$ a dense subgroup of $G'$, then $\Lambda'$ is spanned by $\Lambda' \cap U'$: indeed, the group spanned by $\Lambda' \cap U'$ is dense in $\Lambda'$ for the induced topology.

Now, since the group $G' := G/G^\circ$ is compactly generated and since the image $H'$ of $H$ in $G'$ is open, there exists a finite set $F'$ in $G'$ such that $F'H'$ generates $G'$. Since the image $\Lambda'$ of $\Lambda$ in the group $G'$ is a dense subgroup, we may assume $F'$ is contained in $\Lambda'$. Then, $\Lambda'$ is generated by $F'$ and $\Lambda' \cap H'$. Therefore, $\Lambda'$ is finitely generated and so is $\Lambda$, since $\Lambda \cap G^\circ$ is contained in $\Lambda \cap H$. □

A.2. Structure results, after Ratner. We recall a few definitions and results from [25].

Let $G$ be a weakly regular $S$-adic Lie group and $g = \bigoplus_{p \in S} g_p$ be the Lie algebra of $G$. We let $G^\infty$ denote the connected component of $G$, which is also the analytic Lie subgroup of $G$ whose Lie algebra is the archimedean part of $g$. We also let $G_u$ denote the closure of the subgroup of $G$ spanned by Ad-unipotent one-parameter subgroups of $G$ and $G_{u,f}$ denote the closure of the subgroup of $G$ spanned by Ad-unipotent one-parameter subgroups of $G$ with derivative in $g_f = \bigoplus_{p \neq \infty} g_p$. The groups $G^\infty$, $G_u$ and $G_{u,f}$ are normal in $G$.

A difficulty in the study of $S$-adic Lie groups is that there is in general no normal open subgroup which plays the role of the connected component of real Lie groups. We may however define a weak analogue notion. A standard open subset $\Omega$ is a product of a small open neighborhood $\Omega^\infty$ of $e$ in $G^\infty$ and of a standard compact subgroup $\Omega_f$ of $G$ with Lie algebra $g_f$ (see [4, Sect. 5.1] and note that the archimedean factor $\Omega^\infty$ will play no role in the sequel). For any closed subgroup $H$ of $G$, we define the $\Omega$-semiconnected component of $H$ as its open subgroup $H_{\Omega} := (H \cap \Omega)H_{u,f}H^\infty$. The group $H$ is said to be $\Omega$-semiconnected if $H = H_{\Omega}$. The group $H$ is said to be semiconnected if there exists a standard open subset $\Omega$ such that $H$ is $\Omega$-semiconnected. Note that semiconnected components are not necessarily normal subgroups as, for instance, if $G = \mathbb{Z}_p \ltimes (\oplus_{n \geq 1} \mathbb{Z}^{\mathbb{Z}/p^n\mathbb{Z}})$, where $p$ is a prime number and the group $\mathbb{Z}_p$ of $p$-adic integers acts by translations on each $\mathbb{Z}/p^n\mathbb{Z}$. 
Let us now focus on a particular class of $S$-adic Lie groups. For $p$ in $S$, a $p$-adic Lie group $N$ is said to be algebraic unipotent if it is isomorphic to the group of $\mathbb{Q}_p$-points of a unipotent $\mathbb{Q}_p$-group. These groups are extensively studied in [25, Sect.2] (where they are called quasiconnected).

If $p = \infty$, $N$ is algebraic unipotent if and only if it is connected, simply connected and nilpotent. Then, the exponential map is a diffeomorphism $n \rightarrow N$, whose inverse is denoted by log.

If $p < \infty$, by [25, Prop. 2.1], $N$ is algebraic unipotent if and only if it is weakly regular, spanned by Ad-unipotent one-parameter subgroups and has nilpotent Lie algebra. Then, for any $g$ in $N$, the morphism $\mathbb{Z} \rightarrow N; n \mapsto g^n$ extends as a continuous morphism $\mathbb{Z}_p \rightarrow N$, whose derivative is denoted by $\log(g)$. The map $\log : N \rightarrow n$ is an analytic homeomorphism whose inverse is denoted by exp.

In any case, exp and log are Aut($N$)-equivariant and the map $n \times n \rightarrow n; (X, Y) \mapsto \log(\exp X \exp Y)$ is polynomial (and is therefore given by the Baker-Campbell-Hausdorff formula). In particular, if $N'$ is a closed normal subgroup of $N$ which is spanned by one-parameter subgroups, then $N/N'$ is also an algebraic unipotent group. An $S$-adic Lie group is said to be algebraic unipotent if it is a product of algebraic unipotent $p$-adic Lie groups, $p \in S$.

Let $G$ be any weakly regular $S$-adic Lie group. Then, the group $G_{u,f}$ admits Levi decompositions. More precisely, let the solvable radical $R_{u,f}$ of $G_{u,f}$ be the closure of the subgroup of $G_{u,f}$ spanned by the Ad-unipotent one-parameter subgroups tangent to the solvable radical $\mathfrak{r}_{u,f}$ of $\mathfrak{g}_{u,f}$. We define a Levi subgroup $S_{u,f}$ of $G_{u,f}$ as being the closure of the subgroup of $G_{u,f}$ spanned by the Ad-unipotent one-parameter subgroups tangent to a given Levi subalgebra (i.e. a maximal semisimple Lie subalgebra) $\mathfrak{s}_{u,f}$ of $\mathfrak{g}_{u,f}$. By [22] and [25, Cor. 2.1], the group $S_{u,f}$ has Lie algebra $\mathfrak{s}_{u,f}$, the center of $S_{u,f}$ is finite, the group $R_{u,f}$ has Lie algebra $\mathfrak{r}_{u,f}$, is algebraic unipotent, and one has $G_{u,f} = S_{u,f}R_{u,f}$. In particular, one has $S_{u,f} \cap R_{u,f} = \{e\}$ and the product map $S_{u,f} \times R_{u,f} \rightarrow G_{u,f}$ is a homeomorphism. It follows that every ad-nilpotent element in $\mathfrak{g}_{u,f}$ is tangent to an Ad-unipotent one-parameter subgroup in $G_{u,f}$.

A.3. Unstable subgroups. We will now give conditions for some weakly regular $S$-adic Lie groups to be compactly generated.

Let still $G$ be a weakly regular $S$-adic Lie group with Lie algebra $\mathfrak{g}$ and let $\Gamma$ be a closed subgroup of $G$. An element $v$ of $\mathfrak{g}$ is said to be $\Gamma$-unstable if 0 belongs to the closure of the orbit Ad\$\Gamma(v)$. A one-parameter subgroup of $G$ is said to be $\Gamma$-unstable if its derivative
is a $\Gamma$-unstable vector (such a one-parameter subgroup is necessarily $\text{Ad}$-unipotent). Let $H$ be a closed subgroup of $G$ with Lie algebra $\mathfrak{h}$ that is normalized by $\Gamma$. We will make repeated use of the following fact from [4, Lem. 5.12]:

\begin{equation}
\text{if } \text{Ad}\Gamma^Z \text{ is semisimple, any } \Gamma\text{-unstable vector in } \mathfrak{h} \text{ is tangent to a } \Gamma\text{-unstable one-parameter subgroup of } H.
\end{equation}

**Lemma A.3.** Let $G$ be a weakly regular $S$-adic Lie group, $\Gamma$ be a closed subgroup of $G$ such that $\text{Ad}\Gamma^Z$ is semisimple and $H$ be a closed subgroup of $G$ that is normalized by $\Gamma$. The following are equivalent.

