



Mémoire de Mathématiques en Optimisation

Problem of q -Moment Measures via Optimal Transport and Applications in Convex Geometry

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Abstract. A recent paper by Klartag provides a result on existence of affine hemisphere of elliptic type which depends on a convex function $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$. This convex function comes from problem of q -moment measures, i.e are the measures of the form $(\nabla\varphi)_{\#}\varphi^{-(n+q)}$ for $q > 0$, and existence of φ in Klartag's paper is by a variational method in the class of convex functions. In this mémoire, we propose a purely optimal transport-based method supposed by Santambrogio to retrieve the same result. In new approach, the variational problem becomes the minimization of a local functional and a transport cost among probability measures ϱ and the optimizer ϱ_{opt} turns out to be of form $\varrho_{\text{opt}} = \varphi^{-(n+q)}\mathcal{L}^n + \varrho_{\text{opt}}^{\text{sing}}$. This requires to develop some estimates and some semicontinuity results for the corresponding functionals which are natural in optimal transport.

After that, an application in convex geometry is connected, that is, in special case of $q = 2$ and the probability measure $(\nabla\varphi)_{\#}\varrho_{\text{opt}}^{\text{ac}}$ is the uniform probability measure on a non-empty, open, bounded, convex set $L \subset \mathbb{R}^n$ with the barycenter of L lies at the origin, we prove that

$$\mathcal{M} = \left\{ \left(\frac{1}{\varphi(x)}x, \frac{1}{\varphi(x)} \right) \in \mathbb{R}^n \times \mathbb{R} : x \in \text{dom}(\varphi) \right\}$$

is an affine hemisphere in \mathbb{R}^{n+1} with anchor $\mathcal{K} = L^\circ \times \{0\}$, which is centered at the origin. As an extension, we finally show that for any n -dimensional, compact, convex set $\mathcal{K} \subset \mathbb{R}^{n+1}$, we can always associate an $(n+1)$ -dimensional, compact, convex set $\mathcal{K}_1 \subset \mathbb{R}^{n+1}$ whose boundary consists of two parts: the convex set \mathcal{K} itself is a facet, and the rest of the boundary is an affine hemisphere with anchor \mathcal{K} , which is centered at the Santaló point of \mathcal{K} . Moreover, the affine hemisphere in this case is uniquely determined up to transformations.

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Introduction

With any positive, convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that φ goes to $+\infty$ at infinity, and a positive real number $q > 0$, we define the q -moment measure of φ to be the push-forward of the log-concave measure $\varphi^{-(n+q)}dx$ under the differential of φ . In other words, a Borel probability measure μ on \mathbb{R}^n is the q -moment measure of φ if for any bounded, continuous function $b : \mathbb{R}^n \rightarrow \mathbb{R}$, the following transport

equation is satisfied

$$\int_{\mathbb{R}^n} b(y) d\mu(y) = \int_{\mathbb{R}^n} b(\nabla\varphi(x)) \varphi^{-(n+q)}(x) dx.$$

This notion of q -moment measure finds applications in differential geometry, convex geometry and partial differential equations (in particular Monge–Ampère equation as 19), ...

In recent paper [13], Klartag shows that a result on existence of an affine hemisphere, which relies upon analysis of the equation with the constraint

$$\begin{cases} \det \nabla^2 \varphi &= C \cdot \varphi^{-(n+2)} & \text{in } \mathbb{R}^n \\ \nabla \varphi(\mathbb{R}^n) &= K^\circ \end{cases} \quad (1)$$

where $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$ is a smooth, convex function, $K \subset \mathbb{R}^n$ is a convex body whose Santaló point is at the origin and K° is polar body of the set K . Moreover, the solution φ to the equation (1) relate to q -moment measures with $q = 2$; and in [13], these are studied via a variational problem that the arguments for existence and uniqueness of solution rely mostly on convex analysis and functional inequalities. Especially, the Borell-Brascamp-Lieb inequality plays an important role for stating convexity of the functional and also is a key for the proof in [13].

In this mémoire, we want to present another way that is based on Theory of Optimal Transport (replacing functional inequalities techniques with ideas from optimal transport) to prove the existence of solution to the equation (1). In this approach, the variational problem becomes the minimization of a local functional $\mathcal{F}(\rho)$ and a transport cost $\mathcal{T}(\rho, \mu)$ among densities ρ and the absolutely continuous part of the optimizer turns out to be of the form $\rho_{\text{opt}}^{\text{ac}} = (\bar{u} - c)^{-(n+q)} \mathcal{L}^n$, where $\bar{u} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is solution to the dual optimal transport problem of $\mathcal{T}(\rho, \mu)$. In fact that we are interested in the local functional \mathcal{F} , defined as follows

$$\mathcal{F}(\rho) := \int_{\mathbb{R}^n} f(\rho^{\text{ac}}) dx$$

where $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by $f(t) := \frac{-1}{\alpha} |t|^\alpha$ with $\alpha = 1 - \frac{1}{n+q}$ and $q > 1$, $\rho := \rho^{\text{ac}} \mathcal{L}^n + \rho^{\text{sing}}$ is the decomposition of ρ as an absolutely continuous part ρ^{ac} (identified with its density) and a singular part ρ^{sing} w.r.t \mathcal{L}^n ; and consider the maximal transport problem from μ to ρ for cost function $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $c(x, y) = x.y$,

$$\mathcal{T}(\rho, \mu) := \sup \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} (x.y) d\gamma(x, y) \mid \gamma \in \Pi(\mu, \rho) \right\}$$

where $\Pi(\mu, \rho)$ is the set of transport plans between two probably measures μ, ρ on \mathbb{R}^n . The approach of this method consists in considering u as a secondary variable, i.e., for every solution $\bar{\rho}_{\text{opt}}$ we compute the minimum over possible u .

Similarly to [24], the study of the minimization problem in the new approach

$$(P) \quad \min \{ \mathcal{F}(\rho) + \mathcal{T}(\rho, \mu) : \rho \in \mathcal{P}_1(\mathbb{R}^n) \},$$

requires to develop some estimates and some semicontinuity results for functionals $\mathcal{F}(\rho)$ and $\mathcal{T}(\rho, \mu)$. However, many properties including both estimate and semicontinuity results of the maximal correlation functional $\mathcal{T}(\rho, \mu)$ were established by Santambrogio in [24] for studying moment measures. Furthermore, results on the lower semicontinuity of functionals on measures were studied in general case for

$$\mathcal{P}(\Omega) \ni \rho \longmapsto \int_{\Omega} f(\rho^{\text{ac}}) d\lambda(x) + L \cdot \rho^{\text{sing}}(\Omega)$$

where $L = \lim_{t \rightarrow +\infty} (f(t)/t)$ and λ is a fixed finite positive measure on Ω (see [22], chapter 7). In this case, the lower semicontinuity is obtained by representing the functional \mathcal{F} as the supremum of a family of lower semi-continuous functionals for the narrow topology,

$$\mathcal{F}(\rho) := \sup_{a \in C_b(\Omega), \sup a < L} \left\{ \int_{\Omega} a(x) d\rho(x) - \int_{\Omega} f^*(a(x)) d\lambda(x) \right\}.$$

where f^* is the Legendre transform of f .

The mémoire is organized as follows. We present results on the maximal correlation functional and local functionals in section 1. An estimate for the local functional $\mathcal{F}(\rho)$ is also established in this section (Proposition 1.4).

Section 2 presents the variational problem that we need to solve in order to find the log-concave measure $\rho_{\text{opt}}^{\text{ac}} = (\bar{u} - c)^{-(n+q)}$. In fact that we prove that the minimizer ρ_{opt} of the problem (P) cannot be such that $\rho_{\text{opt}}^{\text{ac}} = 0$ a.e., and

$$\rho_{\text{opt}} = \rho_{\text{opt}}^{\text{ac}} = \frac{1}{(\bar{u} - c)^{n+q}}.$$

Theorem 2.1 and Theorem 2.2 are key results of section 2.

In the last section, we retake some results on affine hemispheres of elliptic type as in [13]. The relation between the solution of the equation (1) and existence of an affinely-spherical with center at the origin is established in Theorem 3.1. In subsection 3.3, Theorem 3.3 provides a way to prove that the equation (1) has a solution which is uniquely determined up to transformations. For the convex function φ that is defined by the formula (55), Theorem 3.4 shows that there exists an affine hemisphere in \mathbb{R}^{n+1} which depends on it. Finally, Theorem 3.5 provides a general result on the existence and uniqueness up to transformation of an affine hemisphere.

1 The Maximal Correlation Functional and Local Functionals

The goal of this section is to summarize a few results in theory of optimal transport, some results on estimate and lower semicontinuity of the maximal correlation functional from [24]; establish and prove some results on estimate, lower semicontinuity and displacement convexity of the local functionals that we will use them next parts.

1.1 Some preliminaries on optimal transport

The history of Optimal Transport goes back to 1781, when the French mathematician G. Monge asked the following question: “which is the best way to transport a sand pile into a hole with the same volume?” This question can be formulated mathematically as follows.

Definition 1.1. *Given two metric spaces X, Y , two probability measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ and a cost function $c : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$. Monge’s problem is the following optimization problem*

$$(\mathcal{MP}) := \inf \left\{ \int_X c(x, T(x)) d\mu(x) \mid T : X \rightarrow Y \text{ and } T_{\#}\mu = \nu \right\},$$

where we recall that the measure denoted by $T_{\#}\mu$ is defined through $T_{\#}\mu(A) := \mu(T^{-1}(A))$ for every A and is called the image measure or pushforward of μ through T .

A measurable map $T : X \rightarrow Y$ such that $T_{\#}\mu = \nu$ is also called a *transport map* between μ and ν . Moreover, if T is \mathcal{C}^1 diffeomorphisms between open sets X, Y of \mathbb{R}^n , and assume also that the probability measures μ, ν have continuous densities ρ, σ with respect to the Lebesgue measure. Then, T is a transport plan if and only if the non-linear Jacobian equation holds

$$\rho(x) = \sigma(T(x)) \det(DT(x)).$$

The problem (\mathcal{MP}) exhibits several difficulties, one of which is that both the constraint $T_{\#}\mu = \nu$ and the functional are non-convex.

Definition 1.2. *Given $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ and a cost function $c : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$. Kantorovich’s problem is the following optimization problem*

$$(\mathcal{KP}) := \inf \left\{ \int_{X \times Y} c(x, y) d\gamma(x, y) \mid \gamma \in \Pi(\mu, \nu) \right\}$$

where $\Pi(\mu, \nu)$ is the set of the so-called transport plans, i.e.,

$$\Pi(\mu, \nu) := \left\{ \gamma \in \mathcal{P}(X \times Y) : (\pi_x)_\# \gamma = \mu \text{ and } (\pi_y)_\# \gamma = \nu \right\},$$

and π_x, π_y are the two projections of $X \times Y$ onto X and Y , respectively.

The infimum in Kantorovich's problem is less than the infimum in Monge's problem. Indeed, consider a transport map satisfying $T_\# \mu = \nu$ and the associated transport plan $\gamma_T = (\text{id}, T)_\# \mu$. Then, by the change of variable one has

$$\int_{X \times Y} c(x, y) d(\text{id}, T)_\# \mu(x, y) = \int_X c(x, T(x)) d\mu,$$

thus proving the claim.

Theorem 1.1. (an existence result of solution) *Let X, Y be two Polish spaces (i.e. complete and separable metric spaces) and $c : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous cost function, which is bounded from below. Then Kantorovich's problem admits a minimizer.*

The proof of Theorem 1.1 relies on the direct method in the calculus of variations, i.e. the fact that the minimized functional is lower semicontinuous and the set over which it is minimized is compact. In Polish spaces, the relevant topology here is the topology of narrow convergence². The references of complete proofs of Theorem 1.1 are the textbooks [22, 28, 29].

Another feature that the problem (\mathcal{KP}) has is, that it admits a dual problem. By the means of standard techniques from convex analysis one can show that problem (\mathcal{KP}) and the following problem (\mathcal{DP}) are in duality.

$$(\mathcal{DP}) := \sup \left\{ \int_X \varphi d\mu + \int_Y \psi d\nu : (\varphi, \psi) \in C_b(X) \times C_b(Y) \text{ and } \varphi \oplus \psi \leq c \right\}.$$

The competitors in Problem (\mathcal{DP}) are called *Kantorovich potentials* in the transport of μ onto ν , and the optimal potentials always have the form (φ, φ^c) ; where $\varphi^c : Y \rightarrow \overline{\mathbb{R}}$ is called *c-transform* (analogously to the well-known Legendre-Fenchel transform from convex analysis) and is defined by

$$\varphi^c(y) := \inf_{x \in X} \{c(x, y) - \varphi(x)\}.$$

²we say that a sequence $(\mu_n) \subset \mathcal{P}(X)$ is narrowly convergent to $\mu \in \mathcal{P}(X)$ as $n \rightarrow +\infty$ if

$$\lim_{n \rightarrow +\infty} \int_X f(x) d\mu_n(x) = \int_X f(x) d\mu(x)$$

for every function $f \in C_b(X)$, the space of continuous and bounded real functions defined on X .

Theorem 1.2. (strong duality) *If X, Y are Polish spaces and $c : X \times Y \rightarrow \mathbb{R}$ is uniformly continuous and bounded, then $\min(\mathcal{KP}) = \max(\mathcal{DP})$.*

The idea for proving this theorem is to use a minimizer γ_{opt} of the primal problem to build maximizers (φ, ψ) for the dual problem. The proof of Theorem 1.2 also show that (φ, ψ) can be constructed from the support $\text{spt}(\gamma_{\text{opt}})$ only. A proof of Theorem 1.2 can be found in ([22], Theorem 1.39).

As often, the Kantorovich potentials have an economic interpretation as prices. For instance, imagine that μ is the distribution of sand available at quarries, and ν describes the amount of sand required by construction work. Then, (\mathcal{KP}) can be interpreted as finding the cheapest way of transporting the sand from μ to ν for a construction company. Imagine that this company wants to externalize the transport, by paying a loading coast $\varphi(x)$ at a point x (in a quarry) and an unloading coast $\psi(y)$ at a point y (at a construction place). Then, the constraint $\varphi(x) + \psi(y) \leq c(x, y)$ translates the fact that the construction company would not externalize if its cost is higher than the cost of transporting the sand by itself. Then, Kantorovich's dual problem (\mathcal{DP}) describes the problem of a transporting company: maximizing its revenue $\int \varphi d\mu + \int \psi d\nu$ under the constraint $\varphi \oplus \psi \leq c$ imposed by the construction company. The economic interpretation of the strong duality $(\mathcal{KP}) = (\mathcal{DP})$ is that in this setting, externalization has exactly the same cost as doing the transport by oneself.

For the quadratic cost $c(x, y) := \frac{1}{2}|x - y|^2$, under the additional assumption that $\mu \ll \mathcal{L}^n$, Y. Brenier showed that there exists a unique optimal transport map in the problem (\mathcal{MP}) which is a gradient of a convex function, in addition the relation

$$T(x) = \nabla \left(\frac{1}{2}|x|^2 - \varphi(x) \right) = x - \nabla \varphi(x)$$

holds, for any φ optimal Kantorovich potential from the transport of μ onto ν in the problem (\mathcal{DP}) . Moreover $\gamma_T := (\text{id}, T)_{\#}\mu$ is the optimal plan in the problem (\mathcal{KP}) . Nowadays we usually refer to this result as *Brenier's Theorem*. On the other hand, the value of the problem (\mathcal{KP}) with the quadratic cost may also be used to define a quantity, called Wasserstein distance, over $\mathcal{P}_2(\mathbb{R}^n)$

$$W_2(\mu, \nu) := \left(\min \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\} \right)^{1/2}.$$

This quantity may be proven to be a distance over $\mathcal{P}_2(\mathbb{R}^n)$ and the space $\mathcal{P}_2(\mathbb{R}^n)$ endowed with the distance W_2 is called Wasserstein space of order 2, denoted by $W_2(\mathbb{R}^n)$.

The geodesics³ in the space $W_2(\mathbb{R}^n)$ play an important role in the theory of optimal transport. Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n)$ and $\mu \ll \mathcal{L}^n$, we define $\rho_t := ((1-t)\text{id} + tT)_{\#}\mu$, where T is the optimal transport from μ to ν . This curve ρ_t happens to be a constant speed geodesic for the distance W_2 connecting μ to ν . Once we know the geodesics in $W_2(\mathbb{R}^n)$, one can wonder which functionals $\mathcal{F} : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}$ are geodesically convex (convex along constant speed geodesics) i.e. for every two points $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^n)$ there exists a constant-speed geodesic $\omega : [0, 1] \rightarrow \mathcal{P}_2(\mathbb{R}^n)$ connecting $\omega(0) = \rho_0$ to $\omega(1) = \rho_1$ such that

$$[0, 1] \ni t \longmapsto \mathcal{F}(\omega(t))$$

is convex. This notion, applied to the case of the Wasserstein spaces, is also called *displacement convexity* and has been introduced by McCann in [16]. It is very useful both to provide uniqueness results for variational problems and to provide sufficient optimality conditions.

