Master Thesis

Dynamic Programming Interpretation of Turnpike and Hamilton-Jacobi-Bellman Equation

Hicham Kouhkouh

Supervised by Prof. Enrique Zuazua

Dynamic Control Research Team
Chair of Computational Mathematics
DeustoTech - Deusto Foundation
Bilbao, Basque Country Spain

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Outline

Consider a Lagrange type optimal control problem

$$\text{minimize } J_{t,x}(u) \text{ over all controls } u : [t, T] \to U$$

where

$$J_{t,x}(u) = \int_t^T L(y(s), u(s))ds, \quad \text{where } y(\cdot) = y(\cdot; t, x, u)$$

such that

$$\dot{y} = f(y, u), \quad \text{and } y(t) = x$$

And denote the Hamiltonian

$$\mathcal{H}(x, p) = \max_{u \in U}\{ -p \cdot f(x, u) - L(x, u) \}$$

Consider the control system

$$\dot{x}(s) = Ax(s) + Bu(s), \quad s \in (0,t), \quad x(0) = x_0, \ A \in \mathbb{R}^{n^2}, \ B \in \mathbb{R}^{n \times m} \quad (1)$$

and the cost function

$$J(x,u) := \frac{1}{2} \int_0^t \|u(s)\|^2 + \|Cx(s) - z\|^2 ds, \quad C \in \mathbb{R}^{n^2}, \quad z \in \mathbb{R}^n$$

(LQ-) Optimal Control Problem

Given $x_0, z \in \mathbb{R}^n$, find $u \in L^2([0,t];\mathbb{R}^m)$ such that

- $x(0; x_0, u) = x_0$;
- $x(t; x_0, u)$ is free;
- (x,u) satisfies (1);
- $J(x(\cdot\, ; x_0, u), u)$ as small as possible

Denote $V(x_0, t)$ its <u>value function</u> (initial state & final time)

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From *Bellman*'s optimality principle

$$V(x_0,t) = \min_{u\in L^2(0,\Delta t;\mathbb{R}^m)} \frac{1}{2}\int_0^{\Delta t} |u(s)|^2 + |Cx(s) - z|^2 ds + V(x(\Delta t), t-\Delta t)$$

From the feedback law $u = -B^* p$ and the optimality system

$$\begin{aligned} \dot{x} &= Ax - BB^*p, & x(0) &= 0 \\ -\dot{p} &= A^*p + C^*(Cx - z), & p(t) &= 0 \end{aligned}$$

x, p in $H^1(0,t;\mathbb{R}^n) \Rightarrow x, p$ continuous $\Rightarrow$ (bootstrap) x, p, u are ∞ diff.
Assume smoothness of $V(\cdot,\cdot)$ (it will be proved later)

$$\int_0^{\Delta t} \frac{1}{2}(|u(s)|^2 + |Cx(s) - z|^2) ds = \frac{1}{2}(|u(0)|^2 + |Cx_0 - z|^2)(\Delta t) + o(\Delta t)$$

$$V(x(\Delta t), t-\Delta t) = V(x_0,t) - \frac{\partial V}{\partial t}(x_0,t)\Delta t + \nabla_x V(x_0,t)^\top \dot{x}(0)\Delta t + o(\Delta t)$$

We can now derive the HJB equation

$$\frac{\partial V}{\partial t}(x,t) + \mathcal{H}\Big(x, \nabla_x V(x,t)\Big) = 0$$

Noticing that in the LQ setting

$$\mathcal{H}(x,p) = \frac{1}{2}\|B^* p\|^2 - p^* A x - \frac{1}{2}\|Cx - z\|^2$$

We recover the Cauchy problem in the form

$$\begin{aligned} \frac{\partial V}{\partial t}(x,t) + H\Big(x, \nabla_x V(x,t)\Big) &= \ell(x), \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ V(x,0) &= 0, \qquad x \in \mathbb{R}^n \end{aligned}$$

where $H(x,p) = \frac{1}{2}\|B^* p\|^2 - p^* A x$ and $\ell(x) = \frac{1}{2}\|Cx - z\|^2$