(i) The group $H_{u,f}$ is topologically spanned by $\Gamma$-unstable one-parameter subgroups.

(ii) The Lie algebra $\mathfrak{h}_{u,f}$ of $H_{u,f}$ is spanned by $\Gamma$-unstable vectors.

**Definition A.4.** If either of the two properties of Lemma [A.3] holds, we shall say $H$ is $\Gamma$-unstable.

Note, if $G$ is a quotient of the real Lie group $\widetilde\text{SL}(2, \mathbb{R}) \times \mathbb{T}$ by a discrete central subgroup whose projection on $\mathbb{T}$ is dense, then $G$ is spanned topologically by $G$-unstable one-parameter subgroups, but its Lie algebra is not spanned by $G$-unstable vectors.

**Proof of Lemma [A.3].** (ii)$\Rightarrow$(i) If $\mathfrak{h}_{u,f}$ is topologically spanned by $\Gamma$-unstable vectors, by (A.1), the closure $H'$ of the subgroup of $H_{u,f}$ spanned by $\Gamma$-unstable one-parameter subgroups has Lie algebra $\mathfrak{h}_{u,f}$. By [25, Cor. 2.1], every ad-nilpotent vector in $\mathfrak{h}_{u,f}$ is tangent to some $\text{Ad}$-unipotent one-parameter subgroup in $H'$, that is $H' = H_{u,f}$.

(i)$\Rightarrow$(ii) Conversely, assume $H_{u,f}$ is topologically spanned by $\Gamma$-unstable one-parameter subgroups. Let $\mathfrak{r}_{u,f}$ be the radical of $\mathfrak{h}_{u,f}$ and set $j = \mathfrak{h}_{u,f}/\mathfrak{r}_{u,f}$. We will first prove that the Lie algebra $j$ is spanned by $\Gamma$-invariant vectors. Let $\mathfrak{l}$ be the subalgebra of those $v$ in $j$ such that $\Gamma v$ is a bounded subset of $j$ and $j_1$ be the subalgebra of $j$ spanned by $\Gamma$-unstable vectors, so that, since $\text{Ad}\Gamma^Z$ is semisimple, we get $j = \mathfrak{l} + j_1$. As $[\mathfrak{l}, j_1] \subset j_1$, the subalgebra $j_1$ is an ideal of $j$. By construction, $\Gamma$ has no unstable vector in $j/j_1$, hence, $H_{u,f}$ being topologically spanned by $\Gamma$-unstable one-parameter subgroups, the adjoint map $H_{u,f} \rightarrow \text{Aut}(j/j_1)$ is trivial. Since $j/j_1$ is semisimple, we get $j_1 = j$, that is $j$ is spanned as a Lie algebra by $\Gamma$-unstable vectors.

Now, as $\text{Ad}\Gamma^Z$ is semisimple, the Lie algebra $\mathfrak{h}_{u,f}$ admits a Levi factor $\mathfrak{s}_{u,f}$ which is $\Gamma$-invariant and, since $\mathfrak{s}_{u,f}$ is $\Gamma$-isomorphic to $j$, it is spanned by $\Gamma$-unstable vectors. Let $\mathfrak{h}'$ be the subalgebra of $\mathfrak{h}_{u,f}$ spanned by $\Gamma$-unstable vectors and $\mathfrak{r}' = \mathfrak{h}' \cap \mathfrak{r}_{u,f}$. Since $\mathfrak{s}_{u,f} \subset \mathfrak{h}'$, we get $[\mathfrak{s}_{u,f}, \mathfrak{r}'] \subset \mathfrak{r}'$. Let $H_{u,f} = S_{u,f} R_{u,f}$ be the Levi decomposition of $H_{u,f}$.
associated to the Levi decomposition $\mathfrak{h}_{u,f} = \mathfrak{s}_{u,f} \oplus \mathfrak{r}_{u,f}$ and $R' = \exp(\mathfrak{r}')$ be the unique algebraic unipotent subgroup of $R_{u,f}$ with Lie algebra $\mathfrak{r}'$. Then $S_{u,f}$ normalizes $R'$ and hence $S_{u,f}R'$ is a closed subgroup in $H_{u,f}$. Now, by construction, every $\Gamma$-unstable one-parameter subgroup in $H_{u,f}$ is contained in $S_{u,f}R'$, that is $H_{u,f} = S_{u,f}R'$, hence $\mathfrak{h}_{u,f} = \mathfrak{h}'$, what should be proved. \hfill \Box

We now get a criterion to ensure some of the groups we will encounter have finitely generated lattices:

**Lemma A.5.** Let $G$ be a weakly regular $S$-adic Lie group and $\Gamma$ be a closed compactly generated subgroup of $G$ such that $\mathrm{Ad}\Gamma$ is semisimple. Assume $G$ admits a normal $\Gamma$-unstable semiconnected subgroup $H$ such that $G = \Gamma H$. Then $G$ is compactly generated. In particular, any lattice in $G$ is finitely generated.

**Proof.** Let $v_i$, $1 \leq i \leq \ell$, be $\Gamma$-unstable vectors which span the Lie algebra $\mathfrak{h}_{u,f}$. By (A.1), the lines they generate are tangent to one-paramater subgroups $V_i$ of $H_{u,f}$. By [25, Cor. 2.1], the group $H_{u,f}$ is spanned by $V_1 \cup \ldots \cup V_\ell$. By construction, the group generated by $\Gamma$ and $V_1, \ldots, V_\ell$ is compactly generated. Since this group is equal to $\Gamma H_{u,f}$, since $H$ is a compact extension of $H_{u,f}H_\infty$, and since $H_\infty$ is compactly generated, the group $G = \Gamma H$ is compactly generated. The property on lattices follows by Proposition [A.1]. \hfill \Box

### A.4. Cosolvable radicals

We will construct large compactly generated subgroups in weakly regular semiconnected $S$-adic Lie groups.

Let $\mathfrak{g}$ be a Lie algebra. We shall say an ideal $\mathfrak{h}$ of $\mathfrak{g}$ is cosolvable if the algebra $\mathfrak{g}/\mathfrak{h}$ is solvable. As the intersection of any family of cosolvable ideals of $\mathfrak{g}$ still is a cosolvable ideal, $\mathfrak{g}$ admits a smallest cosolvable ideal $\mathfrak{c}$. We say $\mathfrak{c}$ is the cosolvable radical of $\mathfrak{g}$.

**Lemma A.6.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$. Then the cosolvable radical of $\mathfrak{g}$ is the subalgebra of $\mathfrak{g}$ spanned by $[\mathfrak{s}, \mathfrak{g}]$.

