Paris Saclay Univerity

Master Thesis

Optimization Master

Potential Formulation and Fictitious Play Procedure for Finite Mean Field Game and Mean Field Game of Control

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1 Introduction

Mean Field Game models are really powerful to study interactions between a large number of agents. This class of games stands as a limit case for N player games with $N \mapsto +\infty$. Two kinds of Mean Field Game can be distinguished. In Classical Mean Field Game model agents react to the statistical distribution of position of others. In Mean Field Game of Control (or extended Mean Field Game) agents react to the joint measure of position and strategy of others. Mean Field Game of Control are then well indicated to model price evolution in trading since the price is determined by the net demand. Two examples can be provided: in electricity management [5] where agents are prosumers of electricity and tries to trade electricity optimally; in finance [3] where traders are trying to reach a desired quantity of assets by the end of the day. In both cases players take care of the strategy of the others (buy or sell) through the price of the good. The Mean Field Game system is composed of a Hamilton-Jacobi-Bellman equation which is backward in time and come from a Dynamic Programming Principle, and a Fokker-Planck equation which is forward in time. To solve numerically such problem we present here two approaches. The first one is based on potential formulation. This approach proposes to find a new formulation of the problem such that the first order condition of the Lagrangian leads to the same solution as the original problem. In this thesis we put ourselves in the best condition to derive augmented Lagrangian methods in future works by deriving dual formulations using the Fenchel-Rockafellar theorem. In the case of Classical Mean Field Game we are able to compare our formulation to the literature. The second part of the thesis is based on fictitious play procedure. This method comes from the game theory approach and is about finding equilibrium of the game by iterated best responses. After explaining the method and deriving the convergence of the method for Finite Mean Field Game, we prove that the result also holds for Finite Mean Field Game of Control. Finally we provide a numerical part for this last section. We give the algorithm and some plots concerning the evolution of the measure through time via a simple example.

2 Introduction to Mean Field Games and Mean Field Games of Control

2.1 Mean Field Game Problem

Consider an infinitely many agent problem. Let a process $(X_s)_s$ evolve according to the following SDE:

$$X_t = x_s + \int_s^t b(\tau, X_\tau, \alpha_\tau, m(\tau))\mathrm{d}\tau + \int_s^t \sigma(\tau, X_\tau, \alpha_\tau, m(\tau))\mathrm{d}B_\tau, \tag{1}$$

where B_τ is a M dimensional Brownian motion and x_s is the initial value at time s. This system describes the evolution of the dynamic of each agent. The dynamic is controlled by the agent via its control $(\alpha_s)_s$ and we define $(\mathcal{A}, \delta_{\mathcal{A}})$ to be the metric space where the control takes its values. The dynamics also depends on the rational expectation of the statistical distribution of others $m(s) \in \mathcal{P}^1(\mathbb{R}^d)$. We define the cost function J that each player tries to minimize

$$J(t, x_t, \alpha) := \mathbb{E}\left[\int_t^T L(s, X_s, \alpha_s) + F(X_s, m(s))\mathrm{d}s + g(X_T, m(T))\right].,\tag{2}$$

where we define $L : [0,T] \times \mathbb{R}^d \times \mathcal{A} \mapsto \mathbb{R}$, $F : \mathbb{R}^d \times \mathcal{P}^1(\mathbb{R}^d) \mapsto \mathbb{R}$ and $g : \mathbb{R}^d \times \mathcal{P}^1(\mathbb{R}^d) \mapsto \mathbb{R}$ (instead of $L(s, X_s, \alpha_s) + F(X_s, m(s))$ one can also find only $L(s, X_s, \alpha_s, m(s))$). Then we define the value function as follows: We suppose $b : [0,T] \times \mathbb{R}^d \times \mathcal{A} \times \mathcal{P}^1(\mathbb{R}^d) \mapsto \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{A} \times \mathcal{P}^1(\mathbb{R}^d) \mapsto \mathbb{R}^{d\times M}$ to be smooth enough to have existence of a solution. Each agent looks for minimizing their cost function

$$u(t, x_t) := \inf_{\alpha\in\mathcal{A}} J(t, x_t, \alpha), \tag{3}$$

which leads using a dynamic programming principle to the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} -\partial_t u(t,x) + H(t, x, Du(t,x), D^2u(t,x), m(t)) = F(x, m(t)) & (t,x) \in [0,T] \times \mathbb{R}^d \\ u(T,x) = g(x, m(T)) & x \in \mathbb{R}^d \end{cases} \tag{4}$$

Where the Hamiltonian H is defined by

$$H(t, x, p, M) := \sup_{\alpha\in\mathcal{A}} \left\{ -L(t,x,\alpha) - pb(t,x,\alpha) - \frac{1}{2}\operatorname{Tr}(\sigma\sigma^\star(t,x,\alpha)M) \right\}. \tag{5}$$

Now we introduce $\alpha^\star$ a maximum solution in the specific Hamiltonian $H(t, x, Du(t,x), D^2u(t,x), m(t))$. Then we have:

$$\begin{aligned} H(t, x, Du(t,x), D^2u(t,x), m(t)) = &-L(t, x, \alpha^\star(x,t)) - Du(t,x)b(t, x, \alpha^\star(x,t), m(t)) \\ &- \frac{1}{2}\operatorname{Tr}(\sigma\sigma^\star(t, x, \alpha^\star(x,t), m(t))D^2u(t,x)), \end{aligned} \tag{6}$$

where $\alpha^\star$ depends on $m(t)$. Indeed this dependence comes from the dynamics of the system which evolves according to the measure of agents position. From Hamilton-Jacobi-Bellman equation we also observe that the dynamic of the value function u depends on $m(t)$. Now we study the evolution of statistical distribution of players through time. The two main hypothesis are: players are all involved in the same game with the same system to control ; the noise is independent to the initial position of agents. The distribution of the agent is given by the law of X_s at time s so as people are playing optimally, the statistical distribution of agents is given by the law of $X_s^\star$. We have:

$$\mathrm{d}X_s^\star = b(s, X_s^\star, \alpha_s^\star, m(s))\mathrm{d}s + \sigma(s, X_s^\star, \alpha_s^\star, m(s))\mathrm{d}B_s \tag{7}$$

Now we have to use the part on mean field limit. Indeed if we only have one agent starting from an initial point x at time 0, his position at time $t \geq 0$ will be given by a law $\mathcal{L}(X_t^\star)$. But now we have an infinite number of agents. Then starting from a given distribution of position $\bar{m}_0$ at time 0, the statistical distribution of positions will be also given by the law of $\mathcal{L}(X_t^\star)$ since all the agents are controlling the same system. Thus we consider any test function $\phi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R}^d)$ with value in $\mathbb{R}$ to derive the law of $X_s^\star$

$$0 = \mathbb{E}\left[\phi(T, X_T^\star)\right] = \mathbb{E}\left[\phi(0, X_0^\star)\right] + \mathbb{E}\left[\int_0^T \partial_t \phi(s, X_s^\star) + b(s, X_s^\star, \alpha_s^\star, m(s)) D\phi(s, X_s) + \frac{1}{2}\operatorname{Tr}(\sigma\sigma^\star(s, X_s^\star, \alpha_s^\star, m(s)) D^2\phi(s, X_s)) \mathrm{d}s\right], \tag{8}$$

then we have:

$$0 = \int_{\mathbb{R}^d} \phi(T,x)\bar{m}(T,\mathrm{d}x) = \int_{\mathbb{R}^d} \phi(0,x)\bar{m}(0,\mathrm{d}x) + \int_0^T \int_{\mathbb{R}^d} \Big(\partial_t\phi(s,x) + b(s,x,\alpha_s^\star, m(s))D\phi(s,x) + \frac{1}{2}\operatorname{Tr}(\sigma\sigma^\star(s,x,\alpha_s^\star,m(s))D^2\phi(s,x))\Big)\bar{m}(s,\mathrm{d}x)\mathrm{d}s. \tag{9}$$

By integration by parts we have

$$0 = \int_{\mathbb{R}^d} \bar{m}(T,x)\phi(T,\mathrm{d}x) = \int_{\mathbb{R}^d} \bar{m}(0,x)\phi(0,\mathrm{d}x) + \int_0^T \int_{\mathbb{R}^d} \Big(-\partial_t\bar{m}(s,x) - \operatorname{div}(\bar{m}(s,x)b(s,x,\alpha_s^\star,m(s))) + \frac{1}{2}\sum_{ij} D_{i,j}^2(\bar{m}(s,x)\sigma\sigma_{i,j}^\star(s,x,\alpha_s^\star,m(s)))\Big)\phi(s,\mathrm{d}x)\mathrm{d}s. \tag{10}$$

Finally $\bar{m}$ solves in the weak sense

$$\begin{cases} \partial_t\bar{m}(t,x) + \operatorname{div}(\bar{m}(t,x)b(t,x,\alpha_t^\star,m(t))) - \dfrac{1}{2}\sum_{ij} D_{i,j}^2(\bar{m}(t,x)\sigma\sigma_{i,j}^\star(t,x,\alpha_t^\star,m(t))) = 0 & (t,x) \in [0,T]\times\mathbb{R}^d \\ \bar{m}(0,x) = \bar{m}_0(x) & x \in \mathbb{R}^d \end{cases} \tag{11}$$

We expect the expectation made by the agents to be correct (rational expectation) and we get

$$\begin{cases} \partial_t m(t,x) + \operatorname{div}(m(t,x)b(t,x,\alpha_t^\star,m(t))) - \dfrac{1}{2}\sum_{ij} D_{i,j}^2(m(t,x)\sigma\sigma_{i,j}^\star(t,x,\alpha_t^\star,m(t))) = 0 & (t,x) \in [0,T]\times\mathbb{R}^d \\ m(0,x) = m_0(x) & x \in \mathbb{R}^d \end{cases} \tag{12}$$

Let's denote $a = \sigma\sigma^\star$, then we end up with the following system:

$$\begin{cases} -\partial_t u(t,x) + H(t,x,Du(t,x),D^2u(t,x),m(t)) = F(x,m(t)) & (t,x) \in [0,T]\times\mathbb{R}^d \\ \partial_t m(s,x) + \operatorname{div}(m(s,x)b(s,x,\alpha_s^\star,m(s))) - \dfrac{1}{2}\sum_{ij} D_{i,j}^2(m(s,x)a_{i,j}^\star(s,x,\alpha_s^\star,m(s))) = 0 & (t,x) \in [0,T]\times\mathbb{R}^d \\ m(0,x) = m_0(x), \quad u(T,x) = g(x,m(T)) & x \in \mathbb{R}^d \end{cases} \tag{13}$$

This system is the classical second order Mean Field Game system we can see in [8] and [1].

2.2 Mean Field Game of Control Problem

In the Mean Field Game Problem, agents react to the statistical distribution of others. The Mean Field Game of Control problem slightly differ since agents control their own states and react now to the joint law of positions and instantaneous controls of others. Using the same notation as above, we define μ_t to be the joint density of the pairs (X_t, α_t). Note that it is a probability measure over $(\mathbb{R}^d, \mathcal{A})$ the first marginal of μ_t is m_t. We suppose $b : [0, T] \times \mathbb{R}^d \times \mathcal{A} \times \mathcal{P}^1(\mathbb{R}^d \times \mathcal{A}) \mapsto \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{A} \mapsto \mathbb{R}^{d\times M}$ to be smooth enough to have existence of a solution. Here we present the framework of P.Cardaliaguet and C-H Lehalle [3], the dynamics of $(X_t)_t$ is given by

$$X_t = x + \int_s^t b(\tau, X_\tau, \alpha_\tau; \mu_\tau)\mathrm{d}\tau + \int_s^t \sigma(\tau, X_\tau, \alpha_\tau)\mathrm{d}B_\tau. \tag{14}$$

The cost function is given by:

$$J(t, x, \alpha, \mu) := \mathbb{E}\left[\int_t^T L(s, X_s, \alpha_s; \mu_s)\mathrm{d}s + g(X_T, m(T))\right] \tag{15}$$

So given a product measure μ, the value function of the agents at time t is:

$$u(t, x_t) := \inf_{\alpha\in\mathcal{A}} J(t, x_t, \alpha; \mu) \tag{16}$$

If we define the following Hamiltonian:

$$H(t, x, p, M) := \sup_{\alpha} \{-L(t, x, \alpha; \nu) - pb(t, x, \alpha; \nu)\}, \tag{17}$$

the value function is a viscosity solution of the following Hamilton-Jacobi-Bellman equation:

$$\begin{cases} -\partial_t u(t,x) - \mathrm{Tr}(\sigma\sigma^T(t,x)D^2u(t,x)) + H(t,x,Du(t,x);\mu_t) = 0 & (t,x) \in [0,T]\times\mathbb{R}^d \\ u(T,x) = g(x, m(T)) & x \in \mathbb{R}^d \end{cases} \tag{18}$$

Now if we derive formally the optimal drift

$$b(t, x, \alpha^\star(t, x); \mu_t) = -D_pH(t, x, Du(t, x); \mu_t) \quad \forall(t, x) \in [0, T] \times \mathbb{R}^d, \tag{19}$$

the population density m_t is expected to evolve according to the Kolmogorov equation:

$$\begin{cases} \partial_t m(t,x) + \mathrm{div}(m(t,x)b(t,x,\alpha_t^\star(x,t),\mu_t)) - \frac{1}{2}\sum_{ij} D^2_{i,j}(m(t,x)\sigma\sigma^\star_{i,j}(t,x)) = 0 & (t,x)\in[0,T]\times\mathbb{R}^d \\ m(0,x) = m_0(x) & x\in\mathbb{R}^d \end{cases} \tag{20}$$

For the sake of simplicity, we suppose that the map $\alpha \mapsto b(t, x, \alpha, ; \nu)$ is one-to-one with a smooth inverse. Then from the optimal drift (19) we get that the optimal control of a player is $\alpha^\star(t, x, Du(t, x); \mu_t)$. So in an equilibrium configuration , the measure μ_t has to be the image of m_t by the application $x \mapsto (id, \alpha^\star(t, x, Du(t, x); \mu_t))$, which leads to the fixed point relation

$$\mu_t = (id, \alpha^\star(t, ., Du(t, .); \mu_t))\sharp m_t. \tag{21}$$

This fixed point relation can be understood in the following way: Knowing that $\alpha^\star$ is the optimal control and depends on μ_t, then the first marginal of μ_t is m_t and the second marginal is the measure m_t pushed by the optimal control. So in this second marginal we look at the measure of individuals and given that we know the strategy of agents given a state x, we end up with a distribution of strategy. Thus the second marginal is exactly the statistical distribution of instantaneous strategy of the players. Finally the Mean Field Game of Control problem takes the form:

$$
\begin{cases}
-\partial_t u(t,x) - \mathrm{Tr}(\sigma\sigma^T(t,x)D^2u(t,x)) + H(t,x,Du(t,x),\mu(t)) = 0 & (t,x)\in[0,T]\times\mathbb{R}^d \\
\partial_t m(t,x) + \mathrm{div}(m(t,x)b(t,x,\alpha_t^\star(x,t),\mu_t)) - \dfrac{1}{2}\sum_{ij} D^2_{i,j}(m(t,x)\sigma\sigma^\star_{i,j}(t,x)) = 0 & (t,x)\in[0,T]\times\mathbb{R}^d \\
m(0,x) = m_0(x), \quad u(T,x) = g(x,m(T)) & x\in\mathbb{R}^d \\
\mu_t = (id, \alpha^\star(t,.,Du(t,.);\mu_t))\sharp m_t & t\in[0,T]
\end{cases} \tag{22}
$$

In what we did until now, everything was continuous. In what follows we will consider a finite game, where time and space are discrete. The motivation is to study numerical methods and design algorithms to solve Mean Field Game problems.