Consider the steady problem corresponding to the LQ problem

$$V_s = \min_{u\in\mathbb{R}^m, x\in\mathbb{R}^n} \quad J_s(x,u) := \frac{1}{2}(\|u\|^2 + \|Cx - z\|^2)$$
$$\text{s.t.} \qquad 0 = Ax + Bu$$

its Lagrangian writes

$$\mathcal{L}(x,u,\lambda) = J_s(x,u) + \lambda^*(Ax + Bu)$$

Since the constraint is qualified and the problem is convex, KKT conditions are sufficient and necessary. Denote $(x^\#, u^\#, \lambda^\#)$ optimal solution, so

$$V_s = \frac{1}{2}\|B^*\lambda^\#\|^2 + \frac{1}{2}\|Cx^\# - z\|^2 + \lambda^\# \cdot (Ax^\# - BB^*\lambda^\#)$$

which writes

$$V_s + H(x^\#, \lambda^\#) = \ell(x^\#)$$

Outline

Study convergence of the evolutionary problem

$$\frac{\partial V}{\partial t}(x,t) + H\Big(x, \nabla_x V(x,t)\Big) = \ell(x), \quad (x,t) \in \mathbb{R}^n \times (0,\infty)$$
$$V(x,0) = 0, \qquad x \in \mathbb{R}^n$$

towards the stationary (*ergodic*) problem

$$c + H(x, \nabla\psi(x)) = \ell(x), \quad \forall x \in \mathbb{R}^n$$

where the unknown is $(c,\psi) \in \mathbb{R}^+ \times C(\mathbb{R}^n;\mathbb{R})$ and c is called *ergodic cost*.

This has been proved by G. Barles & al., *"Large Time Behavior of Unbounded Solutions of First-Order HJ Equations in the Whole Space"*,(2010). But their assumptions do not fit the LQ setting.

Outline

Main results I

Riccati operator

We know that for an LQ problem where the target $z = 0$, we have:

- The value function is a quadratic function of the Riccati operator

$$V(x,t) = \frac{1}{2}\langle E(t)x, x\rangle$$

- From HJB equation we can derive DRE;

$$\dot{E}(t) = C^*C + EA + A^\top E - EBB^\top E, \quad t \in (0,\infty), \quad E(0) = 0$$

- The asymptotic behavior of Riccati operator $E(t) \longrightarrow \hat{E}$ such that $\hat{E}$ satifies the ARE

$$0 = C^*C + \hat{E}A + A^\top \hat{E} - \hat{E}BB^\top \hat{E}$$

We will first construct a Riccati operator for the case $z \neq 0$, and then prove the same results.

Let us introduce the well known operator

$$\Phi : \begin{array}{rcl} L^2(0,t;\mathbb{R}^m) & \longrightarrow & L^2(0,t;\mathbb{R}^n) \\ u & \longmapsto & x(\cdot;0,u) \end{array} \quad ; \quad \Phi u(t) = \int_0^t e^{A(t-s)} B u(s) ds$$

Therefore a solution is of the form $x(t;x_0,u) = \Phi u + e^{tA} x_0$.
We rewrite the cost function

$$\|u\|^2 + \|Cx - z\|^2 = \langle u, u \rangle + \langle C\Phi u + Ce^{tA}x_0 - z, C\Phi u + Ce^{tA}x_0 - z \rangle$$

After minimization, one gets

$$V(x_0, t) = \min_u \frac{1}{2}\left(\|u\|^2_{L^2} + \|Cx - z\|^2_{L^2} \right) = \frac{1}{2} \langle E(t) \begin{pmatrix} x_0 \\ z \end{pmatrix}, \begin{pmatrix} x_0 \\ z \end{pmatrix} \rangle$$

where $E(\cdot)$ is a symmetric matrix in $C^1([0,\infty[;\mathcal{M}_{2n}(\mathbb{R}))$.

Now using HJB equation, one recovers the Riccati equation in $\mathcal{M}_{2n}(\mathbb{R})$

$$\frac{d}{dt}E(t) = \tilde{C} + E(t)\tilde{A} + \tilde{A}^* E(t) - E(t)\tilde{B}\tilde{B}^* E(t), \quad \text{in } (0, +\infty)$$
$$E(0) = 0$$

where

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C^* C & -C^* \\ -C & I \end{pmatrix}, \quad E = \begin{pmatrix} E_1 & E_2 \\ E_2^* & E_3 \end{pmatrix}$$

To study its asymptotic behavior, we will assume

A1. The pair (A, B) is stabilizable.

A2. The pair (C, A) is detectable.