**Proof.** Let $\mathfrak{c}$ be the cosolvable radical of $\mathfrak{g}$, $\mathfrak{c}'$ be the subalgebra spanned by $[\mathfrak{s}, \mathfrak{g}]$ and $\mathfrak{l}$ be the centralizer of $\mathfrak{s}$. Since $\mathfrak{s} \subset \mathfrak{c}$, we have $\mathfrak{c}' \subset \mathfrak{c}$. Now, as $\mathfrak{s}$ is semisimple, we have $\mathfrak{g} = \mathfrak{l} \oplus [\mathfrak{s}, \mathfrak{g}]$. As $\mathfrak{l}$ normalizes $[\mathfrak{s}, \mathfrak{g}]$, it normalizes $\mathfrak{c}'$, so that $\mathfrak{c}'$ is an ideal. As $\mathfrak{s} \subset \mathfrak{c}'$, $\mathfrak{c}'$ is cosolvable, that is $\mathfrak{c} \subset \mathfrak{c}'$ and we are done. \hfill \Box

For a weakly regular $S$-adic Lie group $G$, we define the cosolvable radical $\mathfrak{c}_u$ of $G_u$ as the closure of the subgroup of $G_u$ spanned by the Ad-unipotent one-parameter subgroups tangent to the cosolvable radical $\mathfrak{c}_u$ of $\mathfrak{g}_u$. 
Lemma A.7. Let $G$ be a weakly regular $S$-adic Lie group with Lie algebra $\mathfrak{g}$, $\Omega$ be a standard open subset of $G$, $\mathfrak{c}_u$ be the solvable radical of $\mathfrak{g}_u$ and $C_u$ be the solvable radical of $G_u$.  

a) The group $C_{u,f}$ has Lie algebra $\mathfrak{c}_{u,f}$ and the group $N_u := G_{u,f}/C_{u,f}$ is algebraic unipotent.  
b) The group $C_uG_\infty$ is compactly generated; it is the largest compactly generated closed subgroup $H$ of $G$ with $H = H_uH_\infty$.  
c) The group $\Omega C_uG_\infty$ is the largest compactly generated $\Omega$-semiconnected subgroup of $G$.  

Proof. a) Let $\mathfrak{r}_{u,f}$ be the solvable radical of $\mathfrak{g}_{u,f}$, $R_{u,f}$ be the solvable radical of $G_{u,f}$, $\mathfrak{s}_{u,f}$ be a Levi subalgebra of $\mathfrak{g}_{u,f}$ and $S_{u,f}$ be the associated Levi subgroup of $G_{u,f}$. By construction, the Lie algebra $\mathfrak{s}_{u,f}$ has no anisotropic factor and the group $\text{Ad}(S_{u,f})$ has finite index in the connected algebraic subgroup of $\text{GL}(\mathfrak{g}_{u,f})$ with Lie algebra $\mathfrak{s}_{u,f}$, so that $[\mathfrak{s}_{u,f}, \mathfrak{g}_{u,f}]$ is exactly the subspace spanned by $S_{u,f}$-unstable vectors in $\mathfrak{g}_{u,f}$ and, by Lemma A.6, $\mathfrak{c}_{u,f}$ is the subalgebra spanned by $S_{u,f}$-unstable vectors in $\mathfrak{g}_{u,f}$. By (A.1), every such vector is tangent to a $S_{u,f}$-unstable one-parameter subgroup. Hence, by Lemma A.3, the group $C_{u,f}$ has Lie algebra $\mathfrak{c}_{u,f}$. In other terms, if $T_1 = \mathfrak{c}_{u,f} \cap \mathfrak{r}_{u,f}$, we have $C_{u,f} = S_{u,f}R_1$, where $R_1 = \exp(T_1)$ is the unique algebraic unipotent subgroup of $R_{u,f}$ with Lie algebra $T_1$. In particular, $N_u$ is algebraic unipotent since it is isomorphic to $R_{u,f}/R_1$ and a) is proved.  
b) There exists $v_1, \ldots, v_r$ in $\mathfrak{c}_{u,f}$ which span $\mathfrak{c}_{u,f}$ as a Lie algebra and are eigenvectors associated to eigenvalues with modulus $< 1$ of some $\gamma_1, \ldots, \gamma_r$ in $S_{u,f}$. Therefore, if $\Gamma$ denotes the closure of the subgroup of $S_{u,f}$ spanned by $\gamma_1, \ldots, \gamma_r$, the group $C_{u,f}$ is $\Gamma$-unstable. By enlarging $\Gamma$, we can assume $\text{Ad}(\Gamma)$ to be Zariski dense in $\text{Ad}(S_{u,f})$. Thus, by Lemma A.5, $C_{u,f}G_\infty = C_uG_\infty$ is compactly generated. Conversely, let $H$ be a compactly generated closed subgroup of $G$ with $H = H_uH_\infty$, so that $H$ is contained in $G_uG_\infty$. Since the closure of $HC_uG_\infty$ is still compactly generated, we may assume $C_uG_\infty \subset H$. Since the quotient group $G_uG_\infty/C_uG_\infty$ is isomorphic to $N_u$ and, in a non-archimedean algebraic unipotent group, every compactly generated subgroup is compact and every one-parameter subgroup has closed non compact image, the image of $H$ in $N_u$ is trivial as required. We have proved b).  
c) follows easily from b).  

We can now characterize the compactly generated weakly regular semiconnected $S$-adic Lie groups.
Corollary A.8. Let $G$ be weakly regular semiconnected $S$-adic Lie group and $\mathfrak{g}_{u,f}$ be the Lie algebra of the group $G_{u,f}$. The group $G$ is compactly generated if and only if $\mathfrak{g}_{u,f} = [\mathfrak{g}_{u,f}, \mathfrak{g}_{u,f}]$.

Proof. By Lemma A.7, $G$ is compactly generated if and only if $G_{u,f}$ equals its cosolvable radical, that is $\mathfrak{g}_{u,f} = [\mathfrak{g}_{u,f}, \mathfrak{g}_{u,f}]$. □

A.5. Finitely generated subgroups of lattices. The proof of Proposition 2.1 for an $S$-adic Lie group $G$ will rely on studying sets of subgroups of $G$ which play the same role as the sets $T(G, \Delta, \Sigma)$ in the real case. Thanks to Lemma A.5, we will know that the relevant subgroups $\Sigma$ will be finitely generated, hence will vary in a countable set. But the groups $\Delta$ will be lattices in semiconnected groups and might not be finitely generated.

In this section we develop tools for overcoming this difficulty: indeed, given $\Delta$, we will exhibit a subgroup $\Delta'$ of $\Delta$ which is finitely generated and such that the group $\Delta$ is spanned by all the conjugates of $\Delta'$ under the elements of $\Sigma$. This subgroup $\Delta'$ will be constructed as the intersection of $\Delta$ with some large semiconnected compactly generated open subgroup as in Lemma A.7.

Lemma A.9. Let $H$ be a weakly regular $S$-adic Lie group, $\Omega$ be a standard open subset of $H$ and $\Gamma$ be a compactly generated subgroup of $H$ such that $\overline{\text{Ad} \Gamma \Omega}$ is semisimple and equal to $\overline{\text{Ad} \Gamma \Omega^{Z,nc}}$. Let $\Sigma$ be a lattice in $H$ such that $\Gamma$ acts ergodically on $H/\Sigma$. Assume the $\Omega$-semiconnected component $H_{\Omega}$ of $H$ is a normal $\Gamma$-unstable subgroup and $H = \Gamma H_{\Omega}$. Set $H'$ to be the largest compactly generated $\Omega$-semiconnected subgroup of $H$, $\Delta = \Sigma \cap H_{\Omega}$ and $\Delta' = \Delta \cap H'$. Then $\Delta'$ is finitely generated and $\Delta$ is the smallest normal subgroup of $\Sigma$ containing $\Delta'$.

