A survey on divisible convex sets

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Abstract

We report without proof recent advances on the study of open properly convex subsets Ω of the real projective space which are divisible i.e. for which there exists a discrete group Γ of projective transformations which acts cocompactly on Ω.

1 Introduction

The main objects I would like to describe in this lecture[1] are called *divisible convex sets* in [53]. They also come sometimes in other disguises as *compact convex real projective structures* or as *projective tilings in real projective geometry*.

These *divisible convex sets* I want to discuss here are very concrete and their definition is quite easy: those are properly convex open subsets Ω of the n-dimensional real projective space for which there exists a discrete group Γ of projective transformations which acts cocompactly on Ω. To a non specialist, this subject might look very narrow. We will disprove that impression and show how rich this topic is. The history of the convex divisible sets began 50 years ago with Benzecri thesis published in [7]. Since then our understanding of these divisible convex sets has considerably progressed. They are now related to many different kinds of mathematics as *dynamical systems, Coxeter groups, representation theory, differential geometry, geometric group theory, partial differential equations, moduli spaces, quasisymmetric spaces, ...*

Even if the definition of the divisible convex sets is quite easy, the proof that they do exist is not easy at all. Hence the construction of examples is an important part of this topic (see theorems 3.1, 6.4, 8.1 and 8.2). Recall that the same is true for lattices in semisimple Lie group: they do exist but the proof of their existence needs deep arguments. Once we know that convex divisible sets do exist, we study their properties: one will focus

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on the algebraic properties of the group Γ, on the regularity properties of the boundary of the convex set Ω, on the topological properties of the quotient orbifold $\Gamma\backslash\Omega$, on the natural Finsler metric on this orbifold, on the ergodic properties of the corresponding geodesic flow, on the parametrization of the corresponding moduli spaces...

Before begining this survey, just a few words on our notations. Let $m \geq 2$. Instead of working on the real projective m-space $\mathbb{P}^m$, it will be more convenient for us to work on its two-fold cover the projective m-sphere $\mathbb{S}^m$ which is the set of half-lines in the real vector space $V := \mathbb{R}^{m+1}$. The group of projective transformations of $\mathbb{S}^m$ is the group $G = \mathrm{SL}^{\pm}(m+1, \mathbb{R})$ of real matrices of determinant ± 1.

A subset Ω of $\mathbb{S}^m$ is *convex* if its intersection with any great circle is connected. It is *properly convex* if moreover its closure $\overline{\Omega}$ does not contain two opposite points. It is *strictly convex* if moreover its boundary $\partial\Omega$ does not contain any open segment i.e. any open arc of great circle. An open properly convex set is *divisible* if the group $\mathrm{Aut}(\Omega) = \{g \in G \ / \ g(\Omega) = \Omega\}$ of automorphism of Ω contains a discrete subgroup Γ which acts properly and cocompactly on Ω i.e. such that the quotient $\Gamma\backslash\Omega$ is compact. Choosing a compact fundamental domain F for the action of Γ on Ω, the images $\gamma(F)$ of the fundamental domain by the elements of Γ gives then a projective periodic tiling of Ω.

Instead of working on the projective sphere $\mathbb{S}^m$, it is sometimes, but not very often, simpler to work directly in the vector space $V = \mathbb{R}^{m+1}$. Hence one introduces the open convex cone C which is the inverse image of Ω in $V - \{0\}$ and its automorphism group $\mathrm{Aut}(C) = \{g \in \mathrm{GL}(\mathbb{R}^{m+1}) \ / \ g(C) = C\}$. We will say that C is *properly convex*, *strictly convex*, *divisible*... when Ω is. One can prove that C is divisible if and only if there exists a discrete subgroup of $\mathrm{Aut}(C)$ which acts properly and cocompactly on C. Hence those two points of view are truly equivalent.

I did my best to make this survey short. If it were reasonably expanded, this text would fill hundreds of pages. I hope the reader would at least get from it a feeling of the tools used in this topic which inherits not only the flavour of rank one lattices and of higher rank lattices but also a new spiciness...

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2 Symmetric convex sets

The simplest examples of divisible properly convex sets are the symmetric ones. Those are special instances of symmetric spaces. Let us first say a few words on them, after some basic definitions.

2.1 Hilbert distance and properness of the action

Every properly convex open set Ω in $\mathbb{S}^m$ is endowed with a distance d_Ω called *Hilbert distance* and defined by, for every x, x' in Ω, $d_\Omega(x,x') = \log([x;x';a;a'])$ where a and a' are the two points in $\partial\Omega$ such that a, x, x', a' are aligned in this order and where $[x;x';a;a'] = \frac{xa'}{xa}\frac{x'a}{x'a'}$ is the cross-ratio of these four points (see for instance [11], [18] or [38]).

Note that every element of $\mathrm{Aut}(\Omega)$ is an isometry for the Hilbert distance of Ω. Since the balls of d_Ω are compact the action of $\mathrm{Aut}(\Omega)$ on Ω is always proper. Hence any discrete subgroup Γ of $\mathrm{Aut}(\Omega)$ always acts properly on Ω. To prove that such a group divides Ω, one has just to check that the action is cocompact i.e. to find a compact subset F of Ω whose translates by Γ cover Ω.