Outline

Lemma

Assume that **(A1)** *and* **(A2)** *hold. Then the solution* $E(\cdot)$ *of Riccati equation associated to LQ problem with* <u>*non-zero target*</u>*, satisfies the large-time behavior*

$$\|E(t) - \overline{E}(t)\| \leq Ke^{-\mu t}$$

where K*,* $\mu > 0$ *independent of* x, t *and* z*, and* $\overline{E}(t)$ *is uniquely defined as*

$$\overline{E}(t) := \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2^* & t\Lambda_3 + \beta \end{pmatrix}$$

such that $(\Lambda_1, \Lambda_2, \Lambda_3, \beta) \in \mathcal{S}_n^{++}(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R}) \times \mathcal{S}_n(\mathbb{R}) \times \mathcal{S}_n(\mathbb{R})$*, independent of* t, x *and* z*.*

Theorem

The value function $V(x,t)$ of the LQ problem satisfies the HJB equation. In addition, if **(A1)** *and* **(A2)** *are satisfied, the value function has the following large-time behavior*

$$\exists\,!\,(c,\psi)\in\mathbb{R}^{+}\times C(\mathbb{R}^{n}),\ s.t.\ \forall\, x\in\mathbb{R}^{n},\quad |V(x,t)-(\psi(x)+ct)|\leq Ke^{-\mu t}$$

where $C(\mathbb{R}^n)$ is the set of continuous functions defined on $\mathbb{R}^n$ and K,μ positive constants independent of t,x and z.

Main results I

Asymptotic Behavior

Recall the stationary problem

$$c + H(x, \nabla\psi(x)) = \ell(x), \quad \forall x \in \mathbb{R}^n$$

We have

- (c, ψ) is a solution $\forall\, x \in \mathbb{R}^n$, *Proof: computations*;
- $(V_s, \lambda^{\#})$ satisfies the equation for $x = x^{\#}$.

Question: what is the relation between them?

Outline

It states that:

> *The most efficient strategy for an OCP when the time horizon is very large, is to quickly move the optimal path to a level close to the steady one, where it remains until the time is close to end. Then it moves towards its final target.*

This has been studied in the framework of C.V. and Riccati theory.

We will see how it appears in the language of HJB.

From "Large Time versus Steady State Optimal control", A. Poretta and E. Zuazua (2013), we can withdraw the following convergence results:

Assuming **controllability** and **observability**

- $$\frac{1}{T}\min_{u\in U} J^T \xrightarrow{T\to\infty} \min_{u\in\mathbb{R}^m} J_s =: V_s$$
- $$|u^T(s)-\bar{u}| + |x^T(s)-\bar{x}| \leq K(e^{-\lambda s}+e^{-\lambda(T-s)}).$$
- $$|p^T(s)-\bar{p}| \leq C(e^{-\lambda s}+e^{-\lambda(T-s)})$$

Outline

Corollary

Under the same assumptions as in the previous theorem, turnpike property insures that

- $c = V_s$ *the value of the steady problem;*
- ψ *is s.t.* $\nabla\psi(\bar{x}) = \bar{p}$, *where* $(\bar{x}, \bar{p})$ *is the pair steady state, co-state.*
- (c, ψ) *satisfies the stationary HJB equation;*
- *In addition, the pair* (c, ψ) *satisfies*

$$\begin{aligned} c + H(x, \nabla\psi(\bar{x})) &\leq \ell(x), \quad \forall x \in \mathbb{R}^n \\ c + H(\bar{x}, \nabla\psi(\bar{x})) &= \ell(\bar{x}) \\ c + H(\bar{x}, p) &\geq \ell(\bar{x}), \quad \forall p \in \mathbb{R}^n \end{aligned}$$

As a consequence, we can provide an alternative proof of turnpike property.

Corollary

Under the same assumptions as in the previous theorem, if (V_s, ψ) satisfy stationary HJB equation and the inequalities in the previous corollary, then the following stand

$$x^t(s) \xrightarrow{t\to\infty} \bar{x} \text{ exponentially;}$$

$$p^t(s) \xrightarrow{t\to\infty} \bar{p} \text{ exponentially;}$$

$$\frac{1}{t}V(x,t) \xrightarrow{t\to\infty} V_s$$

Outline

Denote $\vec{\alpha}(t) = x(t) - \bar{x}$ and recall for $t \in (0, T - \tau)$ for $\tau > 0$,

$$\frac{d}{dt}(x(t) - \bar{x}) = (A - BB^*\Lambda_1)(x(t) - \bar{x}) + o(e^{-C(T-t)})$$

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  - Tangential component: $\mathbb{A}\vec{\alpha} - \langle \mathbb{A}\frac{\vec{\alpha}}{\|\vec{\alpha}\|}, \frac{\vec{\alpha}}{\|\vec{\alpha}\|}\rangle\vec{\alpha} =: \omega(\theta)\vec{\alpha}$
    $\Rightarrow$ constant angular velocity $\Rightarrow$ periodicity with a constant period

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Therefore $x(t) = \bar{x} + \varepsilon(t)e^{i\frac{2\pi}{T_{per}}t}$, such that $\varepsilon(\cdot)$ exponentially decaying.