The reader can keep in mind the example where $H = \Gamma \ltimes H_{\Omega}$ and $\Sigma = \Gamma \rtimes \Delta$ with $\Gamma = \text{SL}(d, \mathbb{Z}[1/p])$ acting diagonally on the group $H_{\Omega} = \mathbb{Q}_p^d \times \mathbb{R}^d$ and on its lattice $\Delta = \mathbb{Z}[1/p]^d$ (for a prime number $p$). In this case, one can set $H' = \mathbb{Z}_p^d \times \mathbb{R}^d$ and $\Delta' = \mathbb{Z}^d$.

A key step for the proof is the following

Lemma A.10. Let $G$ be a weakly regular $S$-adic Lie group and $C_u$ be the cosolvable radical of $G_u$. Then, if $\Lambda$ is a finite covolume subgroup of $G$, the group $C_u G_{\infty} \Lambda$ contains $G_u$.

Proof. By [4, Lem. 5.27], this statement is true when the group $G_u$ is nilpotent. We will reduce the general case to this one. Indeed, after replacing $G$ by a semiconnected component, we may assume $G$ is semiconnected. By Lemma A.7, the group $N_u := G_{u,f}/C_{u,f}$ is algebraic.
unipotent. Since the quotient $N := G/C_uG_\infty$ is a compact extension of $N_u$, by [25, Prop. 1.2] every one-parameter subgroup of $N$ is contained in $N_u$, hence $N$ is weakly regular. After replacing $G$ by $N$, we may assume $G_u$ is algebraic unipotent and apply [4, Lem. 5.27]. □

Lemma [A.10] yields the following extension of [4, Lem. 5.28]:

**Corollary A.11.** Let $G$ be a weakly regular $S$-adic Lie group and $\Omega$ be a standard open subset of $G$. Assume $G$ is $\Omega$-semiconnected and let $G'$ be the largest compactly generated $\Omega$-semiconnected subgroup of $G$. Then, if $\Lambda$ is a lattice in $G$, one has $G = G'\Lambda$.

**Proof.** Let $C_u$ be the cosolvable radical of $G_u$. By Lemma [A.7] one has $G' = \Omega C_u G_\infty$. As $G'$ is open, the set $G'\Lambda$ is closed. As it contains $C_u G_\infty \Lambda$, by Lemma [A.10] it contains $G_u$. The result follows, since $G = G' G_u$. □

We can now give the proof of Lemma [A.9].

**Proof of Lemma A.9.** Let $C_u$ be the cosolvable radical of $H_u$. By Lemma [A.7] one has $H' = \Omega C_u H_\infty$. Since $H'$ is compactly generated, by Proposition [A.1] its lattice $\Delta'$ is finitely generated.

Let $\Delta''$ be a subgroup of $\Delta$ that contains $\Delta'$ and is normal in $\Sigma$: we have to prove $\Delta'' = \Delta$. To do this, we will prove one has $H_u \subset C_u H_\infty \Delta''$, as in Lemma [A.10] and infer, as in Corollary A.11, this gives $H_\Omega = H' \Delta''$, which in turn yields $\Delta'' = \Delta$.

Let us therefore study the group $C_u H_\infty \Delta''$. By Lemma [A.7] the quotient group $N_u := H_u H_\infty / C_u H_\infty$ is algebraic unipotent. We denote by $M$ the image of $C_u H_\infty \Delta'' \cap H_u H_\infty$ in $N_u$. We want to prove $M = N_u$.

First note that, by Lemma [A.10], the group $C_u H_\infty \Delta$ contains $H_u$. Thus, since $H'$ is open in $H$ and contains $C_u H_\infty$, the group $C_u H_\infty \Delta'$ contains $H' \cap H_u$ and the group $M$ is open in $N_u$. Let respectively $m_u$ and $n_u$ be the Lie algebras of $M_u$ and $N_u$. As $M$ is normalized by $\Sigma$, the map

$$H \to \text{Gr}(n_u); h \mapsto \text{Ad}_h(m_u)$$

factors as a $H$-equivariant continuous map $H/\Sigma \to \text{Gr}(n_u)$. As $H/\Sigma$ is $\Gamma$-ergodic, by Lemma [2,3] this map is constant, that is $m_u$ is a $H$-invariant ideal of $n_u$ and $M_u$ is a $H$-invariant subgroup of $N_u$.

Since the $S$-adic Lie group $N_u$ is algebraic unipotent and non-archimedean, by [25, Prop. 2.1], its closed subgroup $M$ is a compact extension of $M_u$. Hence $P = \log(M/M_u)$ is a $\Sigma H_\infty$-invariant compact open subset of $n_u/m_u$. Now, $\Sigma H_\infty / H_\infty$ is a finite covolume subgroup in the non-archimedean $S$-adic Lie group $H/H_\infty$, hence by [6, Prop. 5.1], the space $H/\Sigma H_\infty$ is compact and the set $Q = \bigcup_{h \in H} h P$ is a compact open
$H$-invariant subset of $n_u/m_u$. As $Q$ is $\Gamma$-invariant, $n_u/m_u$ contains no $\Gamma$-unstable vector. On the other hand, since $H_\Omega$ is $\Gamma$-unstable, the Lie algebra $n_u$ is spanned by $\Gamma$-unstable vectors and so is $n_u/m_u$. This proves $m_u = n_u$, hence $M = N_u$ as required.

Since $H'$ is open in $H$, this gives $H_u \subset H'\Delta''$, hence $H_\Omega = H'\Delta''$. As $\Delta'' \subset \Delta$ and $(\Delta \cap H') \subset \Delta''$, we get $\Delta = \Delta''$. □

Appendix B. Countability of invariant subspaces in the $S$-adic case

We now start the proof of Proposition 2.1. We will first prove an analogue of Lemma 2.5.

B.1. Well-shaped compact open subgroups. One of the difficulties we encounter is to extend Lemma 2.6: indeed, if $G$ is a weakly regular $S$-adic Lie group and $H_1$ and $H_2$ are subgroups of $G$, if $H_1$ normalizes the Lie algebra of $H_2$, there is no reason for $H_1$ to normalize a semiconnected component of $H_2$, as, for instance, when $G = H_1 = SL(d, \mathbb{Q}_p)$ and $H_2 = SL(d, \mathbb{Z}_p)$. In this section, we explain how to chose semiconnected components carefully in order to ensure properties of this kind to hold.

Let $\Gamma$ be a subgroup of $G$ such that $\overline{\text{Ad}\Gamma}^Z$ is semisimple and equal to $\overline{\text{Ad}\Gamma}^{Z, nc}$ and let $\mathfrak{l}$ be the centralizer of $\Gamma$ in $\mathfrak{g}$. Let $\Omega$ be a standard open subset of $G$ with exponential map $\exp_\Omega: O \to \Omega$.

