Internship Report

# Discrete-Time Optimal Control of a Pool of Electric Water Heaters: Testing Decomposition Method and Best-Response Dynamics


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# Abstract

This project is dealing with the efficiency of who decomposition techniques on the problem of a pool of tanks which is a large scale problem. Because of computational cost is increasing exponentially with the number of tanks a global ploting, using a single optimization program, is not rational. Luckily we are having MILP/MIQP problem with separable constraints which allows us to use decomposition-coordination techniques. In this work, we considered and simulated two approaches. 1) Lagrangian decomposition, 2) Best Response Dynamics (or Block Coordinate Descent).

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# 1 Introduction

Energy resources are available to supply the world's expanding needs without environmental detriment. However, today the most used energy sources are based on fossil fuels which cause greenhouse effects. According to the International Electricity Agency's work, by 2014 the World's electricity generation by fuel is 66.7 % [1]. On the other hand, the World's population will continue to grow for several decades at least[2]. Energy demand is likely to increase even faster, and the proportion supplied by electricity will also grow faster still. There are numerous of studies and predictions about the relations between fossil fuels consumption and the effects of this consumption on the global warming. As seen in these studies, the necessity to overcome climate change by reducing greenhouse gas emissions is required for the sustainability of the life. On the other hand, being aware of this necessity, countries have attempt to change their energy resource system from centralized to decentralized for the purpose of using renewable energy sources (sometimes intermittent) allowing greater active participation of consumers by becoming producers themselves and/or by smarter demand response management of their own energy use. This decentralized system allows to apply Demand Side Management [3], which could potentially bring production closer to the point of consumption. It does not necessarily save energy, but rather shifts energy loads around in time. This is very important since it potentially avoids the need to shed excess energy supply at times of low demand or high supply. The management of small local users must be achieved automatically at their level which requires online communications (see fig1).

The energy performance of buildings addresses the energy consumption of buildings, which represents from 25% to 40% of total European energy consumption and is responsible for 40% of total $CO_2$ emissions of the European Union[5]. In this project, we studied on controlling of the electric consumption of the large pool of electric hot water tanks found in homes in many countries appears as very relevant for load shifting applications due to its large storage capacity. Electric hot water tanks are heating the water over relatively long periods of time, for later consumption. To reduce cost of operation, a simple but efficient strategy may consist in using the

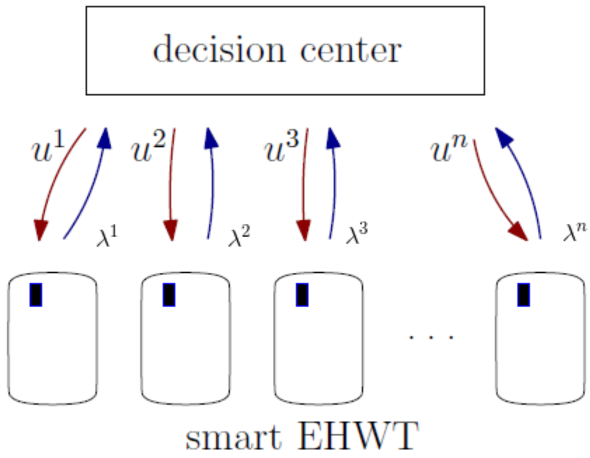


**Figure 1:** Interactions between smart local users and the center [4]

electricity in the night time (a period when electricity price is low), while hot water used in the next day-time. Thanks to Nathanael Beeker et al.[6], their well presented model of optimal control of one or several electric hot water tanks with separable constraints opens a way to apply decomposition-coordination techniques.

However, when the number of hot water tanks is large, we are facing a large scale optimization problem which needs to be decomposed in order to apply classical optimization techniques. On the other hand, the constraints of the problem add non-convexity property to our problem which obstruct us to work in a convex framework. Optimizing non-convex functions is difficult mainly for two reasons. First is that the simple characterization of the critical point such as the KKT conditions may not be applicable. The second is the possible existence of many local optima, a search algorithm that is looking for the global optimum point might be get stuck at a local optimum point. On the other hand it is easy to minimize a convex function over any convex subset of its domain because of any local minimum is automatically a global minimum.

Our aim is to find an algorithm allowing to solve this problem efficiently. We consider two approaches. 1) Lagrangian decomposition, 2) Best Response (or Block Coordinate Descent) in two cases: 1) as a benchmark where our problem is convex and objective function is smooth and drains are deterministic 2) a realistic model.

# 2 Discrete-Time Optimal Control of Electric Hot Water Tanks

In this chapter, first we build a basic structure to open the way to optimization under the general considerations and assumptions. Latter we discuss about system dynamics and system constraints for one tank where we face some conditions on constraints which helps us to understand the problem's characteristics. And at the end we glide into problems with objectives when the number of tanks is $n$.

## 2.1 General Considerations and Modelling Assumptions

A typical electric hot water tank has a shape of vertical cylindrical and filled with water. Cold water is injected at the bottom while hot water is drained from the top at exactly the same flow-rate. Hence the tanks is always full. A heating element, which converts electricity into the heat through the process of resistive is plunged at the bottom of the tank.(see Fig.2)

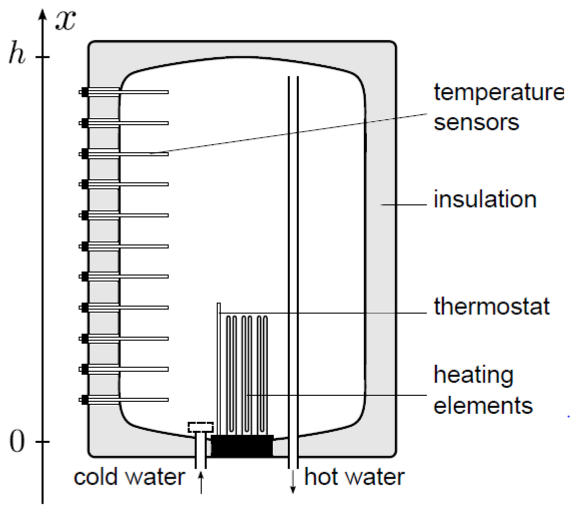


**Figure 2:** Schematic view of an electric hot water tank

The heating element is pole-shaped, and its length is up to one third of the tank. In the tank, layers of water with various temperatures coexists and. These layers are mixed only by heat diffusion. Existence of a non uniform quasi-equilibrium temperature profile which is increasing with height in the tank is called $stratification$. Thanks to the temperature sensors, it is available to measure temperature profile at each height of the tank. And thanks to the stratification that causes; while the hot water near the outlet of the tank is available for consumption, the heating element is able to heat the rest. Because of this stratification effect and the cylindrical symmetry of the tank we can assume that the temperature is homogeneous at each height and continuously increasing function of height.(see Fig.3)

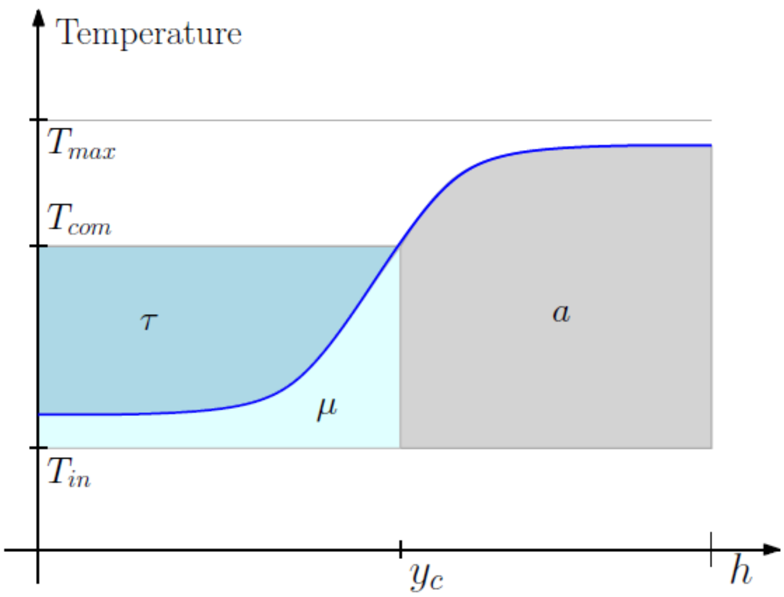


**Figure 3:** Temperature Profile, Available, Delay and Reserve Energy after a Drain

*Temperature Profile:* We assume that the cold water is injected at the bottom has a constant temperature, called $T_{in}$, which represents a lower bound of the temperature profile. The heating process is driven by turbulence generated by buoyancy property during the heating process, which is the cause of a local mixing at the bottom of the tank. We consider that this mixing does not affect the temperature profile in the upper part of the tank. The user can specify a temperature $T_{max}$ at which the heating has to be stopped to prevent overheating via thermostat. As a result of the heating process, if the temperature at the bottom of the tank is $T_{max}$, then the temperature at each height in the tank is uniformly $T_{max}$. Also the user can set a comfort temperature $T_{com}$ for his comfort. Water having temperature higher than

$T_{com}$ is also useful because it can be blended with cold water to reach $T_{com}$, on the other hand water having temperature lower than $T_{com}$ is useless.