Qualitative Results

State behavior

$$\frac{d}{dt}(x(t)-\bar{x}) = (A - BB^*\Lambda_1)(x(t)-\bar{x}) + o(e^{-C(T-t)})$$

Entering Time. Using stability of $A - BB^*\Lambda_1$

one has $\|x-\hat{x}\| \leq e^{-\mu t}\|x_0-\hat{x}\|$,

time to be ε-close to $\bar{x}$: $t_\varepsilon := \frac{1}{\mu}\ln\left(\frac{\|x_0-\bar{x}\|}{\varepsilon}\right)$

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Exit Time. $t \to T^-$, the approximation $P_1(t) \approx \Lambda_1$ is no more valid,

The feedback is expressed with $P_1(t)$;

But $P_1(t) \xrightarrow[t \to T^-]{} 0$ (this is the backward Riccati);

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$\Rightarrow$ the dynamics is less stable when x leaves the steady state towards its final state

In both situations, the speed is high.

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Outline

Qualitative Results

Level set behavior

Denote

$$\hat{x} : t \mapsto \hat{x}(t) := -E_1^{-1}(t)E_2(t)z \xrightarrow[t\to+\infty]{} \hat{x}(t) = -\Lambda_1^{-1}\Lambda_2 z$$

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Level sets write

$$L_c^{\infty}(V) = \{(t,x) \in (0,+\infty) \times \mathbb{R}^n \mid \|\Lambda_1^{\frac{1}{2}}(x - \underline{\hat{x}})\|^2 + \langle \beta z, z\rangle - \|\Lambda_1^{-\frac{1}{2}}\Lambda_2 z\|^2 + t\langle \Lambda_3 z, z\rangle = c\}$$

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and describes elliptic curves in the space (t,x), which center is $\underline{\hat{x}}$.

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Velocity $\vec{v} = \frac{\delta x}{\delta t}$ of propagation is obtained with

$$\langle \bar{E}(t+\delta t)\begin{pmatrix} x+\delta x \\ z \end{pmatrix}, \begin{pmatrix} x+\delta x \\ z \end{pmatrix}\rangle = \langle \bar{E}(t)\begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix}\rangle$$

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and writes

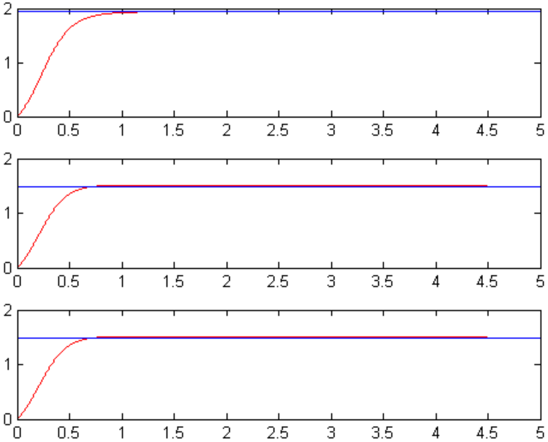
$$\vec{v} = \frac{\langle \Lambda_3 z, z\rangle}{2\|\Lambda_1(x - \underline{\hat{x}})\|}\vec{n}, \quad \text{where } \vec{n} = -\frac{\Lambda_1(x - \underline{\hat{x}})}{\|\Lambda_1(x - \underline{\hat{x}})\|}$$