2.2 Real projective structures

Any group Γ which divides a properly convex set Ω is finitely generated. Then, according to Selberg lemma, it contains a subgroup Γ' of finite index which is torsion free i.e. does not contain elements of finite order. Replacing Γ by Γ' one can suppose that Γ is torsion free. The compact quotient $M := \Gamma\backslash\Omega$ is then a C^∞-manifold.

The identification of the universal cover $\widetilde{M}$ of M with Ω induces a *real projective structure on M* i.e. a maximal atlas of charts with values in the projective sphere $\mathbb{S}^m$

and with transition functions locally given by projective transformations of $\mathbb{S}^m$. Such a projective structure on M is called a *properly convex real projective structure.*

Recall that a real projective structure on M is "equivalent" to the data (h, D) where $h : \pi_1(M) \to G$ is a morphism called *holonomy* and $D : \widetilde{M} \to \mathbb{S}^m$ is an equivariant local diffeomorphism called the *developing map.*

Hence, a properly convex real projective structure is a real projective structure for which the developping map is a diffeomorphism onto a properly convex subset of $\mathbb{S}^m$.

2.3 Hyperbolic spaces

The simplest example of a convex divisible set is the round open disk $\mathbb{H}^2$ in dimension 2. The group Γ is then a discrete cocompact subgroup of $\mathrm{PGL}(2, \mathbb{R})$. The existence of such subgroups and the pictures of the corresponding tilings of the disc are well-known since Klein and Poincaré in the second half of the nineteenth century. They are the starting point of hyperbolic geometry.

More generally, for all $m \geq 2$, the round open ball $\mathbb{H}^m \subset \mathbb{S}^m$ is a divisible convex set. More precisely, this convex set $\mathbb{H}^m$ is the set of half-lines in the Lorentz cone $\Lambda_{m+1} := \{x \in \mathbb{R}^{m+1} \;/\; q(x) > 0 \text{ and } x_1 > 0\}$ where $q(x) = x_1^2 - x_2^2 - \cdots - x_{m+1}^2$. Its group of automorphisms is the group $\mathrm{Aut}(\mathbb{H}^m) = \mathrm{O}^+(1, m)$ of orthogonal transformations of q which preserve $\mathbb{H}^m$. Hence our space $\mathbb{H}^m$, endowed with its Hilbert distance, is nothing else than the hyperbolic m-space or, more precisely, the "projective model of the hyperbolic m-space". The group Γ is then a discrete cocompact subgroup of $\mathrm{O}^+(1, m)$

The existence of such groups Γ was first known only for some small values of m. Those were groups generated by reflections with respect to the faces of some polyhedron in $\mathbb{H}^m$. As was shown later on by Vinberg, this "Coxeter group construction" can not work for m large.

The construction of such groups Γ for any integer $m \geq 2$, goes back to Siegel in [50] in the early fifties and is obtained thanks to arithmetic groups (see for instance the chapter 2 of [6] for a self-contained proof).

2.4 Symmetric convex cones

Hence the hyperbolic m-space $\mathbb{H}^m$ is an example of convex divisible set Ω. Note that for this example the group $\mathrm{Aut}(\Omega)$ of automorphisms of Ω acts transitively on Ω. One says that Ω is *homogeneous.* Even more, for each point x in Ω, there exists a *symmetry* with respect to x, i.e. an automorphism s_x of Ω of order 2 such that x is the only fixed point of s_x in Ω; one says that Ω is *symmetric.* In fact the symmetric convex sets are exactly the homogeneous convex sets for which the automorphism group $\mathrm{Aut}(\Omega)$ is reductive. As before, we will say that C is *homogeneous* or *symmetric* when Ω is.

Those symmetric convex cones where classified by Koecher in the sixties using the classification of Jordan algebras (see [21], [39] and [54]).

A properly convex set Ω in $\mathbb{S}^m$ is said *reducible* if the cone C is *reducible* i.e. if it can be written as the sum $C = C_1 + C_2$ of two convex cones C_i living in proper subspaces V_i of

V. Otherwise, they are said *irreducible*. When $C = C_1 + C_2$, the cone C is symmetric if and only if C_1 and C_2 are symmetric. One can also show that, in this case, C is divisible if and only if C_1 and C_2 are divisible.