As in [4, Sect. 5], we will say $\Omega$ is $\Gamma$-good if, for every $v$ in $O$ and $\gamma$ in $\Gamma$ with $\text{Ad}\gamma(v)$ in $O$, one has

$$\exp_\Omega(\text{Ad}\gamma(v)) = \gamma \exp_\Omega(v)\gamma^{-1}.$$ 

We will say $\Omega$ is $\Gamma$-well shaped if it is $\Gamma$-good and if, for every Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ which is normalized by $\Gamma$, setting $H_{\Gamma,u}$ to be the closure of the subgroup of $G$ spanned by $\Gamma$-unstable one-parameter subgroups in $\mathfrak{h}$, one has

$$\exp_\Omega(\mathfrak{h} \cap O) \subset \exp_\Omega(\mathfrak{h} \cap \mathfrak{t} \cap O)H_{\Gamma,u}H_\infty.$$ 

Example B.1. There may exist $\Gamma$-good standard open subsets which are not $\Gamma$-well-shaped. For instance, fix a prime number $p$ and set

$$G = \Gamma = \mathbb{Z}_p \times SL(2, \mathbb{Q}_p),$$

$$\Omega = \{(t, g) \in \mathbb{Z}_p \times SL(2, \mathbb{Z}_p) \mid g \equiv \begin{pmatrix} 1 & tp \\ 0 & 1 \end{pmatrix} \pmod{p^2} \}.$$ 

One easily checks $\Omega$ is not contained in $\exp_\Omega(\mathfrak{t} \cap O)G_{\Gamma,u} = (p\mathbb{Z}_p) \times SL(2, \mathbb{Q}_p)$. 

Example B.1.
Lemma B.2. Let $G$ be a weakly regular $S$-adic Lie group and $\Gamma$ be a compactly generated subgroup of $G$ such that $\Ad \Gamma$ is semisimple and equal to $\Ad \Gamma^{Z,nc}$. Then $G$ admits arbitrarily small $\Gamma$-well shaped standard open subsets.

Proof. It suffices to deal with the case where $S = \{p\}$ for some $p < \infty$. By [4, Prop. 5.11], there exists a $\Gamma$-good standard open subset $\Omega$. Let $\mathfrak{l}$ be the centralizer of $\Gamma$ in $\mathfrak{g}$ and $\mathfrak{v}$ be the $\Gamma$-invariant complement of $\mathfrak{l}$. As the set of $\Gamma$-invariant subspaces of $\mathfrak{v}$ is compact, as each of these subspaces is spanned by $\Gamma$-unstable vectors and as two nearby such $\mathfrak{v}$'s are $\Gamma$-isomorphic, there exists an open subset $O' \subset O$ such that, for every $\Gamma$-invariant subspace $\mathfrak{w}$ of $\mathfrak{v}$, the sub-$\mathbb{Z}_p$-module spanned by the $\Gamma$-unstable vectors in $\mathfrak{w} \cap O'$ contains $\mathfrak{w} \cap O'$. By shrinking $O'$, we can assume it is arbitrarily small, it is a Lie sub-$\mathbb{Z}_p$-algebra and one has $O' = (\mathfrak{h} \cap \mathfrak{o}') \oplus (\mathfrak{v} \cap \mathfrak{o}')$. We set $\Omega' = \exp_\Omega(O')$. Let us prove that $\Omega'$ is a $\Gamma$-well shaped standard open subset of $G$.

Let $\mathfrak{h}$ be a $\Gamma$-invariant subalgebra of $\mathfrak{g}$ and set $\mathfrak{w} = \mathfrak{h} \cap \mathfrak{v}$. We have $\mathfrak{h} \cap \mathfrak{o}' = (\mathfrak{h} \cap \mathfrak{l} \cap \mathfrak{o}') \oplus (\mathfrak{w} \cap \mathfrak{o}')$, hence, if $\mathfrak{o}'$ is small enough,

$\exp_\Omega(\mathfrak{h} \cap \mathfrak{o}') = \exp_\Omega(\mathfrak{h} \cap \mathfrak{l} \cap \mathfrak{o}') \exp_\Omega(\mathfrak{w} \cap \mathfrak{o}')$,

so that we just have to prove

$\exp_\Omega(\mathfrak{w} \cap \mathfrak{o}') \subset H_{\Gamma,u}$,

where $H_{\Gamma,u}$ is defined as above. Indeed, let $P = \exp_{\Omega}^{-1}(H_{\Gamma,u} \cap \Omega)$. Then $P$ is a sub-$\mathbb{Z}_p$-module of $\mathfrak{o}$ and, since $\Omega$ is $\Gamma$-good, by (A.1), $P$ contains every $\Gamma$-unstable element in $\mathfrak{w}$. Hence, $P$ contains $\mathfrak{o}' \cap \mathfrak{w}$, what should be proved. □

B.2. Subgroups with a given lattice. Let us now introduce the set of subgroups of $G$ that will play in the general case the same role as the one played by the set $T(G, \Delta, \Sigma)$ in the real case.

Let $\Delta \subset \Sigma$ be discrete subgroups of $G$ and $\Omega$ be a standard open subset of $G$ with exponential map $\exp_\Omega : \mathfrak{o} \to \Omega$.

Definition B.3. We let $T_\Omega(G, \Delta, \Sigma)$ denote the set of closed subgroups $H$ of $G$ satisfying the following properties:

(i) $\Sigma$ is contained in $H$ and $\Sigma$ is a lattice in $H$.
(ii) the group $\exp_\Omega(\mathfrak{h} \cap \mathfrak{o})$ is equal to $H \cap \Omega$, the $\Omega$-semiconnected component $H_\Omega$ is a normal subgroup of $H$ and one has $\Delta = \Sigma \cap H_\Omega$.
(iii) there exists a compactly generated subgroup $\Gamma$ of $H$ which acts ergodically on $H/\Sigma$, such that $\Ad \Gamma$ is semisimple and equal to $\Ad \Gamma^{Z,nc}$, $\Omega$ is $\Gamma$-well-shaped, $H_\Omega$ is $\Gamma$-unstable and $H = \Gamma H_\Omega$.

Lemma 2.5 admits the following weak analogue in the $S$-adic case:
Lemma B.4. Let $G$ be a second countable weakly regular $S$-adic Lie group, $\Omega$ be a standard open subset of $G$ and $\Delta \subset \Sigma$ be discrete subgroups of $G$. Then, there exists a countable set $U_\Omega(G, \Delta, \Sigma)$ of closed subgroups of $G$ such that, for any $H$ in $T_\Omega(G, \Delta, \Sigma)$, there exists $J$ in $U_\Omega(G, \Delta, \Sigma)$ with $H_\infty \subset J \subset H$ such that $H/J$ is compact and $H$ virtually normalizes $J$.

We say $H$ virtually normalizes $J$ if some finite index subgroup of $H$ normalizes $J$.

The set $T_\Omega(G, \Delta, \Sigma)$ might not be countable, as, for instance when $G = \Omega = \mathbb{Z}_p^2$ and $\Delta = \Sigma = \{e\}$, since, in this case, $H$ can be any subgroup of the form $G \cap D$, where $D$ is a line in $\mathbb{Q}_p^2$.