3 Finite MFG

3.1 Formulation of the problem

In this subsection we follow the approach of [6] and [7]. Let us define a finite discrete time $\mathcal{T} = \{0,1,\ldots,T-1\}$, $\mathcal{T}+1 = \{0,\ldots,T\}$ and a finite discrete space S set. We define the simplex over S by $\mathcal{P}(S)$ and we define the set of transition kernel $\mathcal{K}_{\mathcal{S},\mathcal{T}}$ to be

$$
\mathcal{K}_{\mathcal{S},\mathcal{T}} := \{P : S\times S\times\mathcal{T}\mapsto[0,1], \sum_{y\in S} P(x,y,t) = 1\}. \tag{23}
$$

Then we define a finite MFG as follows. Given $M_0 \in \mathcal{P}(S)$ and $P \in \mathcal{K}_{\mathcal{S},\mathcal{T}}$, (M_0, P) induces a probability distribution over S^T with marginals recursively given by:

$$
M_P^{M_0}(x,0) := M_0(x) \quad \forall x\in S \tag{24}
$$

$$
M_P^{M_0}(x,t) := \sum_{y\in S} M_P^{M_0}(y,t-1)P(y,x,t-1) \quad \forall x\in S, \forall t\in\{1,\ldots,T\}. \tag{25}
$$

Now let $c : S\times S\times\mathcal{P}(S)\times\mathcal{P}(S)\mapsto\mathbb{R}$, $g : S\times\mathcal{P}(S)\mapsto\mathbb{R}$, $M : \{0,\ldots,T\}\mapsto\mathcal{P}(S)$ and define $J_M : \mathcal{K}_{\mathcal{S},\mathcal{T}}\mapsto\mathbb{R}$ as:

$$
J_M(P) := \sum_{t=0}^{T-1}\sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)c_{xy}(P(x,.,t),M(.,t)) + \sum_{x\in S} M_P^{M_0}(x,T)g(x,M(.,T)). \tag{26}
$$

Then the MFG problem is to find a couple $(\hat{M},\hat{P})$ such that:

$$
\hat{P} := \underset{P\in\mathcal{K}_{S,\mathcal{T}}}{\mathrm{argmin}}\, J_{\hat{M}}(P) \text{ with } \hat{M} = M_{\hat{P}}^{M_0}. \tag{MFG_d}
$$

For every $M : \mathcal{T}\mapsto\mathcal{P}(S)$ the function

$$
U_M(x,t) := \inf_{\mathcal{P}\in\mathcal{K}_{S,\mathcal{T}}} J_M^{x,t}(P) \quad \forall(x,t)\in S\times\mathcal{T}, \quad U_M(x,T) = g(x,M(x,T)) \quad \forall x\in S \tag{27}
$$

satisfies the Dynamic Programming Principle,

$$
U_M(x,t) = \inf_{p\in\mathcal{P}(S)} \sum_{y\in S} p(y)\left[c_{xy}\left(p,M(.,t)\right) + U_M(y,t+1)\right] \quad \forall(x,t)\in S\times\mathcal{T} \tag{28}
$$

Then one can characterize the optimal policy for the state that are reachable. Let us set

$$c_x(p, M(.,t)) := \sum_{y\in S} p(y) c_{xy}\left(p, M(.,t)\right) \tag{29}$$

We define M^q_{xy} to be a transition operator between state x and state y at time q. In our case this transition operator is $P(x, y, q)$ where P is the probability of transition. We define

$$S_x := \cup_{q=t}^{T-t-1}\{y \in S; M^q_{xy} > 0\} \tag{30}$$

to be the set of all reachable state starting from state x. We set $S_x(P)$ to be the set of all accessible state starting from x with policy $P \in \mathcal{K}_{S,\mathcal{T}}$. We set $\hat{S}_x(P) = \{x\} \cup S_x(P)$ and we have the following lemma

Lemma 1. *A policy $P \in \mathcal{K}_{S,\mathcal{T}}$ is optimal, when starting from $x_0 \in S$ if and only if it satisfies the dynamic programming equation over $\hat{S}_{x_0}(P)$, that is to say*

$$\hat{P}(x, ., t) \in \operatorname*{argmin}_{p\in\mathcal{P}(S)} c_x(p, M(.,t)) + \sum_{y\in S} p(y) U_M(y, t+1) \qquad x \in \hat{S}_{x_0}(P),\ \forall t \in \mathcal{T}.$$

Proof. Let $x \in S$ such that

$$c_x(\hat{p}, M(.,t)) + \sum_{y\in S} \hat{p}(y) J^{y,t}_M(p) = \inf_{p\in P(S)} \left\{ c_x(p, M(.,t)) + \sum_{y\in S} p(y) U_M(y, t+1) \right\} \tag{31}$$

Since $U(y, t+1) \leq J^{y,t}_M(p)$ the equality holds if $U(y, t+1) = J^{y,t}_M(p)$ whenever $p(y) > 0$. Then the results follows by induction starting form $x = x_0$. □

Then from this Lemma we deduce that (MFG_d) is equivalent to find $U : S \times \mathcal{T} \mapsto \mathbb{R}$ and $M : \mathcal{T} \mapsto \mathcal{P}(S)$ for reachable state such that

$$U(x,t) = \sum_{y\in S} \dot{P}(x, y, t) \left[c_{xy}\left(\hat{P}(x, ., t), M(.,t) \right) + U(y, t+1) \right] \qquad \forall (t, x) \in \mathcal{T} \times S \tag{MFG'_d}$$

$$M(x,t) = \sum_{y\in S} M(y, t-1) \dot{P}(y, x, t-1) \qquad \forall (t, x) \in \{1, \ldots, T\} \times S \tag{32}$$

$$U(x, T) = g(x, M(., T)), \quad M(x, 0) = M_0(x) \qquad \forall x \in S \tag{33}$$

where $\dot{P} \in \mathcal{K}_{S,\mathcal{T}}$ satisfies:

$$\dot{P}(x, ., t) \in \operatorname*{argmin}_{p\in\mathcal{P}(S)} \sum_{y\in S} p(y) \left[c_{xy}(p, M(.,t)) + U(y, t+1) \right] \quad \forall t \in \mathcal{T}, \quad x \in S. \tag{DPP}$$

The equation (DPP) expresses the dynamic of the value function. In continuous setting, this equation leads to the Hamilton-Jacobi-Bellman equation. The equation (32) is the Fokker-Planck equation. In what follow we assume that the cost has a separable form and can depend on time, that is to say

$$c_{xyt}(p, M) := K_1(p, x, y, t) + K_2(M, x, t) \tag{34}$$

then if p is a measure we can set

$$\sum_{y\in S} p(y) c_{xyt}(p, M) := c_{xt}(p, x) + K_2(M, x, t) \tag{35}$$

with $c_{xt}(p, x) := \sum_{y\in S} p(y) K_1(p, x, y, t)$. We mention that in special case, the cost function can be defined with an entropy term that will ensure the uniqueness of the minimum:

$$c_{xy}\left(p,M\right)=K(x,y,M)+\epsilon\log(p(y))$$

And one can derive by the minimization problem for all $t\in\mathcal{T}$:

$$\hat{P}(x,y,t)=\frac{\exp\left(-(K(x,y,M)+U(y))/\epsilon\right)}{\sum_{y'\in S}\exp\left(-(K(x,y',M)+U(y'))/\epsilon\right)} \tag{36}$$

3.2 Finite MFG with potential structure

In this subsection we are looking for a potential formulation of (MFG'_d). To make some hypothesis on the data, we need first to give two definitions

Definition 1. *Consider $\mathcal{P}_2$ to be the set of Borel probability measure with finite second order moment. We say that $U:\mathcal{P}_2\mapsto\mathbb{R}^k$ is C^1 in the L^2 sens if there exists a continuous map*

$$\frac{\delta U}{\delta m}:\mathcal{P}_2\times\mathbb{T}^d\mapsto\mathbb{R}^k \tag{37}$$

such that for any $m,m'\in\mathcal{P}_2$,

$$U(m')-U(m):=\int_{\mathbb{R}^k}\frac{\delta U}{\delta m}((1-s)m+sm',y)\mathrm{d}(m-m')(y) \tag{38}$$

we say that $\frac{\delta U}{\delta m}$ is the L^2 derivative of U.

Definition 2. *Let $x_0\in dom(f)$. We define the recession function $f_\infty:X\mapsto\bar{\mathbb{R}}$ by*

$$f_\infty(d):=\sup_{\tau>0}\frac{f(x_0+\tau d)-f(x_0)}{\tau} \tag{39}$$

In what follows we set $L:\mathbb{R}\times S\times S\times\mathcal{T}\mapsto\mathbb{R}$ to be a proper l.s.c, convex function with respect to its first variable and such that $L_\infty(d,.,.,.)=+\infty$ if $d>0$. Let $F:\mathcal{P}(S)\mapsto\mathbb{R}$ and $G:\mathcal{P}(S)\mapsto\mathbb{R}$ to be two proper functions with derivatives f and g over the space of measure $\mathcal{P}(S)$. Then we consider the following problem:

$$\min_{P,M}\left\{\sum_{t\in\mathcal{T}}\sum_{x,y\in S}M(x,t)L\left(P(x,y,t),x,y,t\right)+\sum_{t\in\mathcal{T}}F(M(.,t))+G(M(.,T))\right\} \tag{$P_{P,M}$}$$

Under the constraints $C_{P,M}$:

$$\begin{cases}\sum_{y\in S}M(y,t-1)P(y,x,t-1)=M(x,t) & (x,t)\in S\times\{1,\dots,T\}\\ \sum_{y\in S}P(x,y,t)=1 & (x,t)\in S\times\mathcal{T}\\ P(x,y,t)\geq 0 & (x,y,t)\in S\times S\times\mathcal{T}\\ M(x,0)=M_0(x) & x\in S\end{cases} \tag{$C_{P,M}$}$$

This problem is interesting because it is a potential formulation of (MFG'_d). We define h to be equal to f if $t<T$ and g if $t=T$. From the separation assumption (35) we set:

$$c_{xt}(P):=\sum_{y\in S}L(P(x,y,t),x,y,t),\quad K_2(M,x,t):=h(x,M(.,t))\quad\forall(x,t)\in S\times\mathcal{T}+1 \tag{40}$$

We recognize the cost we used in (MFG'_d) (we allow it to be dependent of time here). To fully understand the link between ($P_{P,M}$) and (MFG'_d) we derive the Lagrangian. For all $\mu(x,y,t)\geq 0$ we have:

$$\mathcal{L}(M,P,U,\lambda,\beta,\mu) = \sum_{t\in\mathcal{T}}\sum_{x,y\in S} M(x,t)L\left(P(x,y,t),x,y,t\right) + \sum_{t\in\mathcal{T}} F(M(.,t)) + G(M(.,T))$$
$$+ \sum_{t\in\mathcal{T}}\sum_{x\in S}\left(U(x,t+1)\left(\sum_{y\in S} M(y,t)P(y,x,t) - M(x,t+1)\right) + \lambda(x,t)\left(\sum_{y\in S} P(x,y,t) - 1\right)\right) \quad (41)$$
$$+ \sum_{x\in S}\beta(x)\left(M(x,0) - M_0(x)\right) - \sum_{t\in\mathcal{T}}\sum_{x,y\in S}\mu(x,y,t)P(x,y,t).$$

Then we have the following Lemma.

Lemma 2. *Let $(M,P,U,\lambda,\beta,\mu)$ a saddle point of (41). Then (U,M) is solution of (MFG'_d) and P is the optimal policy.*

Proof. Suppose $(M,P,U,\lambda,\beta,\mu)$ is a saddle point. Then (M,P) satisfies ($C_{P,M}$). Thus it remains to show that the Dynamic Programming equation (DPP) and the terminal boundary condition are satisfied. Since F and G are differentiable and (41) is convex in M, first order condition with respect to M

$$\frac{\partial\mathcal{L}}{\partial M(x,t)} = 0 \quad \forall t\in\mathcal{T} \quad (42)$$

gives

$$\sum_{y\in S}\left[L(P(x,y,t),x,y,t) + U(y,t+1)P(x,y,t)\right] + h(x,M(.,t)) - U(x,t) = 0. \quad (43)$$

Using equation (40) we derive

$$U(x,t) = \sum_{y\in S} P(x,y,t)\left[c_{xyt}\left(P(x,.,t),M(.,t)\right) + U(y,t+1)\right] \qquad \forall (t,x)\in S\times\mathcal{T} \quad (44)$$

Which is exactly equation (DPP). Using first order condition of M at time T

$$\frac{\partial\mathcal{L}}{\partial M(x,T)} = 0, \quad (45)$$

we get the boundary condition (33)

$$U(x,T) = g(x,M(.,T)). \quad (46)$$

Then we look at $P(x,.,t)\in\mathcal{P}(S)$ and derive the optimality condition it should satisfy:

$$P(x,y,t) \in \underset{P(x,.,t)\in\mathcal{P}(S)}{\operatorname{argmin}}\ M(x,t)\sum_{y\in S} L(P(x,y,t),x,y,t)) + M(x,t)\sum_{x\in S} U(y,t+1)P(x,y,t). \quad (47)$$

We can get rid of $M(x,t)$ and add $h(x,M(.,t))$ since it does not change the minimal argument. And if we use the definition (40) we get

$$P(x,.,t) \in \underset{P(x,.,t)\in\mathcal{P}(S)}{\operatorname{argmin}}\ c_{xt}(P) + h(x,M(.,t)) + \sum_{y\in S} P(x,y,t)U(y,t+1) \quad \forall (t,x)\in\mathcal{T}\times S, \quad (48)$$

which leads to

$$P(x,.,t) \in \underset{P(x,.,t)\in\mathcal{P}(S)}{\operatorname{argmin}} \sum_{y\in S} P(x,y,t)\left[c_{xyt}(P(x,.,t),M(.,t)) + U(y,t+1)\right] \quad \forall (t,x)\in\mathcal{T}\times S, \quad (49)$$

using the separable assumption (35). Then we understand that if $(M, P, U, \lambda, \beta, \mu)$ is a saddle point of the Lagrangian associated to the potential formulation, then (U, M) is solution of (MFG'_d) and P is the optimal policy.