Theorem 2.1 (Koecher, 1965) *The irreducible symmetric convex cones in* $\mathbb{R}^{m_1}$ *are given by the following list with* $n \geq 3$.
- *The half-line* $\Lambda_1 := \{x \in \mathbb{R} \ / \ x > 0\}$ *(with* $m_1 = 1$*)*
- *The Lorentz cones* $\Lambda_n := \{x \in \mathbb{R}^n \ / \ x_1^2 - x_2^2 - \cdots - x_n^2 > 0 \text{ and } x_1 > 0\}$ *(with* $m_1 = n$*)*
- $\Pi_n(\mathbb{R}) = \{$positive symmetric $n \times n$ real matrices$\}$ *(with* $m_1 = (n^2+n)/2$*)*
- $\Pi_n(\mathbb{C}) = \{$positive hermitian $n \times n$ complex matrices$\}$ *(with* $m_1 = n^2$*)*
- $\Pi_n(\mathbb{H}) = \{$positive hermitian $n \times n$ quaternion matrices$\}$ *(with* $m_1 = 2n^2 - n$*)*
- $\Pi_3(\mathbb{O})$, a symmetric cone such that $\mathrm{Lie}(\mathrm{Aut}(\Pi_3(\mathbb{O}))) = \mathfrak{e}_{6(-26)} \oplus \mathbb{R}$ *(with* $m_1 = 27$*).*

Any symmetric cone is a product of irreducible symmetric cone.

Around the same time, Borel proved in [8] that every reductive real Lie group contains a discrete cocompact subgroup. As a consequence, all the symmetric convex sets are divisible. Hence Koecher list gives many example of divisible convex sets.

Later on, all the homogeneous convex sets where classified by Vinberg ([55]). No new examples of divisible convex set arise from this list since Vinberg proved that, when Ω is homogeneous, one has the equivalence:

$$\Omega \textit{ is symmetric} \iff \textit{The group } \mathrm{Aut}(\Omega) \textit{ is unimodular.}$$

Hence when Ω is homogeneous but not symmetric, its automorphism group can not contain a discrete cocompact subgroup and Ω is not divisible.

The theory of divisible convex sets splits naturally in two different parts: the strictly convex (see chapter 4) and the non strictly convex case (see chapter 9). Hence it is important to notice that, among the symmetric convex sets the only ones which are strictly convex are the hyperbolic m-spaces. Those are also the only one with a boundary of class C^1.

3 First examples

Up to now all the examples of convex divisible sets that we have seen are symmetric and the theory might look as nothing but a special case of the theory of cocompact lattices in semisimple Lie groups.

3.1 Kac-Vinberg examples

In fact, in his thesis, Benzecri tried to prove that in dimension $m = 2$ all the convex divisible sets are symmetric (i.e. are either the triangle or the hyperbolic disk). However his proof needs a regularity hypothesis on the curve $\partial\Omega$ that bounds Ω. More precisely he needs that locally $\partial\Omega$ is the graph of some convex function F whose derivative f is "absolutely continuous". We will see again this important condition in theorem 4.6.

A few years later, Kac and Vinberg ([33]) were able to construct the first examples of divisible convex set which are not symmetric. Their examples were two-dimensional and the groups Γ were Coxeter groups.

3.2 Coxeter groups

In the early seventies, Vinberg understood the general condition under which a group Γ generated by projective reflections fixing the faces of some convex polyhedron P of the sphere $\mathbb{S}^m$ will divide some properly convex open set Ω with P as a fundamental domain. Quickly stated this theorem says that a convex polyhedron P and its images by a group Γ generated by projective reflections through its faces tile some open convex set Ω of the projective sphere , as soon as two natural necessary local conditions are satisfied. First, "around each 2-codimensional face of P": some rotations must have angle $\frac{2\pi}{m}$. Second, "around each vertex of P": the corresponding Coxeter group must be finite. More precisely :

Theorem 3.1 (Vinberg, 1970) *Let P be a convex polyhedron of $\mathbb{S}^m$ and, for each face s of P, let $\sigma_s = Id - \alpha_s \otimes v_s$ be a projective reflection fixing this face s. Choose the signs of α_s such that P is the intersection of the half-spheres $\alpha_s \leq 0$. Suppose that these projective reflections satisfy, for every faces s, t such that* $\mathrm{codim}(s \cap t) = 2$

$$\alpha_t(v_s) \leq 0 \ \textit{ and } (\ \alpha_t(v_s) = 0 \Leftrightarrow \alpha_s(v_t) = 0\ )$$
$$\alpha_t(v_s)\alpha_s(v_t) = 4\cos^2(\tfrac{\pi}{m_{s,t}}) \ \textit{ with } \ m_{s,t} \geq 2 \ \text{ integer.}$$

Suppose also that for every vertex x of P, the group Γ_x generated by σ_s for s containing x is a finite group.

Then the group Γ generated by all these reflections σ_s is discrete, the polyhedra $\gamma(P)$, for γ in Γ, tile a convex open subset Ω of $\mathbb{S}^m$ and hence Γ divides Ω.

See [56] or chapter one of [6] for more details on this theorem which is a generalization of a famous theorem of Tits (see [9]).

Note that the condition on the finiteness of the groups Γ_x can be easily checked using the list of finite Coxeter groups (see [9] or [57]).

Note also that Ω is properly convex as soon as the vectors v_s generate V and the linear forms α_s generate V^*.

The first explicit examples where this theorem applies and gives a divisible properly convex set Ω which is not symmetric is Kac-Vinberg example and its natural generalisations. In these examples, the polyhedron P is a simplex of dimension $m \leq 4$.

We will see more examples later on.