1.2 The maximal correlation functional

In this paper, we consider the maximal transport problem from μ to ρ for cost function $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $c(x, y) = x.y$,

$$\mathcal{T}(\rho, \mu) := \sup \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) d\gamma(x, y) \mid \gamma \in \Pi(\mu, \rho) \right\}.$$

and the dual problem would become

$$\inf \left\{ \int_{\mathbb{R}^n} u d\rho + \int_{\mathbb{R}^n} v d\mu : u(x) + v(x) \geq x.y \right\}.$$

In this problem it is quite clear that any pair (u, v) can be replaced with (u, u^*) where $u^*(y) := \sup_x \{x.y - u(x)\}$ is the Legendre transform of u and is the smallest function compatible with u in the constraint $u(x) + u^*(y) \geq x.y$. Then, it is easy to see by the primal-dual optimality conditions that the optimal γ and the optimal u satisfy

$$\text{spt}(\gamma) \subset \{(x, y) : u(x) + u^*(y) = x.y\} = \{(x, y) : y \in \partial u(x)\},$$

which shows that γ is concentrated on the graph of a map T (which is one-valued μ -a.e., provided $\mu \ll \mathcal{L}^n$), given by $T = \nabla u$. Thus, for $\mu, \rho \in \mathcal{P}_1(\mathbb{R}^n)$ with

³In a length space, a curve $\omega : [0, 1] \rightarrow X$ is said to be a *constant-speed geodesic* between $\omega(0)$ and $\omega(1) \in X$ if it satisfies $d(\omega(t), \omega(s)) = |t - s|d(\omega(0), \omega(1))$ for all $t, s \in [0, 1]$.

$\int y d\mu(y) = 0$, by the strong duality we can define the following quantity

$$\begin{aligned} \mathcal{T}(\rho, \mu) &= \sup \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} (x \cdot y) d\gamma(x, y) \mid \gamma \in \Pi(\mu, \rho) \right\} \\ &= \inf \left\{ \int_{\mathbb{R}^n} u d\rho + \int_{\mathbb{R}^n} u^* d\mu \mid u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \text{ convex and l.s.c} \right\}. \end{aligned}$$

We notice that, if $\mu, \rho \in \mathcal{P}_2(\mathbb{R}^n)$, then we also have

$$\mathcal{T}(\rho, \mu) = \frac{1}{2} \int_{\mathbb{R}^n} |x|^2 d\rho(x) + \frac{1}{2} \int_{\mathbb{R}^n} |y|^2 d\mu(y) - \frac{1}{2} W_2^2(\rho, \mu).$$

This functional \mathcal{T} is a transport cost, but we also observe that it is the maximal correlation between ρ and μ , in the sense

$$\mathcal{T}(\rho, \mu) = \sup \{ \mathbb{E}[X \cdot Y] \mid X \sim \rho, Y \sim \mu \}.$$

For this reason, \mathcal{T} will be called *maximal correlation functional*. In this mémoire, we are interested in the following properties which were established by Santambrogio in [24].

- Proposition 1.1.** (a) For every $\rho \in \mathcal{P}_1(\mathbb{R}^n)$, we have $0 \leq \mathcal{T}(\rho, \mu) \leq +\infty$.
 (b) If ρ and $\tilde{\rho}$ are one obtained from one another by translation, then $\mathcal{T}(\rho, \mu) = \mathcal{T}(\tilde{\rho}, \mu)$.
 (c) (lower semicontinuity). If $\int x d\rho_n(x) = 0$ and $\rho_n \rightharpoonup \rho$, then

$$\mathcal{T}(\rho, \mu) \leq \liminf_{n \rightarrow +\infty} \mathcal{T}(\rho_n, \mu).$$

- (d) If μ is not supported on a hyperplane then, for every ρ such that $\int x d\rho(x) = 0$, \mathcal{T} satisfies an inequality of the form $\mathcal{T}(\rho, \mu) \geq c \int |x| d\rho(x)$ for $c = c(\mu) > 0$, where

$$c(\mu) := \frac{1}{2n} \inf \left\{ \int_{\mathbb{R}^n} |y \cdot e - l| d\mu(y) : e \in \mathbb{S}^{n-1}, l \in \mathbb{R} \right\}.$$

- (e) (displacement convexity). Let $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^n)$ be absolutely continuous measures, and let $\rho_t = ((1-t)\text{id} + tT)_{\#} \rho_0$ be the unique constant speed geodesic connecting them for the Wasserstein distance W_2 . Then $[0, 1] \ni t \mapsto \mathcal{T}(\rho_t, \mu)$ is convex.

1.3 Local functionals

In this section, we will establish some results of the following functional, including lower semicontinuity, estimate and displacement convexity.

$$\mathcal{F}(\rho) := \int_{\mathbb{R}^n} f(\rho^{\text{ac}}) dx \tag{2}$$

where $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by $f(t) := \frac{-1}{\alpha}|t|^\alpha$ with $\alpha = 1 - \frac{1}{n+q}$ and $q > 1$, $\rho := \rho^{\text{ac}} \mathcal{L}^n + \rho^{\text{sing}}$ is the decomposition of ρ as an absolutely continuous part ρ^{ac} (identified with its density) and a singular part ρ^{sing} w.r.t \mathcal{L}^n . We first need to determine the Legendre transform of f , that is $f^* : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f^*(h) = \sup_{x \geq 0} \{x \cdot h - f(x)\} = \sup_{x \geq 0} \{x \cdot h + \frac{1}{\alpha}x^\alpha\}$. Setting $g(x) = x \cdot h + \frac{1}{\alpha}x^\alpha$. For every $h \leq 0$, we have $g'(x) = h + x^{\alpha-1}$ and second derivative $g''(x) = (\alpha - 1)x^{\alpha-2} \leq 0$, so in this case supremum of the function g over $\{x \in \mathbb{R} : x \geq 0\}$ attains at $x_{\max} = (-h)^{1/(\alpha-1)}$ and $g(x_{\max}) = (\frac{1}{\alpha} - 1)(-h)^{\alpha/(\alpha-1)}$. Thus we can conclude that

$$f^*(h) = \begin{cases} +\infty & \text{if } h > 0, \\ \left(\frac{1}{\alpha} - 1\right) (-h)^{\frac{\alpha}{\alpha-1}} & \text{if } h \leq 0. \end{cases} \quad (3)$$

Proposition 1.2. *The functional \mathcal{F} is defined by (2) is lower semi-continuous. Moreover, this functional can be represented as the supremum of a family of lower semi-continuous functionals for the narrow topology as follows*

$$\mathcal{F}(\rho) = \sup_{a \in C_b(\mathbb{R}^n), \sup a < 0} \left\{ \int_{\mathbb{R}^n} a(x) d\rho(x) - \left(\frac{1}{\alpha} - 1\right) \int_{\mathbb{R}^n} [-a(x)]^{\frac{\alpha}{\alpha-1}} dx \right\}.$$

Proposition 1.2 is as a special case of the following general result when $f'(+\infty) = 0$, $\lambda = \mathcal{L}^n$, $\Omega = \mathbb{R}^n$ and f^* is defined as in (3).

Proposition 1.3. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a convex l.s.c. function, and set*

$$L = f'(+\infty) := \lim_{t \rightarrow +\infty} \left\{ \frac{f(t)}{t} \right\} = \sup_{t > 0} \left\{ \frac{f(t)}{t} \right\} \in \mathbb{R} \cup \{+\infty\}.$$

Let λ be a fixed finite positive measure on Ω . For every measure ρ write $\rho = \rho^{\text{ac}} \cdot \lambda + \rho^{\text{sing}}$, where $\rho^{\text{ac}} \cdot \lambda$ is the absolutely continuous part of ρ and ρ^{sing} is the singular one (w.r.t. λ). Then, the functional defined through

$$\mathcal{F}(\rho) := \int_{\Omega} f(\rho^{\text{ac}}) d\lambda(x) + L\rho^{\text{sing}}(\Omega)$$

(note that if $L = +\infty$, then $\mathcal{F}(\rho) = +\infty$ whenever ρ has a singular part) is l.s.c.

Proof. Since the function f is convex and lower semi-continuous, we have $f = f^{**}$, where $f^{**} := (f^*)^*$ is the biconjugate of f . Moreover, it is easy to see that for $a > L$, we have $f^*(a) = +\infty$: indeed, $f^*(a) = \sup_t \{at - f(t)\} \geq \lim_{t \rightarrow +\infty} t(a - f(t)/t)$. Hence, we can write

$$f(t) = f^{**}(t) = \sup_{a \in \mathbb{R}} \{at - f^*(a)\} = \sup_{a \leq L} \{at - f^*(a)\} = \sup_{a < L} \{at - f^*(a)\}$$

(the last equality is justified by the fact that f^* is continuous on the set where it is finite).

Now we consider the following functional

$$\tilde{\mathcal{F}}(\rho) := \sup_{a \in C_b(\Omega), \sup a < L} \left\{ \int_{\Omega} a(x) d\rho(x) - \int_{\Omega} f^*(a(x)) d\lambda(x) \right\}.$$

$\tilde{\mathcal{F}}$ is obviously l.s.c, since it is the supremum of a family of affine functionals, continuous w.r.t. the weak convergence. We want to prove

$$\mathcal{F} = \tilde{\mathcal{F}}.$$

In order to do so, first note that, thanks to Lusin's theorem ⁴, applied here to the measure $\lambda + \rho$, it is not difficult to replace bounded and continuous functions with measurable bounded functions. By abuse of notation, we denote measurable bounded functions by $L^\infty(\Omega)$ (even if we do not mean functions which are essentially bounded w.r.t. a given measure, but really bounded) and we get

$$\tilde{\mathcal{F}}(\rho) = \sup_{a \in L^\infty(\Omega), \sup a < L} \left\{ \int_{\Omega} a(x) d\rho(x) - \int_{\Omega} f^*(a(x)) d\lambda(x) \right\}.$$

Then take a set A such that $\rho^{\text{sing}}(\Omega \setminus A) = \lambda(A) = 0$, this allows to write

$$\tilde{\mathcal{F}}(\rho) := \sup_{a \in L^\infty(\Omega), \sup a < L} \left\{ \int_{\Omega \setminus A} [a(x) \rho^{\text{ac}}(x) - f^*(a(x))] d\lambda(x) + \int_A a(x) d\rho^{\text{sing}}(x) \right\}.$$

The values of $a(x)$ may be chosen independently on A and $\Omega \setminus A$. Setting

$$\mathcal{G}(a) := \int_{\Omega \setminus A} f^*(a(x)) d\lambda(x) \quad \text{and} \quad \langle a, \rho^{\text{ac}} \rangle := \int_{\Omega \setminus A} a(x) \rho^{\text{ac}}(x) d\lambda(x).$$

⁴Lusin's Theorem in measure theory states that every measurable function f on a reasonable measure space (X, τ) is actually continuous on a set K with $\tau(X \setminus K)$ small. Exactly, if $f : X \rightarrow \mathbb{R}$ is measurable, then for every $\varepsilon > 0$, there exists a compact set $K \subset X$ and a continuous function $g : X \rightarrow \mathbb{R}$ such that $\tau(X \setminus K) < \varepsilon$ and $f = g$ on K .

By Theory of Conjugates of Integral Functionals, we have

$$\begin{aligned}
 & \sup_{a \in L^\infty(\Omega), \sup a < L} \int_{\Omega \setminus A} [a(x) \rho^{\text{ac}}(x) - f^*(a(x))] \, d\lambda(x) \\
 &= \sup_{a \in L^\infty(\Omega), \sup a < L} \{ \langle a, \rho^{\text{ac}} \rangle - \mathcal{G}(a) \} \\
 &= \mathcal{G}^*(\rho^{\text{ac}}) \\
 &= \int_{\Omega \setminus A} f^{**}(\rho^{\text{ac}}(x)) \, d\lambda(x) \\
 &= \int_{\Omega \setminus A} f(\rho^{\text{ac}}(x)) \, d\lambda(x) \\
 &= \int_{\Omega \setminus A} f(\rho^{\text{ac}}(x)) \, d\lambda(x) + \int_A f(\rho^{\text{ac}}(x)) \, d\lambda(x) \\
 &= \int_{\Omega} f(\rho^{\text{ac}}(x)) \, d\lambda(x).
 \end{aligned}$$

On the other hand, for every $a \in L^\infty(\Omega)$ with $\sup a < L$, we have

$$H(a) := \int_A a(x) \, d\rho^{\text{sing}}(x) < \int_A L \, d\rho^{\text{sing}}(x) = L\rho^{\text{sing}}(A) = L\rho^{\text{sing}}(\Omega)$$

Hence,

$$\sup_{a \in L^\infty(\Omega), \sup a < L} \int_A a(x) \, d\rho^{\text{sing}}(x) = \sup_{a \in L^\infty(\Omega), \sup a < L} H(a) = L\rho^{\text{sing}}(\Omega).$$

Thus, $\mathcal{F} = \tilde{\mathcal{F}}$ and the proposition is proven. \square

Proposition 1.4. *The functional $\mathcal{F} : \mathcal{P}_1(\mathbb{R}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$ is well-defined. Moreover, for every $q > 1$ and $\delta \in \left(\frac{n}{n+q-1}, 1\right)$, there exists $C \geq 0$ such that the following estimate holds*

$$\mathcal{F}(\rho) \geq -C - \left(\int_{\mathbb{R}^n} |x| \, d\rho(x) \right)^\delta. \quad (4)$$

Proof. The functional \mathcal{F} can be represented as follows

$$\begin{aligned}
 \mathcal{F}(\rho) &= \int_{\mathbb{R}^n} (f(\rho^{\text{ac}}(x)) + f^*(h(x)) - \rho^{\text{ac}}(x)h(x)) \, dx \\
 &\quad + \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)h(x) \, dx - \int_{\mathbb{R}^n} f^*(h(x)) \, dx.
 \end{aligned}$$

Using the Young-Fenchel inequality, we obtain $f(\rho^{\text{ac}}(x)) + f^*(h(x)) - \rho^{\text{ac}}(x)h(x) \geq 0$, this allows that the integrand

$$\int_{\mathbb{R}^n} (f(\rho^{\text{ac}}(x)) + f^*(h(x)) - \rho^{\text{ac}}(x)h(x)) dx$$

is always nonnegative. It follows that

$$\mathcal{F}(\rho) \geq \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)h(x) dx - \int_{\mathbb{R}^n} f^*(h(x)) dx.$$

Now we need to choose the function $h \leq 0$ such that we can evaluate $\int \rho^{\text{ac}}(x)h(x) dx \geq \text{const} - (\int |x| d\rho(x))^\delta$ and prove $\int f^*(h(x)) dx < +\infty$. Since $\rho \in \mathcal{P}_1(\mathbb{R}^n)$ we can take $h(x) = -(1 + |x|)^\delta$, then

$$\begin{aligned} \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)h(x) dx &= - \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)(1 + |x|)^\delta dx \\ &\geq - \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)(1 + |x|^\delta) dx \quad \left((a+b)^\delta \leq (a)^\delta + (b)^\delta \text{ for } a, b \geq 0 \right) \\ &= - \int_{\mathbb{R}^n} \rho^{\text{ac}}(x) dx - \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)|x|^\delta dx \\ &\geq -1 - \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)|x|^\delta dx \quad \left(\text{since } \int_{\mathbb{R}^n} d\rho^{\text{ac}}(x) \leq \int_{\mathbb{R}^n} d\rho(x) = 1 \right) \end{aligned}$$

Applying Hölder inequality in general case with $p_1 = \frac{1}{\delta}$ and $p_2 = \frac{1}{1-\delta}$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \rho^{\text{ac}}(x)|x|^\delta dx &= \int_{\mathbb{R}^n} (\rho^{\text{ac}}(x)|x|)^\delta (\rho^{\text{ac}}(x))^{1-\delta} dx \\ &\leq \left(\int_{\mathbb{R}^n} (\rho^{\text{ac}}(x)|x|)^{\delta p_1} dx \right)^{1/p_1} \left(\int_{\mathbb{R}^n} (\rho^{\text{ac}}(x))^{(1-\delta)p_2} dx \right)^{1/p_2} \\ &= \left(\int_{\mathbb{R}^n} |x| d\rho^{\text{ac}}(x) \right)^\delta \left(\int_{\mathbb{R}^n} d\rho^{\text{ac}}(x) \right)^{1-\delta} \\ &\leq \left(\int_{\mathbb{R}^n} |x| d\rho^{\text{ac}}(x) \right)^\delta \\ &\leq \left(\int_{\mathbb{R}^n} |x| d\rho(x) \right)^\delta. \end{aligned}$$

Thus we get

$$\int_{\mathbb{R}^n} \rho^{\text{ac}}(x)h(x) dx \geq -1 - \left(\int_{\mathbb{R}^n} |x| d\rho(x) \right)^\delta$$

On the other hand, using the polar coordinates formula

$$\begin{aligned}
 0 \leq \int_{\mathbb{R}^n} f^*(h(x)) dx &= \left(\frac{1}{\alpha} - 1\right) \int_{\mathbb{R}^n} (1 + |x|)^{\frac{-\delta\alpha}{1-\alpha}} dx \\
 &= \left(\frac{1}{\alpha} - 1\right) c. \int_0^{+\infty} r^{n-1} (1+r)^{\frac{-\delta\alpha}{1-\alpha}} dr \\
 &\leq \left(\frac{1}{\alpha} - 1\right) c. \int_0^{+\infty} (1+r)^{n-1} (1+r)^{\frac{-\delta\alpha}{1-\alpha}} dr \\
 &= \left(\frac{1}{\alpha} - 1\right) c. \int_1^{+\infty} r^{n-1-\frac{\delta\alpha}{1-\alpha}} dr
 \end{aligned}$$

and

$$\int_1^{+\infty} r^{n-1-\frac{\delta\alpha}{1-\alpha}} dr < +\infty \iff n-1-\frac{\delta\alpha}{1-\alpha} < -1 \iff \delta > \frac{n}{n+q-1}.$$

The condition above is satisfied by assumption of δ in the proposition. By setting the constant

$$C := 1 + \left(\frac{1}{\alpha} - 1\right) \int_{\mathbb{R}^n} (1 + |x|)^{\frac{-\delta\alpha}{1-\alpha}} dx$$

then $C \in (0, +\infty)$ and this completes the proof of the proposition. \square