□

This is very convenient since this formulation will allow us to derive gradient descent algorithms to solve the initial (MFG'_d) problem.

3.3 Convex formulation

We have motivated the study of problem ($P_{P,M}$) in the previous section. Unfortunately this problem is not convex. We want to show that the problem ($P_{P,M}$):

$$\min_{P,M} \left\{ \sum_{t\in\mathcal{T}} \sum_{x,y\in S} M(x,t) L\left(P(x,y,t), x, y, t\right) + \sum_{t\in\mathcal{T}} F(M(.,t)) + G(M(.,T)) \right\} \tag{$P_{P,M}$}$$

under the constraints $C_{P,M}$:

$$\begin{cases} \sum_{y\in S} M(y,t-1)P(y,x,t-1) = M(x,t) & (x,t) \in S \times \{1,\dots,T\} \\ \sum_{y\in S} P(x,y,t) = 1 & (x,t) \in S \times \mathcal{T} \\ P(x,y,t) \geq 0 & (x,y,t) \in S \times S \times \mathcal{T} \\ M(x,0) = M_0(x) & x \in S \end{cases} \tag{$C_{P,M}$}$$

Can be turned into the convex problem ($P_{W,M}$):

$$\min_{W,M} \left\{ \sum_{t\in\mathcal{T}} \sum_{x,y\in S} \tilde{L}^{\star\star}\left(W(x,y,t), M(x,t), x, y, t\right) + \sum_{t\in\mathcal{T}} F(M(.,t)) + G(M(.,T)) \right\} \tag{$P_{W,M}$}$$

under the constraints $C_{W,M}$:

$$\begin{cases} \sum_{y\in S} W(y,x,t) = M(x,t+1) & (x,t) \in S \times \mathcal{T} \\ \sum_{y\in S} W(x,y,t) = M(x,t) & (x,t) \in S \times \mathcal{T} \\ W(x,y,t) \geq 0 & (x,y,t) \in S \times S \times \mathcal{T} \\ M(x,0) = M_0(x) & x \in S \end{cases} \tag{$C_{W,M}$}$$

where we define:

$$\tilde{L}\left(W(x,y,t), M(x,t), x, y, t\right) := M(x,t) L\left(W(x,y,t)/M(x,t), x, y, t\right), \tag{50}$$

and we use the closure of the perspective function $\tilde{L}^{\star\star} = \overline{\text{conv}}(\tilde{L})$ (see Proposition 3). We denote $F(P)$ the feasible set and $Val(P)$ the value of a problem P. We define

$$\phi : (p,m) \mapsto (pm, m) \quad \forall p, m \geq 0, \tag{51}$$

to be a change of variable function. Notice that $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto (\mathbb{R}^+ \times \mathbb{R}^+_\star) \cup \{(0,0)\}$ is not injective since $\phi(p,0) = (0,0)$ for all $p \in \mathbb{R}^+$, but $\phi : (\mathbb{R}^+ \times \mathbb{R}^+_\star) \cup \{(0,0)\} \mapsto (\mathbb{R}^+ \times \mathbb{R}^+_\star) \cup \{(0,0)\}$ is bijective. Then we have the following proposition,

Proposition 1. $Val(P_{P,M}) = Val(P_{W,M})$.

Proof. Let $(P, M) \in F(P_{P,M})$. Now we use the change of variables

$$W(x, y, t), M(x, t)) = \phi(P(x, y, t), M(x, t)) \quad \forall (x, y, t) \in S \times S \times \mathcal{T}. \tag{52}$$

We have $((W, M) \in F(P_{W,M})$ and one directly check that the associated costs in ($P_{P,M}$) and ($P_{W,M}$) are equals (the non obvious case is $M(x, t) = 0$, where we have $\tilde{L}^{\star\star}(0, 0, x, y, t) = 0$ using Proposition 3).

Now let $(W, M) \in F(P_{W,M})$. If

$$(W(x, y, t), M(x, t)) \in \mathbb{R}^+ \times \{0\}, \quad \forall (x, y, t) \in S \times S \times \mathcal{T} \tag{53}$$

then $W(x, y, t) = 0$ using its positiveness and that $\sum_{y \in S} W(x, y, t) = 0$. Then

$$(W(x, y, t), M(x, t)) \in (\mathbb{R}^+ \times \mathbb{R}^+_\star) \cup \{(0, 0)\}, \quad \forall (x, y, t) \in S \times S \times \mathcal{T}. \tag{54}$$

Using the bijective change of variables

$$(P(x, y, t), M(x, t)) = \phi^{-1}(W(x, y, t), M(x, t)), \quad \forall (x, y, t) \in S \times S \times \mathcal{T} \tag{55}$$

and choosing P such that $\sum_{y \in S} P(x, y, t) = 1$ we have $(P, M) \in F(P_{P,M})$. One also directly check that the associated costs in ($P_{P,M}$) and ($P_{W,M}$) are equals. □

3.4 Dual Problem

We want to determine the dual of the following problem using the Fenchel-Rockafellar theorem. We have

$$\min_{(W,M)} \left\{ \sum_{t \in \mathcal{T}} \sum_{x,y \in S} \tilde{L}^{\star\star}\left(W(x, y, t), M(x, t), x, y, t\right) + \sum_{t \in \mathcal{T}} F(M(., t)) + G(M(., T)) \right\} \tag{$P_{W,M}$}$$

and in order to recover some well-known formulation in the next section (see [2] for example) we formulate the constraints $C_{W,M}$ in the following way:

$$\begin{cases} M(x, t+1) - M(x, t) + \sum_{y \in S} W(x, y, t) - \sum_{y \in S} W(y, x, t) = 0 & (x, t) \in S \times \mathcal{T} \\ \sum_{y \in S} W(x, y, t) = M(x, t) & (x, t) \in S \times \mathcal{T} \\ W(x, y, t) \geq 0 & (x, y, t) \in S \times S \times \mathcal{T} \\ M(x, 0) = M_0(x) & x \in S \end{cases} \tag{$C_{W,M}$}$$

The first constraint represents the evolution of the measure through the divergence of W at (x, t). Then we define

$$K := \{0\}_{\mathbb{R}^{S \times \mathcal{T}}} \times \{0\}_{\mathbb{R}^{S \times \mathcal{T}}} \times \mathbb{R}_+^{S \times S \times \mathcal{T}} \times \{M_0\}, \tag{56}$$

and the operator

$$\Lambda : \mathbb{R}^{S \times \mathcal{T}+1} \times \mathbb{R}^{S \times S \times \mathcal{T}} \mapsto \mathbb{R}^{S \times \mathcal{T}} \times \mathbb{R}^{S \times \mathcal{T}} \times \mathbb{R}^{S \times S \times \mathcal{T}} \times \mathbb{R}^{S}. \tag{57}$$

$$\Lambda(M, W) = \begin{pmatrix} M(x, t+1) - M(x, t) + \sum_{y \in S} W(x, y, t) - \sum_{y \in S} W(y, x, t) \\ \sum_{y \in S} W(x, y, t) - M(x, t) \\ W(x, y, t) \\ M(x, 0) \end{pmatrix} \tag{58}$$

Then we define for any $(M, W) \in \mathbb{R}^{S \times \mathcal{T}+1} \times \mathbb{R}^{S \times S \times \mathcal{T}}$

$$\mathcal{A}(M,W) := \sum_{t\in\mathcal{T}}\sum_{x,y\in S}\tilde{L}^{\star\star}\left(W(x,y,t),M(x,t),x,y,t\right) + \sum_{t\in\mathcal{T}}F(M(.,t)) + G(M(.,T)) + \chi_{\{\Lambda(M,W)\in K\}} \quad (59)$$

such that the problem ($P_{W,M}$) under ($C_{W,M}$) is now

$$\inf_{(M,W)} \mathcal{A}(M,W) \quad (60)$$

Now we define for any $(U,\lambda,\mu,\beta) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$ the set

$$C := \{U(.,T-1) = 0, \mu \geq 0, G^T = G(M(.,T))\}, \quad (61)$$

and for notation purpose we set

$$\Delta_t U(x,t) := U(x,t-1)\mathbb{1}_{t>0} - U(x,t)\mathbb{1}_{t<T} \quad (62)$$

$$\Delta U(x,y,t) := U(y,t) - U(x,t) \quad (63)$$

Then we define the following quantity

$$\mathcal{B}(U,\lambda,\mu,\beta) := -\chi_C - \sum_{x\in S} M_0(x)(-\beta(x) - G^T) \quad (64)$$

$$-\sum_{t\in\mathcal{T}} F^\star\left(\Delta_t U(.,t) - \lambda(.,t)\mathbb{1}_{t<T} + \beta(.)\mathbb{1}_{t=0} + \sum_y L^\star(\Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t), ., y, t)\right)$$

Lemma 3. *Under the assumption $M_0 > 0$ we have*

$$\inf_{(M,W)} \mathcal{A}(M,W) = \max_{(U,\lambda,\mu,\beta)} \mathcal{B}(U,\lambda,\mu,\beta) \quad (65)$$

Proof. We set for any $(M,W) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$

$$\mathcal{F}(M,W) := \sum_{t\in\mathcal{T}}\sum_{x,y\in S}\tilde{L}^{\star\star}\left(W(x,y,t),M(x,t),x,y,t\right) + \sum_{t\in\mathcal{T}}F(M(.,t)), \quad (66)$$

$\mathcal{F}$ is l.s.c and convex. We also define for any $B \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$

$$\mathcal{G}(B) := G(M(.,T)) + \chi_{\{B\in K\}}. \quad (67)$$

Then we have

$$\mathcal{G}(\Lambda(M,W)) := G(M(.,T)) + \chi_{\{\Lambda(M,W)\in K\}}. \quad (68)$$

We can compute the Fenchel transform of $\mathcal{F}$ (see Appendix B for details) for any $(a,b) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$ which is

$$\mathcal{F}^\star(a,b) = \sum_{t\in\mathcal{T}} F^\star\left(a(.,t) + \sum_y L^\star(b(x,y,t), ., y, t)\right) + \chi_{\{a(.,T)=0\}}, \quad (69)$$

and we can compute the Fenchel transform of $\mathcal{G}$ (see Appendix B for details) for any $B = (b_1,b_2,b_3,b_4) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$ which is

$$\mathcal{G}^\star(B) = \chi_{\{b_3\leq 0, G^T = G(M(.,T))\}} + \sum_{x\in S} M_0(x)(b_4(x) - G^T) \quad (70)$$

Now one can define the adjoint operator

$$\Lambda^\star : \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S} \mapsto \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}, \tag{71}$$

where

$$\Lambda^\star(U,\lambda,\mu,\beta) := \begin{pmatrix} \Delta_t U(x,t) - \lambda(x,t)\mathbb{1}_{t<T} + \beta(x)\mathbb{1}_{t=0} \\ \Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t) \end{pmatrix} \tag{72}$$

Now we check the qualification condition. Notice that the application $(M,W) \mapsto \Lambda(M,W)$ is surjective. Let us consider

$$W(x,y,t) = M_0(x)/card(S) \quad \forall (x,y,t) \in S\times S\times\mathcal{T} \tag{73}$$

Then $W > 0$ by hypothesis on M_0, which means that it satisfies strictly the inequality constraint. We want to show that the constraints are qualified thus we consider a perturbation $y \in B(0,\epsilon)$, which can be written $y = (y_1^t, y_2^t, y_3^t, y_4)_t$ then if we consider the measure

$$\hat{M}(.,0) = M_0 - y_4 \tag{74}$$

$$\hat{M}(.,t) = M_0 - y_4 - \sum_{i=0}^{t-1}(y_2^i + y_3^i) \quad \forall t \in \{1,\dots,T\} \tag{75}$$

and W such that

$$\hat{W}(.,x,0) = (M_0 - y_4 - y_2^0)/card(S) \quad \forall x \in S \tag{76}$$

$$\hat{W}(.,x,t) = (M_0 - y_4 - \sum_{i=0}^{t-1}(y_3^i + y_2^i) - y_2^t)/card(S) \quad \forall (x,t) \in S\times\{1,\dots,T-1\} \tag{77}$$

$(\hat{M},\hat{W})$ are admissible in the perturbed problem. Because of the recursive form of the constraint we see that the perturbation at a given time will have an impact on later constraints. Nevertheless for ϵ small enough, $\hat{W}$ still satisfies the inequality constraint strictly, thus the problem is qualified. Finally one can use the Fenchel-Rockafellar theorem:

$$\inf_{(M,W)}\{\mathcal{F}(M,W) + \mathcal{G}(\Lambda(M,W))\} = \max_{(U,\lambda,\mu,\beta)}\{-\mathcal{F}^\star(\Lambda^\star(U,\lambda,\mu,\beta)) - \mathcal{G}^\star(-(U,\lambda,\mu,\beta))\}. \tag{78}$$

with

$$-\mathcal{F}^\star(\Lambda^\star(U,\lambda,\mu,\beta)) - \mathcal{G}^\star(-(U,\lambda,\mu,\beta)) = -\chi_C - \sum_{x\in S} M_0(x)(-\beta(x) - G^T) \tag{79}$$

$$-\sum_{t\in\mathcal{T}} F^\star\left(\Delta_t U(.,t) - \lambda(.,t)\mathbb{1}_{t<T} + \beta(.)\mathbb{1}_{t=0} + \sum_y L^\star(\Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t), ., y, t)\right). \tag{80}$$

□

3.5 Bi-dual Problem

In this section we compare our framework to [2]. Using the same notation as before we have the following lemma

Lemma 4. *If F is coercive then*

$$\sup_{(U,\lambda,\mu,\beta)} \mathcal{B}(U,\lambda,\mu,\beta) = \min_{(M,W)} \mathcal{A}(M,W) \tag{81}$$

Proof. For any $(b_1, b_2, b_3, b_4) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$ we set

$$\mathcal{F}(B) := -\sum_{x\in S} M_0(x)(b_4(x) + G(M(.,T))) + \chi_{\{b_3(x)\leq 0\}}, \tag{82}$$