We shall again need several preparatory lemmas. Thanks to our intricate definition, we have the following analogue of Lemma 2.6:

Lemma B.5. Let $G$ be a weakly regular $S$-adic Lie group, $\Omega$ be a standard open subset of $G$ and $\Delta \subset \Sigma$ be discrete subgroups of $G$. Then, if $H_1$ and $H_2$ are in $T_\Omega(G, \Delta, \Sigma)$, the group $H_1$ normalizes $H_2, \Omega$.

Proof. We will first prove that $H_1, \Omega$ normalizes $H_2, \Omega$. As $\Sigma$ normalizes $\mathfrak{h}_2$, we get a $H_1$-equivariant continuous map

$$H_1/\Sigma \to \text{Gr}(\mathfrak{g}); \ h_1/\Sigma \mapsto \text{Ad} h_1(\mathfrak{h}_2).$$

By Lemma 2.3, this map is constant, that is $H_1$ normalizes $\mathfrak{h}_2$, hence $H_1$ normalizes $H_{2,\infty}$. In the same way, $H_1$ normalizes $\mathfrak{h}_{2,u}$, hence it normalizes $H_{2,u}$. Set $O = \log \Omega$. The normalizer $H'_1$ of the group $H_{2,\Omega} = \exp_\Omega(\mathfrak{h}_2 \cap O)H_{2,u}H_{2,\infty}$ in $H_1$ contains the groups $\exp_\Omega(\mathfrak{h}_1 \cap O)$, $H_{1,\infty}$ and $\Sigma$. In particular, $H'_1$ is open in $H_1$ and, since $\Sigma$ is a lattice in $H_1$, $H'_1$ is a finite index subgroup of $H_1$. Therefore, since for any $p$ in $S$, $\mathbb{Q}_p$ does not admit any proper finite index open subgroup, $H'_1$ contains $H_{1,u}$ and the group $H_{1,\Omega} = \exp_\Omega(\mathfrak{h}_1 \cap O)H_{1,u}H_{1,\infty}$ normalizes $H_{2,\Omega}$.

To conclude, let $\Gamma$ be, as in the definition, a compactly generated subgroup of $H_1$ such that $\overline{\text{Ad}\Gamma^0}$ is semisimple and equal to $\overline{\text{Ad}\Gamma^{Z,nc}}$, that $\Omega$ is $\Gamma$-well-shaped and that $H_1 = H_{1,\Omega}\Gamma$, so that we only have to prove that $\Gamma$ normalizes $H_{2,\Omega}$. Since $H'_1$ has finite index in $H_1$, the group $H_{2,\Omega}$ is normalized by a finite index subgroup of $\Gamma$. Hence, setting $I$ to be the centralizer of $\Gamma$ in $\mathfrak{g}$ and $\mathfrak{v}$ to be the $\Gamma$-invariant complementary subspace of $I \cap \mathfrak{h}_2$ in $\mathfrak{h}_2$, by (A.1), we have $\mathfrak{v} \subset \mathfrak{h}_{2,u}$. Thus, since $\Omega$ is $\Gamma$-well-shaped, we have

$$\exp_\Omega(\mathfrak{h}_2 \cap O) \subset \exp_\Omega(\mathfrak{h}_2 \cap I \cap O)H_{2,u}H_{2,\infty}. $$

But then, since $\Gamma$ commutes with $\exp_\Omega(\mathfrak{h}_2 \cap I \cap O)$ and normalizes $H_{2,u}H_{2,\infty}$, it also normalizes $H_{2,\Omega}$, what should be proved. \qed
The proof of Lemma B.4 also uses an analogue of Lemma 2.7. Note that a second countable S-adic Lie group $G$ may have uncountably many compact normal subgroups, as for instance $G = \mathbb{Z}_p^2$.

**Lemma B.6.** Let $G$ be a second countable S-adic Lie group. Then the set of connected normal compact subgroups of $G$ is countable.

**Proof.** The proof mimics the real case. Let $K$ be a connected normal compact subgroup of $G$, $\mathfrak{k}$ be the Lie algebra of $K$ and $K_\infty$ be the immersed real Lie subgroup of $G$ with Lie algebra $\mathfrak{k}_\infty$. Since the group $\text{Ad}_{\mathfrak{k}_\infty}(K_\infty)$ has compact closure (equal to $\text{Ad}_{\mathfrak{k}_\infty}(K)$), the Lie algebra $\mathfrak{k}_\infty$ may be decomposed in a unique way as a direct sum of ideals $\mathfrak{k}_\infty = \mathfrak{s} \oplus \mathfrak{a}$ where $\mathfrak{s}$ is compact semisimple and $\mathfrak{a}$ is abelian. As $\mathfrak{k}_\infty$ is a $G$-invariant ideal of $\mathfrak{g}_\infty$, so are $\mathfrak{s}$ and $\mathfrak{a}$. As the Lie algebra $\mathfrak{g}_\infty$ contains only finitely many semisimple ideals, we may assume $\mathfrak{s}$ is fixed. As the connected real Lie subgroup $S$ of $G$ with Lie algebra $\mathfrak{s}$ is compact and normal in $G$, after replacing $G$ by $G/S$, we may assume $\mathfrak{s} = \{0\}$.

Now, we have to prove $G$ contains countably many abelian connected compact normal subgroups $K$. Since such a $K$ is compact, $\mathfrak{k}_\infty$ admits a $K$-invariant complementary subspace $\mathfrak{v}$ in $\mathfrak{g}_\infty$. As $\mathfrak{v}$ is $K$-invariant and $\mathfrak{t}_\infty$ is an ideal of $\mathfrak{g}_\infty$, $\mathfrak{t}_\infty$ is contained in the center $\mathfrak{z}_\infty$ of $\mathfrak{g}_\infty$. Note that if $\mathfrak{a}$ and $\mathfrak{a}'$ are vector subspaces of $\mathfrak{z}_\infty$ and $\exp \mathfrak{a}$ and $\exp \mathfrak{a}'$ have compact closures in $G$, so has $\exp (\mathfrak{a} + \mathfrak{a}')$. Hence, if $\mathfrak{t}_\infty$ is the subspace of $\mathfrak{z}_\infty$ spanned by all such subspaces, then $\exp \mathfrak{t}_\infty$ has compact closure $T$ in $G$. Hence we may assume that $G = T$ is a solenoid, i.e. $G$ is isomorphic to a quotient of $\mathbb{R}^{d_\infty} \times \Omega_f$ by a cocompact lattice whose projection on $\Omega_f$ is dense, where $\Omega_f = \prod_{p < \infty} \mathbb{Z}_p^{d_p}$. Such a group admits only countably many closed subgroups. The result follows. □

We will need the following information on the normalizer of a totally discontinuous compact group.

**Lemma B.7.** Let $G$ be an $S$-adic Lie group, $H$ be a compact subgroup of $G$ with $H_\infty = \{e\}$ and $\Gamma$ be a subgroup of $G$ normalizing $H$ such that $\text{Ad}_{\Gamma^Z}$ is semisimple and equal to $\text{Ad}_{\Gamma^Z,\text{nc}}$. Then $\Gamma$ centralizes the Lie algebra $\mathfrak{h}$ of $H$.