In the final part of this section, we assume that ρ is absolutely continuous (i.e. $\rho = \rho^{\text{ac}}$, $\rho^{\text{sing}} = 0$) and we will investigate strict convexity of the functional \mathcal{F} on every geodesic $t \mapsto \rho_t = ((1-t)\text{id} + tT)_{\#}\rho$ via second derivative. Applying Monotone Change of Variables Theorem (see [16], Theorem 4.4), $\mathcal{F}(\rho_t)$ can be represented as follows

$$\begin{aligned}
 \mathcal{F}(\rho_t) &= \int_{\mathbb{R}^n} f\left(\frac{\rho(x)}{\det((1-t)\text{id} + tDT(x))}\right) \det((1-t)\text{id} + tDT(x)) dx \\
 &= \frac{-1}{\alpha} \int_{\mathbb{R}^n} \frac{[\rho(x)]^{\alpha-1} d\rho(x)}{[g(t,x)]^{n(\alpha-1)}},
 \end{aligned}$$

where $g(t,x) := \{\det((1-t)\text{id} + tDT(x))\}^{1/n}$. Let $\{\lambda_i\}$ be the eigenvalues of $DT(x) - I$,

$$\beta_1 := \sum_{i=1}^n \frac{\lambda_i}{1+t\lambda_i} \quad \text{and} \quad \beta_2 := \sum_{i=1}^n \frac{\lambda_i^2}{(1+t\lambda_i)^2}.$$

Then, we have (see [22], Lemma 7.26), $g(t,x) = (\prod_{i=1}^n (1+t\lambda_i))^{1/n}$,

$$\frac{d}{dt} [g(t, x)] = \frac{\beta_1}{n} g(t, x) \quad \text{and} \quad \frac{d^2}{dt^2} [g(t, x)] = \frac{1}{n} \left\{ \beta_1 \frac{d}{dt} [g(t, x)] - \beta_2 g(t, x) \right\}.$$

Proposition 1.5. *The functional $\rho \mapsto \mathcal{F}(\rho)$ is strictly convex on a geodesic $t \mapsto \rho_t = ((1-t)\text{id} + tT)_{\#}\rho$ unless the optimal map T is such that $DT = I$ a.e. on $\{\rho > 0\}$.*

Proof. We compute first derivative

$$\begin{aligned} \frac{d}{dt} [\mathcal{F}(\rho_t)] &= \frac{n(\alpha-1)}{\alpha} \int_{\mathbb{R}^n} \frac{[g(t, x)]^{n(\alpha-1)-1} g'(t, x) [\rho(x)]^{\alpha-1} d\rho(x)}{[g(t, x)]^{2n(\alpha-1)}} \\ &= \frac{n(\alpha-1)}{\alpha} \int_{\mathbb{R}^n} \frac{g'(t, x) [\rho(x)]^{\alpha-1} d\rho(x)}{[g(t, x)]^{n(\alpha-1)+1}}, \end{aligned}$$

and the second derivative

$$\frac{d^2}{dt^2} [\mathcal{F}(\rho_t)] = \frac{n(\alpha-1)}{\alpha} \int_{\mathbb{R}^n} k(t, x) (\rho(x))^{\alpha-1} d\rho(x)$$

where

$$\begin{aligned} k(t, x) &:= \frac{g''(t, x) [g(t, x)]^{n(\alpha-1)+1} - [n(\alpha-1) + 1] [g(t, x)]^{n(\alpha-1)} g'(t, x)^2}{[g(t, x)]^{2\{n(\alpha-1)+1\}}} \\ &= \frac{g''(t, x)}{[g(t, x)]^{n(\alpha-1)+1}} - \frac{[n(\alpha-1) + 1] g'(t, x)^2}{[g(t, x)]^{n(\alpha-1)+2}} \\ &= \frac{1}{[g(t, x)]^{n(\alpha-1)}} \left(\frac{g''(t, x)}{g(t, x)} - \frac{[n(\alpha-1) + 1] g'(t, x)^2}{[g(t, x)]^2} \right) \\ &= \frac{-1}{n} \cdot \frac{1}{[g(t, x)]^{n(\alpha-1)}} \left(\frac{n(\alpha-1) + 1}{n} \cdot \frac{[ng'(t, x)]^2}{[g(t, x)]^2} - \frac{ng''(t, x)}{g(t, x)} \right) \\ &= \frac{-1}{n} \cdot \frac{1}{[g(t, x)]^{n(\alpha-1)}} \left(\frac{n(\alpha-1) + 1}{n} \beta_1^2 - \frac{1}{n} \beta_1^2 + \beta_2 \right) \\ &= \frac{-1}{n} \cdot \frac{1}{[g(t, x)]^{n(\alpha-1)}} (\beta_2 - (1-\alpha) \beta_1^2) \end{aligned}$$

Finally we obtain that the second derivative of the functional \mathcal{F} is

$$\frac{d^2}{dt^2} [\mathcal{F}(\rho_t)] = \frac{(1-\alpha)}{\alpha} \int_{\mathbb{R}^n} \left\{ \frac{1}{[g(t, x)]^{n(\alpha-1)}} (\beta_2 - (1-\alpha) \beta_1^2) \right\} [\rho(x)]^{\alpha-1} d\rho(x) \quad (5)$$

Using the quadratic-arithmetic mean inequality, we have

$$\begin{aligned} \beta_2 &= \sum_{i=1}^n \frac{\lambda_i^2}{(1+t\lambda_i)^2} \geq \frac{1}{n} \left(\sum_{i=1}^n \frac{\lambda_i}{1+t\lambda_i} \right)^2 \geq \frac{1}{n+q} \left(\sum_{i=1}^n \frac{\lambda_i}{1+t\lambda_i} \right)^2 \\ &= (1-\alpha) \beta_1^2. \end{aligned}$$

This shows that the second derivative of $\mathcal{F}(\rho_t)$ is non-negative, and hence \mathcal{F} is displacement convex. Moreover, it shows that $[0, 1] \ni t \mapsto \mathcal{F}(\rho_t)$ is strictly convex unless $\beta_2 - (1-\alpha) \beta_1^2 = 0$. The equality holds if and only if

$$\begin{cases} \lambda_1 = \lambda_2 = \dots = \lambda_n \\ \sum_{i=1}^n \frac{\lambda_i}{1+t\lambda_i} = 0 \end{cases} \iff \lambda_1 = \lambda_2 = \dots = \lambda_n = 0.$$

This implies that $DT = I$ a.e. on $\{\rho > 0\}$. The proposition is proven. \square

2 A Variational Principle for q -Moment Measures

2.1 Attainment of minimum

Theorem 2.1. *Assume that $\mu \in \mathcal{P}_1(\mathbb{R}^n)$ with $\int y d\mu(y) = 0$ and not supported on a hyperplane. Then the following optimization problem*

$$(P) \quad \min \{ \mathcal{F}(\rho) + \mathcal{T}(\rho, \mu) : \rho \in \mathcal{P}_1(\mathbb{R}^n) \}$$

admits a solution.

The idea for proving existence relies on the direct method in the calculus of variations, i.e. the fact that the minimized functional is lower semi-continuous and minimizing sequences are compact (lower bounds of the functional).

Proof of Theorem 2.1. Define $\mathcal{J}(\rho) := \mathcal{F}(\rho) + \mathcal{T}(\rho, \mu)$. Let $\rho_k \in \mathcal{P}_1(\mathbb{R}^n)$ be a minimizing sequence, i.e. $\inf_{\rho \in \mathcal{P}_1(\mathbb{R}^n)} \mathcal{J}(\rho) = \lim_{k \rightarrow +\infty} \mathcal{J}(\rho_k)$. We can suppose that all ρ_k have 0 as their barycenter as translations do not change the value of the two parts of the functional and we may also assume that $\mathcal{J}(\rho_k) \leq C_0 := \mathcal{J}(\rho^0)$ with $\rho^0 \in \mathcal{P}_1(\mathbb{R}^n)$. Using the estimates in Proposition 1.1 and Proposition 1.4, we have

$$\mathcal{J}(\rho) \geq C_1 \int_{\mathbb{R}^n} |x| d\rho(x) - \left(\int_{\mathbb{R}^n} |x| d\rho(x) \right)^\delta - C_2.$$

This implies that the moment

$$\int_{\mathbb{R}^n} |x| \, d\rho_k(x)$$

must be bounded. According to Remark 5.1.5 in [1], this gives tightness of the sequence ρ_k and hence we may extract a subsequence such that $\rho_k \rightharpoonup \bar{\rho}$. On the other hand, by Proposition 1.2 we know that the functional \mathcal{F} is l.s.c for the weak convergence, and the semi-continuity of \mathcal{T} along sequences with $\int x d\rho_k(x) = 0$ is in Proposition 1.1, these results imply that the functional \mathcal{J} is l.s.c with the condition $\int x d\rho_k(x) = 0$. Thus, we have

$$\mathcal{J}(\bar{\rho}) \leq \liminf_{k \rightarrow +\infty} \mathcal{J}(\rho_k) \leq \inf_{\rho \in \mathcal{P}_1(\mathbb{R}^n)} \mathcal{J}(\rho),$$

thus the minimum is indeed attained at $\rho_{\text{opt}} = \bar{\rho}$.

□

2.2 Optimality conditions and characterization of the minimizers

The goal of this section is to prove that each solution of the problem (P) is absolutely continuous, i.e. $\rho_{\text{opt}} = \rho^{\text{ac}}$ and show that if \bar{u} is solution to the minimization problem

$$\inf \left\{ \int_{\mathbb{R}^n} u d\rho_{\text{opt}} + \int_{\mathbb{R}^n} u^* d\mu \mid u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \text{ convex and l.s.c} \right\} \quad (6)$$

then there is a constant c such that $\bar{u} > c$ and $\rho_{\text{opt}} = (\bar{u} - c)^{-(n+q)}$. To do so, we need some lemmas as follows.

Lemma 2.1. *Assume that $\sigma, \sigma_1 \in L^1(\mathbb{R}^n)$ and $\varepsilon \in (0, 1)$. Then we have*

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \frac{f(\sigma + \varepsilon(\sigma_1 - \sigma)) - f(\sigma)}{\varepsilon} dx = \int_{\mathbb{R}^n} f'(\sigma)(\sigma_1 - \sigma) dx.$$

Proof. By convexity of the function f , the inequality

$$\int_{\mathbb{R}^n} \frac{f(\sigma + \varepsilon(\sigma_1 - \sigma)) - f(\sigma)}{\varepsilon} dx \geq \int_{\mathbb{R}^n} f'(\sigma)(\sigma_1 - \sigma) dx \quad (7)$$

is straightforward. For the opposite inequality, we will use Fatou's Lemma. The pointwise convergence of the integrand is trivial and we can get an upper bound by means of the following inequality, which is a consequence, for $\varepsilon < 1$, of the monotonicity of the incremental ratios of convex functions

$$\frac{f(\sigma + \varepsilon(\sigma_1 - \sigma)) - f(\sigma)}{\varepsilon} \leq f(\sigma_1) - f(\sigma).$$

We have $\sigma_1, \sigma \in L^1(\mathbb{R}^n)$ and $f(t) = \frac{-1}{\alpha}|t|^\alpha$ with $\alpha \in (0, 1)$, these imply that $f(\sigma_1), f(\sigma) \in L^{\frac{1}{\alpha}}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ (since $\frac{1}{\alpha} > 1$ ⁵). Therefore, this is sufficient to apply Fatou's Lemma and get

$$\limsup_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \frac{f(\sigma + \varepsilon(\sigma_1 - \sigma)) - f(\sigma)}{\varepsilon} dx \leq \int_{\mathbb{R}^n} f'(\sigma)(\sigma_1 - \sigma) dx. \quad (8)$$

Combining (7) and (8), the lemma is proven. \square

Lemma 2.2. *If $\bar{\rho}$ is a solution to the problem (P), $\bar{\rho} = \bar{\rho}^{\text{ac}} \mathcal{L}^n + \bar{\rho}^{\text{sing}}$ and $\bar{u} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is solution to (6), then*

$$\int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\bar{\rho}^{\text{ac}} + \int_{\mathbb{R}^n} \bar{u} d\bar{\rho}^{\text{sing}} \leq \int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\rho^{\text{ac}} + \int_{\mathbb{R}^n} \bar{u} d\rho^{\text{sing}} \quad (9)$$

for all $\rho \in \mathcal{P}_1(\mathbb{R}^n)$ written as $\rho = \rho^{\text{ac}} \mathcal{L}^n + \rho^{\text{sing}}$.

Proof. Assume that $\bar{\rho}$ is optimal and \bar{u} is a convex function realizing the minimum in the dual definition of $\mathcal{T}(\bar{\rho}, \mu)$, then functional

$$\rho \mapsto \int_{\mathbb{R}^n} f'(\rho^{\text{ac}}) dx + \int_{\mathbb{R}^n} \bar{u} d\rho = \int_{\mathbb{R}^n} f'(\rho^{\text{ac}}) dx + \int_{\mathbb{R}^n} \bar{u} d\rho^{\text{ac}} + \int_{\mathbb{R}^n} \bar{u} d\rho^{\text{sing}}$$

is minimal for $\rho = \bar{\rho}$. Now for every $\varepsilon \in (0, 1)$ and any $\rho \in \mathcal{P}_1(\mathbb{R}^n)$, we define $\rho_\varepsilon := (1 - \varepsilon)\bar{\rho} + \varepsilon\rho$, $\Phi_1(\rho^{\text{ac}}) = \int f'(\rho^{\text{ac}}) dx$, $\Phi_2(\rho^{\text{ac}}) = \int \bar{u} d\rho^{\text{ac}}$ and $\Phi_3(\rho^{\text{sing}}) = \int \bar{u} d\rho^{\text{sing}}$. Using the optimality of $\bar{\rho}$, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\{(\Phi_1 + \Phi_2)(\rho_\varepsilon^{\text{ac}}) - (\Phi_1 + \Phi_2)(\bar{\rho}^{\text{ac}})\} + \{\Phi_3(\rho_\varepsilon^{\text{sing}}) - \Phi_3(\bar{\rho}^{\text{sing}})\}}{\varepsilon} \geq 0. \quad (10)$$

⁵In general case, if μ is a measure on Ω with $\mu(\Omega) < +\infty$ and $1 \leq q \leq p < +\infty$ then $L^p(\Omega) \subset L^q(\Omega)$. Indeed, assume $f \in L^p(\Omega)$. Using Hölder's Inequality for $p_1 = \frac{p}{q}$, $p_2 = \frac{p}{p-q}$, we have

$$\|f\|_{L^q}^q = \int_{\Omega} |f|^q = \int_{\Omega} |f|^q \cdot 1 \leq \| |f|^q \|_{L^{p/q}} \cdot \|1\|_{L^{p/(p-q)}} = \left(\int_{\Omega} |f|^p \right)^{p/q} \cdot \mu(\Omega)^{p/(p-q)} < +\infty.$$

Thus $f \in L^q(\Omega)$.

On the other hand,

$$\begin{cases} \frac{\Phi_2(\rho_\varepsilon^{\text{ac}}) - \Phi_2(\bar{\rho}^{\text{ac}})}{\varepsilon} = \int_{\mathbb{R}^n} \bar{u} d(\rho^{\text{ac}} - \bar{\rho}^{\text{ac}}) \\ \frac{\Phi_3(\rho_\varepsilon^{\text{sing}}) - \Phi_3(\bar{\rho}^{\text{sing}})}{\varepsilon} = \int_{\mathbb{R}^n} \bar{u} d(\rho^{\text{sing}} - \bar{\rho}^{\text{sing}}) \end{cases} \quad (11)$$

and by Lemma 2.1,

$$\lim_{\varepsilon \rightarrow 0} \frac{\Phi_1(\rho_\varepsilon^{\text{ac}}) - \Phi_1(\bar{\rho}^{\text{ac}})}{\varepsilon} = \int_{\mathbb{R}^n} f'(\bar{\rho}^{\text{ac}}) d(\rho^{\text{ac}} - \bar{\rho}^{\text{ac}}). \quad (12)$$

Combining (10), (11) and (12), we obtain (9). □

Lemma 2.3. *The solution $\bar{\rho}$ to the problem (P) cannot be such that $\bar{\rho}^{\text{ac}} = 0$ a.e.*

Proof. Indeed, assume that $\bar{\rho}^{\text{ac}} = 0$, then $\bar{\rho} = \bar{\rho}^{\text{sing}}$, $\mathcal{F}(\bar{\rho}) = \mathcal{F}(\bar{\rho}^{\text{sing}}) = 0$ and hence

$$\mathcal{J}(\bar{\rho}) = \mathcal{T}(\bar{\rho}, \mu) \geq 0.$$

On the other hand, for $\eta > 0$, taking

$$\rho^\eta := \frac{\mathbf{1}_{B(0, \eta)}}{|B(0, \eta)|} \cdot \mathcal{L}^n.$$

then $\rho^\eta \in \mathcal{P}_1(\mathbb{R}^n)$,

$$\mathcal{T}(\rho^\eta, \mu) = \max \left\{ \int_{\text{spt}(\rho^\eta) \times \mathbb{R}^n} x \cdot y d\gamma : \gamma \in \Pi(\mu, \rho^\eta) \right\} \leq \eta \int_{\mathbb{R}^n} |y| d\mu =: C\eta,$$

and value of the local functional

$$\mathcal{F}(\rho^\eta) = -\frac{|B(0, \eta)|}{|B(0, \eta)|^\alpha} = -|B(0, \eta)|^{1-\alpha} = -C_1 \eta^{n(1-\alpha)}.$$

It follows that

$$\mathcal{J}(\rho^\eta) = C\eta - C_1 \eta^{n(1-\alpha)} = C_1 \eta \left(\frac{C}{C_1} - \eta^{n(1-\alpha)-1} \right) = C_1 \eta \left(\frac{C}{C_1} - \eta^{-q/(n+q)} \right).$$

We have

$$\frac{C}{C_1} - \eta^{-q/(n+q)} < 0 \iff \eta < \left(\frac{C_1}{C} \right)^{(n+q)/q}.$$