Outline

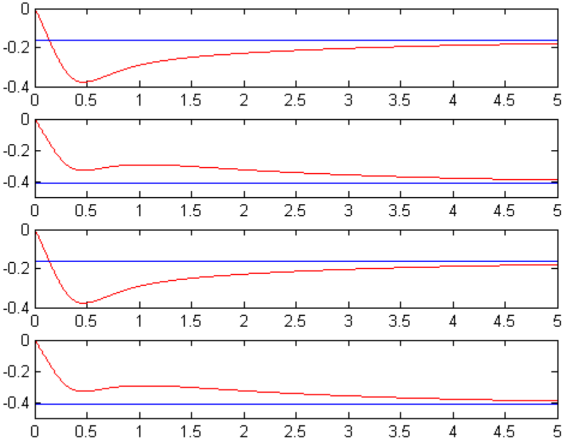
Numerical results

Riccati Convergence

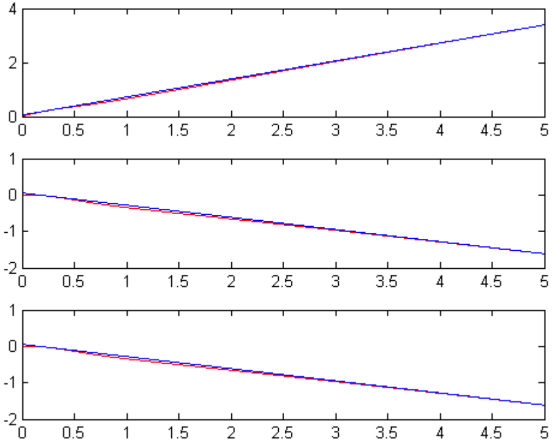
$$(E_{1,i}(t))_{1\leq i\leq 3} \xrightarrow[t\to\infty]{} (\Lambda_{1,i})_{1\leq i\leq 3}$$



$$(E_{2,i}(t))_{1\leq i\leq 4} \xrightarrow[t\to\infty]{} (\Lambda_{2,i})_{1\leq i\leq 4}$$



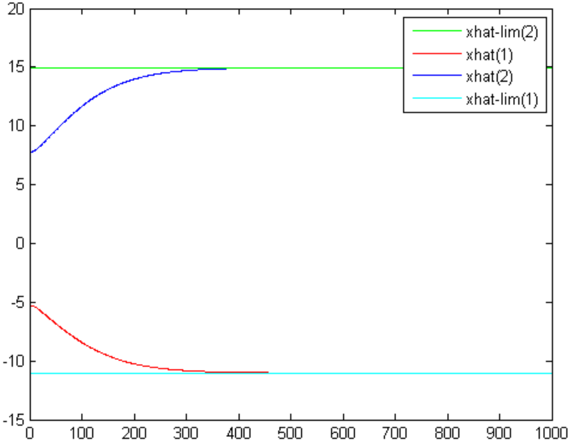
$$(E_{3,i}(t))_{1\leq i\leq 3} \xrightarrow[t\to\infty]{} (\Lambda_{3,i}t+\beta)_{1\leq i\leq 3}$$



Numerical results

Riccati Convergence

$$\hat{x}(t) \xrightarrow[t\to\infty]{} \underline{\hat{x}}$$



Numerical results
Level sets

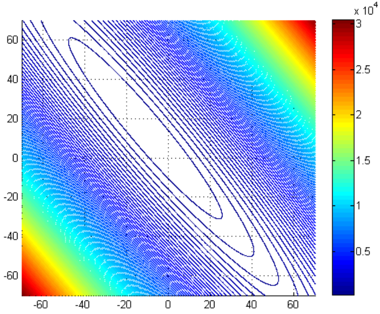


Figure: Level sets for fixed time.

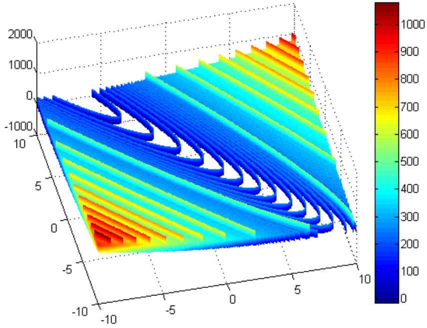
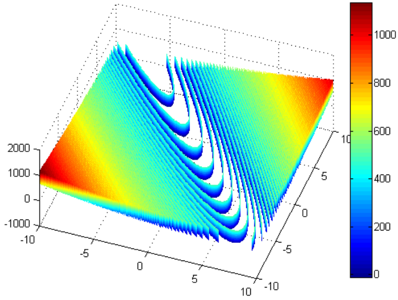


Figure: Level sets for different time.

