Fano varieties; Iskovskih’s classification

Ekaterina Amerik

For details and extensive bibliography, we refer to [2], chapter V, and [1].

A Fano variety is a projective manifold $X$ such that the anticanonical line bundle $K_X^{-1}$ is ample. By Kodaira vanishing, the Hodge numbers $h^{p,0}(X) = h^{0,p}(X)$ are zero for $p \neq 0$. Furthermore, Fano manifolds are simply connected (this is implied for example by their property to be rationally connected; see the main article on rational curves and uniruled varieties).

Simplest examples are obtained by taking smooth complete intersections of type $(m_1, m_2, \ldots, m_k)$ in $\mathbb{P}^n$. By adjunction formula, such a complete intersection is Fano if and only if $\sum m_i \leq n$. A larger class of examples is that of complete intersections in a weighted projective space $\mathbb{P}(a_0, a_1, \ldots, a_n)$ (this is $(\mathbb{C}^{n+1} - 0)/\mathbb{C}^*$, where $\mathbb{C}^*$ acts with weights $a_0, a_1, \ldots, a_n$; it is singular when not isomorphic to a usual projective space, but we consider complete intersections avoiding the singularities): the Fano condition amounts then to $\sum m_i < \sum a_i$. Rational homogeneous varieties $G/H$ ($G$ semisimple, $H$ parabolic) are Fano, too.

A Fano curve is, obviously, $\mathbb{P}^1$. If $n = \dim(X) = 2$ and $X$ is Fano, then $X$ is called a Del Pezzo surface. Such surfaces have been classically studied, and it is well-known that any such $X$ is isomorphic either to $\mathbb{P}^2$, or to $\mathbb{P}^1 \times \mathbb{P}^1$, or to $\mathbb{P}^2$ blown up in $d$ points $(1 \leq d \leq 8)$ in general position, ”general position” meaning here that no three points are on a line and no six on a conic. For $1 \leq d \leq 6$, the anticanonical map is an embedding. It realizes a blow-up of $\mathbb{P}^2$ in $d$ points $(1 \leq d \leq 6)$ as a surface $X_l$ of degree $l = K_X^2 = 9 - d$ in $\mathbb{P}^l$. For $d = 7$, one obtains $X_2$ which is a double cover of $\mathbb{P}^2$ ramified along a quartic, and for $d = 8$, a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$ (which is the same as a double covering of a quadratic cone ramified in its section by a cubic; the double covering is given by the space of sections of $K_X^{-2}$.)

There are many more types of Fano threefolds. Even under the restriction $Pic(X) \cong \mathbb{Z}$, one obtains 18 families. Fano threefolds of Picard number one have been classified by Iskovskih, and Fano threefolds of higher Picard number, by Mori and Mukai (see [1]). Below we mention a few generalities on Fano manifolds and give an outline of Iskovskih’s classification.

A basic invariant of a Fano manifold is its index: this is the maximal integer $r$ such that $K_X$ is divisible by $r$ in $Pic(X)$.

**Theorem 1** ([3]) Let $X$ be Fano, $\dim(X) = n$. Then the index $ind(X)$ is at most $n + 1$; moreover, if $ind(X) = n + 1$, then $X \cong \mathbb{P}^n$, and if $ind(X) = n$, then $X$
is a quadric.

(Note that \( \text{ind}(X) \leq n + 1 \) follows immediately from bend-and-break; but in fact the result of Kobayashi and Ochiai is much older, and the proof of their first statement is quite elementary.)

**Theorem 2** (Kollar-Miyaoka-Mori, [2]): For any positive integer \( n \), there is only finitely many deformation types of Fano manifolds of dimension \( n \).

The number of families probably grows very fast together with \( n \).

A Fano threefold \( X \) can have index 4 (if \( X = \mathbb{P}^3 \)), 3 (if \( X = Q^3 \)), 2 or 1. Suppose that the Picard number of \( X \) is 1. Then \( K_X = H_X^{\text{ind}(X)} \), where \( H_X \) is the ample generator of \( \text{Pic}(X) \). Iskovskih’s classification asserts the following:

If \( \text{ind}(X) = 2 \), then \( 1 \leq H_X^3 \leq 5 \), and:
- if \( H_X^3 = 1 \), \( X \) is a hypersurface of degree 6 in a weighted projective space \( \mathbb{P}(1,1,1,2,3) \);
- if \( H_X^3 = 2 \), \( X \) is a double covering of \( \mathbb{P}^3 \) ramified in a quartic surface (in other words, a hypersurface of degree 4 in a weighted projective space \( \mathbb{P}(1,1,1,1,2) \));
- if \( H_X^3 = 3 \), \( X \) is a cubic in \( \mathbb{P}^4 \);
- if \( H_X^3 = 4 \), \( X \) is a complete intersection of type (2,2) in \( \mathbb{P}^5 \);
- if \( H_X^3 = 5 \), \( X \) is a linear section of the Grassmannian \( G(2,5) \) in the Plücker embedding.

If \( \text{ind}(X) = 1 \), then:
- \( H_X^3 \) takes all even values between 2 and 22, except 18;
- low values of \( H_X^3 \) correspond to double covers: of \( \mathbb{P}^3 \) ramified in a sextic if \( H_X^3 = 2 \), of a quadric ramified in a quartic section if \( H_X^3 = 4 \);
- for \( H_X^3 = 4 \) there is of course one more family - that of quartics in \( \mathbb{P}^4 \);
- and also for all other families, \( H_X \) is very ample and embeds \( V_d \) (that is, a Fano threefold with \( H_X^3 = d \)) in \( \mathbb{P}^{d/2+2} \);
- \( V_6 \) and \( V_8 \) are obvious complete intersections, \( V_{10} \) is a section of a cone over \( G(2,5) \) by three hyperplanes and a quadric, \( V_{14} \) is a linear section of \( G(2,6) \);
- other \( V_d \) (\( d = 12, 16, 18, 22 \)) are more complicated, but there is a relatively simple description in terms of vector bundles on homogeneous varieties, due to Mukai. For instance, a \( V_{22} \) is the zero set of a section of the sum of three copies of \( \Lambda^2 U^* \) on \( G(3,7) \) (where \( U \) is the universal bundle).

There are more families of threefolds of higher Picard number; according to Mori and Mukai, its maximal value is 10.

**References**