$\mathcal{F}$ is l.s.c and convex. We also define for any $(a, b) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$

$$\mathcal{G}(a,b) := \sum_{t\in\mathcal{T}} F^\star \left(a(.,t) + \sum_y L^\star(b(x,y,t),.,y,t) \right) + \chi_{\{a(.,T)=0\}}. \tag{83}$$

we define the operator

$$\Gamma : \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S} \mapsto \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}, \tag{84}$$

where for any $(U, \lambda, \mu, \beta) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$

$$\Gamma(U, \lambda, \mu, \beta) = \begin{pmatrix} \Delta_t U(x,t) - \lambda(x,t)\mathbb{1}_{t<T} + \beta(x)\mathbb{1}_{t=0} \\ \Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t) \end{pmatrix} \tag{85}$$

Then we have

$$\begin{aligned} &\mathcal{G}(\Gamma(U,\lambda,\mu,\beta)) = \\ &\sum_{t\in\mathcal{T}} F^\star \left(\Delta_t U(.,t) - \lambda(.,t)\mathbb{1}_{t<T} + \beta(.)\mathbb{1}_{t=0} + \sum_y L^\star(\Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t),.,y,t) \right). \end{aligned} \tag{86}$$

Now one can define the adjoint operator

$$\Gamma^\star : \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}} \mapsto \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}, \tag{87}$$

$$\Gamma^\star(M,W) = \begin{pmatrix} M(x,t+1) - M(x,t) + \sum_{y\in S} W(x,y,t) - \sum_{y\in S} W(y,x,t) \\ \sum_{y\in S} W(x,y,t) - M(x,t) \\ W(x,y,t) \\ M(x,0) \end{pmatrix} \tag{88}$$

We can compute the Fenchel transform of $\mathcal{F}$ for any $A \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$ which is

$$\mathcal{F}^\star(A) = G(M(.,T)) + \chi_{\{A\in K\}}, \tag{89}$$

and we can compute the Fenchel transform of $\mathcal{G}$ for any $(M, W) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$

$$\mathcal{G}^\star(M,W) = \sum_{t\in\mathcal{T}} \sum_{x,y\in S} \tilde{L}^{\star\star}\left(W(x,y,t), M(x,t), x, y, t\right) + \sum_{t\in\mathcal{T}} F(M(.,t)), \tag{90}$$

The qualification condition is:

$$0 \in \text{int}(\text{dom}(\mathcal{G}) - \Gamma\text{dom}(\mathcal{F})), \tag{91}$$

Using that F is proper and coercive (thus $F^\star$ is finite)

$$\text{dom}(\mathcal{G}) = (\mathbb{R}^{S\times\mathcal{T}} \times 0) \times \mathbb{R}^{S\times S\times\mathcal{T}}, \quad \text{dom}(\mathcal{F}) = \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}_-^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}. \tag{92}$$

In addition we have $\Gamma\text{dom}(\mathcal{F}) = \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$ thus the problem is qualified. Finally one can use the Fenchel-Rockafellar theorem:

$$\inf_{(U,\lambda,\mu,\beta)} \{\mathcal{F}(U,\lambda,\mu,\beta) + \mathcal{G}(\Gamma(U,\lambda,\mu,\beta))\} = \max_{(M,W)} \{-\mathcal{F}^\star(\Gamma^\star(M,W)) - \mathcal{G}^\star(-(M,W))\}. \tag{93}$$

Where we have

$$-\mathcal{F}^\star(\Gamma^\star(M,W)) - \mathcal{G}^\star(-(M,W)) = -\sum_{t\in\mathcal{T}}\sum_{x,y\in S} \tilde{L}^{\star\star}\left(W(x,y,t), M(x,t), x, y, t\right) \tag{94}$$

$$-\sum_{t\in\mathcal{T}} F(M(.,t)) + G(M(.,T)) - \chi_{\{\Gamma^\star(M,W)\in K\}}, \tag{95}$$

□

In [2] there is a lower number of dual variables. Nevertheless we find a formulation which is close, with an operator Γ which gives the dynamics of the value function.

4 Finite MFG of Controls

Mean Field Game of Control is a Mean Field Game where the cost the agents try to minimize depends in addition to the control of the others. More specifically a mean field game of control consists in the joint distribution of the agents and their instantaneous control. A way to feel it is to imagine a trading game where everyone buy or sell. Then the cost of the agent depends on the price of the good which is determined by the market impact of all positions and controls. So in this example a player plays in front of the anticipated statistical distribution of others and their instantaneous strategies. For a continuous example of this kind of games, one can refer to [5] for an application to electricity trading and to [3] for an application to the financial market. We define

$$U(x,t) = \sum_{y\in S} \hat{P}(x,y,t)\left[\phi(t)L(x,y) + U(y,t+1)\right] \qquad \forall (t,x)\in\mathcal{T}\times S \tag{96}$$

$$M(x,t) = \sum_{y\in S} M(y,t-1)\hat{P}(x,y,t-1) \qquad \forall (t,x)\in\mathcal{T}+1\times S \tag{97}$$

$$U(x,T) = g(x,M(.,T)), \quad M(x,0) = M_0(x) \qquad \forall x\in S \tag{98}$$

where $\hat{P}\in\mathcal{K}_{S,\tau}$ satisfies:

$$\hat{P}(x,.,t) \in \operatorname*{argmin}_{p\in\mathcal{P}(S)} \sum_{y\in S} p(y)\left[\phi(t)L(x,y) + U(y,t+1)\right] \quad (t,x)\in\mathcal{T}\times S \tag{$MFGC_d$}$$

Here $\phi(t)$ can represent a price for example. As explained in the motivation for this section it can depend on the distribution of controls:

$$\phi(t) = \Phi\left(\sum_{x,y\in S} M(x,t)P(x,y,t)L(x,y)\right) \tag{99}$$

4.1 Finite MFG of Control with potential structure

Here we motivate and explain how we can derive a potential structure for the previous MFG of control. Let Ψ be a convex l.s.c proper function. Suppose we have $\Phi = \Psi'$ and consider:

$$\min_{P,M}\left\{\sum_{t\in\mathcal{T}} \Psi\left(\sum_{x,y\in S} M(x,t)P(x,y,t)L(x,y)\right) + G(M(.,T))\right\} \tag{$Q_{P,M}$}$$

Under the constraints:

$$\begin{cases} \sum_{y\in S} M(y,t-1)P(y,x,t-1) = M(x,t) \\ \sum_{y\in S} P(x,y,t) = 1 \\ P(x,y,t) \geq 0, \quad M(x,0) = M_0(x) \end{cases} \tag{$K_{P,M}$}$$

One can form the Lagrangian and derive first order conditions. The Lagrangian is:

$$\mathcal{L} = \sum_{t\in\mathcal{T}} \left[\Psi\left(\sum_{x,y\in S} M(x,t)P(x,y,t)L(x,y) \right) + \sum_{x\in S} U(x,t) \left(\sum_{y\in S} M(y,t-1)P(y,x,t-1) - M(x,t) \right) \right. \\ \left. + \sum_{x\in S} \lambda(x,t) \left(\sum_{y\in S} P(x,y,t) - 1 \right) - \sum_{x,y\in S} \mu(x,y,t)P(x,y,t) \right] + \sum_{x\in S} \beta(x)\left(M(x,0) - M_0(x)\right) + G(M(.,T)) \tag{100}$$

Lemma 5. *Suppose $(M, P, U, \lambda, \mu, \beta)$ is a saddle point in (100). Then (U, M) is solution in ($MFGC_d$) and P is in the minimal argument.*

Proof. Suppose $(M, P, U, \lambda, \mu, \beta)$ is a saddle point, then (M, P) satisfy ($K_{P,M}$). Thus it remains to check if the Dynamic Programming equation holds and if we recover the boundary condition. Since Ψ is differentiable and convex, first order condition over M

$$\frac{\partial \mathcal{L}}{\partial M(x,t)} = 0, \tag{101}$$

leads for all $t \in \mathcal{T}$:

$$U(x,t) = \sum_{y\in S} P(x,y,t)\left(\phi(t)L(x,y) + U(y,t+1)\right). \tag{102}$$

and at time $t = T$ first order condition over M leads

$$U(x,T) = g(x, M(.,T)) \tag{103}$$

And then we also have:

$$\frac{\partial \mathcal{L}}{\partial P(x,y,t)} = M(x,t)\left(\phi(t)L(x,y) + U(y,t+1)\right) + \lambda(x,t) - \mu(x,y,t) \tag{104}$$

If $P(x,y,t) > 0$ then $\mu(x,y,t) = 0$ and

$$M(x,t)\left(\phi(t)L(x,y) + U(y,t+1)\right) + \lambda(x,t) = 0 \tag{105}$$

which means for all y such that $P(x,y,t) > 0$ the quantity $M(x,t)\left(\phi(t)L(x,y) + U(y,t+1)\right)$ just depends on x, and is equal to $-\lambda(x,t)$, and for other y we have $M(x,t)\left(\phi(t)L(x,y) + U(y,t+1)\right) \geq -\lambda(x,t)$. So we deduce:

$$U(x,t) = \min_{y\in S}\left(\phi(t)L(x,y) + U(y,t+1)\right) \tag{106}$$

which leads to

$$U(x,t) = \min_{p\in\mathcal{P}(S)} \sum_{y\in S} p(y) \left(\phi(t)L(x,y) + \sum_{y\in S} U(y,t+1) \right) \tag{107}$$

Thus we recover that $P(x,y,t)$ satisfies

$$\hat{P}(x,.,t) \in \operatorname*{argmin}_{p\in\mathcal{P}(S)} \sum_{y\in S} p(y)\left[\phi(t)L(x,y) + U(y,t+1)\right] \quad (t,x)\in\mathcal{T}\times S \tag{108}$$

□

4.2 Convex formulation

We want to have a convex formulation for the following problem:

$$\min_{P,M} \left\{ \sum_{t\in\mathcal{T}} \Psi \left(\sum_{x,y\in S} M(x,t)P(x,y,t)L(x,y) \right) + G(M(.,T)) \right\} \tag{$Q_{P,M}$}$$

Under the constraints:

$$\begin{cases} \sum_{y\in S} M(y,t-1)P(y,x,t-1) = M(x,t) & (x,t)\in S\times\{1,\dots,T\} \\ \sum_{y\in S} P(x,y,t) = 1 & (x,t)\in S\times\mathcal{T} \\ P(x,y,t) \geq 0 & (x,y,t)\in S\times S\times\mathcal{T} \\ M(x,0) = M_0(x) & x\in S \end{cases} \tag{$K_{P,M}$}$$

As before, we define the following convex problem and we will show that the value are the same:

$$\min_{W,M} \left\{ \sum_{t\in\mathcal{T}} \Psi \left(\sum_{x,y\in S} W(x,y,t)L(x,y) \right) + G(M(.,T)) \right\} \tag{$Q_{W,M}$}$$

Under the constraints:

$$\begin{cases} \sum_{y\in S} W(y,x,t) = M(x,t+1) & (x,t)\in S\times\mathcal{T} \\ \sum_{y\in S} W(x,y,t) = M(x,t) & (x,t)\in S\times\mathcal{T} \\ W(x,y,t) \geq 0 & (x,y,t)\in S\times S\times\mathcal{T} \\ M(x,0) = M_0(x) & x\in S \end{cases} \tag{$K_{W,M}$}$$

Then we have the following proposition:

Proposition 2. $Val(Q_{P,M}) = Val(Q_{W,M})$.

Proof. Let $(P,M)\in F(Q_{P,M})$. Now we use the change of variables

$$(W(x,y,t), M(x,t)) = \phi(P(x,y,t), M(x,t)) \quad \forall (x,y,t)\in S\times S\times\mathcal{T}. \tag{109}$$

We have $(W,M)\in F(P_{W,M})$ and one directly check that the associated costs in ($P_{P,M}$) and ($P_{W,M}$) are equals. Now let $(W,M)\in F(Q_{W,M})$. If there exists $(x,y,t)\in S\times S\times\mathcal{T}$ such that

$$(W(x,y,t), M(x,t)) \in \mathbb{R}^+ \times \{0\} \tag{110}$$

then $W(x,y,t)=0$ using its positiveness and that $\sum_{y\in S} W(x,y,t) = 0$. Then

$$(W(x,y,t), M(x,t)) \in (\mathbb{R}^+ \times \mathbb{R}^+_\star) \cup \{(0,0)\}. \tag{111}$$

Using the bijective change of variables

$$(P(x,y,t), M(x,t)) = \phi^{-1}(W(x,y,t), M(x,t)) \tag{112}$$

and choosing P such that $\sum_{y\in S} P(x,y,t) = 1$ we have $(P,M) \in F(P_{P,M})$. One also directly check that the associated costs in ($Q_{P,M}$) and ($Q_{W,M}$) are equals. □

4.3 Dual Problem

We want to determine the dual of the following problem using the Fenchel- Rockafellar theorem. We have

$$\min_{W,M} \left\{ \sum_{t\in\mathcal{T}} \Psi \left(\sum_{x,y\in S} W(x,y,t)L(x,y) \right) + G(M(.,T)) \right\} \tag{$Q_{W,M}$}$$

To be consistent with the dual derivation of MFG in section (3.4) we formulate the constraints in the following way:

$$\begin{cases} M(x,t+1) - M(x,t) + \sum_{y\in S} W(x,y,t) - \sum_{y\in S} W(y,x,t) = 0 & (x,t) \in S\times\mathcal{T} \\ \sum_{y\in S} W(x,y,t) = M(x,t) & (x,t) \in S\times\mathcal{T} \\ W(x,y,t) \geq 0 & (x,y,t) \in S\times S\times\mathcal{T} \\ M(x,0) = M_0(x) & x\in S \end{cases} \tag{$K_{W,M}$}$$

Then we define

$$K := \{0\}_{\mathbb{R}^{S\times\mathcal{T}}} \times \{0\}_{\mathbb{R}^{S\times\mathcal{T}}} \times \mathbb{R}_+^{S\times S\times\mathcal{T}} \times \{M_0\}, \tag{113}$$

and the operator

$$\Lambda : \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}} \mapsto \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}. \tag{114}$$

$$\Lambda(M,W) = \begin{pmatrix} M(x,t+1) - M(x,t) + \sum_{y\in S} W(x,y,t) - \sum_{y\in S} W(y,x,t) \\ \sum_{y\in S} W(x,y,t) - M(x,t) \\ W(x,y,t) \\ M(x,0) \end{pmatrix} \tag{115}$$

Then we define for any $(M,W) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$

$$\mathcal{A}(M,W) := \sum_{t\in\mathcal{T}} \Psi \left(\sum_{x,y\in S} W(x,y,t)L(x,y) \right) + G(M(.,T)) + \chi_{\{\Lambda(M,W)\in K\}} \tag{116}$$

such that the problem ($Q_{W,M}$) under ($K_{W,M}$) is now

$$\inf_{(M,W)} \mathcal{A}(M,W) \tag{117}$$

We define A to be the set of $(U,\lambda,\mu,\beta) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$ such that

$$\begin{cases} \Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t) = \alpha(t)L(x,y) & (x,y,t) \in S\times S\times\mathcal{T} \\ \Delta_t U(x,t) - \lambda(x,t)\mathbb{1}_{t<T} + \beta(x)\mathbb{1}_{t=0} = 0 & (x,t) \in S\times\{0,\dots,T\} \\ \mu(x,y,t) \geq 0 & (x,y,t) \in S\times S\times\mathcal{T} \\ G^T = G(M(.,T)) \end{cases} \tag{118}$$

and the following quantity

$$\mathcal{B}(U,\lambda,\mu,\beta) := -\sum_{t\in\mathcal{T}} \Psi^\star(\alpha(t)) - \sum_{x\in S} M_0(x)(-\beta(x) - G^T) - \chi_A. \tag{119}$$

Then we have the following Lemma.