Even if this construction gives examples of non symmetric divisible convex open sets only in small dimension... it is worth studying the general properties of the divisible convex sets. We will see later on how these properties will help us in constructing examples in every dimension!

4 Strict convexity and regularity of $\partial\Omega$

As Benzecri already understood in his thesis, the regularity of $\partial\Omega$ is an important issue in this topic.

4.1 Strict convexity and Gromov hyperbolicity

Before quoting Benzecri theorem, let us quote a more recent result which relates the regularity of $\partial\Omega$ and the strict convexity of Ω. This result will be a corollary of the following theorem.

Theorem 4.1 ([4].I) *Let Γ be a discrete group which divides some properly convex open set Ω in $\mathbb{S}^m$. Then Ω is strictly convex if and only if the group Γ is Gromov hyperbolic.*

For a precise definition of "Gromov hyperbolicity" see [23] and [27]. Roughly it means that, "*for some $\delta > 0$, the 3 sides of any geodesic triangle in the Cayley graph of Γ meet a mutual ball of radius δ*".

As a consequence of this theorem, the strict convexity of Ω is a property of the abstract group Γ.

Note that Γ divides also the dual convex set Ω^* i.e. the convex set whose inverse image in $\mathbb{R}^{m+1}$ is the dual cone $C^* := \{f \in V^*/\forall x \in \overline{C} - \{0\}\ f(x) > 0\}$. Since the strict convexity of Ω^* is equivalent to the C^1 regularity of $\partial\Omega$, one gets the following criterion

Corollary 4.2 ([4].I) *A divisible properly convex open set has a boundary of class C^1 if and only if it is strictly convex.*

4.2 Closedness of the orbit of Ω

The main tool introduced by Benzecri to study the regularity of the boundary of a divisible properly convex set is to endow the space of properly convex subset of $\mathbb{S}^m$ with the Hausdorff distance and to study the G-orbit of Ω in this metric space.

Theorem 4.3 (Benzecri, 1960) *Let Ω be a divisible properly convex open set in $\mathbb{S}^m$. Then the G-orbit of Ω in the space of properly convex subset of $\mathbb{S}^m$ is closed.*

Since $\mathbb{H}^m$ is in the closure of the G-orbit of any properly convex open subset of $\mathbb{S}^m$ whose boundary is of class C^2, on gets the following corollary (see [7], [24] and also [31],[51]).

Corollary 4.4 (Benzecri, 1960) *The only divisible properly convex open set in $\mathbb{S}^m$ whose boundary is of class C^2 is the hyperbolic space $\mathbb{H}^m$.*

4.3 The geodesic flow and the regularity theorem

One can describe more precisely the regularity of $\partial\Omega$. The important tool one has to introduce for that is the geodesic flow of the Hilbert metric.

Theorem 4.5 ([4].I) *Let Γ be a torsion free discrete group which divides some strictly convex open set Ω in $\mathbb{S}^m$. Then the geodesic flow φ_t of the Hilbert metric on the quotient manifold $M = \Gamma\backslash\Omega$ is Anosov.*

Note that this flow φ_t is of class C^1 and that the geodesics of the projective structure, i.e. the straight lines, and the .

See [29] for a precise definition of Anosov flows. Roughly it means that "*the normal bundle to the flow is a direct sum of two continuous subbundles, one contracted and one expanded by the flow φ_t.*

One can show that this flow φ_t is topologically transitive i.e. it has a dense orbit in M. However, one can show that when Ω is not $\mathbb{H}^m$, then the flow φ_t does not preserve any finite measure on M which is absolutely continuous with respect to the Lebesgue measure.

Using then the thermodynamical formalism for this Anosov flow φ_t as in [44], using the Zariski density of Γ (theorem 5.2 below), and using some properties of Zariski dense subgroups of G as in [2], one gets as corollary the following regularity theorem for the boundary of Ω.

Theorem 4.6 ([4].I) *Let Ω be a divisible strictly convex open set in $\mathbb{S}^m$. Suppose that Ω is not the hyperbolic space $\mathbb{H}^m$. Then*
a) There exists $\alpha \in (0,1)$ such that the boundary $\partial\Omega$ is $C^{1+\alpha}$.
b) The maximum α_{max} of these α satisfies $\alpha_{max} < 1$.
c) The curvature of $\partial\Omega$ is concentrated on a subset of zero measure.

The condition $C^{1+\alpha}$ means that the normal map $n : \partial\Omega \to \mathbb{S}^{m-1}$ is α-Hölder. The *curvature* is the measure on $\partial\Omega$ pull-back of the Lebesgue measure on $\mathbb{S}^{m-1}$ by this normal map.

In dimension $m = 2$, the point a) is due to Kuiper in [41] and the point c) is due to Benzecri in [7]: this is the absolute continuity assumption that we already mentioned in section 3.1.

Recently, O.Guichard ([28]) has given a formula for α_{max} thanks to the eigenvalues of the elements of Γ. As a consequence he has proven that this constant α_{max} is the same for $\partial\Omega$ and for $\partial\Omega^*$.