Thus, for every $\bar{\eta}$ such that $0 < \bar{\eta} < \left(\frac{C_1}{C} \right)^{(n+q)/q}$, $\mathcal{J}(\rho^{\bar{\eta}}) < 0 \leq \mathcal{J}(\bar{\rho})$. This gives a contradiction because of the optimality of $\bar{\rho}$. □

Theorem 2.2. *If $\bar{\rho}$ is a solution to the problem (P) and \bar{u} is a solution to (6) (with ρ_{opt} is replaced by $\bar{\rho}$) then there exists a constant c such that $\bar{u} \geq c$ and*

$$\begin{cases} \bar{u} + f'(\bar{\rho}^{\text{ac}}) = c \text{ a.e on } \{\bar{\rho}^{\text{ac}} > 0\} \\ \bar{u} + f'(\bar{\rho}^{\text{ac}}) \geq c \text{ a.e on } \{\bar{\rho}^{\text{ac}} = 0\} \end{cases} \quad (13)$$

and $\bar{\rho}^{\text{sing}}$ is concentrated on $\{\bar{u} = c\}$. Moreover, the set $\{\bar{u} = c\}$ is empty and thus $\bar{\rho}^{\text{sing}} = 0$, the optimal $\bar{\rho}$ is absolutely continuous and

$$\bar{\rho} = \bar{\rho}^{\text{ac}} = \frac{1}{(\bar{u} - c)^{n+q}}. \quad (14)$$

Proof. Define the essential infimum

$$c = \text{ess inf } \{\bar{u} + f'(\bar{\rho}^{\text{ac}})\} := \sup \{l : \mathcal{L}^n(\{x : \bar{u}(x) + f'(\bar{\rho}^{\text{ac}}(x)) < l\}) = 0\}.$$

Then, we have $\bar{u} + f'(\bar{\rho}^{\text{ac}}) \geq c$ a.e, and so $\bar{u} \geq \bar{u} + f'(\bar{\rho}^{\text{ac}}) \geq c$ a.e (since $f'(\bar{\rho}^{\text{ac}}) = -(\bar{\rho}^{\text{ac}})^{\alpha-1} \leq 0$). It follows that

$$\int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\bar{\rho}^{\text{ac}} + \int_{\mathbb{R}^n} \bar{u} d\bar{\rho}^{\text{sing}} \geq \int_{\mathbb{R}^n} c d\bar{\rho}^{\text{ac}} + \int_{\mathbb{R}^n} c d\bar{\rho}^{\text{sing}} = c. \quad (15)$$

Take any $c' > c$. By definition of essential infimum, the set $\{\bar{u} + f'(\bar{\rho}^{\text{ac}}) < c'\}$ has positive Lebesgue measure, and so (by the arbitrariness of ρ in Lemma 2.2) we can choose $\rho \in \mathcal{P}_1^{\text{ac}}(\mathbb{R}^n)$ such that ρ concentrated on the set $\{\bar{u} + f'(\bar{\rho}^{\text{ac}}) < c'\}$. Then we get

$$\int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\rho^{\text{ac}} + \int_{\mathbb{R}^n} \bar{u} d\rho^{\text{sing}} = \int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\rho < c'. \quad (16)$$

Combining (9) in Lemma 2.2 and (16), we obtain

$$c' > \int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\bar{\rho}^{\text{ac}} + \int_{\mathbb{R}^n} \bar{u} d\bar{\rho}^{\text{sing}} \quad (17)$$

Letting $c' \rightarrow c$ in (17), we get

$$c \geq \int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\bar{\rho}^{\text{ac}} + \int_{\mathbb{R}^n} \bar{u} d\bar{\rho}^{\text{sing}} \quad (18)$$

From (15) and (18), we deduce that

$$\begin{cases} \int_{\mathbb{R}^n} (\bar{u} + f'(\bar{\rho}^{\text{ac}})) d\bar{\rho}^{\text{ac}} = \int_{\mathbb{R}^n} c d\bar{\rho}^{\text{ac}} \\ \int_{\mathbb{R}^n} \bar{u} d\bar{\rho}^{\text{sing}} = \int_{\mathbb{R}^n} c d\bar{\rho}^{\text{sing}} \end{cases}$$

It follows that

$$\begin{cases} \bar{u} + f'(\bar{\rho}^{\text{ac}}) = c & \bar{\rho}^{\text{ac}}\text{-a.e} \\ \bar{u} = c & \bar{\rho}^{\text{sing}}\text{-a.e} \end{cases}$$

This implies that (13) is satisfied and $\bar{\rho}^{\text{sing}}$ is concentrated on $\{\bar{u} = c\}$.

We next prove that $\{\bar{u} = c\} = \emptyset$. Firstly, the condition (13) implies that if $\bar{u} = +\infty$ then $\bar{\rho}^{\text{ac}} = 0$. Thanks to Lemma 2.3, we get $\{\bar{u} < +\infty\} \neq \emptyset$, hence the interior of $\{\bar{u} < +\infty\}$ is also non-empty (since if the interior is empty then $\bar{\rho}^{\text{ac}} = 0$)⁶.

Now, we suppose that there exists x_0 such that $\bar{u}(x_0) = c$, we will consider two cases:

- *First case:* x_0 is in the interior of $\{\bar{u} < +\infty\}$, then there exists a neighborhood of x_0 , $\mathcal{N}(x_0) \subset \{\bar{u} < +\infty\}$ where \bar{u} is locally Lipschitz, i.e. $\bar{u} - c < L|x - x_0|$ for all $x \in \mathcal{N}(x_0)$, it follows that

$$+\infty > \int_{\mathcal{N}(x_0)} \bar{\rho}^{\text{ac}} dx \geq \int_{\mathcal{N}(x_0)} (L|x - x_0|)^{-(n+q)} dx = +\infty,$$

This is a contradiction. Therefore we exclude this case.

- *Second case:* if x_0 is on the boundary of $\{\bar{u} < +\infty\}$, then we take n points x_1, x_2, \dots, x_n in the interior of $\{\bar{u} < +\infty\}$ and obtain a simplex with non-empty interior Δ whose vertices are $x_0, x_1, x_2, \dots, x_n$. On this simplex, \bar{u} is finite and $\bar{u} - c < L|x - x_0|$ for all $x \in \Delta$. Hence we also get a contradiction and this case is also excluded.

Thus $\{\bar{u} = c\} = \emptyset$ and final conclusions of the theorem are proven. \square

2.3 Uniqueness result

Theorem 2.3. *The solution to the problem (P) is unique up to translation.*

Proof. Uniqueness of minimizer is deduced from strict convexity on the geodesic ρ_t connecting ρ_0 to ρ_1 in Proposition 1.5. For the case $DT = I$ a.e on $\{\rho_0 > 0\}$, uniqueness result is deduced from Lemma 2.4. \square

⁶Our goal here is to need to exclude the case where the interior of $\{\bar{u} < +\infty\}$ is empty. The way to deduce this: if the interior of $\{\bar{u} < +\infty\}$ is empty, since the set $\{\bar{u} < +\infty\}$ is a convex set, then it is negligible. In this case $\bar{u} = +\infty$ a.e., and $\bar{\rho}^{\text{ac}} = 0$ a.e.. This means $\bar{\rho} = \bar{\rho}^{\text{sing}}$ but we already saw that it is impossible, Lemma 2.3.

Lemma 2.4. ([24], Lemma 4.2). *Suppose that ρ_0 and ρ_1 are two absolutely continuous densities in \mathbb{R}^n with $K_i = \{x \in \mathbb{R}^n : \rho_i(x) > 0\}$ convex for $i = 0, 1$, and let $T = \nabla u$ be the optimal transport from ρ_0 to ρ_1 and $S = \nabla u^*$ the optimal transport from ρ_1 to ρ_0 . Suppose $DT = D^2u = I$ a.e on K_0 and $DS = D^2u^* = I$ on K_1 . Then both T and S are translations, i.e. there exists a vector $v \in \mathbb{R}^n$ such that $T(x) = x + v$ and $S(y) = y - v$.*

3 Applications in Convex Geometry

In this section, we want to establish some results in differential geometry, convex geometry related to q -moment measures, that are, existence and uniqueness up to affine transformations of affine hemispheres of elliptic type, i.e. affine hemispheres such that the ray from an arbitrary point of it to the origin always passes through the convex side of it.

3.1 Some definitions in differential geometry

We say that a smooth, connected hypersurface $M \subset \mathbb{R}^{n+1}$ is locally strongly-convex if the second fundamental form is a definite symmetric bilinear form at any point $y_0 \in M$. Let $P := T_{y_0}M$ be the tangent space to M at the point $y_0 \in M$, i.e. the linear space that best approximates M at y_0 .

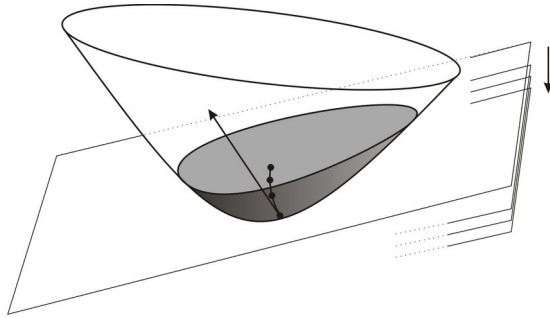


Figure 1⁷. The affine normal vector $\zeta'(0)$.

Definition 3.1. (affine normal vector). Let $(P_t)_{t \geq 0}$ be a sequence of parallel transformations of the tangent space P , defined by

$$\begin{cases} P_0 = T_{y_0}M & \text{if } t = 0 \\ P_t = T_{y_0}M + tv & \text{if } t > 0 \end{cases}$$

where $v \notin T_{y_0}M$ is a vector pointing to the convex side of M at the point y_0 . For every t the plane P_t cuts M and selects a compact sector Ω_t of finite volume inside M . Then let $\zeta(t) = \text{bar}(\Omega_t)$ be the barycenter of Ω_t . As $t \rightarrow 0$ the point $\zeta(t)$ draws a curve $t \mapsto \zeta(t)$ which ends at $\zeta(0) = y_0$. The tangent to the curve at y_0 selects a special direction, that is $\zeta'(0)$, which is the direction of the affine normal line; and the vector $\zeta'(0)$ is called the affine normal vector of M at the point y_0 (which is also denoted by $\zeta'(0)(y_0)$).

⁷the original of this figure is in Minguzzi's paper [18].

Definition 3.2. (i) The line $\ell_M(y_0)$ is called the affine normal line at $y_0 \in M$ if it passes through y_0 in the direction of the non-zero vector $\zeta'(0)$.

(ii) M is affinely-spherical with center at a point $p \in \mathbb{R}^{n+1}$ if all of the affine normal lines of M meet at p . In the case where all of the affine normal lines are parallel, we say that M is affinely-spherical with center at infinity.

(iii) An affine sphere is an affinely-spherical hypersurface which is complete, i.e., it is a closed subset of \mathbb{R}^{n+1} .

Affine spheres were introduced by the Romanian geometer Tzitzéica in 1909. All convex quadratic hypersurfaces in \mathbb{R}^{n+1} are affine spheres, as well as the hypersurface

$$M = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \text{for every } i, x_i > 0 \text{ and } \prod_{i=1}^n x_i = 1 \right\}$$

found by Tzitzéica [26] and Calabi [7].

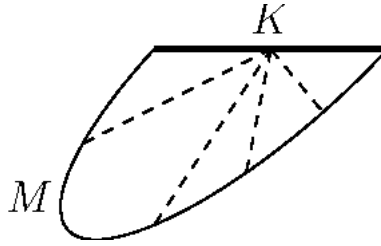


Figure 2⁸. Half of an ellipse, which is an affine one-dimensional hemisphere in \mathbb{R}^2 .

Definition 3.3. A smooth, connected, locally strongly-convex hypersurface $M \subset \mathbb{R}^{n+1}$ is called an affine hemisphere if the following conditions are satisfied

- (a) there exist compact, convex sets $K, \tilde{K} \subset \mathbb{R}^{n+1}$, with $\dim(K) = n$ and $\dim(\tilde{K}) = n + 1$ such that M does not intersect the affine hyperplane spanned by K and

$$\partial \tilde{K} = M \cup K$$

- (b) it is affinely-spherical with center at the relative interior of K .

In Definition 3.3, the dimension $\dim(K)$ is the maximal number N such that K contains $N + 1$ affinely-independent vectors. And when $M \subset \mathbb{R}^{n+1}$ is an affine hemisphere then its anchor K is the compact, convex set enclosed by $\overline{M} \setminus M$, where \overline{M} is the closure of M . In particular, K is convex hull⁹ of $\overline{M} \setminus M$, i.e., $K = \text{Conv}(\overline{M} \setminus M)$.

⁸the original of this figure is in Klartag's paper [13].

⁹The convex hull of a set X is the smallest convex set that contains X .

3.2 The affine hemisphere equations

For a smooth function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, we recall that $\nabla^2 g(x_0)$ is the Hessian matrix of g at the point $x_0 \in \mathbb{R}^n$. A smooth function $g : L \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is called strongly-convex if $\nabla^2 g(x)$ is positive-definite for any $x \in L$. Assume that $L \subset \mathbb{R}^n$ is a non-empty, open, bounded, convex set. In many problems of mathematics, especially in geometry, one is interested in smooth, convex solutions $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$ to the equation with the constraint

$$\begin{cases} \det \nabla^2 \varphi = C/\varphi^{n+2} & \text{in } \mathbb{R}^n \\ \nabla \varphi(\mathbb{R}^n) = L \end{cases} \quad (19)$$

where C is a positive number and $\nabla \varphi(\mathbb{R}^n) := \{\nabla \varphi(x) : x \in \mathbb{R}^n\}$. The equation (19) is a form of the *Monge–Ampère second boundary value problem*, and it has a solution if and only if the barycenter of L lies at the origin, this result will be proven in next subsection. In this part, we want to establish the relation between the solution of the equation (19) and existence of an affinely-spherical with center at the origin. Our main result is the following.

Theorem 3.1. *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface and let $L \subset \mathbb{R}^n$ be a non-empty, open, convex set. Suppose that $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex function whose restriction to the set L is finite, smooth, strongly convex; and assume that*

$$M = \text{Graph}_L(\psi) := \{(x, \psi(x)) : x \in L\} \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}.$$

Denote $\varphi = \psi^*$ and $\Omega = \nabla \psi(L) = \{\nabla \psi(x) : x \in L\}$. Then, we have

- (i) *The set $\Omega \subset \mathbb{R}^n$ is open and the function φ is smooth in Ω .*
- (ii) *The hypersurface M is affinely-spherical with center at the origin if and only if there exists $C \in \mathbb{R} \setminus \{0\}$ such that $\varphi^{n+2} \cdot \det \nabla^2 \varphi = C$ in the entire set Ω .*

The statement (i) of Theorem 3.1 is deduced from some results in convex analysis and functional analysis. The main idea to prove the part (ii) is thanks to the equation of the affine normal line at an arbitrary point $(x, \psi(x)) \in M$, which depends on derivatives of ψ . After that, using the relations between $(\psi, \nabla \psi, \nabla^2 \psi)$ and $(\varphi, \nabla \varphi, \nabla^2 \varphi)$, we prove that this is equivalent to

$$\nabla (\varphi^{n+2} \cdot \det \nabla^2 \varphi)(y) = 0 \quad \text{for every } y \in \Omega. \quad (20)$$

Thus, one of key challenges in proving Theorem 3.1 is finding the formula of the affine normal vector $\zeta'(0)$ at an arbitrary point $(x, \psi(x)) \in M$. In fact we prove that

$$\begin{cases} \zeta'(0)((x, \psi(x))) &= \frac{1}{2n+4} \cdot (a(x), b(x)) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \\ b(x) \cdot x - \psi(x) \cdot a(x) &= 0 \iff (0_{\mathbb{R}^n}, 0_{\mathbb{R}}) \in \ell_M((x, \psi(x))) \end{cases} \quad (21)$$

where $(a(x), b(x))$ is defined by

$$\begin{cases} a(x) := -(\nabla^2 \psi(x))^{-1} \cdot \nabla(\log \det \nabla^2 \psi)(x) \\ b(x) := n + 2 - \langle (\nabla^2 \psi(x))^{-1} \cdot \nabla(\log \det \nabla^2 \psi)(x), \nabla \psi(x) \rangle. \end{cases} \quad (22)$$

To do so, we first consider the following special case.

Lemma 3.1. *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly-convex hypersurface. Let $L \subset \mathbb{R}^n$ be an open, convex set containing the origin. Assume that $U \subset \mathbb{R}^{n+1}$ is an open set such that*

$$M \cap U = \text{Graph}_L(\psi)$$

where $\psi : L \rightarrow \mathbb{R}$ is a smooth, strongly-convex function with $\psi(0) = 0, \nabla \psi(0) = 0$ and $\nabla^2 \psi(0)$ is the identity matrix.

Then for $y_0 = (0, 0) = (0, \psi(0)) \in M$, the affine normal vector at the point y_0 is the vector

$$\zeta'(0)(y_0) = \frac{1}{2n+4} \cdot (a(0), b(0)) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}. \quad (23)$$

Sketch of proof of Lemma 3.1. Step 1: determine the representation of the set Ω_t (that is defined as in Definition 3.1); in this case, we will prove the formula (26). Step 2: find the barycenter of the set $\Omega_{t/2}$ by using the spherical-coordinates

representation of $\Omega_{t/2}/\sqrt{t}$ as in (29); note that the formula of the barycenter \bar{x} ¹⁰ of a set in form $\{r\theta : \theta \in \mathbb{S}^{n-1} \text{ and } 0 \leq r \leq r(\theta)\}$ is given by

$$\bar{x} = \frac{1}{A} \int_{\mathbb{S}^{n-1}} \int_0^{r(\theta)} (r\theta) r^{n-1} dr d\theta \quad \text{where} \quad A = \int_{\mathbb{S}^{n-1}} \int_0^{r(\theta)} r^{n-1} dr d\theta \quad (24)$$

Step 3: deduce the integral form of the affine normal vector $\zeta'(0)(y_0)$ as in (30). Step 4: using properties of the Gaussian random vector to prove that (30) is equivalent to (23). If $X = (X_1, X_2, \dots, X_n)$ is a standard Gaussian random vector in \mathbb{R}^n then $\mathbb{E}X_i^k = 0$ for $k = 2l + 1$, $\mathbb{E}X_i^k = 1.3.5 \dots (k - 1)$ for $k = 2l$, and

$$\int_{\mathbb{S}^{n-1}} p(\theta) d\sigma_{n-1}(\theta) = \frac{1}{n(n+2)} \mathbb{E}p(X) \quad (25)$$

for any homogeneous polynomial p of degree 4 in n real variables.