To avoid the control of the entire temperature profile, in order to simplify the system, we define a few variables of interests which is called state variables.

*The available energy* $a$ is defined as the energy contained in the zone having temperature greater than the comfort temperature $T_{com}$. If $a$ reaches the value 0 and a water drain is applied, it means that the user is trying to consume hot water when none is available, and therefore that the comfort constraints are broken.

*The delay energy* $\tau$ is defined as the energy required by the system to reach the temperature $T_{com}$. If the tank is heated at constant maximum power, and without drains and heat losses, $\tau$ has a tendency to decrease in the state $a$ can effectively be increased.

*The reserve energy* $\mu$ is defined as the energy contained in the tank under $T_{com}$ that is currently unavailable for consumption. When, during the heating process, $\tau$ reaches the value 0, the energy $\mu$ becomes available to consumption. This generates an immediate increase of $a$ and decrease of $\mu$ up to 0.

The idea of these definitions is that to plan the heating before the energy reserves embodied by $a$ (in the total energy $a + \mu$ is consumed) and the time necessary to provide new hot water which is embodied by $\tau$.

A drain is mainly characterized by a decrease of $a$ and an increase of $\tau$ with a insignificant raise of $\mu$ due to an energy transfer from $a$.

Note that h the height of the tank, S its cross-section and T() is the increasing temperature profile of the water defined on $[0, h]$. Then, the definitions yield the following expressions of $a$, $\tau$, $\mu$ as

$$a = S\rho c_p \int_{y_c}^{h} T(y)dy$$

$$\tau = S\rho c_p \int_{0}^{y_c} (T_{com} - T(y))dy$$

$$\mu = S\rho c_p \int_{0}^{y_c} T(y)dy$$

where $\rho$ and $c_p$ are the density and the heat capacity of water, respectively, and $y_c$ is defined as

$$y_c := min\,\{y \mid T(y) = T_{com}\}$$

Hot water consumption is an (uncontrolled) input of our problem and represented by drain in the system. For his comfort, the user consumes certain quantities of energy each day. On the other hand, the heat injected via the heating element in the tank is a control variable ($(u)$ see fig4). The control have a number of objectives. The most obvious one is cost reduction for a single unit in response to a price signal and second one is reaching a load profile for the aggregated consumption in where the pool of tanks exist.

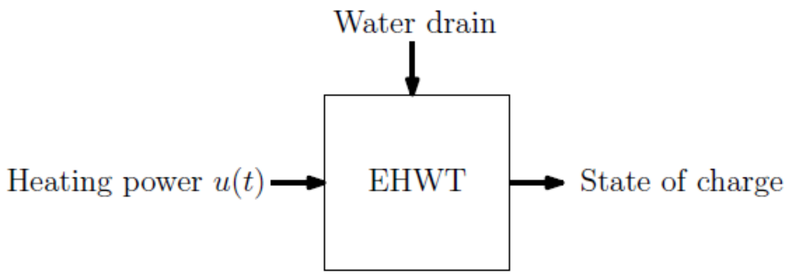


**Figure 4:** Schematic view of inputs and output

## 2.2 Discrete-Time Dynamics

We establish a discrete-time model for the dynamics of the triplet $x = (a, \tau, \mu)$ over a finite horizon which is discretized into uniform time-steps $[0, ..., p-1]$. Note that $(a_0, ..., a_{p-1})$ , $(\tau_0, ..., \tau_{p-1})$ and $(\mu_0, ..., \mu_{p-1})$ respectively state the values of $a, \tau, \mu$ at each of these time-steps.

At each time-step $t\epsilon\,[0, ..., p-1]$, energy is consumed by the user via draining by means of an amount $d_t$ that is already known and considered as an input of our programming. Energy is introduced via the heating element by an amount $u_t \in [0, u_{max}]$ in terms of at each step the heating element's utilization is limited. We divide this energy into two parts $v_t$ and $w_t$ representing respectively the share introduced in $a_t$ and $\mu_t$ . And a flow of energy is defined with the variable $\Phi_t$ from $\mu_t$ to $a_{t+1}$ under some conditions which is explained below.

*Balance Equations:* For any t, the dynamics of $x_t$ is given by the energy balance,

$$a_{t+1} = (1-r)a_t - \alpha d_t + v_t + \phi_t \tag{1}$$

$$\tau_{t+1} = \tau_t + r\mu_t + \beta_t d_t - w_t \tag{2}$$

$$\mu_{t+1} = (1-r)\mu_t - (1-\alpha)d_t + w_t - \phi_t \tag{3}$$

The heat losses at the state $x_{t+1}$ is modeled with an exponential decay that is characterized by the erosion of $a$ and $\mu$ at a rate $r$. This erosion has a positive effect on the increase of $\tau$. The energy consumed by the user $d_t$ during one time-step is split between $a$ and $\mu$ with a coefficient $\alpha$ , $1-\alpha$ and affects $\tau$ with a coefficient $\beta$.

*Definition of the heat source terms $v_t$ , $w_t$:* The energy is injected at the bottom of the tank via the heating element. When $\tau > 0$ the heating has no impact on the available energy $a$, but instead tends to reduce $\tau$ and increase $\mu$ until $\tau = 0$. When the value of $\tau$ is 0 , the injected energy becomes immediately available. This can be modeled by dividing $u_t$ into two shares $v_t$ and $w_t$ representing, respectively, the part of the injected energy going into $a$ and $\mu$ , and subject to the following conditions,

$$u_t = v_t + w_t \tag{4}$$

$$0 = v_t \tau_{t+1} \tag{5}$$

$$0 \leq v_t, w_t \leq u_{max} \tag{6}$$

Given these conditions, if $\tau_{t+1} > 0$ then no energy can be introduced in $a_{t+1}$. Moreover $v_t = 0$ and $w_t = u_t$ , and if $\tau_{t+1} = 0$ the value of $w_t$ and energy drain $\beta d_t$ have to compensate for heat losses $r\mu_t$ in the balance equations and while the remainder is introduced in $a_{t+1}$.

*Definition of internal energy flow $\phi_t$:* The flow $\phi_t$ from $\mu_t$ to $a_{t+1}$ is generally equal to 0, except when $\tau = 0$. Then, the value of $\phi$ is defined by the fact that all the energy $\mu$ suddenly becomes available. This can be described as follows

$$0 = \phi_t \tau_{t+1} \tag{7}$$

$$0 \leq \phi_t \tag{8}$$

$$\tau_{t+1} = 0 \Rightarrow \mu_{t+1} = 0 \tag{9}$$

*Bounds and Temperature Comfort Constraints:* Here we determine some bounds to the state $x_t$ for all $t$. This defines the admissible controls that allow each $x_t$ to respect these constraints. Moreover, to ensure that the dynamics are properly defined, we assume $\tau_t \geq 0$ for all $t$.The case where $a_t$ or $\mu_t$ is negative, corresponds to the drain of non-existence of the energy in the tank and therefore the user's comfort constraints are broken. For this reason, for all $t$, we require $a_t \geq 0$ and $\mu_t \geq 0$. Let $m = S\rho c_p(T_{max} - T_{in})$ is the maximal energy that can be contained in the tank, and

$$\lambda := \frac{T_{com} - T_{in}}{T_{max} - T_{in}}$$

Then from the integral definitions of $a$, $\tau$ and $\mu$ we have $a_t \leq m$, $\tau_t \leq \lambda m$ and $\mu_t \leq \lambda m$ for all $t$.