Numerical results

Level sets



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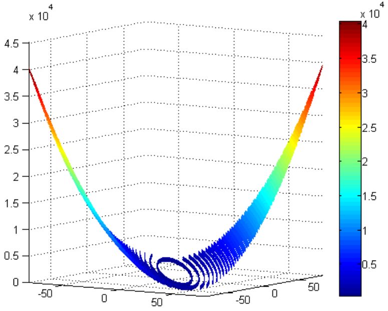

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Numerical results
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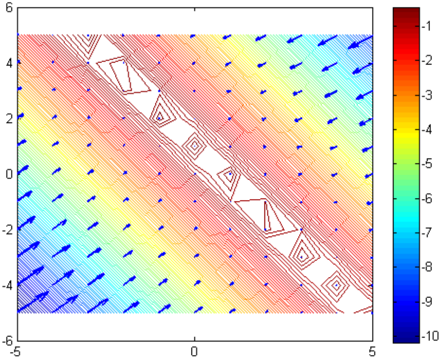
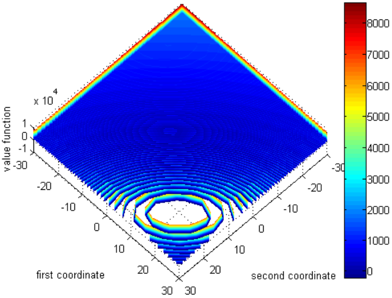


Figure: Velocity field of propagation of level sets.



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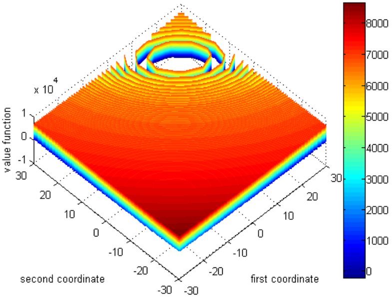

Figure: Level sets for different time.

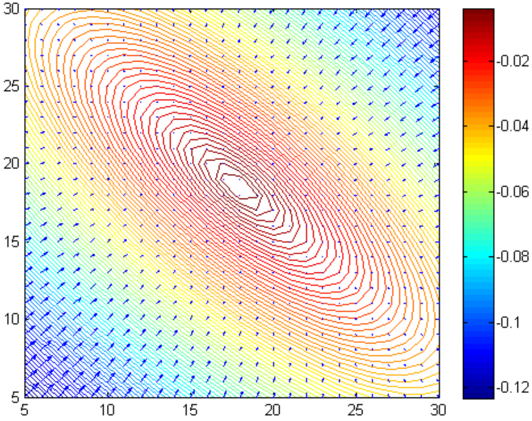

Figure: Velocity field of propagation of level sets.

DyCon

Level sets for time-dependent problem

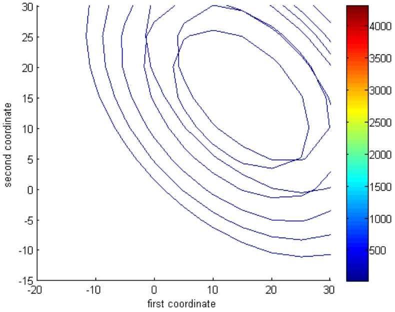

Figure: Level sets with value in $[0, 100]$.

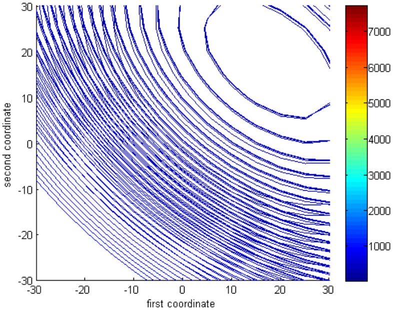

Figure: Level sets with value in $[100, 500]$.

comparison between level of time-dependent and stationary problem

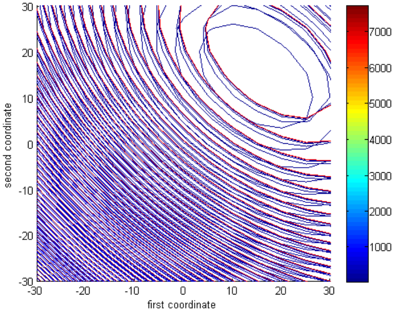

Figure: Level sets for the time-dependent problem.

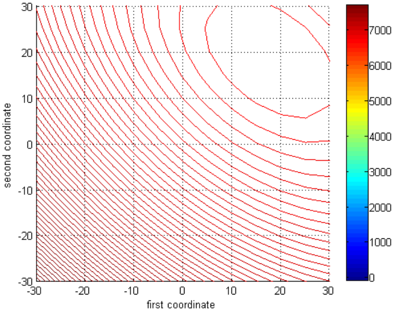

Figure: Level sets for the stationary problem.

Difference between level sets of the time-dependent and the stationary pb.