Lemma 6. *Under the assumption $M_0 > 0$ we have*

$$\inf_{(M,W)} \mathcal{A}(M,W) = \max_{(U,\lambda,\mu,\beta)} \mathcal{B}(U,\lambda,\mu,\beta) \tag{120}$$

Proof. For any $(W,M) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}}$ we set

$$\mathcal{F}(M,W) := \sum_{t\in\mathcal{T}} \Psi\left(\sum_{x,y\in S} W(x,y,t)L(x,y)\right), \tag{121}$$

$\mathcal{F}$ is l.s.c and convex. We also define for any $B \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^S$

$$\mathcal{G}(B) := G(M(.,T)) + \chi_{\{B\in K\}}. \tag{122}$$

Then we have

$$\mathcal{G}(\Lambda(M,W)) := G(M(.,T)) + \chi_{\{\Lambda(M,W)\in K\}}. \tag{123}$$

We can compute the Fenchel transform of $\mathcal{F}$ for any $(a,b) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$ which is

$$\mathcal{F}^\star(a,b) = \sum_{t\in\mathcal{T}} \Psi^\star(\alpha(t)) + \chi_{\{a=0, b(.,.,t)=\alpha(t)L\}}. \tag{124}$$

Where $\alpha(t) \in \mathbb{R}^\star$. We define $\mathcal{G}$

$$\mathcal{G}(\Lambda(M,W)) := G(M(.,T)) + \chi_{\{\Lambda(M,W)\in K\}}, \tag{125}$$

and for any $(b_1,b_2,b_3,b_4) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^S$ we recall that the Fenchel transform of $\mathcal{G}$ is

$$\mathcal{G}^\star(B) = \chi_{\{b_3\le 0, G^T = G(M(.,T))\}} + \sum_{x\in S} M_0(x)(b_4(x) - G^T) \tag{126}$$

Now we check the qualification condition (same argument as in the MFG case). Notice that the application $(M,W) \mapsto \Lambda(M,W)$ is surjective. Let us consider

$$W(x,y,t) = M_0(x)/card(S) \quad \forall (x,y,t) \in S\times S\times\mathcal{T} \tag{127}$$

Then $W > 0$ by hypothesis on M_0, which means that it satisfies strictly the inequality constraint. We want to show that the constraints are qualified thus we consider a perturbation $y \in B(0,\epsilon)$, which can be written $y = (y_1^t, y_2^t, y_3^t, y_4)_t$ then if we consider the measure

$$\hat{M}(.,0) = M_0 - y_4 \tag{128}$$

$$\hat{M}(.,t) = M_0 - y_4 - \sum_{i=0}^{t-1}(y_2^i + y_3^i) \quad \forall t \in \{1,\dots,T\} \tag{129}$$

and W such that

$$\hat{W}(.,x,0) = (M_0 - y_4 - y_2^0)/card(S) \quad \forall x \in S \tag{130}$$

$$\hat{W}(.,x,t) = (M_0 - y_4 - \sum_{i=0}^{t-1}(y_3^i + y_2^i) - y_2^t)/card(S) \quad \forall (x,t) \in S \times \{1,\ldots,T-1\} \tag{131}$$

$(\hat{M},\hat{W})$ are admissible in the perturbed problem. Because of the recursive form of the constraint we see that the perturbation at a given time will have an impact on future constraints. Nevertheless for ϵ small enough, $\hat{W}$ still satisfies the inequality constraint strictly, thus the problem is qualified. So we use the Fenchel-Rockafellar theorem:

$$\inf_{W,M}\{\mathcal{F}(W,M) + \mathcal{G}(\Lambda(W,M))\} = \max_{(U,\lambda,\mu,\beta)}\{-\mathcal{F}^\star(\Lambda^\star(U,\lambda,\mu,\beta)) - \mathcal{G}^\star(-(U,\lambda,\mu,\beta))\}. \tag{132}$$

Then if we denote A to be the set of (U,λ,μ,β) such that

$$\begin{cases} \Delta_t U(x,t) - \lambda(x,t)\mathbb{1}_{t<T} + \beta(x)\mathbb{1}_{t=0} = 0 & (x,t) \in S \times \{0,\ldots,T\} \\ \Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t) = \alpha(t)L(x,y) & (x,y,t) \in S \times S \times \mathcal{T} \\ \mu(x,y,t) \geq 0 & (x,y,t) \in S \times S \times \mathcal{T} \\ G^T = G(M(.,T)) \end{cases} \tag{133}$$

we can express

$$-\mathcal{F}^\star(\Lambda^\star(U,\lambda,\mu,\beta)) - \mathcal{G}^\star(-(U,\lambda,\mu,\beta)) = -\sum_{t\in\mathcal{T}}\Psi^\star(\alpha(t)) - \sum_{x\in S} M_0(x)(-\beta(x) - G^T) - \chi_A. \tag{134}$$

□

4.4 Bi-dual problem

In this section we derive a duality result where the dynamics of the value function is explicit as in the previous section. Using the same notation as before we have the following lemma

Lemma 7. *If Ψ is coercive we have*

$$\sup_{(U,\lambda,\mu,\beta)} \mathcal{B}(U,\lambda,\mu,\beta) = \min_{(M,W)} \mathcal{A}(M,W) \tag{135}$$

Proof. For any $(b_1,b_2,b_3,b_4) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^S$ we set

$$\mathcal{F}(B) := -\sum_{x\in S} M_0(x)(b_4(x) + G(M(.,T))) + \chi_{\{b_3(x)\leq 0\}}, \tag{136}$$

$\mathcal{F}$ is l.s.c and convex. We also define for any $(a,b) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$

$$\mathcal{G}(a,b) := \sum_{t\in\mathcal{T}} \Psi^\star(\alpha(t)) + \chi_{\{a=0, b(.,.,t)=\alpha(t)L\}}. \tag{137}$$

we define the operator

$$\Gamma : \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^S \mapsto \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}, \tag{138}$$

where for any $(U,\lambda,\mu,\beta) \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^S$

$$\Gamma(U,\lambda,\mu,\beta) = \begin{pmatrix} \Delta_t U(x,t) - \lambda(x,t)\mathbb{1}_{t<T} + \beta(x)\mathbb{1}_{t=0} \\ \Delta U(x,y,t) + \lambda(x,t) + \mu(x,y,t) \end{pmatrix} \tag{139}$$

Then we have

$$\mathcal{G}(\Gamma(U,\lambda,\mu,\beta)) = \sum_{t\in\mathcal{T}} \Psi^\star(\alpha(t)) + \chi_A. \tag{140}$$

Now one can define the adjoint operator

$$\Gamma^\star : \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}} \mapsto \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}, \tag{141}$$

$$\Gamma^\star(M,W) = \begin{pmatrix} \sum_{y\in S} M(x,t+1) - M(x,t) + \sum_{y\in S} W(x,y,t) - \sum_{y\in S} W(y,x,t) \\ \sum_{y\in S} W(x,y,t) - M(x,t) \\ W(x,y,t) \\ M(x,0) \end{pmatrix} \tag{142}$$

We can compute the Fenchel transform of $\mathcal{F}$ for any $A \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}$ which is

$$\mathcal{F}^\star(A) = G(M(.,T)) + \chi_{\{A\in K\}}, \tag{143}$$

and we can compute the Fenchel transform of $\mathcal{G}$ for any $(M,W) \in \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$

$$\mathcal{G}^\star(M,W) = \sum_{t\in\mathcal{T}} \Psi\left(\sum_{x,y\in S} W(x,y,t)L(x,y)\right), \tag{144}$$

The qualification condition is:

$$0 \in \operatorname{int}(\operatorname{dom}(\mathcal{G}) - \Gamma\operatorname{dom}(\mathcal{F})), \tag{145}$$

using that Ψ is coercive and proper we have (thus $\Psi^\star$ is bounded)

$$\operatorname{dom}(\mathcal{G}) = (\mathbb{R}^{S\times\mathcal{T}} \times 0) \times \mathbb{R}^{S\times S\times\mathcal{T}}, \quad \operatorname{dom}(\mathcal{F}) = \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}_-^{S\times S\times\mathcal{T}} \times \mathbb{R}^{S}. \tag{146}$$

In addition we have $\Gamma\operatorname{dom}(\mathcal{F}) = \mathbb{R}^{S\times\mathcal{T}+1} \times \mathbb{R}^{S\times S\times\mathcal{T}}$ thus the problem is qualified. Finally one can use the Fenchel-Rockafellar theorem:

$$\inf_{(U,\lambda,\mu,\beta)} \{\mathcal{F}(U,\lambda,\mu,\beta) + \mathcal{G}(\Gamma(U,\lambda,\mu,\beta))\} = \max_{(M,W)} \{-\mathcal{F}^\star(\Gamma^\star(M,W)) - \mathcal{G}^\star(-(M,W))\}. \tag{147}$$

Where we have

$$-\mathcal{F}^\star(\Gamma^\star(M,W)) - \mathcal{G}^\star(-(M,W)) = -\sum_{t\in\mathcal{T}} \Psi\left(\sum_{x,y\in S} W(x,y,t)L(x,y)\right) - G(M(.,T)) - \chi_{\{\Gamma^\star(W,M)\in K\}} \tag{148}$$

□

5 Fictitious Play

The second part of this thesis deals with Fictitious Play. Fictitious Play is a learning procedure in game theory. The goal of this method is to find Nash equilibrium by iterated best responses. We study the convergence problem of the sequence of $P_n \in \mathcal{K}_{S,\mathcal{T}}$ and $M_n \in \mathcal{P}(S)$ defined by:

$$P_n := \underset{P \in \mathcal{K}_{S,\mathcal{T}}}{\operatorname{argmin}} J_{\bar{M}_n}(P) \tag{149}$$

$$M_{n+1} := M_{P_n}^{M_0} \tag{150}$$

$$\bar{M}_{n+1} := \frac{n}{n+1}\bar{M}_n + \frac{1}{n+1}M_{n+1} \tag{151}$$

The fictitious play procedure goes as follow: We define the optimal transition kernel by choosing P_n to be the minimal argument of the cost J when the statistical distribution of players is given by M_n (equation (149)). Then one update the sequence M_n starting from M_0 using the Fokker-Planck equation and the previous optimal kernel P_n (equation (150)). Finally one computes the average of best-responses $\bar{M}_{n+1}$ which at the limit is the expected best-response of the Mean Field Game (equation (151)).

5.1 Generalized Fictitious Play

This part and the following one comes from [7]. Let $\mathcal{X}, \mathcal{Y}$ be two Polish spaces. We denote by $\mathcal{P}(\mathcal{X})$ the set of Borel probability measures on $\mathcal{X}$. We define d to be the metric on $\mathcal{X}$ and we denote $\mathcal{P}_p(\mathcal{X})$ the set of Borel probability measures μ such that $\int_{\mathcal{X}} d(x,x_0)^p d\mu(x) < +\infty$. Now let $\mathcal{C} \subseteq \mathcal{X}$ be a compact set and define $F : \mathcal{C} \times \mathcal{P}(\mathcal{X}) \mapsto \mathbb{R}$ a continuous function. Given $n \geq 1$, $x_1 \in \mathcal{C}$ and $\eta_1 := \delta_{x_1}$ the dirac mass at x_1 we define:

$$x_{n+1} \in \underset{x \in \mathcal{C}}{\operatorname{argmin}} F(x, \eta_n) \tag{152}$$

$$\eta_{n+1} = \frac{1}{n+1}\sum_{t=1}^{n+1} \delta_{x_t} = \frac{n}{n+1}\eta_n + \frac{1}{n+1}\delta_{x_{n+1}} \tag{153}$$

And we consider now the convergence problem of the sequence (η_n) to some limit $\bar{\eta} \in \mathcal{P}(\mathcal{C})$, satisfying:

$$supp(\bar{\eta}) \subseteq \underset{x \in \mathcal{C}}{\operatorname{argmin}} F(x, \bar{\eta}) \tag{154}$$

Such measure $\bar{\eta}$ is called an equilibrium of the game. Indeed an equivalent statement is:

$$\int_{\mathcal{C}} F(x,\bar{\eta})\mathrm{d}\bar{\eta}(x) = \inf_{\eta \in \mathcal{P}(\mathcal{C})} \int_{\mathcal{C}} F(x,\bar{\eta})\mathrm{d}\eta(x) \tag{155}$$

A first result is the convergence of η_n under a monotonicity and unique minimizer condition for F.

Definition 3. (Monotonicity) *We say that F is monotone if*

$$\int_{\mathcal{C}} (F(x,\eta_1) - F(x,\eta_2))\mathrm{d}(\eta_1 - \eta_2)(x) \geq 0, \quad \forall \eta_1, \eta_2 \in \mathcal{P}(\mathcal{C}), \quad \eta_1 \neq \eta_2 \tag{156}$$

and F is strictly monotone if the inequality holds strictly.

Definition 4. (Unique minimizer condition) *We say that F satisfies the unique minimizer condition if for any $\eta \in \mathcal{P}(\mathcal{C})$ the optimization problem $\inf_{x \in \mathcal{C}} F(x,\eta)$ admits a unique solution.*

Theorem 1. *Assume that*

- *F is monotone and satisfies the unique minimizer condition*
- *F is Lipschitz when $\mathcal{P}(\mathcal{C})$ is endowed with the distance d_1 and there exists $C > 0$ such that:*

$$|F(x_1,\eta_1) - F(x_1,\eta_2) + F(x_2,\eta_1) - F(x_2,\eta_2)| \leq C|x_1 - x_2| d_1(\eta_1,\eta_2) \tag{157}$$

for all $x_1, x_2 \in \mathcal{C}$, $\eta_1, \eta_2 \in \mathcal{P}(\mathcal{C})$.

Then there exists $\bar{x} \in \mathcal{C}$ such that $\bar{\eta} = \delta_{\bar{x}}$ is unique equilibrium and $(\bar{x}, \delta_{\bar{x}})$ is the limit of the sequence (x_n, η_n) defined above.