Moreover, from physical constraints on the total energy we get,

$$\lambda a_t + \tau_t + \mu_t \leq \lambda m$$

$$\lambda m \leq a_t + \tau_t + \mu_t$$

Given these relations, we define the following domain

$$\Omega = \left\{ (a, \tau, \mu) \in \mathbb{R}^3_+ \mid \begin{array}{c} \lambda m \leq a + \tau + \mu, \text{and} \\ \lambda a + \tau + \mu \leq \lambda m \end{array} \right\}$$

Then, $\forall t$ , $x_t$ is subject to the constraint

$$x_t \in \Omega \tag{10}$$

This ensures that no energy is drained more than the tank can provide and the tank is not overheated.

*Admissible Controls:* For given $(a_0, \tau_0, \mu_0) \in \Omega$ and $d = (d_0, ..., d_{p-1}) \in \mathbb{R}^p_+$ and for chosen control sequence $u = (u_0, ..., u_{p-1})$ the relations of constraints for $t \in [0, ..., p-1]$ uniquely define $(a, \tau, \mu, v, w, \phi)$. This allows us to define the admissible set $U$:

$$U(a_0, \tau_0, \mu_0, d) = \{u \in \mathbb{R}^{7n} \mid \begin{array}{c} a, \tau, \mu, v, w, \phi \text{ defined by } (1) - (9), \forall t \in [0, ..., p-1] \\ \text{and satisfy } (10), \forall t \in [0, ..., p] \end{array} \}$$

With given initial conditions and a drain sequence; it constitutes the set of heating periods that do not break the comfort constraints of the user.

*Remark:* Different tanks have different amount of drains ($d$) which causes different admissible sets for each tanks.

*Strengthening of (9):*

Except the two product conditions (5) and (7) and the condition (9), the relations from (1) to (10) are linear equalities and inequalities in the variables $(u_t, a_t, \tau_t, \mu_t, v_t, w_t, \phi_t)$. and therefore define a polytope of $\mathbb{R}^{7n}$ for each tank. In order to solve linear/quadratic program via commercial software we need to modify the condition (9) since its structure is not suitable for linear programming yet. Given the set

$$A = \{(\tau, \mu) \in [0, \lambda m]^2 \mid \tau = 0 \Rightarrow \mu = 0\}$$

then equivalently,

$$A = [0, \lambda m]^2 \setminus \{ \ (\tau, \mu) \in [0, \lambda m]^2 \mid \tau = 0, \mu > 0\}$$

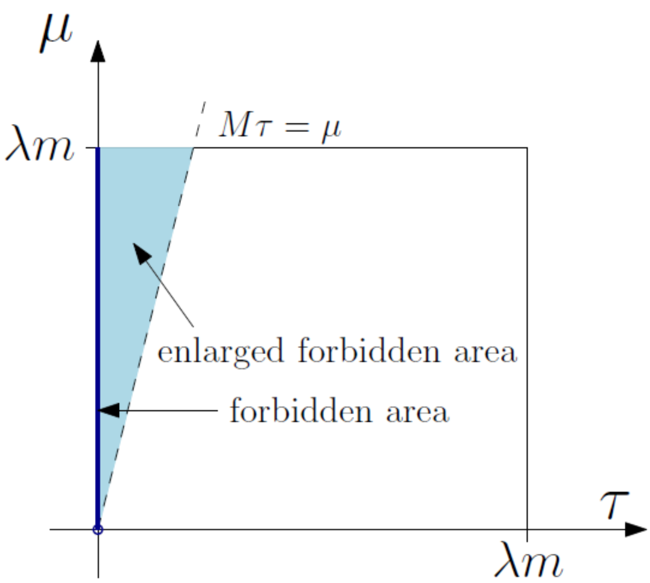


**Figure 5:** Strengthening of (9)

A possible adaptation is

$$\{(\tau, \mu) \in [0, \lambda m]^2 \mid M\tau_t \geq \mu_t\} \subset A$$

where $M > 0$ as seen in the Figure 9. Instead of considering (9) we consider (11) which has the linear form

$$M\tau_t \geq \mu_t \tag{11}$$

choosing $M > 0$ large. This new relation is compatible with a linear programming formulation, and if $M$ is sufficiently large, only a small feasible regime which is be negligible is left out of the optimization problem.

Given this new relation, finally we define the polytope

$$\chi(a_0, \tau_0, \mu_0, d) = \{\chi = (u, a, \tau, \mu, v, w, \phi) \in \mathbb{R}^{7n} \mid \begin{array}{c} (1) - (8), (11), \forall t \in [0, ..., n-1] \\ \text{and } (10), \forall t \in [0, ..., n] \text{ are satisfied.} \end{array}\} \tag{12}$$

*Remark for (5) and (7):* The product conditions (5) and (7) contains the variables that can not be positive at the same time. We will take into account them which is called binary or integer variables by using SOS (Special-Ordered Set) features in order to apply Branch and Bound Algorithm efficiently in terms of the speed. Moreover this SOS constraints make our admissible sets non-convex. So we face with a non-convex problem.

## 2.3 Objective Function and Formulation of an Problem

### 2.3.1 Objective Function

We consider that $(a_0, \tau_0, \mu_0, d_j)$ are given for each tank. In practice, the electricity producer has two objectives; while minimizing the cost of heating for a given a price signal over time $(c_0, ..., c_{p-1})$, reaching a target profile and minimizing the deviation cost.

1) Given a price signal for electricity over time $(c_0, ..., c_{p-1})$ for each tank $j$, the optimal control of cost of heating is defined as follow

$$min_{u_j \in U(a_0,\tau_0,\mu_0,d_j)} \sum_{t=0}^{p-1} \sum_{j=1}^{n} c_{j,t} u_{j,t} \tag{13}$$

The meaning of this objective is therefore to search for $n$ numbers of electric hot water tanks, the heating strategy that minimizes the cost of heating while ensuring a required supply of hot water to the users. Notice that this is separable criterion.

2) In order to reach the target profile, the cost function can be defined as a quadratic distance to an objective for aggregated consumption $(P_t)_{t \in [0,...,p-1]}$ where $\gamma_t$ is the

penalty cost. Then, the problem is

$$min_{u_1,...,u_n \in U(a_0,\tau_0,\mu_0,d_j)} \sum_{t=0}^{p-1} \gamma_t (P_t - \sum_{j=1}^{n} u_{j,t})^2 \tag{14}$$

Notice that this aggregated objective is non-separable.

### 2.3.2 Controlling the Aggregate Consumption of $n$ Water Heaters to Reach a Target Profile

Control of the power injection $u_t = (u_{1,t}, ..., u_{n,t})$ of a group of $n$ tanks during time period $[0, p-1]$. The electricity producer desires to minimize a given objective function, while ensuring user's needs. For a given tank $j$, the fact that a strategy $u_j$ ensures user's comfort will be noted $u_j \in U_j$ ; $U_j$ being the set of admissible controls for tank $j$. Given a price signal for electricity over time $c_j$ an optimal control problem can be,

$$min_{u_1,...,u_n \in U(a_0,\tau_0,\mu_0,d_j)} \sum_{t=0}^{p-1} \sum_{j=1}^{n} (c_{j,t} u_{j,t} + \gamma_t (P_t - \sum_{j=1}^{n} u_{j,t})^2) \tag{15}$$

with $(\gamma_0, ..., \gamma_{p-1}) \in \mathbb{R}_+^n$ allows to compute the minimum cost for each tank, while approaching some global consumption for the times $t$ with positive $\gamma_t$, representing the relative importance of the consumption goal to the price.

*Peak Periods for n Tanks:* We are going to consider 3 time periods.
$i$) Adjustment period: Here our consumption target is zero, $P_t = \sum_j u_j^{max}$ When there is no demand, (off-peak hours)
$ii$) Effacement Period: $P_t = 0$ when the demand is high and we want them to consume less (peak hours)
$iii$) Outside the periods of ($i$) and ($ii$), where we have no consumption goal.

for $\forall u_j \in U(a_0, \tau_0, \mu_0, d_j)$ we can define our problem in following way

$$min \sum_{t \in Adjustment}^{p-1} (c_t u_{j,t} + \gamma_t (P_t - \sum_{j=1}^{n} u_{j,t})^2) +$$

$$\sum_{t \in Effacement}^{p-1} (c_t u_{j,t} + \gamma_t (\sum_{j=1}^{n} u_{j,t})^2) +$$

$$\sum_{t\in Outside}^{p-1} c_t u_{j,t}$$

Considering these time periods, we finally have for $\forall u_j$

$$min_{u_j} \sum_{j=1}^{n}\sum_{t=0}^{p-1}(c_{j,t}u_{j,t} + \gamma_t \sum_{t\in T}(P_t - \sum_{j=1}^{n} u_{j,t})^2)$$

where T=adjustment+effacement and subject to the constraints defined in the previous part (2.2.1).
As a result, we are facing a MILP/MQLP with separable constraints. Water drain profiles for each tank causes different admissible sets on control. Moreover, non-convexity caused by the integer constraints and high dimensionality because of the large number of tanks make the problem even harder to solve and we can not apply the classical optimization techniques directly. In the next chapters we are trying to find an algorithm allowing to solve this problem efficiently. For a benchmark, we omit the SOS constraints and consider the strong convex and smooth case of the problem.