Proof of Lemma 3.1. From the hypothesis of the function $\psi: \psi(0) = 0, \nabla\psi(0) = 0$ and ψ is strongly-convex function, we have $x_0 = 0 \in L$, is unique global minimizer of the function ψ . Hence, the tangent space to M at the point y_0 is

$$P_0 = T_{y_0}M = \{(x, 0) : x \in \mathbb{R}^n\}$$

and every vector of the form $(0, \beta) \in \mathbb{R}^n \times \mathbb{R}$ with $\beta > 0$, is pointing to the convex side of the set M at the point y_0 . Taking vector $v = (0, 1) \in \mathbb{R}^n \times \mathbb{R}$ and define the plane P_t as in Definition 3.1, then the section $P_t \cap M$ encloses an n -dimensional convex body¹¹ $\Omega_t \subset P_t$ given by

$$\Omega_t = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \psi(x) \leq t\}.$$

For every $k \leq n$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i \in \{0, 1, 2, \dots\}$, we denote

$$a_{i_1 i_2 \dots i_k} = \partial^{i_1 i_2 \dots i_k} \psi(0) = \frac{\partial^k \psi}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_k}}(0) := \frac{\partial}{\partial x_{i_1}} \left(\frac{\partial}{\partial x_{i_2}} \dots \left(\frac{\partial \psi}{\partial x_{i_k}} \right) \right) (0)$$

and

$$\partial^\alpha \psi(x) := \frac{\partial^{|\alpha|} \psi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}(x)$$

¹⁰in the formula (24), $\int_{\mathbb{S}^{n-1}} \int_0^{r(\theta)} (r\theta) r^{n-1} dr d\theta$ is a vector in \mathbb{R}^n ,

$$\int_{\mathbb{S}^{n-1}} \int_0^{r(\theta)} (r\theta) r^{n-1} dr d\theta := \left(\int_{\mathbb{S}^{n-1}} \int_0^{r(\theta)} (r\theta_1) r^{n-1} dr d\theta, \dots, \int_{\mathbb{S}^{n-1}} \int_0^{r(\theta)} (r\theta_n) r^{n-1} dr d\theta \right)$$

¹¹is a compact convex set with non-empty interior.

where $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\alpha! := \alpha_1! \alpha_2! \dots \alpha_n!$ and $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Note that for every non-negative integer m , $m! := 1.2.3 \dots (m-1).m$. Applying the Taylor's expansion formula for the function ψ at $x_0 = 0$, we have

$$\psi(x) = \sum_{|\alpha| \leq 3} \frac{\partial^\alpha \psi(0)}{\alpha!} x^\alpha + \sum_{|\alpha|=4} \frac{\partial^\alpha \psi(\eta x)}{\alpha!} x^\alpha$$

for some $\eta \in (0, 1)$. We need some computations to simplify the formula above.

- the first, $\sum_{|\alpha|=0} \frac{\partial^\alpha \psi(0)}{\alpha!} x^\alpha = \psi(0) = 0$,

- next

$$\sum_{|\alpha|=1} \frac{\partial^\alpha \psi(0)}{\alpha!} x^\alpha = \sum_{|\alpha|=1} \partial^\alpha \psi(0) \cdot x^\alpha = \sum_{i=1}^n \frac{\partial \psi}{\partial x_i}(0) \cdot x_i = \langle \nabla \psi(0), x \rangle = 0,$$

-

$$\begin{aligned} \sum_{|\alpha|=2} \frac{\partial^\alpha \psi(0)}{\alpha!} x^\alpha &= \sum_{|\alpha|=2: \exists \alpha_i=1} \frac{\partial^\alpha \psi(0)}{\alpha!} x^\alpha + \sum_{|\alpha|=2: \exists \alpha_i=2} \frac{\partial^\alpha \psi(0)}{\alpha!} x^\alpha \\ &= 0 + \frac{1}{2!} \sum_{i=1}^n \frac{\partial^2 \psi}{\partial x_i^2}(0) \cdot x_i^2 = \frac{1}{2!} \sum_{i=1}^n x_i^2 = \frac{|x|^2}{2} \text{ (since } \nabla^2 \psi(0) = \text{Id)} \end{aligned}$$

-

$$\begin{aligned} &\sum_{|\alpha|=3} \frac{\partial^\alpha \psi(0)}{\alpha!} x^\alpha \\ &= \sum_{|\alpha|=3: \alpha!=1} \partial^\alpha \psi(0) \cdot x^\alpha + \frac{1}{2} \left\{ \sum_{|\alpha|=3: \alpha!=2} \partial^\alpha \psi(0) \cdot x^\alpha \right\} + \frac{1}{6} \left\{ \sum_{|\alpha|=3: \alpha!=6} \partial^\alpha \psi(0) \cdot x^\alpha \right\} \\ &= \frac{1}{3!} \cdot \left\{ \sum_{i \neq j, i \neq k, j \neq k} a_{ijk} x_i x_j x_k \right\} \\ &\quad + \frac{1}{2} \cdot \left\{ \frac{1}{3} \cdot \left[\sum_{i=j \neq k} a_{ijk} x_i x_j x_k + \sum_{i=k \neq j} a_{ijk} x_i x_j x_k + \sum_{i \neq j=k} a_{ijk} x_i x_j x_k \right] \right\} + \frac{1}{6} \cdot \sum_{i=1}^n a_{iii} x_i^3 \\ &= \frac{1}{6} \cdot \sum_{i,j,k=1}^n a_{ijk} x_i x_j x_k \end{aligned}$$

- Finally, we have $x_i \leq \sqrt{x_i^2} \leq \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = |x|$, it follows that $x^\alpha \leq |x|^\alpha$ with $|\alpha| = 4$. Thus, the remainder can be represented as follows

$$\sum_{|\alpha|=4} \frac{\partial^\alpha \psi(\eta x)}{\alpha!} x^\alpha = \mathcal{O}(|x|^4)$$

where $\mathcal{O}(|x|^4)$ is an abbreviation for an expression that is bounded in absolute value by $C|x|^4$ where C depends only on M .

Now we can rewrite the representation of the function ψ at the point $x_0 = 0$,

$$\psi(x) = \frac{|x|^2}{2} + \frac{1}{6} \cdot \sum_{i,j,k=1}^n a_{ijk} x_i x_j x_k + \mathcal{O}(|x|^4),$$

and then we also have the representation of Ω_t

$$\Omega_t = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : \frac{|x|^2}{2} + \frac{1}{6} \cdot \sum_{i,j,k=1}^n a_{ijk} x_i x_j x_k + \mathcal{O}(|x|^4) \leq t \right\}. \quad (26)$$

To simplify in determining the barycenter, we next consider the set

$$\frac{\Omega_{t/2}}{\sqrt{t}} = \left\{ \left(\frac{1}{\sqrt{t}} \cdot x, \frac{\sqrt{t}}{2} \right) \in \mathbb{R}^n \times \mathbb{R} : \frac{|x|^2}{2} + \frac{1}{6} \sum_{i,j,k=1}^n a_{ijk} x_i x_j x_k + \mathcal{O}(|x|^4) \leq \frac{t}{2} \right\}.$$

Then, using the spherical-coordinates representation of $\Omega_{t/2}/\sqrt{t}$,

$$\frac{1}{\sqrt{t}} \cdot x = r\theta \quad \text{with } \theta \in \mathbb{S}^{n-1} := \{s \in \mathbb{R}^n : |s| = 1\} \quad \text{and } r = \frac{|x|}{\sqrt{t}}$$

From the conditions on x , we have

$$\frac{t \cdot r^2}{2} + \left(\frac{1}{6} \sum_{i,j,k=1}^n a_{ijk} \theta_i \theta_j \theta_k \right) t^{3/2} r^3 + \mathcal{O}(t^2) \leq \frac{t}{2}$$

It follows that

$$\frac{r^2}{2} + \left(\frac{\sum_{i,j,k=1}^n a_{ijk} \theta_i \theta_j \theta_k}{6} \sqrt{t} \right) r^3 + \mathcal{O}(t) \leq \frac{1}{2}$$

and so

$$r - 1 + \left(\frac{\sum_{i,j,k=1}^n a_{ijk} \theta_i \theta_j \theta_k}{6} \sqrt{t} \right) \cdot \frac{2r^3}{r+1} + \mathcal{O}(t) \leq 0 \quad (27)$$

The inequality above holds for all $t > 0$ and $\mathcal{O}(t)$ is an abbreviation for an expression that is bounded by $C.t$, where C depends only on M . First we have $r \leq 1 + \mathcal{O}(\sqrt{t})$, and then for a sufficiently small $t > 0$,

$$\begin{aligned} \frac{2r^3}{r+1} &= \frac{2r^3(r-1)}{r^2-1} \approx \frac{2(1+\mathcal{O}(\sqrt{t}))^3 \cdot (1+\mathcal{O}(\sqrt{t})-1)}{(1+\mathcal{O}(\sqrt{t}))^2-1} \\ &\approx \frac{2\mathcal{O}(\sqrt{t}) \cdot (1+\mathcal{O}(\sqrt{t}))^3}{1+2\mathcal{O}(\sqrt{t})-1} = (1+\mathcal{O}(\sqrt{t}))^3 \approx 1 + \mathcal{O}(\sqrt{t}) \end{aligned}$$

Therefore, for a sufficiently small $t > 0$, the inequality (27) implies that

$$r \leq 1 - \frac{\sum_{i,j,k=1}^n a_{ijk}\theta_i\theta_j\theta_k}{6}\sqrt{t} \left[1 + \mathcal{O}(\sqrt{t})\right] + \mathcal{O}(t) = 1 - \frac{\sum_{i,j,k=1}^n a_{ijk}\theta_i\theta_j\theta_k}{6}\sqrt{t} + \mathcal{O}(t) \quad (28)$$

Then, $\Omega_{t/2}/\sqrt{t}$ is presented in the form

$$\frac{\Omega_{t/2}}{\sqrt{t}} = \left\{ \left(r\theta, \frac{\sqrt{t}}{2} \right) : \theta \in \mathbb{S}^{n-1} \text{ and } 0 \leq r \leq r_t(\theta) = 1 - \frac{\sum_{i,j,k=1}^n a_{ijk}\theta_i\theta_j\theta_k}{6}\sqrt{t} + \mathcal{O}(t) \right\}. \quad (29)$$

Consequently, the barycenter of $\Omega_{t/2}$ satisfies $\zeta(t) = \text{bar}(\Omega_{t/2}) = (x_t, t/2)$ for

$$\begin{aligned} x_t &= \frac{1}{\int_{\mathbb{S}^{n-1}} \left(\int_0^{r_t(\theta)} r^{n-1} dr \right) d\theta} \cdot \int_{\mathbb{S}^{n-1}} \left(\int_0^{r_t(\theta)} (\sqrt{t}.r.\theta) r^{n-1} dr \right) d\theta \\ &= \sqrt{t} \cdot \frac{n}{\int_{\mathbb{S}^{n-1}} r_t(\theta)^n d\theta} \cdot \int_{\mathbb{S}^{n-1}} \theta \frac{r_t(\theta)^{n+1}}{n+1} d\theta = \sqrt{t} \cdot \frac{n \int_{\mathbb{S}^{n-1}} \theta r_t(\theta)^{n+1} d\theta}{(n+1) \int_{\mathbb{S}^{n-1}} r_t(\theta)^n d\theta}, \end{aligned}$$

here we use the formula (24) to compute x_t . On the other hand, we have ¹²

$$\left(1 - \frac{\sum_{i,j,k=1}^n a_{ijk}\theta_i\theta_j\theta_k}{6}\sqrt{t} + \mathcal{O}(t) \right)^{n+1} = 1 - \sqrt{t} \cdot \frac{n+1}{6} \cdot \sum_{i,j,k=1}^n a_{ijk}\theta_i\theta_j\theta_k + \mathcal{O}(t)$$

¹²here we use the formula $(a+b)^n = a^n + na^{n-1}b + \dots + \frac{n!}{k!(n-k)!}a^k b^{n-k} + \dots + b^n$ for $a, b \in \mathbb{R}$ and $n \in \{1, 2, \dots\}$.

and note that $\int_{\mathbb{S}^{n-1}} \theta d\theta = 0$,

$$x_t = -t \cdot \frac{n}{6} \cdot \int_{\mathbb{S}^{n-1}} \theta \left(\sum_{i,j,k=1}^n a_{ijk} \theta_i \theta_j \theta_k \right) d\sigma_{n-1}(\theta) + \mathcal{O}\left(t^{3/2}\right),$$

where σ_{n-1} is the uniform probability measure on \mathbb{S}^{n-1} . Moreover, $\left. \frac{d}{dt} \left(\mathcal{O}\left(t^{3/2}\right) \right) \right|_{t=0} = \mathcal{O}\left(\sqrt{t}\right) \Big|_{t=0} = 0$, it follows that the affine normal vector at the point y_0 is

$$\zeta'(0)(y_0) = \left(-\frac{n}{6} \cdot \int_{\mathbb{S}^{n-1}} \theta \left(\sum_{i,j,k=1}^n a_{ijk} \theta_i \theta_j \theta_k \right) d\sigma_{n-1}(\theta), \frac{1}{2} \right) \in \mathbb{R}^n \times \mathbb{R}. \quad (30)$$

Let $X = (X_1, X_2, \dots, X_n)$ be a standard Gaussian random vector in \mathbb{R}^n , and recall that $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}X_i^4 = 3$ for all i . Using the formula (25), we have

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \theta \left(\sum_{i,j,k=1}^n a_{ijk} \theta_i \theta_j \theta_k \right) d\sigma_{n-1}(\theta) &= \frac{1}{n(n+2)} \cdot \mathbb{E}X \left[\sum_{i,j,k=1}^n a_{ijk} X_i X_j X_k \right] \\ &= \frac{1}{n(n+2)} \cdot \mathbb{E}X \left[\sum_{i,j,k=1}^n \partial^{ijk} \psi(0) X_i X_j X_k \right]. \end{aligned}$$

Setting $N_a := \{1, 2, \dots, n\}$ and $N_a^3 = N_a \times N_a \times N_a$. For every $l \in N_a$, we define

$$\begin{cases} S_1(l) := \{(i, j, k) \in N_a^3 : l \in \{i, j, k\}\} \\ S_2(l) := \{(i, j, k) \in N_a^3 : l \notin \{i, j, k\}\} \\ S_3(l) := \left\{ (i, j, k) \in S_1(l) : \begin{cases} i = j \neq k = l \\ j = k \neq i = l \\ k = i \neq j = l \end{cases} \right\} \\ S_4(l) := S_1(l) \setminus (S_3(l) \cup \{l, l, l\}) \end{cases}$$

Then, we obtain

$$\begin{aligned} &\mathbb{E}X_l \left[\sum_{i,j,k=1}^n \partial^{ijk} \psi(0) X_i X_j X_k \right] \\ &= \mathbb{E}X_l \left[\begin{aligned} &\sum_{(i,j,k)=(l,l,l)} \partial^{ijk} \psi(0) X_i X_j X_k + \sum_{(i,j,k) \in S_3(l)} \partial^{ijk} \psi(0) X_i X_j X_k \\ &+ \sum_{(i,j,k) \in S_4(l)} \partial^{ijk} \psi(0) X_i X_j X_k + \sum_{(i,j,k) \in S_2(l)} \partial^{ijk} \psi(0) X_i X_j X_k \end{aligned} \right] \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E}X_l \left[\sum_{(i,j,k)=(l,l,l)} \partial^{ijk} \psi(0) X_i X_j X_k + \sum_{(i,j,k) \in S_3(l)} \partial^{ijk} \psi(0) X_i X_j X_k \right] \\
 &= 3 \cdot \frac{\partial}{\partial x_l} \left(\sum_{i=1}^n \frac{\partial^2 \psi}{\partial x_i^2} \right) (0).
 \end{aligned}$$

Thus, we get

$$\int_{\mathbb{S}^{n-1}} \theta \left(\sum_{i,j,k=1}^n a_{ijk} X_i X_j X_k \right) d\sigma_{n-1}(\theta) = \frac{3}{n(n+2)} \cdot \nabla \left(\sum_{i=1}^n \frac{\partial^2 \psi}{\partial x_i^2} \right) (0).$$

On the other hand, by the hypothesis of ψ , $\nabla^2 \psi(0) = \text{Id}$, this implies that $(\psi^{ij})_{i,j}(0) := (\nabla^2 \psi(0))^{-1}$ is also the identity matrix. Hence,

$$\sum_{k=l} \psi^{lk}(0) \cdot \partial^{ikl} \psi(0) = \sum_{k=1}^n \partial^{ikk} \psi(0) \quad \text{and} \quad \sum_{k \neq l} \psi^{lk}(0) \cdot \partial^{ikl} \psi(0) = 0$$