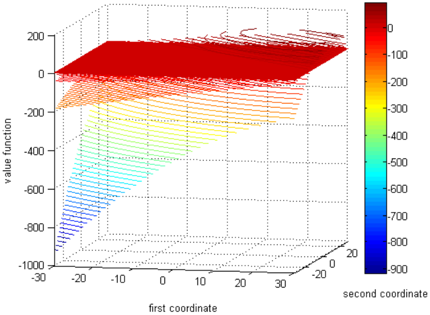

Figure: For $t \in [0, 100]$.

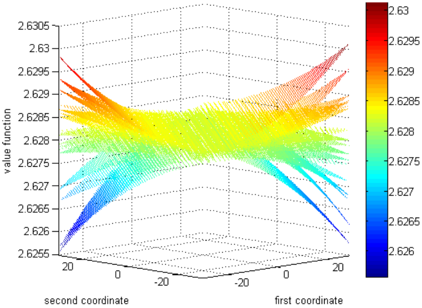

Figure: For $t \in [80, 100]$.

Each curve represents one level set. It is the projection on one coordinate.

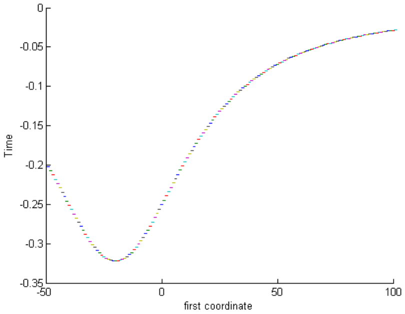

Figure: Ellipses at each time gets bigger when time $T - t$ gets smaller.

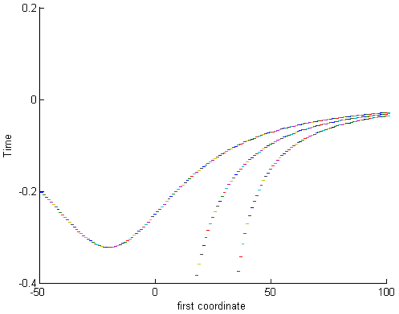

Figure: 3 different level sets (the one on the top is the smallest value).

Outline

Consider a Lagrange type optimal control problem:

minimize $J_{t,x}(u)$ over all controls $u : [t, T] \to U$

$$J_{t,x}(u) = \int_t^T L(y(s), u(s))ds, \quad \text{where } y(\cdot) = y(\cdot; t, x, u)$$

step.1/7 Show that the control problem with unbounded control space is equivalent to the one with compact control space.

$$\inf_{u \in L^2([t,T],U)} J_{t,x}(u) = \inf_{\|u\|_\infty \le \mu_R} J_{t,x}(u)$$

using mainly the superlinearity assumption of the running cost and Lipschitz continuity of the dynamics + Gronwall Lemma.

step.2/7 Show that the value function is the unique viscosity solution of the Cauchy problem:

- The value function is a viscosity solution of HJ equation:
  - Pick an element of the super/sub-differential, and show it is a viscosity sub/super-solution.
- It satisfies a comparison principle for a state-dependent Hamiltonian in unbounded space:
  - Using an auxiliary function.

step.3/7 Show that the value function is SCL_{loc} and Lip_{loc} on $[0, T] \times \mathbb{R}^n$.

step.4/7 Under strict convexity assumption on the Hamiltonian, the value function is differentiable at all $(\tau, y(\tau)), \forall\, \tau \in]t, T[$ and where $y(\cdot)$ is an optimal trajectory.

- *In the LQ case, the Hamiltonian is strictly convex if BB^* is positive definite.*

Now, if we consider special cases of Lagrange problem, we can get better regularity.

Solutions of HJB equation

Regularity – Lagrange case vs. LQ case

step.5/7 For a dynamic $f(x,u) = Ax + Bu$, the value function is convex in x, semiconvex in (t,x), and if in addition the running cost L is $C^{1,1}_{\text{loc}}$, then the value function is in $C^{1,1}_{\text{loc}}([0,\infty[\times\mathbb{R}^n)$.

step.6/7 If in addition the cost is quadratic, i.e. LQ case, we can show that $\exists\ T^* > 0$ such that the value function of an LQ problem is in $C^2([0, T^*[\times\mathbb{R}^n)$.

Step.7/7 If in addition the pair (A,B) is stabilizable, the value function is in $C^2([0,\infty[\times\mathbb{R}^n)$.

Summary

- Traveling wave asymptotic behavior of the value function for an LQ problem under different assumptions than those in literature

$$V(x,t) \approx \psi(x) + ct$$

- It satisfies an ergodic problem where the ergodic cost is nothing but the value of the steady problem

$$c + H(x, \nabla\psi(x)) = \ell(x)$$

- Turnpike property characterizes this ergodic problem.
- Describe the turnpike property using the value function.
- Investigate regularity of the value function as a solution of HJ.