# 3 Lagrangian Decomposition

From optimization point of view, the classical solving techniques can not be applied to the original-problem because of the high dimensionality and non-convexity of the problem. When we ignore the non-convexity of the problem caused by SOS constraints, we still have to cope with the large dimensionality. On the other hand, from organizational point of view, the major motivation to rely on decomposition is to provide a decentralized optimization approach. Indeed, for privacy reasons, we wish to avoid communicating local constraints to a central planner.

In this chapter, the aim is to decompose the initial system into smaller sub-systems or into smaller dimensions. The idea is; at first the decomposition phase; to formulate an optimization sub-problem for each sub-systems, and to solve it using classical optimization techniques (in a convex or non-convex framework) and second the coordination phase; to establish a process of information exchange between the sub-systems, based on the solutions to these sub-problems.

Under certain assumptions like convexity, the primal and dual problems have the same optimal objective function value; hence it is possible to solve the primal problem indirectly by solving its corresponding dual problem. This can lead to important computational advantages. On the other hand in non-convex problem, when the Lagrangian relaxation techniques are applied two unfortunate circumstances arise. First, the optimal solution of the dual problem is different than the optimal solution of the primal problem, the difference being the duality gap. Second, the solution of the primal problem associated with the optimal solution of the dual problem is generally in-feasible for the primal problem. On the other hand, thanks to the Weak Duality Theorem , in the non-convex case the objective function value at the optimal solution of the dual problem provides a lower bound to the objective function value at the optimal solution of the primal problem. Which is still useful for us. Hence in the convex case primal and dual values are the equal which is the case where duality gap is zero.

## 3.1 Problem Definition

Original (non-decomposed) problem is the following

$$min_u \sum_{j=1}^{n} \sum_{t=0}^{p-1} (c_{j,t} u_{j,t} + \gamma_t \sum_{t \in T} (P_t - \sum_{j=1}^{n} u_{j,t})^2)$$

Since $\gamma_t$ is a quadratic penalization cost and constant, we can take it $\gamma_t \Rightarrow \frac{\gamma_t}{2}$ to be well-adjusted to the form of MIQP Programming.

$$min_u \sum_{j=1}^{n} \sum_{t=0}^{p-1} (c_{j,t} u_{j,t} + \frac{\gamma_t}{2} \sum_{t \in T} (P_t - \sum_{j=1}^{n} u_{j,t})^2) \tag{16}$$

The problem (16) is not decomposable at this step because of the coupling term coming from the quadratic penalization. By adding new term $v_t$ (we can consider $v_t$ as a new tank added to the system) and $P_t - \sum_{j=1}^{n} u_{j,t} = v_t$, we have the following problem .

$$min_u \sum_{j=1}^{n} \sum_{t=0}^{p-1} (c_{j,t} u_{j,t} + \gamma_t \sum_{t \in T} (v_t)^2) \tag{17}$$

$$st.$$

$$v_t = P_t - \sum_{j=1}^{n} u_{j,t}$$

We consider this problem without SOS constraints, by this way we alter the non-convexity of the problem. Moreover, we assume that we have strong convexity and smoothness (the objective function is continuously differentiable and gradient is Lipschitz continuous) on the objective function. To have this smoothness we need to add quadratic penalization to each individual tanks. So we have the following

$$min_u \sum_{j=1}^{n} \sum_{t=0}^{p-1} (c_{j,t} u_{j,t} + \frac{g_{j,t}}{2} (u_{j,t})^2 + \frac{\gamma_t}{2} \sum_{t \in T} (v_t)^2) \tag{18}$$

$$st.$$

$$v_t = P_t - \sum_{j=1}^{n} u_{j,t}$$

where $g_{j,t} > 0$ and $\frac{g_{j,t}}{2}(u_{j,t})^2$ the quadratic penalization for each tank at each time-step.

Lagrangian relaxation method is most wide spread procedure for solving the problems. It is a relaxation method which approximates a difficult problem of constrained optimization by a simpler problem. A solution to the relaxed problem is an approximate solution to the original problem, and provides useful information. Within the Lagrangian Relaxation method, there are two main approaches: the classical Lagrangian relaxation method and the Augmented Lagrangian relaxation method. In this chapter, we are going to use the classical Lagrangian Relaxation Method.

If we relax the problem (18) using classical Lagrangian Method where $\lambda_{j,t}$ is called Lagrangian multiplier and is representing the coordination signal, we get the following Lagrangian function for $\forall j \in [1, n]$ and $\forall t \in [1, p-1]$

$$\mathfrak{L}(\lambda, u, v) = \sum_{j,t} c_{j,t} u_{j,t} + \sum_{j,t} \frac{g_{j,t}}{2}(u_{j,t})^2 + \sum_{t \in T} \frac{\gamma_t}{2}(v_t)^2 + \sum_{t \in T} \lambda_{j,t}(v_t + \sum_{j=1}^{n} u_{j,t} - P_t) \quad (19)$$

Note that, now we don't have any coupling term and problem (19) is ready to decompose.

$$\mathfrak{L}_{\mathrm{j}}(\lambda, u_j, v) = \sum_t (c_{j,t} + \lambda_{j,t}) u_{j,t} + \sum_t \frac{g_{j,t}}{2}(u_{j,t})^2 + \sum_{t \in T} \frac{\gamma_t}{2}(v_t)^2 + \sum_{t \in T} \lambda_{j,t} v_t - \sum_{t \in T} \lambda_{j,t} P_t \quad (20)$$

where $\mathfrak{L} = \sum_j L_j$. For each tank $j$ we have $n+1$ sub-problems included the extra tank $v$. Under the hypothesis of the existence of a Lagrangian saddle point of a problem (18), the concatenation of the solutions of the sub-problems(20) provides the solution of the global problem and the complicating constraint is satisfied. But since we have duality gap, because of our original problem's non-convexity property, the sum value of dual decomposition problems can be used only as a lower bound to the global problem from Weak Duality, nevertheless it is still useful for the comparison. Now, we would like to find the values of these dual-decomposed sub-problems.

- First, fix all $\lambda_{j,t}$ in order to solve sub-problems for $u_{j,t}$ and $v_t$

$$W(\lambda) := min_{v_t, u_{j,t}} \mathfrak{L}_{\mathrm{j}}(\lambda, u, v)$$

- To find the dual problem's value , $max_\lambda W(\lambda)$

$$max_\lambda min_{v_t,u_{j,t}} \sum_{j,t}((c_{j,t}+\lambda_t)u_{j,t}+\frac{g_{j,t}}{2}(u_{j,t})^2)+\sum_{t\in T}\frac{\gamma_t}{2}(v_t)^2+\sum_{t\in T}\lambda_t v_t$$

- From Weak Duality, the value of the dual problem is lower than the value of the primal problem. In order to find this lower bound we will use Steepest Descent Algorithm.

Here is the algorithm :
Dual Iteration on $\lambda_t$ , at iteration $k+1$ , for $\forall j,t$

- $(i)$
$$u_{j,t}^{k+1} \in argmin_{u_{j,t}} \sum_t (c_{j,t}+\lambda_t^k)u_{j,t}+\sum_t \frac{g_{j,t}}{2}(u_{j,t})^2$$

- $(ii)$
$$v_t^{k+1} \in argmin_{v_t} \sum_t \lambda_t^k v_t+\sum_{t\in T}\frac{\gamma_t}{2}(v_t)^2$$

- $(iii)$
$$\lambda_t^{k+1}=\lambda_t^k+\rho\ (v_t^{k+1}+\sum_{j=1}^{n}u_{j,t}^{k+1}-P_t)$$

  where $\rho > 0$ is a constant gradient step and must be chosen carefully. If it is too small the algorithm runs slow and it is too big the algorithm is not converging.