Note that the analogue of this Lemma is not true for real Lie groups, as, for instance, when $G = \Gamma \rtimes H$ with $\Gamma = \text{SL}(2, \mathbb{Z})$ acting on $H = \mathbb{T}^2$.

**Proof.** Assume by contradiction the action of $\Gamma$ on $\mathfrak{h}$ is not trivial. In this case, by assumption, $\mathfrak{h}$ contains non-zero $\Gamma$-unstable vectors. Hence by [A.1], there exists a non trivial one-parameter subgroup $\varphi : \mathbb{Q}_p \to H$. Since $H_\infty = \{e\}$, by [25] Prop. 1.2, $\varphi$ is proper, which contradicts the fact that $H$ is compact. The result follows. □
Finally, we shall also need

**Lemma B.8.** Let $G$ be an $S$-adic Lie group and let $\mathcal{K}$ be a set of normal compact subgroups of $G$.  

(i) Assume $G_\infty = \{e\}$ and the set $\bigcup_{K \in \mathcal{K}} K$ spans a dense subgroup of $G$. Then, the Lie algebra of $G$ is the linear span of the Lie algebras of the elements of $\mathcal{K}$.

(ii) Assume, for any $K$ in $\mathcal{K}$, one has $\text{Ad} K = \text{Ad} G$ and the intersection of the Lie algebras of the elements of $\mathcal{K}$ is zero. Then the Lie algebra $g$ of $G$ is abelian.

**Proof.** (i) Let $K$ be a compact normal subgroup of $G$ which is generated by finitely many elements of $\mathcal{K}$ and whose Lie algebra is maximal. After having replaced $G$ by $G/K$, we can assume the elements of $\mathcal{K}$ are finite, and we have to show that $G$ is discrete.

Since $G_\infty = \{e\}$, there exists an open neighborhood $U$ of $e$ in $G$ such that $U$ does not contain any non trivial torsion element. Then $G$ contains a dense subgroup which meets $U$ only at $e$, hence $G$ is discrete.

(ii) Let $\mathfrak{z}$ be the center of $g$. By assumption, for any $K$ in $\mathcal{K}$ with Lie algebra $\mathfrak{k}$, one has $g = \mathfrak{k} + \mathfrak{z}$, so that $g/\mathfrak{k}$ is abelian. Pick $K_1, \ldots, K_r$ in $\mathcal{K}$ whose Lie algebras have zero intersection. Then, the natural map $g \to \bigoplus_{i=1}^r g/\mathfrak{k}_i$ is injective. Hence $g$ is abelian. □

Another difficulty in the $S$-adic case is that the quotient $G/H$ of a weakly regular $p$-adic Lie group $G$ by a closed normal subgroup might not be weakly regular, as for instance when $G = \mathbb{Q}_p^2$ and $H = \mathbb{Z}_p \times \{0\}$.

This difficulty will weigh the following

**Proof of Lemma B.4.** We can assume the set $\bigcup_{H \in T_\Omega(G, \Delta, \Sigma)} H$ spans a dense subgroup of $G$. Then, in particular, by Lemma B.5, the group $H_0 = \bigcap_{H \in T_\Omega(G, \Delta, \Sigma)} H_\Omega$ is normal in $G$. We set $G' = G/H_0$. Since, for any $H$ in $T_\Omega(G, \Delta, \Sigma)$, $\Delta$ is a lattice in both $H_\Omega$ and $H_0$, the image $H'_\Omega$ of $H_\Omega$ in $G'$ is a compact normal subgroup. Now, since, by Lemma B.6, the set of connected normal compact subgroups of $G'$ is countable, we can fix such a compact subgroup $K$ and restrict our attention to the set $T^K_\Omega(G, \Delta, \Sigma)$ of those $H$ in $T_\Omega(G, \Delta, \Sigma)$ such that $\overline{(H_\Omega)}_\infty = K$. We set $K' = \bigcap_{H \in T^K_\Omega(G, \Delta, \Sigma)} H'_\Omega$ and $G'' = G'/K'$ and, for any $H$ in $T^K_\Omega(G, \Delta, \Sigma)$, we let $H''$ be the image of $H$ in $G''$, and $H''_\Omega$ and $\Sigma''$ be the ones of $H_\Omega$ and $\Sigma$. As $\Sigma$ is a lattice in $H$ and $\Delta$ is a lattice in $H_0$, $\Sigma''$ is a lattice in $H''$. Let $M$ be the closure of the subgroup of $G''$ spanned by the normal compact totally discontinuous subgroups $H''_\Omega$ as $H$ varies in $T^K_\Omega(G, \Delta, \Sigma)$.

We claim the Lie algebra $\mathfrak{m}$ of $M$ is abelian. Indeed, first note that, since $M$ is non-archimedean, by Lemma B.8, $\mathfrak{m}$ is the linear span of the
Lie algebras of the normal compact subgroups $H''_\Omega$, $H \in T^K_\Omega(G, \Delta, \Sigma)$. Moreover, if $O = \log \Omega$, since, for any $H \in T^K_\Omega(G, \Delta, \Sigma)$, one has $\exp_\Omega(\mathfrak{h} \cap O) \subset H_\Omega$, the Lie algebras of the $H''_\Omega$, $H \in T^K_\Omega(G, \Delta, \Sigma)$, have trivial intersection. Fix $H \in T^K_\Omega(G, \Delta, \Sigma)$ and let $\Gamma$ be a compactly generated subgroup of $H$ such that $\overline{\text{Ad}\Gamma}$ is semisimple and equal to $\overline{\text{Ad}G}$, the action of $\Gamma$ on $H/\Sigma$ is ergodic and $H = \Gamma H_\Omega$. By Lemma [B.7] one has $\text{Ad}_\mathfrak{m} \Gamma = \{e\}$, and therefore $\text{Ad}_\mathfrak{m} H = \text{Ad}_\mathfrak{m} H_\Omega$. Besides, since the action of $\Gamma$ on $H/\Sigma$ is ergodic, there exists $h$ in $H$ such that $H = h\Gamma h^{-1}\Sigma$, so that $\text{Ad}_\mathfrak{m} H = \text{Ad}_\mathfrak{m} \Sigma$ and hence $\text{Ad}_\mathfrak{m} H''_\Omega = \text{Ad}_\mathfrak{m} M$. By Lemma [B.8] ii, the Lie algebra $\mathfrak{m}$ is thus abelian.

Now, let us prove, for any $H$ in $T^K_\Omega(G, \Delta, \Sigma)$, $H''$ virtually normalizes a finite index subgroup of $\Sigma''$. Indeed, let $\Gamma$ be as above. Since the adjoint action of $\Gamma$ on the Lie algebra of $H''$ is trivial, there exists an open subgroup $U$ of $H''_\Omega$ such that $\Gamma$ centralizes $U$. As the Lie algebra of $H''$ is abelian, we can assume $U$ to be abelian. Since $H'' = \Gamma'' H''_\Omega$ and $H''_\Omega$ is compact, $\Gamma'' U$ has finite index in $H''$. Hence, the centralizer of $U$ in $H''$ has finite index in $H''$ and the centralizer $\Sigma''_U$ of $U$ in $\Sigma''$ has finite index in $\Sigma''$. The group $\Sigma''_U$ being a lattice in $H''$, the group $H''_U := \Sigma''_U U$ has finite index in $H''$ and normalizes $\Sigma''_U$, as required.