  Outlook

  - Withdraw more information from the ergodic problem to characterize turnpike property (without Riccati!).
  - Investigate systems affine in the control and use tools from geometric control.

For Further Reading I

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arXiv:1709.O8387v2, 2010.

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- Martino Bardi, Italo Capuzzo Dolcetta,
  Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations,
  Birkhauser Boston Inc., 2008.

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  Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control,
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Thank you!

Proof (sketch).

Study $Q = E - \overline{E}$ and the nice structure it yields, using DRE and ARE.

Step.1 Existence and uniqueness[a] of Λ_1 such that $E_1(t) \xrightarrow[t\to+\infty]{} \Lambda_1$.

Λ_1, Q_1 satisfy a Riccati equation, independent of other terms.

Step.2 Existence and uniqueness of Λ_2 and Λ_3.

Stability + Λ_2 (resp. Λ_3) is expressed with Λ_1 (resp. Λ_2).

Step.3 Convergence $E_2(t) \xrightarrow[t\to+\infty]{} \Lambda_2$.

Stability + exp. decay of Q_1 + estimates.

Step.4 Convergence $E_3(t) - \Lambda_3 t \xrightarrow[t\to+\infty]{} \beta$.

Stability + exp. decay of Q_2 + Duhamel's formula + estimates.

□

[a] Using "Long Time vs. Steady State OC", by A.Poretta & E.Zuazua, 2013.

Proof.

- From DPP: the value function satisfies the Cauchy problem.
- To get the convergence: use $V(x,t) = \frac{1}{2}\langle E(t)\begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix}\rangle$ where $E(\cdot)$ converges exponentially towards $\overline{E}(\cdot)$ by previous Lemma.
- Setting

$$c = \frac{1}{2}\langle \Lambda_3 z, z\rangle, \quad \text{and} \quad \psi(x) = \frac{1}{2}\left(\langle \Lambda_1 x, x\rangle + 2\langle x, \Lambda_2 z\rangle + \langle \beta z, z\rangle \right)$$

  we recover the desired inequality together with the existence of $(c, \psi) \in \mathbb{R}^+ \times C(\mathbb{R}^n)$.
- Uniqueness is also insured by uniqueness of $\Lambda_1, \Lambda_2, \Lambda_3, \beta$.

□

Appendix

H0 The control set U is compact.

H1 $\exists\ K_1 > 0$ such that
$|f(x_2, u) - f(x_1, u)| \leq K_1|x_2 - x_1|,\ \forall\ x_1, x_2 \in \mathbb{R}^n,\ \forall\ u \in U.$

H2 f_x exists and is continuous, in addition there exists $K_2 > 0$ such that
$\|f_x(x_2, u) - f_x(x_1, u))\| \leq K_2|x_2 - x_1|,\ \forall x_1, x_2 \in \mathbb{R}^n,\ u \in U.$

L1 $\forall\ R > 0,\ \exists\ \gamma_R$ such that
$|L(x_2, u) - L(x_1, u)| \leq \gamma_R|x_2 - x_1|,\ \forall x_1, x_2 \in \mathcal{B}_R,\ u \in U.$

L2 $\forall\ R > 0,\ \exists\ \lambda_R$ such that
$L(x, u) + L(y, u) - 2L\left(\frac{x+y}{2}, u\right) \leq \lambda_R|x - y|^2,\ \forall\ x, y \in \mathcal{B}_R,\ u \in U \cap \mathcal{B}_R.$

L3 $\forall\ R > 0,\ \exists\ \lambda_R$ such that
$L(x, u) + L(y, u) - 2L\left(\frac{x+y}{2}, u\right) \leq \lambda_R|x - y|^2,\ \forall\ x, y \in \mathcal{B}_R,\ u \in U.$

H* $\exists\ K_0$ such that $|f(x, u)| \leq K_0(1 + |x| + |u|),\ \forall x \in \mathbb{R}^n,\ u \in U.$

L* $\exists\ l_0, M_0 \geq 0, m_0 > 0$, a function $l : [0, \infty[\to [0, \infty[$ satisfying
$l(r) \geq \frac{r^2}{m_0},\ \forall r \geq M_0$, and s.t. $L(x, u) \geq l(|u|) - l_0,\ \forall\ x \in \mathbb{R}^n, u \in U.$