During this iteration, the tanks all use the same price $\lambda_t^k$ and calculate based on this price their optimal level of production $\sum_{j=1}^{n} u_{j,t}^{k+1}$. The price is then updated to reflect the difference between the total output of the iteration and the required production $P_t$. The constrained on the total production of units is satisfied that the convergence of the iterative process.

## 3.2 Appendix

Moreover we assumed our objective function is smooth, but in the original problem our objective function is not smooth. One can use Bundle Method or Sub-Gradient Method without assuming differentiability on the dual function. On the other hand, we could use Augmented Lagrangian Relaxation instead of Classical Lagrangian Relaxation with the assumption of non-convexity of the original problem. One advantage of

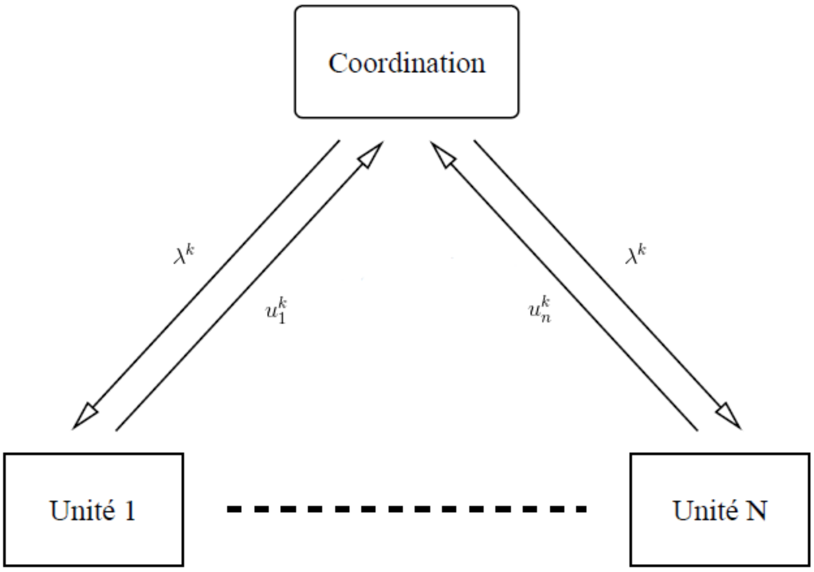


**Figure 6:** Schematic view of the price decomposition [7]

the Augmented Lagrangian Relaxation method over the Classical Lagrangian Relaxation method is that the augmented Lagrangian may obtain a feasible primal solution in cases where the classical Lagrangian presents a duality gap. Augmented Lagrangian Relaxation modifies the classical Lagrangian dual by appending a nonlinear penalty function on the violation of the dualized constraints in order to reduce the duality gap. Another advantage of the Augmented Lagrangian Relaxation method is that the dual function associated to the augmented Lagrangian is differentiable in cases where the Classical Lagrangian Relaxation method presents a non-differentiable dual function; However, when solving non-convex problems the Augmented Lagrangian Relaxation method may obtain a local optimizer without measure of its quality. In this case, the dual bound given by the Classical Lagrangian Relaxation method can be used as a quality measure of a solution.

One of the main drawbacks of the Augmented Lagrangian method, apart from obtaining a local optimizer, is that the quadratic term introduced by the augmented Lagrangian is not separable. The quadratic term has coupling terms (product of $v_t$ and $u_{(j,t)}$) which makes the problem non-separable.

If we apply Augmented Lagrangian Relaxation to our problem (17) we get the following; $\forall j, t$

$$L(u,v,\lambda) = \sum_t (c_{j,t}+\lambda_{j,t})u_{j,t}+\sum_t \lambda_{j,t}v_t+\frac{\gamma_t}{2}\sum_{t\in T}(v_t)^2+\frac{\xi_{j,t}}{2}\|(v_t+\sum_{j=1}^{n} u_{j,t}-P_t)\|^2 \quad (21)$$

where $\lambda_{j,t}$ Lagrangian multiplier , $(v_t + \sum_{j=1}^{n} u_{j,t} - P_t)$ is the Lagrangian term and the penalty term $\frac{\xi_{j,t}}{2}\|(v_t - P_t + \sum_{j=1}^{n} u_{j,t})\|^2$ to the objective function; in addition $\xi$ is a positive constant.
The associated dual objective function is

$$W(\lambda) := min_{u_i,v} L(u,v,\lambda)$$

Then the dual problem ,

$$max_{\lambda} W(\lambda)$$

To decompose (20) we use Alternating Direction Method of Multipliers. Alternating Direction Method of Multipliers is an approximation to ALR by sequentially updating each of the primal variables. It blends the separability of dual decomposition with the superior convergence properties of the method of multipliers in the convex case (which is based on augmented Lagrangian). However, discrete variables make MIPs nonconvex, which destroys the convergence properties of Alternating Direction Method of Multipliers.

The problem at iteration k we have $n+1$ block sub-problems:

- $$min_{u_j^k} \sum_t (c_{j,t} + \lambda_t^{k-1})u_{j,t}^k + \frac{\xi}{2}\|(v_t^{k-1} + \sum_{j=1}^{n} u_{j,t}^k - P_t)\|^2$$

- $$min_{v^k} \sum_t \lambda_t^{k-1} v_t + \gamma \sum_{t\in T}(v_t^k)^2 + \frac{\xi}{2}\|(v_t^k + \sum_{j=1}^{n} u_{j,t}^{k-1} - P_t)\|^2$$

where $\lambda^k \leftarrow \lambda^{k-1} + \xi(v_t^k + \sum_{j=1}^{n} u_{j,t}^k - P_t)$

# 4 Best Response Dynamics

In evolutionary game theory, best response dynamics represents a class of strategy updating rules, where players strategies in the next round are determined by their best responses to some subset of the population. From this point of view, we can consider the tanks as players and the control as their strategies. Each players (participants) select the best (locally optimal) response to what others are currently doing. For selecting the best response each player need only know his own utility function (each individual sub-problem), as his best response does not depend on other players' utility functions, but only on their actions. And this choosing 'best response' is running until the system 'converges' to an equilibrium point (it can also converge to stationary points) and the equilibrium point is a stable state from which none of the players wish to deviate. On the other hand convergence of repeated best response is unfortunately not guaranteed in general. Moreover the infinity on the strategy profiles makes our game infinite and take away us from the convergence speeches. However, in coordinate descent method when we have strong convexity and smoothness we have the convergence. First,we decompose our global problem into n+1 sub-problems, then we use the 'best response algorithm' to find locally optimal for each player, finally we will discuss about the convergence.

## 4.1 Problem Definition

Let us remember the non-decomposed global problem that we want to decompose into sub-problems to be able to plot the consumption of each local actors (tanks), for $\forall i, t$

$$min_u \sum_{i=1}^{n} \sum_{t=0}^{p-1} (c_{i,t} u_{i,t} + \gamma_t \sum_{t \in T} (P_t - \sum_{i=1}^{n} u_{i,t})^2)$$

If we consider each tanks $i$ where $i \in [1, n]$ we have the local problem,
for each $i$, and $\forall t$

$$min_{u_i} \sum_{t=0}^{p-1}(c_{i,t}u_{i,t} + \gamma_t \sum_{t \in T}(P_t - \sum_{i=1}^{n} u_{i,t})^2)$$

Like in the previous chapter; First of all, again for the smoothness we will add quadratic cost penalty for each individual tanks. One can see that again there is a coupling term coming from the quadratic term.

Secondly, this time we apply a small trick on aggregated consumption $(\sum_{i=1}^{n} u_{i,t})$ . If we separate the others aggregate consumption profile and *ith* tank's consumption profile $\sum_{i=1}^{n} u_{i,t} = \sum_{j \neq i}^{n} u_{j,t} + u_{i,t}$ we get the followings,

for $g_i \geq 0$ and for each i, $\forall t$

$$min_{u_i} \sum_{t=0}^{p-1}(c_{i,t}u_{i,t} + \frac{g_{i,t}}{2}(u_{i,t})^2 + \sum_{t \in T} \frac{\gamma_t}{2}(P_t - \sum_{i=1}^{n} u_{i,t})^2)$$

for each i, and $\forall t$

$$min_{u_i} \sum_{t=0}^{p-1}(c_{i,t}u_{i,t} + \frac{g_{i,t}}{2}(u_{i,t})^2 + \sum_{t \in T} \frac{\gamma_t}{2}(P_t - \sum_{i \neq j}^{n} u_{j,t} - u_{i,t})^2)$$

If we arrange all the terms we get the following ,
for each i, and $\forall t$

$$min_{u_i} \sum_{t}(c_{i,t} - 2\gamma_t(P_t - S_{i,t}))u_{i,t} + \sum_{t} \frac{g_{i,t}}{2}(u_{i,t})^2 + \sum_{t} \frac{\gamma_t}{2}(u_{i,t})^2 + \frac{\gamma_t}{2} \sum_{t}(P_t - S_{i,t})^2$$

where $S_{i,t} = \sum_{j \neq i} u_{i,t}$. called aggregate consumption profile of others. Some important remarks about this aggregate consumption profile is, it is a constant term for each tank $i$ and at each time-step $t$. Moreover this is the term that contains other tanks (or players) information about their decision (or strategy) to be exchanged. And ensures the coordination between the tanks.