5 Irreducibility of Ω and irreducibility of Γ

Let us now study some algebraic properties of Γ. A subgroup Γ of G is said *irreductible* if there are no Γ-invariant non trivial subspaces in $\mathbb{R}^{m+1}$. It is said *strongly irreducible* if all subgroups of finite index are also irreducible.

5.1 Vey irreducibility theorem

It is clear that if a strongly irreducible group Γ divides some properly convex set Ω then Ω is irreducible (see the definition in 2.4). Vey proved in [53] the converse of this assertion:

Theorem 5.1 (Vey, 1970) *Let Γ be a discrete group which divides some properly convex open set Ω in $\mathbb{S}^m$. If Ω is irreducible then Γ is strongly irreducible.*

This theorem is very useful, since every convex divisible cone is a product of irreducible convex divisible cones.

The main point in the proof is to check that *the representation of Γ in V is semisimple* i.e. that every Γ-invariant subspace of V has a Γ-invariant supplementary subspace.

Once this is checked, it is easy to conclude, since every Γ-invariant decomposition $V = V_1 \oplus V_2$ gives a one parameter group a_t of elements of G acting by homotheties on each factor. These transformations commute with Γ. For t small, a compact fundamental domain F of Γ in Ω has still its image $a_t(F)$ inside Ω. Hence one has $a_t(\Omega) = \Omega$ which contradicts the irreducibility of Ω.

5.2 The density theorem

According to Vey irreducibility theorem, the Zariski closure S of Γ is semisimple. The following statement describes this Zariski closure.

Theorem 5.2 ([4].II) *Let Γ be a discrete group which divides some properly convex open set Ω in $\mathbb{S}^m$. If Ω is irreducible and is not symmetric then Γ is Zariski dense in* $\mathrm{SL}(m+1, \mathbb{R})$.

This means that Γ is not contained in any proper algebraic subgroup of $\mathrm{SL}(m+1, \mathbb{R})$.

As a corollary, Γ does not preserve any non zero bilinear form on $\mathbb{R}^{m+1}$.

The main point in the proof is to check that the Zariski closure S of Γ has an open orbit in the vector space $\mathbb{R}^{m+1}$. One uses then Kimura-Sato classification of prehomogeneous vector space ([37]).

To finish this chapter, let us mention [47] which gives other algebraic properties for important families of groups preserving properly convex sets.

6 Moduli spaces of representations

Let us now describe an important feature of this topic. It will give us, for our groups Γ, a geometric interpretation of some connected components of the space of representations of Γ. This kind of interpretation is quite exceptional. Except for rigidity phenomena, the only other known examples are for π_1 of compact surfaces as in [10] and [43].

6.1 Koszul openness theorem

Let Γ_0 be a finitely generated group. One wants to understand the space

$$\mathcal{F}_{\Gamma_0} := \{\rho \in \mathrm{Hom}(\Gamma_0, G) \text{ faithful with discrete image } \Gamma := \rho(\Gamma_0) \\ \text{dividing a properly convex open set } \Omega_\rho \text{ in } \mathbb{S}^m \ \}$$

and more precisely the moduli space X_{Γ_0} which is the quotient

$$X_{\Gamma_0} = G \backslash \mathcal{F}_{\Gamma_0}$$

for the action of G on $\mathcal{F}_{\Gamma_0}$ by conjugation on the target.

Recall that from the begining, one has $G = \mathrm{SL}^{\pm}(m+1, \mathbb{R})$.

Theorem 6.1 (Koszul, 1970) $\mathcal{F}_{\Gamma_0}$ *is open in* $\mathrm{Hom}(\Gamma_0, G)$.

Note that this theorem is a combination of Koszul original theorem in [40] and of Thurston holonomy theorem for (G, X)-structure (see [25] for this theorem).

To prove Theorem 6.1, Koszul gives a necessary and sufficient condition for a real projective structure on a compact manifold M to be properly convex. Note that M inherits from $\mathbb{S}^m$ a tautological oriented real line bundle $\mathcal{L}$. This condition is the existence of a positive section s of $\mathcal{L}$ whose graph is "convex with positive hessian" (see also [42] and [45]). The theorem 6.1 is then a corollary since this condition is open.

Thanks to his openness theorem, Koszul was also able to construct a few examples of divisible properly convex sets which are non symmetric in dimension $m \leq 4$. Even if these examples are obtained by a method different from Vinberg Coxeter group construction, they are very similar.

However, we will see in section 6.3 how this theorem 6.1 will allow us to construct examples of non symmetric convex divisible sets in any dimension $m \geq 2$.

6.2 The closedness theorem

What is quite surprising is that the converse of Koszul theorem is true under a very mild hypothesis. Choose some ρ_0 in $\mathcal{F}_{\Gamma_0}$.

Theorem 6.2 ([4].III) *If ρ_0 is strongly irreducible, then $\mathcal{F}_{\Gamma_0}$ is closed in* $\mathrm{Hom}(\Gamma_0, G)$.
Hence $\mathcal{F}_{\Gamma_0}$ is a union of connected components of $\mathrm{Hom}(\Gamma_0, G)$.