Since, by Lemma [A.5] $\Sigma$ is finitely generated, the set of finite index subgroups of $\Sigma$ is countable, hence we can fix a finite index subgroup $\Theta$ of $\Sigma$ and restrict our attention to the set of those $H$ in $T^K_\Omega(G, \Delta, \Sigma)$ such that some finite index open subgroup of $H$ normalizes the image $\Theta''$ of $\Theta$ in $G''$. For such a $H$, the inverse image $J$ of $\Theta''$ in $G$ is cocompact in $H$, contains $H_\infty$, and is normalized by an open finite index subgroup in $H$. The result follows. \]


Proof of Proposition [2.1] in the general case. We fix once for all a standard open subset $\Omega_0$ in $G$, we set $O_0 = \log \Omega_0$ and we let $I$ denote the Lie algebra of $L$. Let $Y$ be in $S_X(\Gamma)$, fix $g$ in $G$ such that $x = g\Lambda$ belongs to $Y$ and $\overline{\Gamma x} = Y$, and let $\mathfrak{h}$ denote the Lie algebra of $g^{-1}G_Y g$. By Lemma [B.2] the group $g^{-1}\Gamma g$ admits a well-shaped standard open set $\Omega = \exp_{\Omega_0}(O) \subset \Omega_0$ such that $\exp_{\Omega}(\mathfrak{h} \cap O) \subset gG_Y g^{-1}$. As the compact group $O_{0,\mathfrak{h}}$ admits only countably many open subgroups, we can suppose $\Omega$ to be fixed.

We first construct an open compactly generated subgroup $H$ of $g^{-1}G_Y g$ which contains $g^{-1}\Gamma g$: we let $H_\infty$ denote the analytic subgroup of $G$ with Lie algebra $\mathfrak{h}_\infty$, $H_a$ denote the closed group spanned by the $g^{-1}\Gamma g$-unstable one-parameter subgroups in $g^{-1}G_Y g$, $I_g := \text{Ad}g^{-1}I$ and $H := g^{-1}\Gamma g \exp(I_g \cap \mathfrak{h} \cap O)H_a H_\infty$. 


This subgroup $H$ of $g^{-1}G_Yg$ is open hence closed and, since $\Omega$ is $g^{-1}\Gamma g$-well-shaped, the $\Omega$-semiconnected component $H_\Omega$ of $H$ satisfies

$$H_\Omega = \exp_\Omega(lg \cap h \cap O)H_uH_\infty = \exp_\Omega(h \cap O)H_uH_\infty.$$  

By construction, the subgroup $H_\Omega$ is normal in $H$ and is $g^{-1}\Gamma g$-unstable. By Lemma [A.5], $H$ is compactly generated, hence the lattice $\Sigma := \Lambda \cap H$ is finitely generated. Since the countable group $\Lambda$ admits countably many finitely generated subgroups, we can assume $\Sigma$ to be fixed.

The subgroup $\Delta := \Lambda \cap H_\Omega$ is a lattice in $H_\Omega$ that might not be finitely generated. Therefore, let us introduce the largest $\Omega$-semiconnected compactly generated subgroup $H'$ of $H$, as in Lemma [A.7]. By Lemma [A.9], the group $\Delta' := \Delta \cap H'$ is finitely generated and the group $\Delta$ is the smallest normal subgroup of $\Sigma$ containing $\Delta'$. Hence we can assume $\Delta$ is fixed.

By construction, $H$ belongs to $T_\Omega(G, \Delta, \Sigma)$. Let $U_\Omega(G, \Delta, \Sigma)$ be as in Lemma [B.4], then there exists some $J$ in $U_\Omega(G, \Delta, \Sigma)$ such that $H$ contains $J$, $H$ virtually normalizes $J$ and, $H/J$ is compact. As $U_\Omega(G, \Delta, \Sigma)$ is countable, we can assume $J$ to be fixed. Now, let $Y_1$ be another element of $S_X(\Gamma)$, proceed to the same construction, so that we get $x_1 = g_1 \Lambda \in Y_1$ with $\Gamma x_1 = Y_1$ and a subgroup $H_1 \supset J$, virtually normalizing $J$, with $H_1/J$ compact and $\Gamma \subset g_1H_1g_1^{-1} \subset G_Y$. By Lemma [B.9] below, we can assume there exists $\ell$ in $L$ with $g_1J = \ell gJ$. Thus, we have $\ell x \in Y_1$, hence $\ell Y = \Gamma \ell x \subset Y_1$. In the same way, since $\ell ^{-1}x_1$ belongs to $Y$, we get $\ell ^{-1}Y_1 \subset Y$, hence $Y_1 = \ell Y$. The result follows.  

As the conclusion of Lemma [B.4] is weaker than the one of Lemma [2.5] we needed in the $S$-adic case the following result that is stronger than Lemma [2.2].

**Lemma B.9.** Let $G$ be a second countable $S$-adic Lie group, $J$ be a closed subgroup of $G$, $\Gamma$ be a closed compactly generated subgroup of $G$ such that $\overline{\Ad \Gamma^Z}$ is semisimple and equal to $\overline{\Ad \Gamma^{Z,nc}}$, and $L$ be the centralizer of $\Gamma$ in $G$. Then the set

$$Y = \{y = gJ \in G/J \mid \Gamma \text{ virtually normalizes } gJg^{-1} \text{ and } \Gamma y \text{ is compact}\}$$

is a countable union of $L$-orbits.

**Proof.** Let $N$ be the normalizer of $J$ in $G$ and, for $y = gJ$ in $G/J$, set $J_y = gJg^{-1}$ and $N_y = gNg^{-1}$. By assumption, for $y$ in $Y$, the group $\Gamma_y = \Gamma \cap N_y$ is an open finite index subgroup of $\Gamma$. Since $\Gamma$ is compactly generated, the set of such subgroups is countable and since $\Gamma$ and $\Gamma_y$ have the same centralizer in $g$, the centralizer of $\Gamma$ in $G$ is
open in the centralizer of $\Gamma_y$ in $G$. Thus, after having replaced $\Gamma$ by $\Gamma_y$, it suffices to prove that the set of $y$ in $G/J$ such that $\Gamma \subset N_y$ and $\overline{\Gamma y}$ is compact is a countable union of $L$-orbits. For such a $y$, as the image of $\Gamma$ in the group $N_y/J_y$ is relatively compact and $\overline{\text{Ad}\Gamma^{\mathbb{Z}}} = \overline{\text{Ad}\Gamma^{\mathbb{Z},nc}}$, the adjoint action of $\Gamma$ on the Lie algebra of $N_y/J_y$ is trivial. As $\overline{\text{Ad}\Gamma^{\mathbb{Z}}}$ is semisimple, this means that $(L \cap N_y)J_y$ is an open subgroup of $N_y$, hence that $N_y y$ is contained in a countable union of $L$-orbits in $G/J$. The result follows since, by Lemma 2.2, the set of fixed points of $\Gamma$ in $G/N$ is a countable union of $L$-orbits.

References


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