Because the objective function that we want to minimize has a constant term $\frac{\gamma_t}{2} \sum_t (P_t - S_{i,t})^2$, we can omit it during the minimization procedure to be added after the minimization computation. Once again, what we have in the hand is; $n$ local decomposed problems, for each $i$, and $\forall t$

$$min_{u_i} \sum_t (c_{i,t} - 2\frac{\gamma_t}{2}(P_t - S_{i,t}))u_{i,t} + \frac{\gamma_t}{2}\sum_t (u_{i,t})^2 + \sum_t \frac{g_{i,t}}{2}(u_{i,t})^2 \tag{22}$$

where $S_{i,t} = \sum_{j \neq i} u_{i,t}$.

## 4.2 Best Response Algorithm

Consider the problem above (22). The objective function ( we will call cost function) not only depends on *ith* action $u_{i,t}$, but also on the one of the others $S_{i,t} = \sum_{j \neq i} u_{i,t}$. We can define the others action : $\forall t \;\; u_{-i,t} = (u_{1,t}, ..., u_{i-1,t}, u_{i+1,t}, ..., u_{n,t})$ to simplify the indication we ignore $t$ time-step at this point. $u_{-i} = (u_1, ..., u_{i-1}, u_{i+1}, ..., u_n)$

In words, $i$ best-responds to given actions of other players $u_{-i}$. And in the definition of best response this best response is indicated with $u_i^*$

$$BR_i(u_{-i}) = \{u_i^* \text{ s.t } cost_i(u_i^*, u_{-i}) = min_{u_i \in U_i} cost_i(u_i, u_{-i})\}$$

Notice that here players only have their own payoff information and can only observe the action of the opponent, then myopic dynamics, such as sequential-best response dynamic can be used for strategy adaptation. In the below we present this sequential strategy adaptation.

**Best-Response Algorithm:** [8]

**Input**$(u_i^{k=0})_i, \eta, K$ for stopping criterion ,**initial iteration** $k = 0$

**While** $\sum_i (u_i^{(k)} - u_i^{(k-1)})^2 \geq \eta$ **and** $k \leq K$ **do**

**Next Iteration** $k = k + 1$

**for** $i \in \{1, ..., n\}$**do**

(I)Get $\text{BR}_i(u_1^{k+1}, ..., u_{i-1}^{k+1}, u_{i+1}^k, ..., u_n^k)$by solving an optimization pb for i

(II)Choose $u_i^{(k+1)} \in \text{BR}_i(u_{-i}^{(k)})$

**end for**

**end while**

## 4.3 Best-Response Dynamics Convergence Analysis

this sequential best response dynamic able to converge a stable state? One can be sure that when the algorithm halts it reaches to a equilibrium so the existence can be shown. However equilibrium existence does not guarantee the convergence.

In game theory vocabulary there are games that Nash Equilibrium can exist even when the pay-off functions are non-convex. On the other hand in our problem even with the convex case we can not talk discuss about convergence of the algorithm since each tanks strategy profiles are infinite $u_{j,t} \in [0, u_{max}]$ for $\forall j, t$. There is no convergence proof yet when each players have infinitely many strategy.

On the other hand Best Response Algorithm perfectly fits on the Coordinate Descent Method in the optimization vocabulary and under some assumptions it is proven that a sequence of iterations generated by the algorithm is converging.

The general idea of Coordinate Descent Method is that line searching along each coordinate direction at the current point in each iteration for finding a local minimum of a function and if at some point, the objective is not decreasing at every coordinate direction, then we have reached the optimum. The basic procedure for coordinate descent method is shown below [9] :

$$\textbf{Input}(u_i^{k=0})_i, \textbf{initialize } k = 0$$

$$\text{pick coordinate i from } \ 1,...,\text{n}$$

$$u_i^{k+1} = argmin_{u_i \in U} f(u_i, u_{-i}^k)$$

$$\textbf{end}$$

where $f(u_i)$ represents the objective function in minimization problem (21) and $u_{-i}$ represent all other coordinates. There are many ways to choose the coordinates to update at each iteration. We choose Gauss-Seidel style which it fixes the rest coordinates to be the most up-to-date solution.

$$u_{-i}^k = (u_1^{k+1}, ..., u_{i-1}^{k+1}, u_{i+1}^k, ..., u_n^k)$$

Moreover we pick the coordinates in a cyclic order. First, select coordinate number 1 , then coordinate number 2 and so on, until n at each iteration k. To summarize we have a Gauss-Seidel iteration whereby the cost function $f$ is successively minimized with respect to the coordinate $u_i$ over $U$ with the other coordinates are fixed. for $i = 1, 2, ..n$ ,

$$u_i^{k+1} = argmin_{u_i \in U} f(u_1^{k+1}, ..., u_{i-1}^{k+1}, u_i, u_{i+1}^k, ..., u_n^k) \tag{23}$$

This is exactly the same with the Best Response Algorithm(I).
Moreover with the assumptions on objective function that is strongly convex and twice differentiable everywhere the below theorem holds.

Theorem [10] : If $u_i^k$ is a sequence of iterates generated by (22) , then $u_i^k$ converges to an element of $U^*$, where $U^*$ is the set of optimal solutions for (21).

In the next section, we will see our theoretical results' numerical solutions.

# 5 Simulations

Exact comparisons of our optimization algorithms with the true optimums for general models of the tanks is of course impossible, due to the complexity of calculating the optimum. For fairness and to compare the two decomposition techniques we take all the initial variables of the states the same, except drain profiles. Drain profiles of each tank is studied in two cases: the first one is deterministic and the second one is random, to randomize the profiles we produced a generator. Moreover the tanks parameters correspond to an Atlantis ATLANTIC VMRSEL 200L water tank. The computations were made using the CPLEX 12.7 commercial solver via OPTI Tool Box. Moreover for smoothness we add quadratic penalization for each individual tanks. We accept that each value that we get from computations has a unit 'euro'. And we took the maximum number of iteration large enough to prevent undesirable stops. Here, at each subsection we investigate different situations and we have different interpretation.

**A**) This is the benchmark part. As a benchmark, we assumed we don't have SOS constraints in our MILP/MIQP which helps us to have convex problem and easy to compute the solution. Moreover, the optimization has been computed in a 48h horizon for a 24h application of the control signal. We computed the optimization respectively for n=3 , n=10, n=50 and n=100 number of tanks and for Lagrangian and Best Response.

1) Where n=3 tanks and taken g=50 (smoothness parameter), rho=20 (step-size), eta=0.001 (stopping criterion where $\sum_i (u_i^k - u_i^{k-1})^2 \geq \eta$)

First of all, Lagrangian Decomposition:

The figure (fig.7) shows the concatenation of sub-problems solutions provides the solution of the global problem provided that the aggregate production constraint is satisfied.When we run the algorithm the constraint $v_t - P_t + \sum_{j=1}^{n} u_{j,t}$ is converging to zero in which we expect in a smooth and convex case.

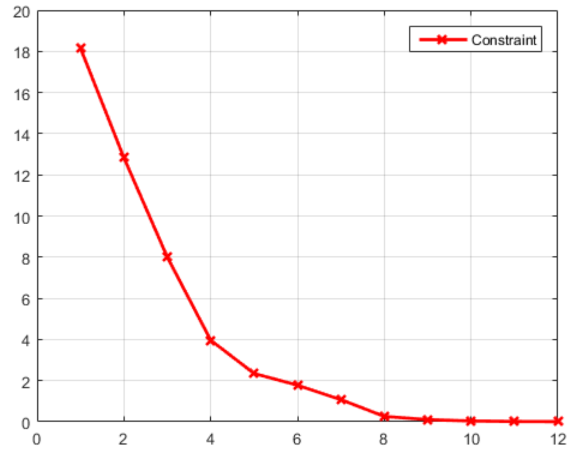


**Figure 7:** Constraint with Lagrangian for n=3

The next, we draw the figure of dual function (fig.8). Because we have strong convexity and the dual gap is zero, what we expect is, the dual function exactly opposite of the global cost function.