The proof of the theorem relies on the following lemma:

Lemma 8. *Let (ϕ_n) be a real sequence such that $\liminf_{n\to\infty} \phi_n \geq 0$. If there exists a sequence (ϵ_n) such that $\lim_{n\to\infty} \epsilon_n = 0$ and*

$$\phi_{n+1} - \phi_n \leq -\frac{1}{n+1}\phi_n + \frac{\epsilon_n}{n}, \quad \forall n \in \mathbb{N}^\star \tag{158}$$

then $\lim_{n\to\infty} \phi_n = 0$.

To prove the lemma, one just has to prove that the sequence $(n\phi_n)$ is bounded by its first term and a finite sum of epsilon. Then one concludes by dividing by n both sides of the inequality and letting n goes to infinity. Now we can prove Theorem 1.

Proof. Let

$$\phi_n := \int_{\mathcal{C}} F(x, \bar{\eta_n}) \mathrm{d}\bar{\eta_n}(x) - F(x_{n+1}, \bar{\eta_n}) \quad \forall n \in \mathbb{N}^\star \tag{159}$$

so we want to prove that $\phi_n \to 0$. If we have such result, then we end up with equation (8) which is an equivalent formulation of the equilibrium property. Let us recall that $x_{n+1} \in \operatorname{argmin}_{x\in\mathcal{C}} F(x,\eta_n)$ is uniquely defined. Also the definition of x_{n+1} provides $\phi_n \geq 0$. Let us define

$$A := \int_{\mathcal{C}} F(x, \bar{\eta}_{n+1}) \mathrm{d}\bar{\eta}_{n+1}(x) - \int_{\mathcal{C}} F(x, \bar{\eta}_n) \mathrm{d}\bar{\eta}_n(x), \tag{160}$$

and

$$B := F(x_{n+1}, \bar{\eta}_n) - F(x_{n+2}, \bar{\eta}_{n+1}). \tag{161}$$

Thus we have:

$$\phi_{n+1} - \phi_n = A + B \tag{162}$$

So first using the definition of x_{n+1} then using the Lipschitz property of F we have:

$$\begin{aligned} B &\leq F(x_{n+2}, \bar{\eta}_n) - F(x_{n+2}, \bar{\eta}_{n+1}) && (163)\\ &\leq F(x_{n+1}, \bar{\eta}_n) - F(x_{n+1}, \bar{\eta}_{n+1}) + C|x_{n+2} - x_{n+1}| d_1(\bar{\eta}_{n+1}, \bar{\eta}_n) && (164)\\ &\leq F(x_{n+1}, \bar{\eta}_n) - F(x_{n+1}, \bar{\eta}_{n+1}) + \frac{C}{n+1}|x_{n+2} - x_{n+1}| d_1(\delta_{x_{n+1}}, \bar{\eta}_n) && (165) \end{aligned}$$

Using similar arguments one has:

$$A \leq \int_{\mathcal{C}} F(x, \bar{\eta}_{n+1}) - F(x, \bar{\eta}_n) \mathrm{d}\bar{\eta}_n(x) - \frac{1}{n+1}\phi_n + \frac{C}{(n+1)^2} d_1(\bar{\eta}_n, \delta_{x_{n+1}}) \tag{166}$$

Using a reformulation of equation (153) we have $-(n+1)(\bar{\eta}_{n+1} - \bar{\eta}_n) = \bar{\eta}_n - \delta_{x_{n+1}}$ and one can derive:

$$\int_{\mathcal{C}} F(x, \bar{\eta}_{n+1}) - F(x, \bar{\eta}_n) \mathrm{d}\bar{\eta}_n(x) + F(x_{n+1}, \bar{\eta}_n) - F(x_{n+1}, \bar{\eta}_{n+1}) \tag{167}$$

$$= -(n+1) \int_{\mathcal{C}} (F(x, \bar{\eta}_{n+1}) - F(x, \bar{\eta}_n)) \mathrm{d}(\bar{\eta}_{n+1} - \bar{\eta}_n)(x) \leq 0 \tag{168}$$

Which is negative by monotonicity of F. Then we deduce:

$$\phi_{n+1} - \phi_n \leq -\frac{1}{n+1}\phi_n + \frac{C}{n+1} d_1(\delta_{x_{n+1}}, \bar{\eta}_n)(\frac{1}{n+1} + |x_{n+2} - x_{n+1}|)$$

By compactness of $\mathcal{P}(\mathcal{C})$ we have $d_1(\delta_{x_{n+1}}, \bar{\eta}_n)$ which is bounded and so:

$$\phi_{n+1} - \phi_n \leq -\frac{1}{n+1}\phi_n + \frac{\epsilon_n}{n}, \quad \forall n \in \mathbb{N} \tag{169}$$

Using the fact that the application $\mathcal{P}(\mathcal{C}) \ni \eta \mapsto x_\eta := \operatorname{argmin} F(x, \eta)$ is uniformly continuous and

$$\lim_{n \to \infty} d_1(\eta_{n+1}, \eta_n) = 0 \tag{170}$$

(since $d_1(\eta_{n+1}, \eta_n) = \frac{d_1(\delta_{n+1}, \eta_n)}{n+1}$) then we have $\lim_{n\to\infty} |x_{n+2} - x_{n+1}| = 0$. Thus $\lim_{n\to\infty} \epsilon_n = 0$ and the result follows from Lemma 8. □

5.2 Convergence of the Fictitious Play for finite MFG

In this part we apply the convergence result we get above to our case. Let us define the cost:

$$c_{xy}(P, M) = K(x, y, p) + f(x, M) \quad \forall x, y \in S, P, M \in \mathcal{P}(S) \tag{171}$$

Where $K : S \times S \times \mathcal{P}(S) \mapsto \mathbb{R}$ is the cost the agent face from moving from x to y with probability P and $f : S \times \mathcal{P}(S) \mapsto \mathbb{R}$ is a running cost from being at x when the statistical distribution of others is M. Given a measure $\eta \in \mathcal{P}(\mathcal{K}_{S,\mathcal{T}})$ on transition kernel. We define $M_\eta : \mathcal{T} \mapsto \mathcal{P}(S)$ such that:

$$M_\eta(t) := \int_{\mathcal{K}_{\mathcal{S},\mathcal{T}}} M_P^{M_0}(t) \mathrm{d}\eta(P) \tag{172}$$

At state t, $M_\eta(t)$ is the average measure from starting at measure M_0 given a measure η on all possible transition kernel. Now one can define $F : \mathcal{K}_{S,\mathcal{T}} \times \mathcal{P}(\mathcal{K}_{S,\mathcal{T}}) \mapsto \mathbb{R}$:

$$F(P, \eta) := J_{M_\eta}(P) \tag{173}$$

Under the **hypothesis** that F is continuous and F satisfies the unique minimizer property we have that the minimum is reached for a dirac measure according to previous results. Now we want to apply theorem 1, so we need F to satisfies the assumptions. So we assume f, g to be monotone and to be Lipschitz with respect to their second arguments. Then we have the following result.

Lemma 9. *If f, g are monotone, then F is monotone.*

Proof. Consider two measures η and η' in $\mathcal{P}(C)$. Using the definition of the cost and using that the sum over y of $P(x, y, t)$ is unitary one can derive:

$$\int_{\mathcal{K}_{S,\mathcal{T}}} (F(P,\eta) - F(P,\eta'))\, \mathrm{d}(\eta - \eta')(P) = \int_{\mathcal{K}_{S,\mathcal{T}}} \sum_{t=0}^{T-1} \sum_{x \in S} M_P^{M_0}(x,t)\, (f(x, M_\eta(t)) - f(x, M_{\eta'}(t)))$$

$$+ \sum_{x \in S} M_P^{M_0}(x,T)\, (g(x, M_\eta(T)) - g(x, M_{\eta'}(T)))\, \mathrm{d}(\eta - \eta')(P)$$

$$= \sum_{t=0}^{T-1} \sum_{x \in S} (f(x, M_\eta(t)) - f(x, M_{\eta'}(t)))\, (M_\eta(t) - M_{\eta'}(t))$$

$$+ \sum_{x \in S} (g(x, M_\eta(T)) - g(x, M_{\eta'}(T)))\, (M_\eta(T) - M_{\eta'}(T)) \geq 0$$

Where we have used the definition of M_η and the monotonicity assumption of f and g. □

Under some continuity assumptions over the cost function and the function g, uniqueness of the minimizer of the dynamic programming problem at each period (satisfied by the cost with entropy term) and Lipschitz assumptions over f, g we have the uniqueness of the equilibrium. Now to check the assumption that F is Lipschitz in order to apply Theorem 1 we need the following result:

Lemma 10. *There exists a constant $C > 0$ such that:*

$$|M_\eta^{M_0}(t) - M_{\eta'}^{M_0}(t)| \leq C d_1(\eta, \eta') \quad \forall \eta, \eta' \in \mathcal{P}(\mathcal{K}_{\mathcal{S},\mathcal{T}}), t \in \mathcal{T} \tag{174}$$

In particular we have:

$$|M_P^{M_0}(x, t+1) - M_{P'}^{M_0}(x, t+1)| \leq C|P - P'|_\infty \tag{175}$$

Proof. For all $t = \{0, 1, \ldots, T-1\}$ and $x \in S$ we have:

$$|M_P^{M_0}(x, t+1) - M_{P'}^{M_0}(x, t+1)| = \sum_{y \in S} M_P^{M_0}(y,t) P(y,x,t) - \sum_{y \in S} M_{P'}^{M_0}(y,t) P'(y,x,t)$$

$$\leq \sum_{y \in S} M_P^{M_0}(y,t)(P(y,x,t) - P'(y,x,t))$$

$$+ |M_P^{M_0}(t) - M_P^{M_0}(t)|_\infty \sum_{y \in S} P'(y,x,t)$$

$$\leq |P - P'|_\infty + |S||M_P^{M_0}(t) - M_{P'}^{M_0}(t)|$$

Then by induction we have:

$$|M_P^{M_0}(x, t+1) - M_{P'}^{M_0}(x, t+1)| \leq C|P - P'|_\infty \tag{176}$$

Now Consider $\gamma \in \Pi(\eta, \eta')$ with marginal law given by η and η'. For any $x \in S$, $t \in \mathcal{T}$, $P, P' \in \mathcal{K}_{S,\mathcal{T}}$ we have:

$$|M_\eta(t+1) - M_{\eta'}(t+1)| = |\int_{\mathcal{K}_{\mathcal{S},\mathcal{T}}} M_P^{M_0}(t) \mathrm{d}\eta(P) - \int_{\mathcal{K}_{\mathcal{S},\mathcal{T}}} M_{P'}^{M_0}(t) \mathrm{d}\eta'(P')| \tag{177}$$

$$= |\int_{\mathcal{K}_{\mathcal{S},\mathcal{T}} \times \mathcal{K}_{\mathcal{S},\mathcal{T}}} (M_P^{M_0}(t) - M_{P'}^{M_0}(t)) \mathrm{d}\gamma(P, P')|$$

$$\leq C|P - P|_\infty d_1(\eta, \eta')$$

the last line is obtain by first using the above inequality and then taking the infimum. □

Then we have the following result:

Lemma 11. *Assume that f, g are Lipschitz. Then $\exists C > 0$ such that:*

$$|F(P,\eta) - F(P,\eta') + F(P',\eta) - F(P',\eta')| \leq C|P - P'|_\infty d_1(\eta,\eta') \tag{178}$$
$$|F(P,\eta) - F(P,\eta')| \leq C d_1(\eta,\eta') \tag{179}$$

for all $P, P' \in \mathcal{K}_{S,\mathcal{T}}$ and $\eta, \eta' \in \mathcal{P}(S)$.

Then combining Lemma 11, Lemma 9 and Theorem 1, one has the following Theorem:

Theorem 2. *Let $(P_n, M_n, \bar{M}_n)$ be a sequence generated by the fictitious play procedure. Under continuity and unique minimizer property of F, under continuity of g, continuity of the expected cost function and unique minimizer of the dynamic programming equation assumptions we have $(P_n, M_n, \bar{M}_n) \to (\hat{P}, M_{\hat{P}}^{M_0}, M_{\hat{P}}^{M_0})$.*

5.3 Finite MFGC

In this section we consider the problem of Mean Field Game of Control. Using the same notation as before, the problem can be written,

$$M_p^{M_0}(x,0) := M_0(x) \quad \forall x \in S \tag{180}$$
$$M_P^{M_0}(y,t) := \sum_{x\in S} M_P^{M_0}(x,t-1)P(x,y,t-1) \quad \forall y \in S, \forall t \in \mathcal{T} \tag{181}$$

We define $\mathcal{W} = (\mathbb{R}_+)^{S\times S\times\mathcal{T}}$ to be the set of all maps $W : S \times S \times \mathcal{T} \mapsto \mathbb{R}_+$. We define $\mathcal{W}_t = (\mathbb{R}_+)^{S\times S}$ the set of elements of $\mathcal{W}$ at time t for all $t \in \mathcal{T}$ and we denote $W(t)$ an element of this set. Now let $\Phi : \mathcal{W}_t \mapsto \mathbb{R}$, $L : S \times S \mapsto \mathbb{R}$, $g : S \times \mathcal{P}(S) \mapsto \mathbb{R}$ and define $J_W : \mathcal{K}_{S,\mathcal{T}} \mapsto \mathbb{R}$ as:

$$J_W(P) := \sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)\Phi(W(t)) + \sum_{x\in S} M_P^{M_0}(x,T)g(x,M(T)) \tag{182}$$

We define:

$$W_P(x,y,t) := P(x,y,t)M_P^{M_0}(x,t) \quad \forall x,y \in S, t \in \mathcal{T} \tag{183}$$

Then the MFGC problem is to solve:

$$\hat{P} := \underset{P\in\mathcal{K}_{S,\mathcal{T}}}{\operatorname{argmin}}\, J_{\hat{W}}(P) \text{ with } \hat{W} = W_{\hat{P}} \tag{184}$$

In particular Φ is not a separable function. Here we suppose the following form on Φ:

$$\Phi(W(t)) := \phi\left(\sum_{x,y\in S} W(x,y,t)L(x,y)\right) \tag{185}$$

L can have the following structure (see [5] for example):

$$L(x,y) = (y-x) + \gamma(y-x)^2 \tag{186}$$

In this example ϕ can represent the price of a good depending on the net demand, and the quadratic form of L can be understand as a penalization on the trading speed.