Recall that the condition "ρ_0 is strongly irreducible" means that the restriction of ρ_0 to any subgroup of finite index is still irreducible. One can check, thanks to Vey irreducibility theorem (theorem 5.1), that this condition does not depend on the choice of ρ_0 and is equivalent to the fact that all the subgroups of finite index of Γ_0 have trivial center.

This converse was proven by Choi and Goldman when $m = 2$ in [16] and by Inkang Kim when $m = 3$ and Γ_0 is hyperbolic in [36].

The proof of the closedness theorem uses old and recent results on Zariski dense subgroups of a semisimple Lie groups as in [1], [2] and [48] and [49].

6.3 The existence theorem

In the middle of the eighties, Johnson and Millson in [32] constructed interesting deformation of discrete cocompact subgroups of $O^+(1,m)$

Theorem 6.3 (Johnson, Millson, 1980) *For any $m \geq 2$, there exist discrete cocompact subgroups Γ_0 of $O^+(1,m)$ such that the identity representation $\rho_0 : \Gamma_0 \to G$ can be included in a continuous family $\rho_s : \Gamma_0 \to G$ of representations whose images $\rho_s(\Gamma_0)$, for $s \neq 0$ is Zariski dense.*

It was only ten years later, that this construction was linked with Koszul openness theorem in the following existence statement.

Theorem 6.4 ([3]) *For any $m \geq 2$, there exist discrete subgroups Γ of G which divide some non symmetric strictly convex open set Ω in $\mathbb{S}^m$.*

In this construction Γ is isomorphic to a cocompact lattice in $O^+(1,m)$, that is why, by theorem 4.1, this divisible convex set Ω is strictly convex.
We will see in section 8.2 a geometric interpretation of these examples.

7 Parametrization in dimension 2

In this section, we will suppose $m = 2$ and Γ torsion free. Hence Γ is isomorphic to the fundamental group Γ_0 of a compact surface Σ_g of genus g. We will suppose that $g > 1$. The case $g = 1$ being very easy. One wants to parametrize all the properly convex open sets Ω in $\mathbb{S}^2$ which are divided by a group isomorphic to Γ_0. Hence one wants to describe the moduli space X_{Γ_0} defined in section 6.1.

7.1 Goldman parametrization

The first description of X_{Γ_0} for $\Gamma_0 := \pi_1(\Sigma_g)$ with $g > 1$ is due to Goldman in [26].

Theorem 7.1 (Goldman, 1990) *The moduli space X_{Γ_0} is diffeomorphic to $\mathbb{R}^{16g-16}$.*

Note that the dimension of this moduli space is $8|\chi|$ where $\chi = 2-2g$ is the Euler characteristic of Σ_g, instead of $3|\chi|$ for the Teichmüller space: the number $3 = \dim(SL(2,\mathbb{R}))$ has been replaced by $8 = \dim(SL(3,\mathbb{R}))$.

Goldman parametrization of X_{Γ_0} in [26] is very similar to the Fenchel-Nielsen parametrization of the Teichmüller space. The main point is to check that if one fixes $3(g-1)$ disjoint free homotopy classes of curves on Σ_g, then one can cut Σ_g along the unique embedded geodesics in these classes and get that way a decomposition of Σ_g in $2(g-1)$ pairs of pants endowed with a projective structure with *hyperbolic* geodesic boundary. Here *hyperbolic* means that the holonomy of the geodesic is a diagonalizable matrix with three positive distinct eigenvalues and that the corresponding lift in Ω of the geodesic connects an attractive fixed point to a repulsive fixed point of this holonomy.

One checks that the projective structure on each pair of pants with fixed hyperbolic holonomy for the three boundary components is parametrized by $\mathbb{R}^2$. The holonomy of each geodesic is parametrized by $\mathbb{R}^2$ and the glueing along each geodesic is also parametrized by $\mathbb{R}^2$. Hence one gets $2(2g-2)+4(3g-3)=8|\chi|$ parameters.

Goldman has also constructed a natural symplectic structure on this moduli space and even, with Darvishzadeh in [20], an almost Kähler structure. Hong Chan Kim has described in [35] this natural symplectic structure.

7.2 Choi classification

Let us quote now Choi classification theorem in [13] which allows to reduce the classification of all real projective structures on a compact surface Σ_g to Goldman's parametrization of properly convex real projective surface with hyperbolic geodesic boundary, parametrization we described in the previous section.

Theorem 7.2 (Choi, 1995) *Every real projective compact surface Σ_g, with $g > 1$ decomposes uniquely along disjoint embedded closed geodesics with hyperbolic holonomy into maximal compact surfaces with convex interior.*

These pieces are either properly convex or are annuli.

The real projective annuli with convex interior and hyperbolic geodesic boundaries are very easy to describe: they are quotients of a closed half plane with origin removed.

Hence, this theorem tells us that, in dimension 2, among the real projective manifolds, the most interesting ones are those obtained as quotient of a properly convex divisible set Ω. One does not know if the same is true in dimension 3.

7.3 Hitchin parametrization

Hitchin has parametrized in [30] one component, called Hitchin component, of the moduli space of representations of $\Gamma_0 := \pi_1(\Sigma_g)$ in $SL(m+1,\mathbb{R})$.