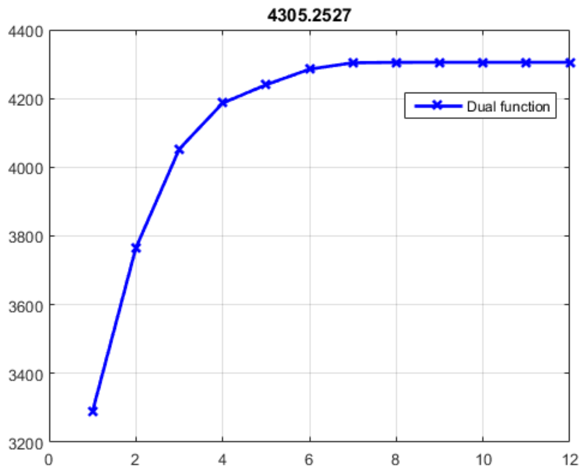


**Figure 8:** Dual Function with Lagrangian for n=3

In the (fig.9), the Lagrangian dual iteration stopped after 12 iterations and reached 4305.2673 which is the global optimum cost of the system. And dual function maximum value is 4305.2527 which is also the lower bound of the problem. One remark about the dual function and Lagrangian, is almost the same. We can ignore small

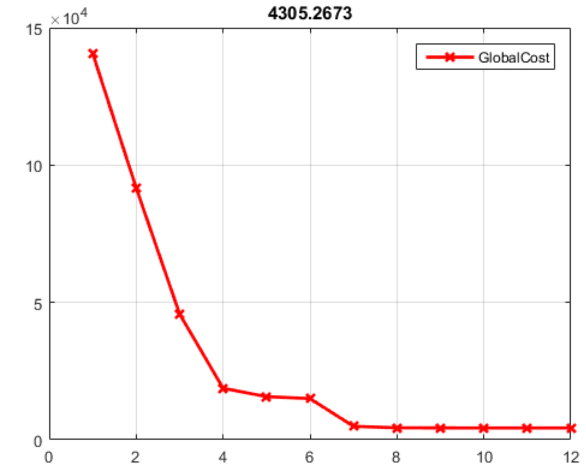


**Figure 9:** Global Cost with Lagrangian for n=3

computational difference.

Secondly, Best Response Dynamics:

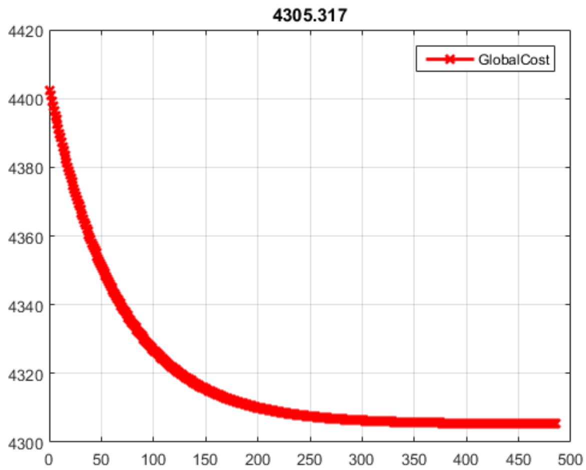


**Figure 10:** Global Cost with Best Response Algorithm for n=3

The Best Response Algorithm stopped after 486 iterations and reached 4305.317 (fig.10). What we expect is the value reached via Lagrangian is smaller than the value reached via Best Response, because of the weak duality. But in any case with given these parameter and assumptions Lagrangian is reaching the global minimum with less computational costs in term of iteration numbers.

2) Where n=10 tanks and taken g=n*10 , rho=100/n , eta=0.01.

First of all, Lagrangian Decomposition:

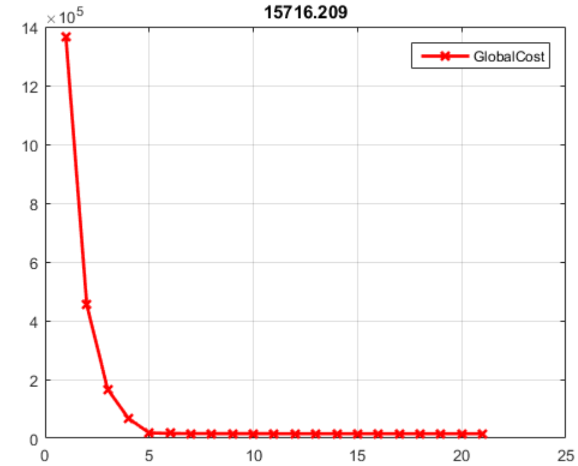


**Figure 11:** Global Cost with Lagrangian for n=10

(fig.11) where the Lagrangian for n=10 tanks stopped after 21 iterations and reached 15716.209.

Secondly, Best Response Dynamics:

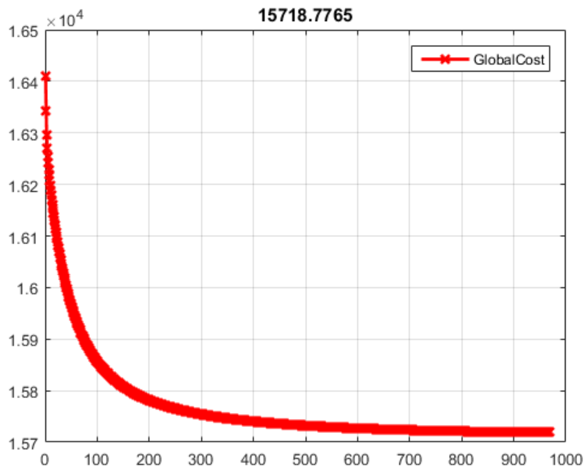


**Figure 12:** Global Cost with Best Response Algorithm for n=10

(fig.12) where the Best Response for n=10 tanks stopped after 969 iterations and reached 15718.7765.

3) Where n=50 tanks and taken g=n*10 , rho=100/n , eta=0.1
Notice that here we chose eta=0.1 which is the iteration criterion. We decreased eta in order to reach the equilibrium with less number of iteration.

First of all, Lagrangian Decomposition:

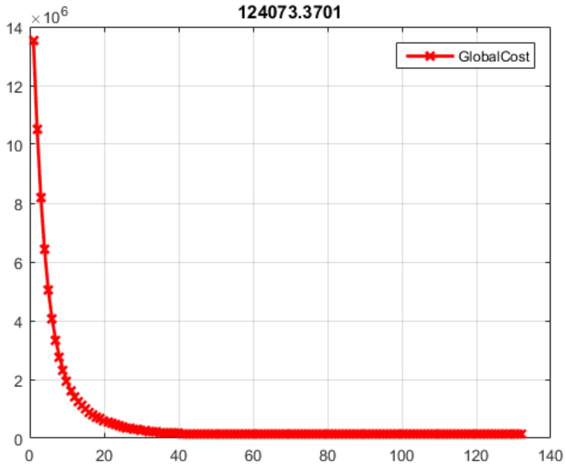


**Figure 13:** Global Cost with Lagrangian for n=50

(fig.13) where the Lagrangian for n=50 tanks stopped after 132 iterations and reached 124073.3701.

Secondly, Best Response Dynamics:

(fig.14 the next page) where the Best Response Algorithm for n=50 tanks stopped after 3085 iterations and reached 124192.96.

4) Where n=100 tanks and taken g=n*10 , rho=100/n , eta=0.1

First of all, Lagrangian Decomposition:

(fig.15 the next page) where the Lagrangian for n=100 tanks stopped after 198 iterations and reached 355740.6925.