It follows that for every $i = 1, 2, \dots, n$,

$$\begin{aligned}
 \frac{\partial}{\partial x_i} \left(\sum_{i=1}^n \frac{\partial^2 \psi}{\partial x_i^2} \right) (0) &= \sum_{k=1}^n \partial^{ikk} \psi(0) \\
 &= \sum_{k=l} \psi^{lk}(0) \cdot \partial^{ikl} \psi(0) + \sum_{k \neq l} \psi^{lk}(0) \cdot \partial^{ikl} \psi(0) \\
 &= \sum_{k,l=1}^n \psi^{lk}(0) \cdot \partial^{ikl} \psi(0) \\
 &= \frac{\partial}{\partial x_i} (\log \det \nabla^2 \psi) (0).
 \end{aligned}$$

Noting that the function ψ is smooth and strongly convex; these properties show that $\det(\nabla^2 \psi(x)) > 0$ for all $x \in L$ and maps $x \mapsto \log(\det(\nabla^2 \psi))(x)$, $x \mapsto \nabla(\log \det(\nabla^2 \psi))(x)$ are well defined. Therefore,

$$\nabla \left(\sum_{i=1}^n \frac{\partial^2 \psi}{\partial x_i^2} \right) (0) = \nabla(\log \det \nabla^2 \psi)(0) = (\nabla^2 \psi(0))^{-1} \cdot \nabla(\log \det \nabla^2 \psi)(0),$$

and

$$\zeta'(0)(y_0) = \left(\frac{-1}{2(n+2)} (\nabla^2 \psi(0))^{-1} \cdot \nabla (\log \det \nabla^2 \psi)(0), \frac{1}{2} \right) = \frac{1}{2n+4} (a(0), b(0)).$$

□

Lemma 3.2. *(the affine normal vector in the general case). Let M, L and ψ be as in Theorem 3.1. Suppose that $x_0 \in L$ and $y_0 = (x_0, \psi(x_0)) \in M$. Then, the affine normal vector $\zeta'(0)(y_0)$ of M at the point y_0 is determined as in (21).*

Proof. By translating the coordinate axes, we may assume that $x_0 = 0$ and $\psi(0) = 0$. Moreover, the vector in (21) does not depend on the choice of the Euclidean structure in \mathbb{R}^n , hence we may switch to a Euclidean structure for which $\nabla^2 \psi(0) = \text{Id}$. We next consider the linear map $\mathcal{A} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$, which is defined by

$$\mathcal{A}((x, t)) := (x, t - \langle x, \nabla \psi(0) \rangle) \in \mathbb{R}^n \times \mathbb{R}, \text{ for all } (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Note that if $\nabla \psi(0) = 0$ then \mathcal{A} is the identity map. Define

$$\psi_1(x) := \psi(x) - \langle x, \nabla \psi(0) \rangle,$$

then the function ψ_1 satisfies all the hypothesis of the convex function in Lemma 3.1: $\psi_1(0) = 0$, $\nabla \psi_1(0) = \nabla \psi(0) - \nabla \psi(0) = 0$ and $\nabla^2 \psi_1(0) = \nabla^2 \psi(0) = \text{Id}$. Moreover, we have

$$\mathcal{A}(M) = \mathcal{A}(\text{Graph}_L(\psi)) = \text{Graph}_L(\psi_1) =: M_1$$

and

$$\mathcal{A}(\zeta'(0)(y_0)) = \frac{1}{2n+4} \left(-(\nabla^2 \psi_1(0))^{-1} \cdot \nabla (\log \det \nabla^2 \psi_1)(0), n+2 \right) =: \zeta_1'(0)(y_0)$$

Applying Lemma 3.1, we have that $\zeta_1'(0)(y_0)$ is the affine normal vector of M_1 at the point y_0 . Define that \mathcal{A}^{-1} is inverse map of the linear map \mathcal{A} . Then \mathcal{A}^{-1} is also the linear map in \mathbb{R}^{n+1} and $M = \mathcal{A}^{-1}(M_1)$, $\zeta'(0)(y_0) = \mathcal{A}^{-1}(\zeta_1'(0)(y_0))$. Thus, $\zeta'(0)(y_0)$ is the affine normal vector of M at the point y_0 .

□

To pass to the proof of the main result of this section, we recall a few well-known results in the theory of Legendre transforms. Given a non-empty, open, convex set $L \subset \mathbb{R}^n$ and $\psi : L \rightarrow \mathbb{R}$ is a convex function, $\varphi(y) = \psi^*(y) := \sup_{x \in L} (\langle y, x \rangle - \psi(x))$. Then, for all points x where ψ is differentiable we know

that the gradient inequality of ψ , $\psi(z) \geq \psi(x) + \nabla\psi(x) \cdot (z - x)$ for all $z \in L$, which expresses the geometrical fact that the whole graph of ψ lies above its tangent hyperplane at the point x . The subdifferential $\partial\psi$ of the convex function ψ is a set-valued application and is defined by

$$y \in \partial\psi(x) \iff \psi(z) \geq \psi(x) + \langle y, z - x \rangle \text{ for all } z \in L.$$

By using the Hahn-Banach separation theorem, one can show that for all $x \in \text{int}(\text{dom}(\psi))$, the subdifferential $\partial\psi(x)$ is non-empty. Moreover, ψ is differential at a point x if and only if $\partial\psi(x) = \{\nabla\psi(x)\}$. If ψ is lower semi-continuous, then the subdifferential mapping $\partial\psi$ is always continuous on the whole of \mathbb{R}^n , in the sense that

$$\left. \begin{array}{l} x_k \rightarrow x \\ \partial\psi(x_k) \ni y_k \rightarrow y \end{array} \right\} \implies y \in \partial\psi(x).$$

To obtain the result on "characterization of subdifferential" as in the proposition 2.4 of [28], we extend the function ψ to a function on all of \mathbb{R}^n in a suitable way (see [9], page 178) and for this, we have to restrict to the case when L is convex. In fact that let us define ψ in $\mathbb{R}^n \setminus L$ as

$$\psi(x) := \begin{cases} \liminf_{L \ni z \rightarrow x} \psi(z) & \text{if } x \in \partial L \\ +\infty & \text{if } x \in \mathbb{R}^n \setminus \bar{L}. \end{cases}$$

Since L is convex, it follows that $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous convex function and

$$\varphi(y) = \psi^*(y) = \sup_{x \in \mathbb{R}^n} (\langle y, x \rangle - \psi(x)) = \sup_{x \in \bar{L}} (\langle y, x \rangle - \psi(x)).$$

In this case, $\psi^{**} = \psi$ (see [28], Proposition 2.5) and the subdifferentials of φ and ψ are inverse to each other

$$x.y = \psi(x) + \varphi(y) \iff y \in \partial\psi(x) \iff x \in \partial\varphi(y) \quad (31)$$

(see [20], Corollary 23.5.1 or [28], Proposition 2.4). In the special case, if ψ is strictly convex in the neighborhood of some $x \in \mathbb{R}^n$, then φ is differentiable on $\partial\psi(x)$ and the end of (31) implies that $x = \nabla\varphi(y)$ for all $y \in \partial\psi(x)$. If ψ is differentiable and strictly convex, then so is φ and $\nabla\psi$ is one-to-one ([20], Theorem 26.5). Using the end of (31), one finds that

$$y = \nabla\psi(\nabla\varphi(y)) \quad \text{or equivalently} \quad (\nabla\psi)^{-1} = \nabla\varphi \quad (32)$$

Furthermore, if ψ and φ are twice differentiable, by differentiating the relation (32), one obtains

$$\nabla^2\psi(\nabla\varphi(y)) \cdot \nabla^2\varphi(y) = \text{Id} \quad \text{or equivalently} \quad \nabla^2\varphi = (\nabla^2\psi)^{-1} \circ \nabla\varphi \quad (33)$$

In particular, taking the determinant on both sides of the relation (33), if $\det \nabla^2 \psi = f$, then

$$\det \nabla^2 \varphi = \frac{1}{f \circ \nabla \varphi} \quad (34)$$

Proof of Theorem 3.1. (i). The function ψ is smooth and strongly convex in the open, convex set L . We recall that if a function is strongly convex then the Hessian matrix of it is positive definite, hence it is also strictly convex function. According to Theorem 26.5 in Rockafellar [20], the smooth map $\nabla \psi : L \rightarrow \Omega$ is one-to-one. Moreover, the differential of the smooth map $\nabla \psi : L \rightarrow \Omega$ is non-singular, and by the Inverse Function Theorem from calculus, the set $\Omega = \nabla \psi(L)$ is open and the map $\nabla \psi : L \rightarrow \Omega$ is a diffeomorphism. By Corollary 23.5.1 in [20], the inverse of the map $\nabla \psi$ is the smooth map $\nabla \varphi : L \rightarrow \Omega$ and hence the relation (33) is satisfied. Thus (i) is proven.

(ii) In this proof, we denote that ∇_z is the gradient with respect to $z \in \mathbb{R}^n$. Taking arbitrary $x_0 \in L$, then $y_0 = (x_0, \psi(x_0)) \in M$. According to Lemma 3.2, the affine normal vector of M at the point y_0 is

$$\zeta'(0)(y_0) = \frac{1}{2n+4} (a(x_0), b(x_0))$$

where $a(x_0)$ and $b(x_0)$ are defined as in the formula (22) (replacing x by x_0). It follows that the affine normal line $\ell_M(y_0)$ of M at the point y_0 is the line passing through y_0 in the direction of the vector $(a(x_0), b(x_0))$, and hence the equations of $\ell_M(y_0)$ are

$$((x_1, x_2, \dots, x_n), x_{n+1}) \in \ell_M(y_0) \iff b(x_0) \cdot (x_1, x_2, \dots, x_n) - x_{n+1} \cdot a(x_0) + c_\ell = 0_{\mathbb{R}^n} \quad (35)$$

where the constant c_ℓ depends only on y_0 .

(\implies) Assume first that M is affinely-spherical with center at the origin. In this case, we have the point $(0_{\mathbb{R}^n}, 0_{\mathbb{R}}) \in \ell_M(y_0)$, and from the equations (35),

$$b(x_0) \cdot 0_{\mathbb{R}^n} - 0_{\mathbb{R}} \cdot a(x_0) + c_\ell = 0_{\mathbb{R}^n},$$

this implies that $c_\ell = 0_{\mathbb{R}^n}$ and

$$((x_1, x_2, \dots, x_n), x_{n+1}) \in \ell_M(y_0) \iff b(x_0) \cdot (x_1, x_2, \dots, x_n) - x_{n+1} \cdot a(x_0) = 0_{\mathbb{R}^n}$$

On the other hand, the point y_0 belongs of course to the affine normal line $\ell_M(y_0)$, which means that

$$b(x_0) \cdot x_0 - \psi(x_0) \cdot a(x_0) = 0_{\mathbb{R}^n}, \quad (36)$$

Since arbitrariness of $x_0 \in L$, the relation (36) implies that

$$\psi(x) \cdot a(x) = b(x) \cdot x \quad \text{for all } x \in L \quad (37)$$

Thanks to the properties of the smooth map $\nabla\psi : L \rightarrow \Omega$ that are proven in the part (i), we may replace x by $\nabla\varphi(y)$ for $y \in \Omega$. Applying the relation (31), we have $y = \nabla\psi(x)$ and

$$\psi(x) = \nabla\varphi(y) \cdot y - \varphi(y) \quad (38)$$

Furthermore, using the relations (33) and (34), we obtain

$$\begin{aligned} \nabla_{\mathbf{y}} (\log \det \nabla^2 \varphi)(y) &= \nabla_{\mathbf{y}} \left(\log \frac{1}{\det \nabla^2 \psi \circ \nabla \varphi} \right) (y) \\ &= -\nabla_{\mathbf{y}} [\log (\det \nabla^2 \psi \circ \nabla \varphi)] (y) \\ &= -\frac{\nabla_{\mathbf{y}} (\det \nabla^2 \psi \circ \nabla \varphi)(y)}{(\det \nabla^2 \psi \circ \nabla \varphi)(y)} \\ &= -\frac{[\nabla^2 \varphi \cdot (\nabla_{\nabla \varphi} (\det \nabla^2 \psi \circ \nabla \varphi))](y)}{(\det \nabla^2 \psi \circ \nabla \varphi)(y)} \\ &= -\frac{\nabla^2 \varphi(y) \cdot \nabla_{\mathbf{x}=\nabla \varphi} [\det (\nabla^2 \psi \circ \nabla \varphi)](y)}{\det (\nabla^2 \psi) (\nabla \varphi(y))} \\ &= -\frac{\nabla^2 \varphi(y) \cdot \nabla_{\mathbf{x}} (\det (\nabla^2 \psi))(x)}{\det (\nabla^2 \psi)(x)} \\ &= -(\nabla^2 \psi)^{-1}(x) \cdot \nabla_{\mathbf{x}} [\log \det (\nabla^2 \psi)](x) \end{aligned}$$

Therefore,

$$a(x) = \nabla_{\mathbf{y}} (\log \det \nabla^2 \varphi)(y) =: A_1(y) \quad (39)$$

Combining the relations (37), (38) and (39), then (37) is equivalent to

$$[\nabla\varphi(y) \cdot y - \varphi(y)] \cdot A_1(y) = [n + 2 + \langle A_1(y), y \rangle] \cdot \nabla\varphi(y) \quad \text{for all } y \in \Omega \quad (40)$$

The function ψ is smooth and strongly convex, hence the set $\{x \in L : \psi(x) \neq 0\}$ is an open, dense set in L . Denote $U_\psi = \{y \in \Omega : \psi(\nabla\varphi(y)) \neq 0\}$, an open, dense set in Ω . For any $y \in U_\psi$, we may define

$$A_2(y) := \frac{n+2 + \langle A_1(y), y \rangle}{\nabla\varphi(y) \cdot y - \varphi(y)} = \frac{n+2 + \langle A_1(y), y \rangle}{\psi(\nabla\varphi(y))} \in \mathbb{R}.$$

According to (40), $A_1 = A_2 \cdot \nabla\varphi$ throughout the set U_ψ . Moreover, the following holds in the set U_ψ ,

$$\nabla\varphi \cdot y \cdot A_1 = \nabla\varphi \cdot y \cdot [A_2 \cdot \nabla\varphi] = [A_2 \cdot \nabla\varphi] \cdot y \cdot \nabla\varphi = A_1 \cdot y \cdot \nabla\varphi \quad (41)$$

From (40) and (41), we have

$$-\varphi(y) \cdot A_1(y) = (n+2) \cdot \nabla\varphi(y) \quad (42)$$

The validity of (42) in the dense set $U_\psi \subset \Omega$ implies by continuity that (42) holds true in the entire open set Ω . Multiplying (42) by $\varphi^{n+1} \cdot \det \nabla^2\varphi$ we obtain that in all of Ω ,

$$\nabla(\varphi^{n+2} \cdot \det \nabla^2\varphi) = 0. \quad (43)$$

The set Ω is connected, being the image of the connected set L under a smooth map. Hence, $(\det \nabla^2\varphi) \cdot \varphi^{n+2} \equiv C$ in Ω . This constant C cannot be zero according to the relations (33) and (34), because φ is not the zero function and $\det \nabla^2\varphi$ never vanishes in Ω ,

$$\det \nabla^2\varphi(y) = \frac{1}{\det \nabla^2\psi \circ \nabla\varphi}(y) = \frac{1}{\det \nabla^2\psi}(\nabla\varphi(y)) = \frac{1}{\det \nabla^2\psi(x)} > 0$$

for all $y \in \Omega$, $x = \nabla\varphi(y)$.

(\Leftarrow) Assume that there exists $C \in \mathbb{R} \setminus \{0\}$ such that $(\det \nabla^2\varphi) \cdot \varphi^{n+2} \equiv C$ in all of Ω . Then we have

$$\begin{aligned} 0 &= \nabla(C) \\ &= \nabla(\varphi^{n+2} \cdot \det \nabla^2\varphi) \\ &= (n+2) \cdot \varphi^{n+1} \cdot \det \nabla^2\varphi \cdot \nabla\varphi + \varphi^{n+2} \cdot \nabla(\det \nabla^2\varphi) \\ &= (\varphi^{n+1} \cdot \det \nabla^2\varphi) \cdot \left[(n+2) \cdot \nabla\varphi + \varphi \cdot \frac{\nabla(\det \nabla^2\varphi)}{\det \nabla^2\varphi} \right] \\ &= (\varphi^{n+1} \cdot \det \nabla^2\varphi) \cdot [(n+2) \cdot \nabla\varphi + \varphi \cdot \nabla(\log \det \nabla^2\varphi)] \end{aligned}$$

It follows that

$$[(n+2) \cdot \nabla \varphi + \varphi \cdot \nabla (\log \det \nabla^2 \varphi)](y) = 0 \quad (44)$$

for all $y \in U_\varphi := \{y \in \Omega : \varphi(y) \neq 0\}$. The function φ is also smooth and strongly convex, hence the set U_φ is an open, dense set in Ω . By continuity, the validity of (44) in the dense set $U_\varphi \subset \Omega$ implies that (44) holds true in the entire open set Ω ,

$$[(n+2) \cdot \nabla \varphi + \varphi \cdot \nabla (\log \det \nabla^2 \varphi)](y) = 0 \quad \text{for all } y \in \Omega \quad (45)$$

On the other hand, for any $y \in U_\varphi$ we may define

$$A_3(y) := -\frac{n+2}{\varphi(y)} \in \mathbb{R}.$$

and according to (44), for every $y \in U_\varphi$ we have

$$A_1(y) := \nabla (\log \det \nabla^2 \varphi)(y) = A_3(y) \cdot \nabla \varphi(y).$$

Moreover, the following holds for every $y \in U_\varphi$,

$$\begin{aligned} \nabla \varphi(y) \cdot y \cdot A_1(y) &= \nabla \varphi(y) \cdot y \cdot [A_3(y) \cdot \nabla \varphi(y)] = A_3(y) \cdot \nabla \varphi(y) \cdot y \cdot \nabla \varphi(y) \\ &= A_1(y) \cdot y \cdot \nabla \varphi(y). \end{aligned}$$

By continuity,

$$\nabla \varphi(y) \cdot y \cdot A_1(y) = A_1(y) \cdot y \cdot \nabla \varphi(y) \quad \text{for all } y \in \Omega. \quad (46)$$

Combining the relations (45) and (46), we obtain the relation (40) and equivalently to (37). Thus, M is affinely-spherical with center at the origin. □

Remark 3.1. In both directions in proof of Theorem 3.1.(ii), we need to have to prove that $\nabla \varphi(y) \cdot y \cdot A_1(y) = A_1(y) \cdot y \cdot \nabla \varphi(y)$ for all $y \in \Omega$. Differences here are that we use the relation (40) to deduce it in the part (\Rightarrow), however it's deduced from (45) in (\Leftarrow). Geometrical property is used here is that if x, y, z are three vectors in \mathbb{R}^n such that $z = \lambda x$ with $\lambda \in \mathbb{R}$ then

$$\langle x, y \rangle \cdot z = \langle x, y \rangle \cdot (\lambda x) = (\langle x, y \rangle \cdot \lambda) \cdot x = \langle \lambda x, y \rangle \cdot x = \langle z, y \rangle \cdot x$$

For our proof, $\lambda = A_2$ in (\Rightarrow) and $\lambda = A_3$ in (\Leftarrow).