5.4 Convergence of MFGC

To prove the convergence of the Finite Mean Field Game of Control, we use the Generalized Fictitious Play part. We adapt the function F to fit the new framework:

$$F(P,\eta) := J_{W_\eta}(P) \tag{187}$$

where

$$W_\eta(x,y,t) := \int_{\mathcal{K}_{\mathcal{S},\mathcal{T}}} W_P(x,y,t)\mathrm{d}\eta(P) \tag{188}$$

Lemma 12. *If ϕ and g are monotone then F is monotone.*

Proof. Consider two measures η and η' in $\mathcal{P}(C)$.

$$\begin{aligned} F(P,\eta) - F(P,\eta') =& J_{W_\eta}(P) - J_{W_{\eta'}}(P) \\ =& \sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)\left[\Phi(W_\eta(t)) - \Phi(W_{\eta'}(t))\right] \\ &+ \sum_{x\in S} M_P^{M_0}(x,T)\left[g(x,M_\eta(T)) - g(x,M_{\eta'}(T))\right] \end{aligned} \tag{189}$$

Let us call for presentation convenience:

$$A := \int_{\mathcal{K}_{S,\mathcal{T}}} \left(F(P,\eta) - F(P,\eta')\right)\mathrm{d}(\eta-\eta')(P) \tag{190}$$

Then we have:

$$A = \int_{\mathcal{K}_{S,\mathcal{T}}} \sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)\left[\Phi(W_\eta(t)) - \Phi(W_{\eta'}(t))\right] \tag{191}$$

$$+ \sum_{x\in S} M_P^{M_0}(x,T)\left(g(x,M_\eta(T)) - g(x,M_{\eta'}(T))\right)\mathrm{d}(\eta-\eta')(P)$$

$$= \int_{\mathcal{K}_{S,\mathcal{T}}} \sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)\left[\Phi(W_\eta(t)) - \Phi(W_{\eta'}(t))\right]\mathrm{d}(\eta-\eta')(P) \tag{192}$$

$$+ \sum_{x\in S} \left(g(x,M_\eta(T)) - g(x,M_{\eta'}(T))\right)\left(M_\eta(x,T) - M_{\eta'}(x,T)\right)$$

Using that g is monotone we have:

$$\sum_{x\in S} \left(g(x,M_\eta(T)) - g(x,M_{\eta'}(T))\right)\left(M_\eta(T) - M_{\eta'}(T)\right) \geq 0 \tag{193}$$

Now we only need to deal with:

$$\int_{\mathcal{K}_{S,\mathcal{T}}} \sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)\left[\Phi(W_\eta(t)) - \Phi(W_{\eta'}(t))\right]\mathrm{d}(\eta-\eta')(P) \tag{194}$$

Using (185) and the following notation:

$$W_\eta^L(t) := \sum_{x,y\in S} W_\eta(x,y,t)L(x,y) \tag{195}$$

We have by monotony of ϕ:

$$\sum_{t=0}^{T-1} \left(W_\eta^L(t) - W_{\eta'}^L(t)\right)\left[\phi\left(W_\eta^L(t)\right) - \phi\left(W_{\eta'}^L(t)\right)\right] \geq 0 \tag{196}$$

□

Lemma 13. *There exists a constant $C > 0$ such that:*

$$|W_\eta(t) - W_{\eta'}(t)| \leq C d_1(\eta,\eta') \quad \forall \eta,\eta' \in \mathcal{P}(\mathcal{K}_{S,\mathcal{T}}), t \in \mathcal{T} \tag{197}$$

In particular we have:

$$|W_P(x,y,t) - W_{P'}(x,y,t)| \leq C|P - P'|_\infty \quad \forall P,P' \in \mathcal{K}_{S,\mathcal{T}} \tag{198}$$

Proof. For any $x \in S$, $t \in \mathcal{T}$, $P, P' \in \mathcal{K}_{S,\mathcal{T}}$ we have:

$$\begin{aligned} |W_P(x,y,t+1) - W_{P'}(x,y,t+1)| &= |P(x,y,t+1)M_P^{M_0}(x,t+1) - P'(x,y,t+1)M_{P'}^{M_0}(x,t+1)| \\ &\leq \Big(|P-P'|_\infty + |S||M_P^{M_0}(t) - M_P'^{M_0}(t)|_\infty\Big) \end{aligned} \tag{199}$$

Using the Lemma 10:

$$|W_P(x,y,t+1) - W_{P'}(x,y,t+1)| \leq |P - P'|_\infty \left(1 + |S|C\right) \tag{200}$$

Consider $\gamma \in \Pi(\eta,\eta')$ with marginal law given by η and η'. Then we deduce for all $P \in \mathcal{K}_{\mathcal{S},\mathcal{T}}$:

$$|W_\eta(x,y,t+1) - W_{\eta'}(x,y,t+1)| = |\int_{\mathcal{K}_{S,\mathcal{T}}} W_P(x,y,t+1) - W_{P'}(x,y,t+1)\mathrm{d}\gamma(P,P')| \tag{201}$$

$$\leq |\int_{\mathcal{K}_{S,\mathcal{T}}} |P-P'|_\infty \left(1+|S|C\right)\mathrm{d}\gamma(P,P')| \tag{202}$$

Taking the infimum over P we get the result.

□

Lemma 14. *Assume that ϕ, g are Lipschitz and for all $x,y \in S, |L(x,y)| < +\infty$. Then $\exists C > 0$ such that:*

$$|F(P,\eta) - F(P,\eta')| \leq C d_1(\eta,\eta') \tag{203}$$

$$|F(P,\eta) - F(P,\eta') + F(P',\eta) - F(P',\eta')| \leq C|P-P'|_\infty d_1(\eta,\eta') \tag{204}$$

for all $P, P' \in \mathcal{K}_{S,\mathcal{T}}$ and $\eta, \eta' \in \mathcal{P}(S)$.

Proof. First we prove relation (203). Fix $\eta,\eta' \in \mathcal{P}(S)$ and $P \in \mathcal{K}_{\mathcal{S},\mathcal{T}}$. We set c to be the maximum between the ϕ and g Lipschitz constant and we set $\bar{L} = \sum_{x,y\in S}|L(x,y)|$, then we have:

$$|F(P,\eta) - F(P,\eta')| = |A + B| \tag{205}$$

With:

$$A := \sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)\left(\Phi(W_\eta(t)) - \Phi(W_{\eta'}(t))\right) \tag{206}$$

$$\leq \sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)|\phi(W_\eta^L(t)) - \phi(W_{\eta'}^L(t))| \tag{207}$$

$$\leq c\sum_{t=0}^{T-1} \sum_{x,y\in S} M_P^{M_0}(x,t)P(x,y,t)L(x,y)|\sum_{x,y\in S} L(x,y)(W_\eta^L(t) - W_{\eta'}^L(t))| \tag{208}$$

Then using Lemma (13) we derive:

$$A \leq cd_1(\eta,\eta')\bar{L}^2C\sum_{t=0}^{T-1}\sum_{y\in S} M^{M_0}(y,t+1) \tag{209}$$

$$\leq cd_1(\eta,\eta')(T-1)\bar{L}^2C \tag{210}$$

Now we have:

$$B := \sum_{x\in S} M_P^{M_0}(x,T)\left(g(x,M_\eta(T)) - g(x,M_{\eta'}(T))\right) \tag{211}$$

$$\leq \sum_{x\in S} M_P^{M_0}(x,T)|g(x,M_\eta(T)) - g(x,M_{\eta'}(T))| \tag{212}$$

$$\leq cd_1(\eta,\eta') \tag{213}$$

Then (203) follows. Now we prove (204), let us set:

$$|F(P,\eta) - F(P,\eta') + F(P',\eta) - F(P',\eta')| = |E + F| \tag{214}$$

Where:

$$E := \sum_{t=0}^{T-1}\sum_{x,y\in S}(M_P^{M_0}(x,t)P(x,y,t) - M_{P'}^{M_0}(x,t)P'(x,y,t))L(x,y)\left(\Phi(W_\eta(t)) - \Phi(W_{\eta'}(t))\right) \tag{215}$$

$$\leq cd_1(\eta,\eta')\bar{L}^2C\sum_{t=0}^{T-1}\sum_{x,y\in S}|W_P(x,y,t) - W_{P'}(x,y,t)| \tag{216}$$

$$\leq cd_1(\eta,\eta')|P - P'|_\infty(\bar{L}C)^2|S|^2(T-1) \tag{217}$$

And we also have:

$$F := \sum_{x\in S}(M_P^{M_0}(x,T) - M_{P'}^{M_0}(x,T))\left(g(x,M_\eta(T)) - g(x,M_{\eta'}(T))\right) \tag{218}$$

$$\leq C|S||P - P|_\infty d_1(\eta,\eta') \tag{219}$$

Then (204) follows. $\square$

Then combining Lemma 14, Lemma 12 and Theorem 1, one has the following Theorem:

Theorem 3. *Let $(P_n, M_n, \bar{M}_n)$ be a sequence generated by the fictitious play procedure. Under continuity and unique minimizer property of F, under continuity of g, continuity of the expected cost function and unique minimizer of the dynamic programming equation assumptions we have $(P_n, M_n, \bar{M}_n) \to (\hat{P}, M_{\hat{P}}^{M_0}, M_{\hat{P}}^{M_0})$.*

5.5 Numerical results

Implementation of algorithm to find the equilibrium of a Mean Field Game can be find in [7] for example. Here we provide results concerning Mean Field Game of Control problem. We made some simplifications, for example we used an entropy term to have an explicit formula for the optimal transition probability

$$\hat{p}(x,y) = \frac{e^{-\frac{\phi(t)L(x,y)+u(y,t)}{\epsilon}}}{\sum_y' e^{-\frac{\phi(t)L(x,y')+u(y',t)}{\epsilon}}}.$$

The cost function we use is

$$L(x,y) = \frac{y-x}{\Delta t} + (\frac{y-x}{\Delta t})^2, \tag{220}$$

(the same as in [5]). With this kind of cost function we penalize the speed of the agents. As price function we take

$$\phi(t) = \Phi\left(\sum_{x,y\in S} P(x,y,t)M(x,t)L(x,y)\right) = \frac{\exp^{\sum_{x,y\in S} P(x,y,t)M(x,t)L(x,y)}}{\exp^{\sum_{x,y\in S} P(x,y,t)M(x,t)L(x,y)} + 1} \tag{221}$$

For the Simulation we also need to give an initial measure and a final reward. Here we give two bell shapes. The initial measure is centered in 0, and the final reward is quadratic with minimum at -0.2. Then we should observe the most of the mass to move from 0 to -0.2. In the algorithm we add a condition (the red part line (24)) that increases the speed of convergence. This condition says that if the measure generated by the best response is to far from the average measure, then we can replace the average by the this measure. This allows to increase the speed of convergence since the procedure is very dependent of a bad starting measure. We will provide an example of this effect.

Algorithm 1 Solving MFGC

Initialization:
Choose T the last time of the game
Construct $X \subseteq \mathbb{R}^n$ the space of actions
Choose Δt a time discretization on $[0, T]$
Choose a discretized action space S of X
Set $u(x, T) = g(x, T) \, \forall \, x \in S$
Set $M^n(x, 0) = M_0^n(x) \, \forall \, x \in S \, , \forall n \geq 0$
Initialize a first path $M^0(x) = \big(M^0(x, 0), M^0(x, \Delta t), \dots, M^0(x, T)\big)$
Initialize a first price sequence $\phi = (\phi(0), \phi(\Delta t), \dots, \phi(T))$
Choose a stopping criteria δ
$\bar{M}^0 = M^0$
Choose an initial Constant C
Condition = True
while Condition = True **do**
  for $t_k \in [T - \Delta t, \dots, \Delta t, 0]$ **do**
  $P^n(x, ., t_k) \in \underset{p \in \mathbb{P}(S)}{\operatorname{argmin}} \sum_{y \in S} p(y)(\phi(t) L(\frac{x - y}{\Delta t}) \Delta t + u(y, t_{k+1})) + \epsilon \varepsilon(p)$
  $u(y, t_k) = \sum_{y \in S} P^n(y, x, t_k) L(\frac{y - x}{\Delta t}) \Delta t + u(y, t_{k+1}) + \epsilon \varepsilon(p)$
  end for
  for $t_k \in [0, \Delta t, \dots, T - \Delta t]$ **do**
    $M^{n+1}(y, t_{k+1}) = \sum_{x \in S} P^n(x, y, t_k) M^{n+1}(x, t_k)$
  end for
  $M^{n+1}(y) = \big(M^{n+1}(y, 0), M^{n+1}(y, \Delta t), \dots, M^{n+1}(y, T)\big)$
  $\bar{M}^{n+1} = \frac{1}{n+1} M^{n+1} + \frac{n}{n+1} \bar{M}^n$
  if $\|M^n - \bar{M}^{n+1}\| < C$ **then**
    $\bar{M}^{n+1} = M^n$
    $C = \|M^n - \bar{M}^{n+1}\|$
  end if
  Condition = $|\bar{M}^{n+1} - \bar{M}^n| > \delta$
  Actualization of price for each time t
  $\phi(t) = \sum_{x, y \in S} \bar{M}^{n+1}(x, t) P^n(x, y, t)(x - y)$
  $n = n + 1$
end while

Then we obtain the following equilibrium measure

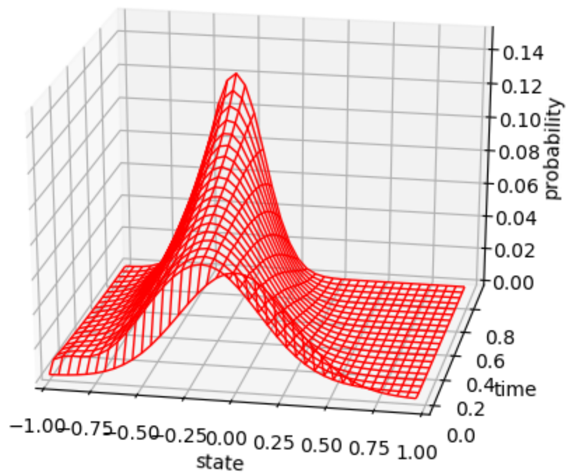

Figure 1: Evolution of the density of players

where we can observe the movement of the mass which start from a given measure to a final measure concentrated around -0.2. The price evolution is the following

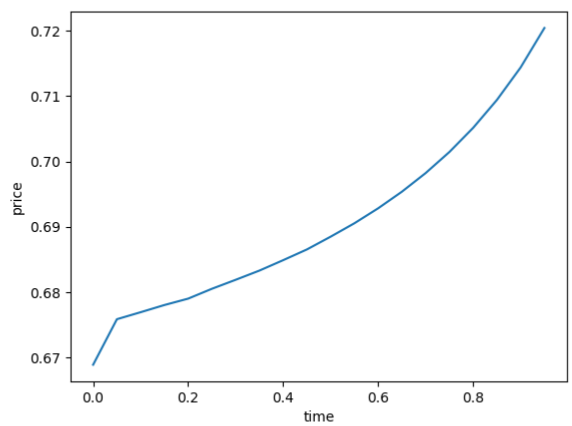

Figure 2: Price evolution

which is not surprising. Indeed we observe that their is a first 'quik' movement at time 0 to the negative number in figure (1). Since the negative numbers stand for demand, the price increases. Then the mass move very slowly to the target -0.2 which keeps the price increasing. We also observe that the mass displacement and the price evolution are smooth. Indeed no agent would have a strategy that consists in buying or selling a lot in a single period since we penalized the speed of trading with a quadratic term.