Secondly, Best Response Dynamics:

where the Best Response Algorithm for n=100 tanks stopped after 4696 iterations

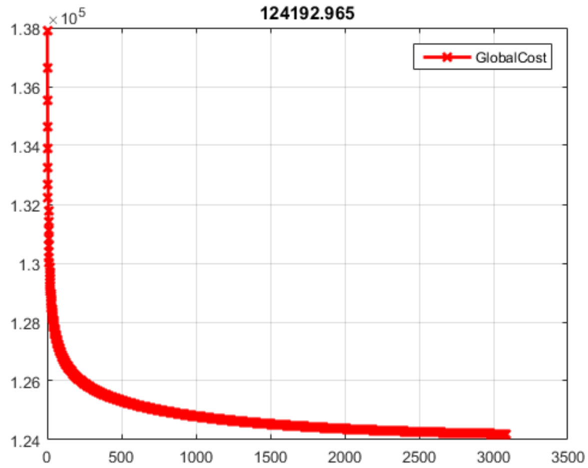


**Figure 14:** Global Cost with Best Response Algorithm for n=50

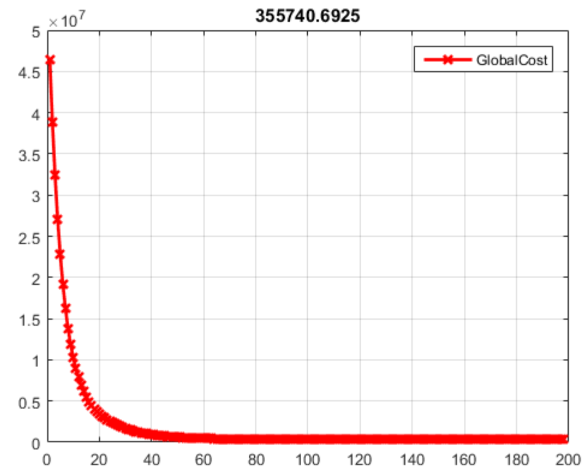


**Figure 15:** Global Cost with Lagrangian for n=100

and reached 355965.3601.

**B**) This the the part, where take into account SOS constraints and compare with the previous non-SOS constraints part. When we have SOS constraints we have non-convex problem and our expectation is, the results achieved from convex case will be a lower bound for non-convex case. Here are the simulations.

1) Where n=3 tanks and taken g=50 (smoothness parameter), rho=20 (step-size),

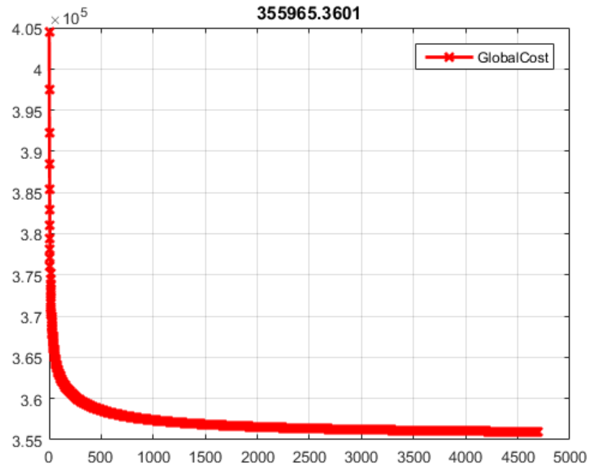


**Figure 16:** Global Cost with Best Response Algorithm for n=100

eta=0.001 (stopping criterion where $\sum_i (u_i^k - u_i^{k-1})^2 \geq \eta$)

First of all, Best Response :

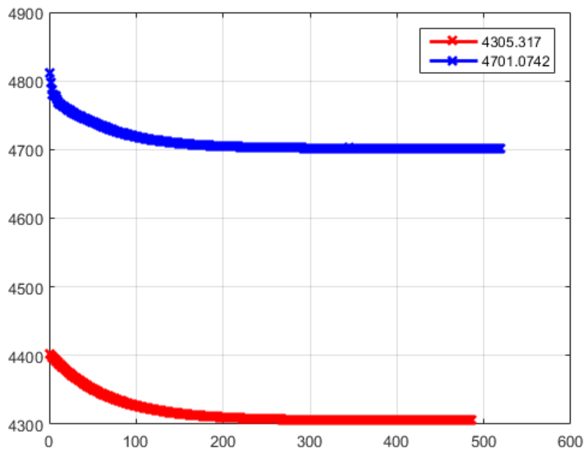


**Figure 17:** The Global Cost Gap with Best Response for n=3

The global cost for convex case(red) is 4305.317 , where the global cost for non-convex case(blue) is 4701.07419.

**B**) This the the part, we run the drain profiles generator and draw the profiles for each tank. Typically we consider two types of user depending on the time period:

either users have two separated time periods or they have one long time period. In figures; y-axis represents magnitude of the drain in percentage of total energy , x-axis represents 0-24 hour time horizon.

Remark: Total Consumption is drawn from truncated probability distribution with the mean value 80. The magnitude of the drain is allowed , either 0.1% or 0.01% of total energy in the tank or no drain per time.

1) The users who have an access to use the tank in a long period.

In the figure (fig18) it is shown a random drain profile for 1 tank with total consumption 77kWh between 6AM to 16PM.

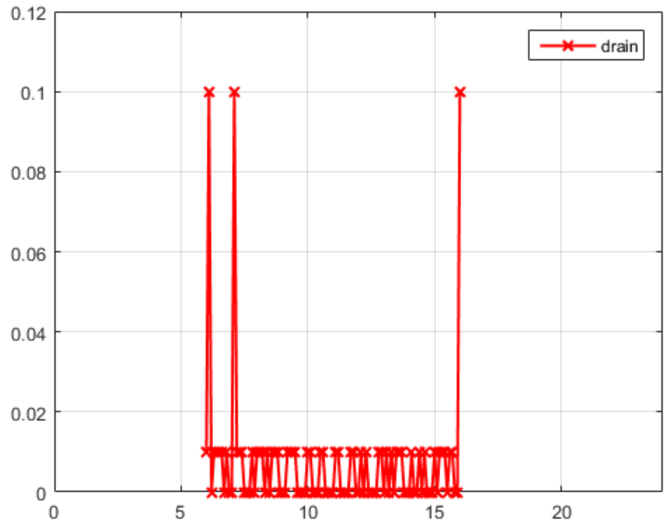


**Figure 18:** Drain profile for a tank 6AM to 16PM

2) The users who have an access to use the tank in 2 short time period during the day.

In the figure (fig19) it is shown a random drain profile for 1 tank with total consumption 59kWh between 5-11AM and 17-21PM.

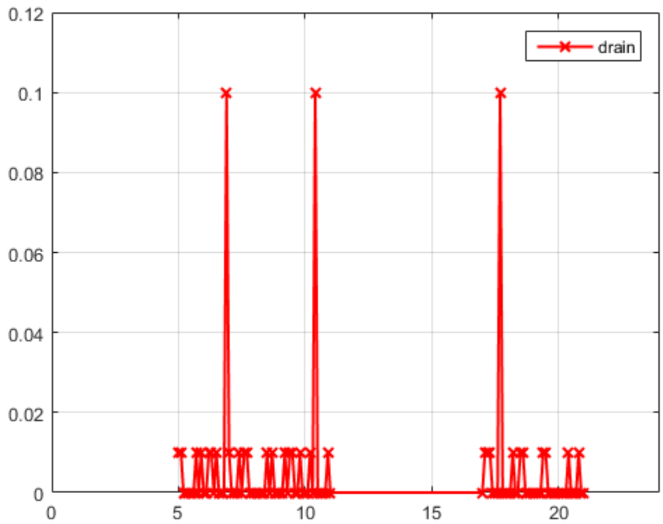


**Figure 19:** Drain profile for a tank 5-11AM, 17-21PM

# 6 Conclusion and Future Works

In this work, we are trying to cope with the large-scale optimization problem with centralized system, in order to get rid of high computational costs and the lack of using already known basic optimization techniques. To do this, we decomposed the global problem into sub-problems in a decentralized system. By this way, each decentralized local users solved their own problems locally, by having less computational cost, and by only knowing their constraints which lets more privacy. The only information in between local users were via central. As seen in the simulations, Lagrangian decomposition is more efficient when we consider our problem is convex. On the other hand Best Response algorithm is more robust, because of the nature of the algorithm, we don't require the convexity of the problem.

Future works would be, running generator and taking into account stochastic drain profiles to have more realistic solutions. Moreover in order to improve Lagrangian decomposition one can use Augmented Lagrangian Method instead of classical Lagrangian Method or in order to have more efficient algorithm one can combine these two algorithms, starting first with Best Response and continue with Lagrangian Decomposition because of Best Response attempt to run faster at the beginning of the runs as seen in the figures.

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