When $m=2$, according to the closedness theorem which is, in this case, due to Choi and Goldman (see also [14]), this component is exactly X_{Γ_0}.

If one fixes a complex structure on Σ_g, Hitchin parametrization, which uses Higgs bundles, identifies X_{Γ_0} with the vector space $H^0(\Sigma_g, K^{\otimes 2}\oplus K^{\otimes 3})$ of couples: a quadratic and a cubic differential form on the Riemann surface Σ_g. Note that, according to Riemann-Roch theorem, this complex vector space is of complex dimension $3(g-1)+5(g-1)=4|\chi|$.

7.4 Labourie-Loftin parametrization

Independently, Labourie in [42] and Loftin in [45] , obtained another parametrization of X_{Γ_0}. Let Ω be a strictly convex subset of $\mathbb{S}^2$ divided by a group isomorphic to Γ_0.

The main idea is to use a deep result of Cheng and Yau in [12]: the existence and unicity up to homothety of an affine sphere asymptotic to the cone C above Ω. As explained in

[42], one can give an easier proof of Cheng Yau theorem in this special case. By unicity, this affine sphere is Γ_0-invariant. The "Blaschke metric" and the "Pick invariant" of this affine sphere are also Γ_0-invariant. They induce on Σ_g first a conformal structure on Σ_g and then a cubic differential form on this Riemann surface.

Theorem 7.3 (Labourie, Loftin, 2000) *The above "Blaschke-Pick" map is a bijection from X_{Γ_0} to the space of complex structures on Σ_g endowed with a cubic differential form.*

Hence the space X_{Γ_0} is a fiber bundle over the Teichmüller space. The fiber is a $5(g-1)$-dimensional complex vector space and the Teichmüller space is a ball in a $3(g-1)$-dimensional vector space. Hence we can check again that X_{Γ_0} is of dimension $8|\chi|$.

8 More examples

Up to now, all the known non symmetric irreducible divisible properly convex open sets Ω of $\mathbb{S}^m$ were strictly convex and the corresponding group Γ was isomorphic to a cocompact lattice in $O^+(1,m)$.

8.1 Groups generated by reflections

It is natural to compute more examples of groups generated by reflections along the faces of a given convex polyhedron P as in theorem 3.1, than those for which P is a simplex.

Using polyhedra combinatorially equivalent to product of simplices, one constructs a few interesting new examples:

Example 8.1 ([4].IV) *For $m = 3$, 4, 5 and 6, there exist non symmetric irreducible divisible properly convex open sets Ω of $\mathbb{S}^m$ which are not strictly convex.*

For $m = 4$, one can also construct with this method (see [5]) a divisible strictly convex open set Ω with Γ not isomorphic to a lattice in $\mathrm{O}^+(1,4)$.

8.2 The bending construction

One can interpret geometrically the examples constructed in theorem 6.4 as a *bending deformation* of a hyperbolic compact n-manifold M along a compact embedded totally geodesic hypersurface N.

Before explaining this construction, let us mention that similar bending deformations are useful to construct flat conformal structures on M: think of $O^+(1,n)$ as being inside $O^+(1,n+1)$ instead of being inside G.

Let us explain now this *bending construction*, (see [26] for the case of surfaces) : One starts with a fixed real projective compact n-manifold M which contains a connected compact embedded totally geodesic hypersurfaces N. Note that the holonomy of N preserves some great hypersphere $\mathbb{S}^{m-1}$ inside $\mathbb{S}^m$. We assume that it also preserves some point p outside this hypersphere. This assumption is clearly satisfied in the hyperbolic case with p equal to the "orthogonal" of $\mathbb{S}^{m-1}$. This assumption allows us to select a system of charts near N such that the glueing maps preserves $\mathbb{S}^{m-1}$ and p.

One can construct for every real $t > 0$ a projective transformation φ_t between two open neighborhoods U and V of N such that in these charts φ_t is given by the diagonal matrix $a_t := \mathrm{diag}(t, \dots, t, t^{-m}) \in G$. Note that the restriction of φ_t to N is the identity.

The bending construction depends on this real $t > 0$ called the *bending parameter*:
- One first cuts M along N, getting that way a real projective manifold M' whose boundary is totally geodesic and equal to two copies of N.
- One then "enlarges" M' near the first copy of N thanks to the open neighborhood U and also near the second copy of N thanks to the open neighborhood V.
- Finally, one glues back U and V thanks to φ_t.

One can check, for example using theorem 6.2, that a real projective structure obtained by bending from a properly convex real projective structure is also properly convex.