3.3 Properties of q -moment measures and solvability of the equation (19)

As explained in the previous subsection, we know that the solution of the Monge-Ampère second boundary value problem (19) relates to an affinely-spherical with center at the origin. Therefore, studying for solutions to the equation (19) plays an important role in analyzing affinely-sphericals. In fact that this is related to the problem of 2-moment measures. Recall that in the definition of q -moment measures, we need to have the integrability condition

$$0 < \int_{\mathbb{R}^n} \varphi^{-(n+q)}(x) dx < +\infty \tag{47}$$

after that to be normalized to equal to 1, and the following transport equation is satisfied

$$\int_{\mathbb{R}^n} b(y) d\mu(y) = \int_{\mathbb{R}^n} b(\nabla\varphi(x)) \cdot \varphi^{-(n+q)}(x) dx \tag{48}$$

for any bounded, continuous function $b : \mathbb{R}^n \rightarrow \mathbb{R}$. Questions that we will address are the following:

- Are the assumptions on the measure μ in Theorem 2.1 sufficient conditions ? (i.e., can we deduce that the measure μ is not supported on a hyperplane and its barycenter is 0 if μ is the q -moment measure of the convex function φ ?)
- When does the equation (19) have a solution ?

The following Lemma gives a equivalent condition of (47).

Lemma 3.3. *Let $q > 0$ and let $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$ be a convex function. Then the function $\varphi^{-(n+q)}$ is integrable if and only if it satisfies the coercivity condition, i.e.,*

$$\lim_{|x| \rightarrow +\infty} \varphi(x) = +\infty.$$

Proof. (\implies) Assume first that the condition (47) is satisfies. Denote $\rho(x) := \varphi^{-(n+q)}(x)$ for every $x \in \mathbb{R}^n$. We know that φ is a positive, convex function on \mathbb{R}^n , hence the function ρ is log-concave ¹³. Since $\int \rho(x) dx > 0$, then there exists

¹³A function $f : \mathbb{R}^n \rightarrow [0, +\infty)$ is log-concave if $\log(f)$ is a concave function, or equivalently

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda \cdot f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$.

$\varepsilon \in (0, 1)$ such that $I_1 := \{x \in \mathbb{R}^n : \rho(x) > \varepsilon\}$ satisfies $\text{Vol}_n(I_1) > 0$, where we write $\text{Vol}_n(\cdot)$ for the n -dimensional volume. Concavity of the function $\log(f)$ implies that the set I_1 is convex set. Therefore I_1 has a non-empty interior, since it is a convex set of positive volume. By translating ρ , we may assume that $r\overline{B} \subset I_1$ for some $r > 0$ and $\overline{B} := \{x \in \mathbb{R}^n : |x| \leq 1\}$. Denote $I_2 := \{x \in \mathbb{R}^n : \rho(x) > \frac{\varepsilon}{2}\}$. Similarly, I_2 is also convex and then we have $\text{Vol}_n(I_2) \geq \text{Vol}_n(I_1) > 0$, $\text{Vol}_n(I_2) < +\infty$ since $\int \rho(x) dx < +\infty$. Consequently, I_2 is bounded, as a convex set of a finite, positive volume. Thus for some $R > 0$ we have $I_2 \subset \frac{R}{2}\overline{B}$. Now we consider two case:

- *Case 1:* Taking an arbitrary point x that does not belong to $R\overline{B}$. Then, we have a convex combination

$$R \frac{x}{|x|} = \left(\frac{|x| - R}{|x| - r} \right) \cdot \left(r \frac{x}{|x|} \right) + \left(\frac{R - r}{|x| - r} \right) \cdot x \quad (49)$$

Note that $R \frac{x}{|x|} \notin I_2$ and $r \frac{x}{|x|} \in I_1$, and these follow that

$$\rho \left(R \frac{x}{|x|} \right) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \rho \left(r \frac{x}{|x|} \right) > \varepsilon$$

Since ρ is the log-concave function, we deduce that

$$\frac{\varepsilon}{2} \geq \rho \left(R \frac{x}{|x|} \right) \geq \rho \left(r \frac{x}{|x|} \right)^{\frac{|x| - R}{|x| - r}} \cdot \rho(x)^{\frac{R - r}{|x| - r}} \geq \varepsilon^{\frac{|x| - R}{|x| - r}} \cdot \rho(x)^{\frac{R - r}{|x| - r}}.$$

In particular,

$$\rho(x) \leq \varepsilon \cdot \left(\frac{1}{2} \right)^{\frac{|x| - R}{R - r}} \leq A_1 e^{-B_1 |x|}$$

for some $A_1, B_1 > 0$ depending on ρ .

- *Case 2:* Taking an arbitrary point x_0 that is in $R\overline{B}$. We will prove that $\rho(x_0) < +\infty$. Indeed, if we assume by contradiction that $\rho(x_0) = +\infty$ then for any $x \in I_1$, we have a convex combination in \mathbb{R}^n

$$x = \lambda x_0 + (1 - \lambda) x_1 \quad \text{where} \quad x_1 = \frac{1}{1 - \lambda} (x - \lambda x_0) \in \mathbb{R}^n$$

and for any $\lambda \in (0, 1)$. The log-concavity of the function ρ on \mathbb{R}^n implies that $\rho(x) \geq \rho(x_0)^\lambda \cdot \rho(x_1)^{1 - \lambda} \geq +\infty$. Since arbitrariness of $x \in I_1$, $\rho \equiv +\infty$ on I_1 , contradicting $\int \rho(x) dx < +\infty$.

Combining both cases, we conclude that an estimate of the form

$$\rho(x) \leq A_2 e^{-B_2 |x|} \quad (50)$$

for some numbers $A_2, B_2 > 0$, holds in the entire \mathbb{R}^n . Taking natural logarithm both sides of the inequality (50), we deduce that there exists $A \in \mathbb{R}$ and $B > 0$ such that $\varphi(x) \geq A + B|x|$ for all $x \in \mathbb{R}^n$. Thus

$$\lim_{|x| \rightarrow +\infty} \varphi(x) \geq \lim_{|x| \rightarrow +\infty} (A + B|x|) = +\infty.$$

(\Leftarrow) Conversely, assume that $\varphi(x)$ tends to infinity as $|x| \rightarrow +\infty$. Then there exists $R > 0$ such that $\varphi(x) \geq \varphi(0) + 1$ whenever $|x| \geq R$. By convexity, for any $x \in \mathbb{R}^n$ such that $|x| > R$, we have

$$\varphi(0) + 1 \leq \varphi\left(\frac{R}{|x|}x\right) = \varphi\left[\left(1 - \frac{R}{|x|}\right) \cdot 0 + \left(\frac{R}{|x|}\right) \cdot x\right] \leq \left(1 - \frac{R}{|x|}\right)\varphi(0) + \frac{R}{|x|}\varphi(x)$$

Therefore,

$$\varphi(x) \geq \varphi(0) + \frac{1}{R} \cdot |x| \quad \text{for all } |x| > R. \quad (51)$$

On the other hand, by continuity, $\min_{|x| \leq R} \varphi(x) = \varphi(x_0)$ is positive and then

$$\varphi(x) \geq \varphi(x_0) = \frac{1}{2}\varphi(x_0) + \frac{1}{2}\varphi(x_0) = \frac{1}{2}\varphi(x_0) + \frac{1}{2R}\varphi(x_0) \cdot |x| \quad \text{for all } |x| \leq R \quad (52)$$

Combining the inequalities (51) and (52), we obtain

$$\begin{aligned} \varphi(x) &\geq \min\left\{\varphi(0), \frac{1}{2}\varphi(x_0)\right\} + \min\left\{\frac{1}{R}, \frac{1}{2R}\varphi(x_0)\right\} \cdot |x| \\ &= \frac{1}{2}\varphi(x_0) + \min\left\{\frac{1}{R}, \frac{1}{2R}\varphi(x_0)\right\} \cdot |x| =: \alpha + \beta \cdot |x| \end{aligned}$$

for all $x \in \mathbb{R}^n$. This implies that the function $x \mapsto \varphi^{-(n+q)}(x)$ is integrable,

$$0 < \int_{\mathbb{R}^n} \varphi^{-(n+q)}(x) \, dx \leq \int_{\mathbb{R}^n} (\alpha + \beta \cdot |x|)^{-(n+q)}(x) \, dx < +\infty.$$

□

In the sequel we recall that the support of a measure μ in \mathbb{R}^n is the closed set $\text{Supp}(\mu)$ that consists of all points $x \in \mathbb{R}^n$ with the following property: $\mu(U) > 0$ for any open set U containing x .

Theorem 3.2. (*characterization of q -moment measures*). Let $q > 1$ and let μ be a compactly-supported Borel probability measure on \mathbb{R}^n . Suppose that μ is the q -moment measure of a convex function $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$, which satisfies the integrability condition (47). Then

- (a) The barycenter of μ lies at the origin (in particular, μ has finite first moments)¹⁴.
- (b) The measure μ is not supported in a hyperplane. Thus, the origin belongs to the interior of $\text{Conv}(\text{Supp}(\mu))$, where we write $\text{Conv}(\cdot)$ for the convex hull.

Proof. (a) We may substitute $b(y) = y_i$ in the transport equation (48), since the function b is bounded on $\text{Supp}(\mu)$. This shows that for $i = 1, 2, \dots, n$, we have

$$\int_{\mathbb{R}^n} y_i d\mu(y) = \int_{\mathbb{R}^n} \varphi^{-(n+q)}(x) \cdot \frac{\partial \varphi}{\partial x_i}(x) dx = \frac{-1}{n+q-1} \int_{\mathbb{R}^n} \partial_i \left(\frac{1}{\varphi^{n+q-1}} \right)(x) dx. \quad (53)$$

Our goal is to prove that $\int y_i d\mu(y) = 0$ for any $i \in \{1, 2, \dots, n\}$. Define $\rho_1 := \varphi^{-(n+q-1)}$, then the function ρ_1 is log-concave. Moreover, since the integrability condition (47) holds, we have $\lim_{|x| \rightarrow +\infty} \varphi(x) = +\infty$ and hence the function ρ_1 is also integrable. From the equality (53), it is sufficient to show that for $i = 1, 2, \dots, n$,

$$\int_{\mathbb{R}^n} \partial_i(\rho_1) dx = 0$$

Without loss of generality, we fix $i = n$ and for every $x \in \mathbb{R}^n$, rewrite

$$x = (y, t) \quad \text{where } y = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \quad \text{and } t = x_n \in \mathbb{R}.$$

We split the argument into a few steps.

Step 1: the mapping $x \mapsto |\nabla \rho_1(x)|$ is integrable. Setting

$$C_{\min}(y) := \inf_{t \in \mathbb{R}} \varphi(y, t).$$

For any fixed $y \in \mathbb{R}^{n-1}$, the function $t \mapsto \varphi(y, t)$ is convex, locally-Lipschitz and tends to $+\infty$ as $t \rightarrow \pm\infty$. Hence, the one-dimensional convex function $t \mapsto \varphi(y, t)$ attains its minimum at a certain point $t_0 \in \mathbb{R}$, is non-decreasing on $[t_0, +\infty)$ and non-increasing on $(-\infty, t_0]$. It follows that

$$\frac{\partial \varphi(y, t)}{\partial t} \geq 0 \quad \text{for all } t \in [t_0, +\infty); \quad \frac{\partial \varphi(y, t)}{\partial t} \leq 0 \quad \text{for all } t \in (-\infty, t_0]$$

¹⁴Note that in the theory of measures and integration, a function f is integrable if and only if $|f|$ is integrable. Here if the barycenter of the measure μ lies at the origin then identity function is integrable, $\int y d\mu(y) = 0$, and hence the first moment $\int |y| d\mu(y) < +\infty$.

and $\rho_1(\pm\infty) = 0$. Therefore, for any $y \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
 & \int_{-\infty}^{+\infty} \left| \frac{\partial \rho_1(y, t)}{\partial t} \right| dt \\
 &= |1 - n - p| \int_{-\infty}^{+\infty} \varphi^{-(n+q)} \left| \frac{\partial \varphi(y, t)}{\partial t} \right| dt \\
 &= |1 - n - p| \left\{ \int_{-\infty}^{t_0} \varphi^{-(n+q)} \left| \frac{\partial \varphi(y, t)}{\partial t} \right| dt + \int_{t_0}^{+\infty} \varphi^{-(n+q)} \left| \frac{\partial \varphi(y, t)}{\partial t} \right| dt \right\} \\
 &= |1 - n - p| \left\{ - \int_{-\infty}^{t_0} \varphi^{-(n+q)} \frac{\partial \varphi(y, t)}{\partial t} dt + \int_{t_0}^{+\infty} \varphi^{-(n+q)} \frac{\partial \varphi(y, t)}{\partial t} dt \right\} \\
 &= \int_{-\infty}^{t_0} \frac{\partial \rho_1(y, t)}{\partial t} dt - \int_{t_0}^{+\infty} \frac{\partial \rho_1(y, t)}{\partial t} dt = 2\rho_1(y, t_0) = 2(C_{\min}(y))^{-(n+q-1)}.
 \end{aligned}$$

Since the function φ is convex, the function $\mathbb{R}^{n-1} \ni y \mapsto C_{\min}(y)$ is also convex. On the other hand, we have

$$\lim_{|y| \rightarrow +\infty} C_{\min}(y) = \lim_{|y| \rightarrow +\infty} (\inf_{t \in \mathbb{R}} \varphi(y, t)) = \lim_{|y| \rightarrow +\infty} \varphi(y, t_0) = +\infty.$$

Applying Lemma 3.3 (by replacing q by $q - 1$), the function $\mathbb{R}^{n-1} \ni y \mapsto C_{\min}(y)$ is integrable on \mathbb{R}^{n-1} . Finally, using the Tonelli's Theorem (see [27], Théorème III-50), we get

$$\int_{\mathbb{R}^n} |\partial_i \rho_1| dx = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{+\infty} \left| \frac{\partial \rho_1(y, t)}{\partial t} \right| dt dy = 2 \int_{\mathbb{R}^{n-1}} (C_{\min}(y))^{-(n+q-1)} dy < +\infty \tag{54}$$

for all $i \in \{1, 2, \dots, n\}$.