The convergence is very slow with the first procedure, specially when the stopping criteria is small. For example we used $\delta = 0.1$ and we get

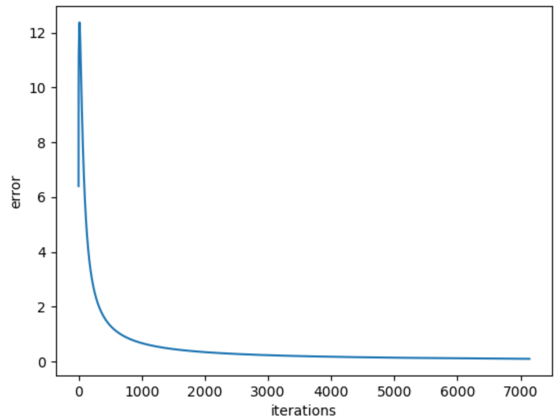

Figure 3: Convergence of the Fictitious Play Procedure for $\delta = 0.1$

convergence in approximately 7000 iterations. Using the "Upgraded" procedure we obtain

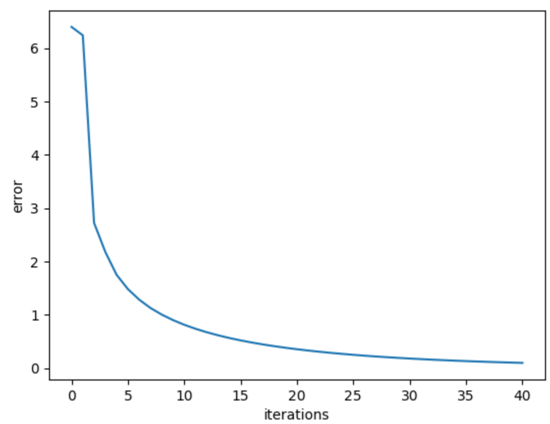

Figure 4: Convergence of the Upgraded Fictitious Play Procedure for $\delta = 0.1$

then we observe a great improvement using condition (24).

6 Conclusion

In this report we have worked on two kinds of Mean Field Game: Classical one and Mean Field Game of Control. The difference between the two problems leads to different treatment because assumptions are different but the same tools can be used. In a first time we dealt with Potential formulations. We expect in future work to derive Augmented Lagrangian methods to solve this games. This approach should be more efficient than the Fictitious Play procedure we present in the second part of this thesis. Nevertheless the assumption that a game admits a Potential form is not always true. Then it is relevant to study a second kind of method. The fictitious play procedure, who has been proved to converge in the case of Mean Field Game under separable cost assumption by F.Silva and S. Hadikhanloo, also converges in the Mean Field Game of Control. For future work, we should prove that the solutions of a finite Mean Field Game of Control converges to the solutions of continuous Mean Field Game of Control.

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Appendices

A Perspective function

According to [4] we define the perspective function and give some properties:

Definition 5. Perspective function
Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ *and* $x \in \mathbb{R}^d, y \in \mathbb{R}_+^\star$*, then the perspective function* $\tilde{f}$ *is defined by:*

$$\tilde{f}(x, y) = y f(\frac{x}{y})$$

Lemma 15. *Let* $f : \mathbb{R}^d \mapsto \mathbb{R}$*, and* $x \in \mathbb{R}^d, y \in \mathbb{R}_+^\star$*, then the perspective function* $\tilde{f}$ *is convex if and only if* f *is convex*

Proof. If for all $x \in \mathbb{R}^d, y \in \mathbb{R}_+^\star$ $\tilde{f}$ is convex then we choose $y = 1$ and we get f convex. If f is convex, then fix $x_1, x_2 \in \mathbb{R}^d, y_1, y_2 \in \mathbb{R}_+^\star$ and $\lambda \in [0, 1]$ and we have:

$$\begin{aligned}
\tilde{f}(\lambda x_1 + (1-\lambda)x_2), \lambda y_1 + (1-\lambda)y_2) &= (y_1 + (1-\lambda)y_2) f\left(\frac{\lambda x_1 + (1-\lambda)x_2}{y_1 + (1-\lambda)y_2}\right) \\
&= (y_1 + (1-\lambda)y_2) f\left(\frac{\lambda y_1}{y_1 + (1-\lambda)y_2}\frac{x_1}{y_1} + \frac{(1-\lambda)y_2}{y_1 + (1-\lambda)y_2}\frac{x_2}{y_2}\right) \\
&\leq (1-\lambda)y_1 f\left(\frac{x_1}{y_1}\right) + (1-\lambda)y_2 f\left(\frac{x_2}{y_2}\right) \\
&\leq \lambda \tilde{f}(x_1, y_1) + (1-\lambda)\tilde{f}(x_2, y_2)
\end{aligned}$$

□

Lemma 16. *If* f *is a proper l.s.c convex function then the conjugate function of its perspective function defined by* $\tilde{f}^\star$ *on* $X^\star \times \mathbb{R}$ *is the indicator function of the non-empty set* $C = \{(x^\star, y^\star) \in X^\star \times \mathbb{R}, f^\star(x^\star) \leq -y^\star\}$

Proof. For all $(x^\star, y^\star) \in X^\star \times \mathbb{R}$ we have:

$$\tilde{f}^\star(x^\star, y^\star) = \sup_{(x,y) \in X^\star \times \mathbb{R}_+^\star} \langle x^\star, x\rangle + \langle y^\star, y\rangle - \tilde{f}(x, y)$$

If for some $(x^\star, y^\star) \in X^\star \times \mathbb{R}$ we have $\langle x^\star, x\rangle + \langle y^\star, y\rangle - \tilde{f}(x, y) \leq 0$ then the supremum is 0. Now if for some $(x^\star, y^\star) \in X^\star \times \mathbb{R}$ we have $\langle x^\star, x\rangle + \langle y^\star, y\rangle - \tilde{f}(x, y) > 0$ then by homogeneity of the perspective function $\tilde{f}$ the supremum is infinite. Then if we define $C = \{(x^\star, y^\star) \in X^\star \times \mathbb{R}, \langle x^\star, x\rangle + \langle y^\star, y\rangle \leq \tilde{f}(x, y)\}$ we have that $\tilde{f}^\star(x^\star, y^\star) = I_C$. This formulation provide that C is non empty since f is proper l.s.c and convex so it admits at least one affine minorant of the form $\langle x_0, x^\star\rangle - f^\star(x_0^\star)$. Then one observe that $(x_0^\star, -f^\star(x_0^\star)) \in C$. Finally we need to prove that $C = \{(x^\star, y^\star) \in X^\star \times \mathbb{R}, f^\star(x^\star) \leq -y^\star\}$. We have:

$$\begin{aligned}
\sup_{(x,y) \in X^\star \times \mathbb{R}_+^\star} \langle x^\star, x\rangle + \langle y^\star, y\rangle - \tilde{f}(x, y) &= \sup_{y \in \mathbb{R}_+^\star} y^\star y + y\left(\sup_{x' \in X} \langle x', x^\star\rangle - f(x')\right) \\
&= \sup_{y \in \mathbb{R}_+^\star} y(y^\star + f^\star(x^\star))
\end{aligned}$$

Finally:

$$\tilde{f}^\star(x^\star, y^\star) = I_C$$

Where $C = \{(x^\star, y^\star) \in X^\star \times \mathbb{R}, f^\star(x^\star) \leq -y^\star\}$ and C is non-empty. □

Lemma 17. *If $f : X \mapsto \mathbb{R}$ is a proper l.s.c convex function then the bi-conjugate function of its perspective function defined by $\tilde{f}^{\star\star}$ on $X \times \mathbb{R}$ is defined by:*

$$\tilde{f}^{\star\star}(x,y) = \begin{cases} \tilde{f}(x,y) \ \textit{if } y > 0 \\ f_\infty(x) \ \textit{if } y = 0 \\ +\infty \ \textit{if } y < 0 \end{cases}$$

Theorem 4. Fenchel-Moreau-Rockafellar *let $f : X \mapsto \bar{\mathbb{R}}$. We have the following alternative: either*

- *$f^{\star\star} = -\infty$ identically, $\overline{conv}(f)$ has no infinite value, and has value $-\infty$ at some point, or*
- *$f^{\star\star} = \overline{conv}(f)$ and $\overline{conv}(f)(x) > -\infty$, for all $x \in X$*

Proof. If f is identically equal to $+\infty$, then the conclusion is obvious. So we may assume that $\text{dom}(f) \neq \emptyset$. Now if there exists a point x_0 such that $f(x_0) = -\infty$ then f admits no affine minorant and then $f^{\star\star} = -\infty$. In addition we know that $\overline{\text{conv}}(f)$ is the function with epigraph $\overline{\text{conv}}epi(f)$ which means that it is the supremum of l.s.c convex minorant of f. Then for any $x_\lambda = \lambda x + (1-\lambda)x_0$ we have:

$$\overline{\text{conv}}(f)(x) \leq \lim_{\lambda \mapsto 1} \overline{\text{conv}}(f)(x_\lambda) \leq \lim_{\lambda \mapsto 1} \lambda \overline{\text{conv}}(f)(x) + (1-\lambda)\overline{\text{conv}}(f)(x_0) = -\infty$$

Then the first point is proved. Now if the first part of the theorem is not true, then f admits an affine minorant, so that $\overline{\text{conv}}(f) > -\infty$. Using that f is proper, l.s.c and convex, $\overline{\text{conv}}(f)$ is the supremum of its affine minorants which coincide with $f^{\star\star}$. □

Proposition 3. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a proper l.s.c convex function and let $\tilde{f} : \mathbb{R}^n \times \mathbb{R}_+^\star$ be its perspective function. Then the convex closure of its perspective function is given by its biconjugate:*

$$\overline{conv}(\tilde{f})(x,y) = \begin{cases} \tilde{f}(x,y) \ \textit{if } y > 0 \\ f_\infty(x) \ \textit{if } y = 0 \\ +\infty \ \textit{if } y < 0 \end{cases}$$

Proof. The proof follows from Lemma 17 and Theorem 4 □

B Fenchel Transform

Let $(a,b) \in \mathbb{R}^{S \times \mathcal{T}+1} \times \mathbb{R}^{S \times S \times \mathcal{T}}$

$$\mathcal{F}^\star(a,b) = \sup_{(M,W)} \sum_{t \in \mathcal{T}+1} \sum_{x \in S} a(x,t)M(x,t) + \sum_{t \in \mathcal{T}} \sum_{x,y \in S} b(x,y,t)W(x,y,t) \tag{222}$$

$$- \sum_{t \in \mathcal{T}} \sum_{x,y \in S} \tilde{L}^{\star\star}(W(x,y,t), M(x,t), x, y, t) - \sum_{t \in \mathcal{T}} F(M(.,t)) \tag{223}$$

First case: if there exists some $(x,t) \in S \times \mathcal{T}$ such that $M(x,t) < 0$ is in the maximal argument then $\mathcal{F}^\star \equiv -\infty$. Second case: if there exists some $(x,y,t) \in S \times S \times \mathcal{T}$ such that $M(x,t) = 0$ and $W(x,y,t) > 0$ is in the maximal argument then we also have $\mathcal{F}^\star \equiv -\infty$ by hypothesis on L_∞. Last case: for all $(x,y,t) \in S \times S \times \mathcal{T}$ such that $(W(x,y,t), M(x,t)) \in (\mathbb{R}^+ \times \mathbb{R}_\star^+) \cup \{(0,0)\}$ we have:

$$\mathcal{F}^\star(a,b) = \sup_M \sum_{t \in \mathcal{T}} \sum_{x \in S} a(x,t)M(x,t) + a(x,T)M(x,T) - \sum_{t \in \mathcal{T}} F(M(.,t)) \tag{224}$$

$$+ \left(\sup_W \sum_{y \in S} b(x,y,t)W(x,y,t) - M(x,t) \sum_{y \in S} L(W(x,y,t), M(x,t), x, y, t) \right) \tag{225}$$

We use that $\phi : (\mathbb{R}^+ \times \mathbb{R}^+_\star) \cup \{(0,0)\} \mapsto (\mathbb{R}^+ \times \mathbb{R}^+_\star) \cup \{(0,0)\}$ is bijective to define the change of variables $(P(x,y,t), M(x,t)) = \phi^{-1}(W(x,y,t), M(x,t))$. Then we derive for equation (225)

$$= M(x,t)(\sup_P \sum_{y\in S} b(x,y,t)P(x,y,t) - \sum_{y\in S} L(P(x,y,t),x,y,t)) \tag{226}$$

$$= M(x,t) \sum_{y\in S} L^\star(b(x,y,t),x,y,t) \tag{227}$$

$$\mathcal{F}^\star(a,b) = \chi_{\{a(.,T)=0\}} + \sup_M \sum_{t\in\mathcal{T}} \sum_{x\in S} M(x,t)(a(x,t) + \sum_{y\in S} L^\star(b(x,y,t),x,y,t)) - \sum_{t\in\mathcal{T}} F(M(.,t)) \tag{228}$$

$$= \sum_{t\in\mathcal{T}} F^\star \left(a(.,t) + \sum_y L^\star(b(x,y,t),.,y,t) \right) + \chi_{\{a(.,T)=0\}}. \tag{229}$$

Let $A, B \in \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times\mathcal{T}} \times \mathbb{R}^{S\times S\times\mathcal{T}} \times \mathbb{R}^S$ with $A = (a_1,a_2,a_3,a_4)$ and $B = (b_1,b_2,b_3,b_4)$

$$\mathcal{G}^\star(B) = \sup_{A\in K} \sum_{t\in\mathcal{T}} \sum_{x,y\in S} a_3(x,y,t)b_3(x,y,t) + \sum_{x\in S} a_4(x)b_4(x) - G(M(.,T)) \tag{230}$$

$$= \chi_{\{b_3\le 0\}} + \sup_{a_4=M_0} \sum_{x\in S} a_4(x)b_4(x) - \sum_{x\in S} M_0(x)G(M(.,T)) \tag{231}$$

$$= \chi_{\{b_3\le 0\}} + \sum_{x\in S} M_0(x)(b_4(x) - G(M(.,T))) \tag{232}$$

$$= \chi_{\{b_3\le 0, G^T=G(M(.,T))\}} + \sum_{x\in S} M_0(x)(b_4(x) - G^T) \tag{233}$$

$$\tag{234}$$