Let us now describe the holonomy h_t of the bended structure thanks to the initial holonomy h_1. To compute the fundamental group Γ of M thanks to the fundamental group Γ_0 of N, there are two cases:
- When M' has two connected components M_1 and M_2 with fundamental groups Γ_1 and Γ_2, the group Γ is the amalgamated product $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$. In this case the holonomy representation h_t of the bended structure is equal to h_1 on Γ_1 but is obtained from h_1 by conjugation with a_t on Γ_2.
- When M' is connected with fundamental group Γ', the group Γ is the HNN-extension of Γ' with respect to the two morphisms from Γ_0 into Γ'. Hence Γ is generated by Γ' and an extra element c that correspond to a curve in M' connecting a base point in the two copies of N. In this case the holonomy representation h_t of the bended structure is equal to h_1 on Γ' and the holonomy of c is $h_t(c) = a_t h_1(c)$.

8.3 Kapovich examples

Using a variation of a construction of Gromov-Thurston in [34], Kapovich has constructed new examples in any dimension $m \geq 4$.

Theorem 8.2 (Kapovich, 2005) *For all $m \geq 4$, there exist divisible strictly convex open sets Ω of $\mathbb{S}^m$ for which the group Γ is not isomorphic to a lattice in $\mathrm{O}^+(1, m)$.*

The idea is to improve the bending construction of section 8.2.

One starts with an hyperbolic compact n-manifold M with a dihedral group D_{2d} of isometries with $d \geq 4$ such that this group contains d reflections along totally geodesic hypersurfaces $M_1, \dots, M_d$ which intersect along a codimension 2 submanifold M_0 and such that one can cut M along these hypersurfaces in isometric pieces.

One constructs a n-manifold N by glueing "isometrically" $2(d+1)$ of these pieces around M_0. One shows then that there exists on N a projective structure whose restriction to each of the pieces is the one given by the hyperbolic structure. Otherwise stated, *there is* *a way to adjust the $2(d+1)$ real bending parameters along each half-hypersurface in such a way that the charts fit together around M_0.*

Using a generalization of Vinberg theorem 3.1 one proves that this projective structure is properly convex.

Using Mostow rigidity, Gromov and Thurston have checked that the π_1 of N is not isomorphic to a lattice in $\mathrm{O}^+(1,4)$ eventhough it is Gromov hyperbolic.

According to theorem 4.1, this structure is strictly convex.

9 Dimension 3

Let us now describe the structure of the 3-dimensional properly convex divisible sets which are not strictly convex.

9.1 Properly embedded triangles

Let us begin by quoting another application of theorem 4.3 which is true in any dimension.

Corollary 9.1 (Benzecri, 1960) *Let Ω be a divisible properly convex open set in $\mathbb{S}^m$. If Ω is not strictly convex, then Ω contains a properly embedded triangle.*

A properly embedded triangle (PET) of Ω is a 2-dimensional triangle T included in Ω whose boundary ∂T is included in $\partial\Omega$.

Let us see now how, in dimension 3, these PETs allow us to describe Ω, Γ and M.

9.2 Totally geodesic JSJ-decomposition of M

When Ω is strictly convex, according to theorem 4.1, the group Γ is Gromov hyperbolic and hence does not contain any $\mathbb{Z}^2$-subgroup. Moreover, according to Perelman result, the quotient manifold M should be diffeomorphic to a quotient of $\mathbb{H}^3$ by a cocompact subgroup of $\mathrm{O}^+(1,3)$.

The following theorem relates, when Ω is not strictly convex, the PETs of Ω, the subgroups of Γ isomorphic to $\mathbb{Z}^2$ and the Jaco-Shalen-Johannson decomposition of M.

Theorem 9.2 ([4].IV) *Let Γ be a discrete group which divides an irreducible properly convex set Ω of $\mathbb{S}^m$. Suppose $m=3$ and Ω is not strictly convex. Then :*
Every $\mathbb{Z}^2$-subgroup of Γ stabilizes a unique PET.
Every PET is stabilized by a unique maximal $\mathbb{Z}^2$-subgroup of Γ. The PETs are disjoints.
Every segment of $\partial\Omega$ is on an edge of a PET. The vertices of the PETs are dense in $\partial\Omega$.

Hence, when Γ is torsion free, the PETs project in the compact quotient $M := \Gamma\backslash\Omega$ onto an union of finitely many embedded disjoint tori and Klein bottles. The decomposition of M along these totally geodesic embedded tori or Klein bottles is nothing but a geometric realization of the Jaco-Shalen-Johannson decomposition of M.

As a consequence of this theorem and of Thurston's geometrization theorem ([52]), the interior of all the components of this decomposition are diffeomorphic to a quotient of $\mathbb{H}^3$ by a subgroup of $O^+(1,3)$ of finite covolume. Moreover this decomposition is unique.

This part of the statement is analogous to Goldman's decomposition of convex real projective surface in pairs of pants with geodesic boundaries used in theorem 7.1.

The main point when proving Theorem 9.2 is to check that the image in M of each PET is compact. For that one proves that the lamination on M given by these PETs has a transverse measure and one studies the action of Γ on the corresponding $\mathbb{R}$-tree using Rips theory.

One can find pictures of these divisible convex sets Ω in [4].IV.

Some moduli spaces of convex projective structures on non closed surfaces have been recently studied in [17], [22] and in [46].

Some moduli spaces of convex projective structures on 3-manifolds have also been recently studied in [15] and in [19].

But very few is known in higher dimension.

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