Step 2: deduce that $\int \partial_i \rho_1 dx = 0$. For almost any $y \in \mathbb{R}^{n-1}$, the function $t \mapsto \rho_1(y, t)$ is continuous, vanishes at infinity, and it is locally-Lipschitz in the interior of its support. Therefore, for almost any $y \in \mathbb{R}^{n-1}$, we have

$$\int_{-\infty}^{+\infty} \frac{\partial \rho_1(y, t)}{\partial t} dt = \rho_1(y, +\infty) - \rho_1(y, -\infty) = 0.$$

Thanks to the condition (54), we may use the Fubini's Theorem (see [27], Théorème III-50), and conclude that

$$\int_{\mathbb{R}^n} \partial_i(\rho_1) dy = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{+\infty} \frac{\partial \rho_1(y, t)}{\partial t} dt \right) dy = 0.$$

(b) Assume by contradiction that $\text{Supp}(\mu) \subset \theta^\perp$ for a vector $\theta \in \mathbb{R}^{n-1}$, where θ^\perp is the hyperplane orthogonal to θ . Without loss of generality, we assume that $\theta = e_n = (0, \dots, 0, 1)$. Using the transport equation (48), take $b(x) = |x_n|$, we have

$$\begin{aligned} \frac{1}{n+p-1} \int_{\mathbb{R}^n} \left| \frac{\partial \rho_1(y, t)}{\partial t} \right| dx &= \int_{\mathbb{R}^n} \varphi^{-(n+q)}(y, t) \left| \frac{\partial \varphi(y, t)}{\partial t} \right| dx \\ &= \int_{\mathbb{R}^n} |z_n| d\mu(z) = \int_{\text{Supp}(\mu)} |\langle z, e_n \rangle| d\mu(z) = 0 \end{aligned}$$

recall that $\rho_1 = \varphi^{-(n+q-1)}$ and $(y, t) = ((x_1, \dots, x_{n-1}), x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Therefore, for almost any $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have

$$\frac{\partial \rho_1(y, t)}{\partial t} = \left| \frac{\partial \rho_1(y, t)}{\partial t} \right| = 0.$$

On the other hand, for almost any $y \in \mathbb{R}^{n-1}$, the function $t \mapsto \rho_1(y, t)$ is continuous in \mathbb{R} and locally-Lipschitz in the interior of the interval $\{t : \rho_1(y, t) > 0\}$. Therefore, for almost any $y \in \mathbb{R}^{n-1}$, the log-concave function $t \mapsto \rho_1(y, t)$ is constant in \mathbb{R} and so

$$\int_{-\infty}^{+\infty} \rho_1(y, t) dt \in \{0, +\infty\}.$$

By Tonelli's Theorem, we get

$$\int_{\mathbb{R}^n} \rho_1(y, t) dx = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{+\infty} \rho_1(y, t) dt \right) dy \in \{0, +\infty\}.$$

This is contradiction with the integrability condition of the function $x \mapsto \rho_1(x)$ that is explained in the proof of the part (a). Thus, $\text{Supp}(\mu) \not\subset \theta^\perp$ for any $\theta \in \mathbb{S}^{n-1}$. □

Theorem 3.3. *Let $L \in \mathbb{R}^n$ be a non-empty, open, bounded, convex set. Then there exists a smooth, positive, convex function $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$ solving the problem (19) if and only if the barycenter of L lies at the origin. Moreover, this convex function φ is uniquely determined up to translation.*

Proof. (\implies) Assume first that the barycenter of L lies at the origin. To apply the problem of q -moment measures in this case, we need to construct a probability measure μ on L that satisfies the assumptions of the Theorem 2.1. According to Theorem 3.2, if μ is the q -moment measure of a convex function then barycenter of the measure μ has to lie at the origin (which is also the barycenter of the convex

set L). By these reasons, we take μ to be the uniform measure on L , normalized to be a uniform probability measure. As explained, we apply Theorem 2.1 with $q = 2$, then there exists a lower semi-continuous and convex function $\bar{u} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is solution to the dual optimal transport problem

$$\inf \left\{ \int_{\mathbb{R}^n} u d\rho_{\text{opt}} + \int_{\mathbb{R}^n} u^* d\mu \mid u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \text{ convex and l.s.c} \right\}$$

where ρ_{opt} is the absolutely continuous solution to the optimization problem (P) in Theorem 2.1. Moreover, using the Theorem 2.2, there exists a constant $c \in \mathbb{R}$ such that $\bar{u} > c$ and $\rho_{\text{opt}} = (\bar{u} - c)^{-(n+2)}$. Now, we define that

$$\varphi(x) := \bar{u}(x) - c \quad \text{for all } x \in \mathbb{R}^n \quad (55)$$

then $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$ is a positive, convex function. Furthermore, by the Brenier's Theorem in the theory of Optimal Transport (see [28], Theorem 2.12), the measure μ is push-forward of the measure ρ_{opt} by the transport map $x \mapsto \nabla\varphi(x)$ (note that $\nabla\varphi = \nabla(\bar{u} - c) = \nabla\bar{u}$), which means that

$$\mu = \nabla\varphi_{\#}\rho_{\text{opt}} \quad \text{where} \quad d\rho_{\text{opt}} = \varphi^{-(n+2)} dx.$$

This implies that the transport equation (48) is satisfied in the sense

$$\int_L b(y) dy = C_L \int_{\mathbb{R}^n} \varphi^{-(n+2)}(x) \cdot b(\nabla\varphi(x)) dx, \quad (56)$$

where $C_L = \text{Vol}_n(L)$. Caffarelli's regularity theory (see [9], Theorem 4.23 and Remark 4.25) implies that φ is C^∞ -smooth in \mathbb{R}^n . Using the change-of-variables formula, the equation (56) implies that for any bounded, continuous function $b : L \rightarrow \mathbb{R}$, we have

$$\int_{\mathbb{R}^n} b(\nabla\varphi(x)) \cdot \frac{C_L}{\varphi^{n+2}(x)} dx = \int_L b(y) dy = \int_{\mathbb{R}^n} b(\nabla\varphi(x)) \cdot \det \nabla^2\varphi(x) dx \quad (57)$$

and hence

$$\det \nabla^2\varphi(x) = \frac{C_L}{\varphi^{n+2}(x)}. \quad (58)$$

The function φ is smooth, convex function. This implies that the Hessian matrix of φ is positive semi-definite for any $x \in \mathbb{R}^n$. On the other hand, the equality (58) shows that $\det \nabla^2\varphi(x) > 0$ for all $x \in \mathbb{R}^n$. Hence, the Hessian matrix of φ is positive definite for any $x \in \mathbb{R}^n$. It follows that the smooth function φ is strongly convex on \mathbb{R}^n . According to Theorem 26.5 in Rockafellar [20], the set $\nabla\varphi(\mathbb{R}^n)$ is open and convex. From (57), we obtain that $\nabla\varphi(\mathbb{R}^n) = L$. Thus, φ is a solution to (19).

For uniqueness up to translation, we assume that φ_1 is also a solution to the equation (19). Then φ_1 is also smooth, strongly convex function, and consequently $\nabla\varphi$ is a diffeomorphism between \mathbb{R}^n and the convex, open set $\nabla\varphi(\mathbb{R}^n) = L$. From (19) and the change-of-variables formula, we deduce that the transport equation (56) is also satisfied for the convex function φ_1 . This implies that μ is 2-moment measure of the function φ_1 . Applying Theorem 2.3, we get that both φ and φ_1 are translations.

(\Leftarrow) Assume that φ is a smooth, positive, convex solution to (19). Let μ be measure as in the part (\Rightarrow). Similarly, we have that μ is q -moment measure of the function φ with $q = 2$. Applying Theorem 3.2, the barycenter of μ lies at the origin. Since μ is the uniform probability measure on the open, bounded and convex set L , the barycenter of L coincides the barycenter of μ , which lies at the origin.

□

The following result is a direct corollary from Theorem 3.3 and Theorem 3.1.

Corollary 3.1. *Let $L \subset \mathbb{R}^n$ be an open, bounded, convex set containing the origin. Then the following are equivalent:*

- (a) *The barycenter of L lies at the origin.*
- (b) *There exists a proper, convex function $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\overline{\text{dom}(\psi)} = \overline{L}$ such that*

$$\text{Graph}_L(\psi) = \{(x, \psi(x)) : x \in L\}$$

is affinely-spherical with center at the origin, and such that ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$ and $\psi(0) < 0$.

Moreover, assuming (a) or (b), the function ψ from (b) is uniquely determined up to a translation.

Proof. (a) \implies (b). Assume that the barycenter of the set L lies at the origin. According to Theorem 3.3, there exists a smooth, positive, convex function $\varphi : \mathbb{R}^n \rightarrow (0, +\infty)$ with $\nabla\varphi(\mathbb{R}^n) = L$ such that

$$\det \nabla^2\varphi(x) = \frac{C}{\varphi^{n+2}(x)} \quad \text{for all } x \in \mathbb{R}^n \tag{59}$$

for some constant $C > 0$. Note that the function φ in this case is defined by (55), so that it is l.s.c, convex function and hence $\varphi^{**} = \varphi$ (see [28], Proposition 2.5). Denote

$$\psi = \varphi^*.$$

According to Theorem 26.5 in Rockafellar [20], we know that $\overline{\text{dom}(\psi)} = \bar{L}$ and that ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$. Applying Theorem 3.1, the equation (59) implies that $\text{Graph}_L(\psi)$ is affinely-spherical with center at the origin. Since the function φ is positive, smooth, strongly convex, the infimum of φ is attained and $\varphi(x_{\min}) > 0$. Thus, we get

$$\begin{aligned} \psi(0) = \varphi^*(0) &= \sup_{x \in \mathbb{R}^n} (\langle x, 0 \rangle - \varphi(x)) = \sup_{x \in \mathbb{R}^n} (-\varphi(x)) \\ &= -\inf_{x \in \mathbb{R}^n} \varphi(x) = -\varphi(x_{\min}) < 0. \end{aligned}$$

Moreover, Theorem 3.3 states that the function φ is uniquely determined up to translation. Therefore, ψ is uniquely determined up to translation.

(b) \implies (a). Assume (b). Denote $\varphi = \psi^*$. We have $\overline{\text{dom}(\varphi^*)} = \overline{\text{dom}(\psi)} = \bar{L}$ is a bounded set, this implies that $\text{dom}(\varphi) = \mathbb{R}^n$ (see [20], Corollary 13.3.3). Moreover, the function ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$ and $\psi(0) < 0$, hence the function φ is smooth, strongly convex in \mathbb{R}^n with $\nabla\varphi(\mathbb{R}^n) = L$ and for all $y \in \mathbb{R}^n$, we have

$$\varphi(y) = \psi^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \psi(x)) \geq \langle 0, y \rangle - \psi(0) = -\psi(0) > 0.$$

Since $\text{Graph}_L(\psi)$ is affinely-spherical with center at the origin, Theorem 3.1 implies that the equation (59) has a solution. Applying Theorem 3.3, the barycenter of L lies at the origin. □

3.4 Affine hemispheres of elliptic type: existence and uniqueness

The aim of this section is to derive a few results on affine hemispheres of elliptic type. Our first result is the following.

Theorem 3.4. *Let $L \subset \mathbb{R}^n$ be a non-empty, open, bounded, convex set. Assume that the barycenter of L lies at the origin and φ is convex function which is defined by the relation (55). Then,*

$$M := \left\{ \left(\frac{x}{\varphi(x)}, -\frac{1}{\varphi(x)} \right) : x \in \text{dom}(\varphi) \right\} \subset \mathbb{R}^n \times \mathbb{R}$$

is an affine hemisphere in \mathbb{R}^{n+1} with anchor $K = L^\circ \times \{0\}$, which is centered at the origin, where L° is the polar body of the set L and is defined as in (A.72).

The proof of Theorem 3.4 requires several steps that are detailed in the next lemmas.

Let $M \subset \mathbb{R}^{n+1}$ be a smooth, connected hypersurface which is locally strongly convex. When the origin does not belong to $T_y M$ (the tangent space to M at y) for any $y \in M$, we may define a vector $\nu_y \in \mathbb{R}^{n+1}$ via the requirements that

$$\langle \nu_y, y \rangle = 1 \quad \text{and} \quad \nu_y \perp T_y M.$$

When ν_y is well-defined for any $y \in M$, we refer to $\nu : M \rightarrow \mathbb{R}^{n+1}$ as the *polarity map*. In this case the *polar hypersurface* M^* is defined as

$$M^* := \nu(M) = \{\nu_y : y \in M\}.$$

It is well-known that M^* is always a smooth, connected, locally strongly-convex hypersurface, that the polarity map $\nu : M \rightarrow M^*$ is a diffeomorphism, and its inverse is the polarity map associated with M^* . In particular, $(M^*)^* = M$.

Lemma 3.4. *Let $L \subset \mathbb{R}^n$ be an open, bounded, convex set containing the origin. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex function with $\psi(0) < 0$ such that $\bar{L} = \overline{\text{dom}(\psi)}$. Assume that ψ is smooth and strongly-convex in L with $\nabla\psi(L) = \mathbb{R}^n$. Denote $M = \text{Graph}_L(\psi)$ and $\tilde{K} = \text{Epigraph}(\psi)^\circ$. Then, we have*

(a) *The Legendre transform $\varphi = \psi^* : \mathbb{R}^n \rightarrow \mathbb{R}$ is positive, convex function.*

(b) *$\partial \tilde{K} \cap \mathcal{H}^- = I^+(\text{Graph}_{\mathbb{R}^n}(\varphi))$.*

(c) *ν_y is well defined and*

$$\nu_y = I^+ \{(\nabla\psi(x), \langle x, \nabla\psi(x) \rangle - \psi(x))\}.$$

(d) *$\partial \tilde{K} \cap \mathcal{H}^- = M^*$.*

(e) *$\partial \tilde{K} \setminus \mathcal{H}^- = L^\circ \times \{0\}$.*

(f) *$\dim(\tilde{K}) = n + 1$.*

(g) *\tilde{K} is compact.*

Proof. (a) $\varphi = \psi^*$ and $\nabla\psi(L) = \mathbb{R}^n$. According to Corollary 13.3.3 in Rockafellar [20], we have $\text{dom}(\varphi) = \mathbb{R}^n$. On the other hand, for any $y \in \mathbb{R}^n$,

$$\varphi(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \psi(x)) \geq \langle 0, y \rangle - \psi(0) = -\psi(0) > 0.$$

(b) Denote $A = \text{Epigraph}(\varphi) \subset \mathcal{H}^+$ and $B = \tilde{K} \cap \mathcal{H}^-$. According to Proposition A.1,

$$B = \text{Epigraph}(\psi)^\circ \cap \mathcal{H}^- = I^+(\text{Epigraph}(\varphi)) = I^+(A). \quad (60)$$

Since $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a positive, convex function, we may assert that $\partial A \cap \mathcal{H}^+ = \partial A = \text{Graph}_{\mathbb{R}^n}(\varphi)$. Moreover, the map I^+ is a homeomorphism, and hence it transforms the relativeboundary of $A \subset \mathcal{H}^+$, which is the set $(\partial A) \cap \mathcal{H}^+$, to the relative-boundary of $B \subset \mathcal{H}^-$, which is the set $(\partial B) \cap \mathcal{H}^-$. Consequently,

$$\partial \tilde{K} \cap \mathcal{H}^- = \partial B \cap \mathcal{H}^- = I^+(\partial A \cap \mathcal{H}^+) = I^+(\text{Graph}_{\mathbb{R}^n}(\varphi)). \quad (61)$$

(c) Since ψ is smooth in L , the identity $\psi(x) + \varphi(\nabla\psi(x)) = \langle x, \nabla\psi(x) \rangle$ holds for all $x \in L$. The fact $\nabla\psi(L) = \mathbb{R}^n$ thus implies that

$$\text{Graph}_{\mathbb{R}^n}(\varphi) = \{(\nabla\psi(x), \langle x, \nabla\psi(x) \rangle - \psi(x)) : x \in L\} \quad (62)$$

Note that $\langle x, \nabla\psi(x) \rangle - \psi(x) = \varphi(\nabla\psi(x)) > 0$ for all $x \in L$, and hence ν_y is well defined. Furthermore, for any $x \in L$ and $y = (x, \psi(x)) \in \text{Graph}_L(\psi)$, from the definition of ν_y , we deduce that

$$\nu_y = \frac{(\nabla\psi(x), -1)}{\langle x, \nabla\psi(x) \rangle - \psi(x)} = I^+\{(\nabla\psi(x), \langle x, \nabla\psi(x) \rangle - \psi(x))\}. \quad (63)$$

(d) Since $M = \text{Graph}_L(\psi)$, by (61), (62) and (63), we have

$$M^* = \nu(M) = \nu(\text{Graph}_L(\psi)) = I^+(\text{Graph}_{\mathbb{R}^n}(\varphi)) = \partial \tilde{K} \cap \mathcal{H}^-. \quad (64)$$

(e) According to Proposition A.1, $\tilde{K} \subset \overline{\mathcal{H}^-}$ and

$$(\partial \tilde{K}) \setminus \mathcal{H}^- = \tilde{K} \setminus \mathcal{H}^- = \{(x, 0) : x \in \text{dom}(\psi)^\circ\} = L^\circ \times \{0\}. \quad (65)$$

(f) It follows from (64) and (65) that $\dim(\tilde{K}) = n + 1$, since the convex set \tilde{K} affinely-spans the hyperplane $\partial \mathcal{H}^-$ while it also contains points outside this hyperplane.

(g) Since $0 \in L$ and $\psi(0) < 0$, the convex set $\text{Epigraph}(\psi)$ contains a neighborhood of the origin in \mathbb{R}^{n+1} . Therefore the closed set $\tilde{K} = \text{Epigraph}(\psi)^\circ$ is bounded, and hence it is compact.

□

Proof of Theorem 3.4. Denote $\psi = \varphi^*$. According to Corollary 3.1, we have that $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, $\text{dom}(\psi) = \bar{L}$, $\nabla\psi(L) = \mathbb{R}^n$, $\psi(0) < 0$ and

$$M_1 := \text{Graph}_L(\psi)$$

is affinely-spherical with center at the origin. Denote $\tilde{K} = \text{Epigraph}(\psi)^\circ$. According to Corollary A.1, the hypersurface M_1^* is affinely-spherical with center at the origin. Moreover, Lemma 3.4 shows that $\tilde{K} \subset \mathbb{R}^{n+1}$ is an $(n+1)$ -dimensional, compact convex set and

$$M_1^* = \left(\partial \tilde{K}\right) \cap \mathcal{H}^- \quad \text{while} \quad \left(\partial \tilde{K}\right) \setminus \mathcal{H}^- = L^\circ \times \{0\} =: K.$$

Consequently, $M_1^* \subset \mathcal{H}^-$ does not intersect the hyperplane $\partial\mathcal{H}^-$ that contains K , while $\partial \tilde{K} = M_1^* \cup K$. According to Definition 3.3, the hypersurface $M := M_1^*$ is an affine hemisphere with anchor K , which is centered at the origin. Moreover, by Lemma 3.4 (d) and (b), we obtain

$$M = I^+(\text{Graph}_{\mathbb{R}^n}(\varphi)) = \left\{ \left(\frac{x}{\varphi(x)}, -\frac{1}{\varphi(x)} \right) : x \in \text{dom}(\varphi) \right\} \subset \mathbb{R}^n \times \mathbb{R}.$$

□

Remark 3.2. The mapping $(x, t) \mapsto (x, -t) \in \mathbb{R}^n \times \mathbb{R}$ is linear. Thus, the set

$$M_2 := \left\{ \left(\frac{x}{\varphi(x)}, \frac{1}{\varphi(x)} \right) : x \in \text{dom}(\varphi) \right\} \subset \mathbb{R}^n \times \mathbb{R}$$

is also an affine hemisphere with anchor K , which is centered at the origin.

Theorem 3.5. *Let $K \subset \mathbb{R}^{n+1}$ be an n -dimensional, compact, convex set. Then there exists an affine hemisphere $M \subset \mathbb{R}^{n+1}$ with anchor K , uniquely determined up to transformation. The affine hemisphere M is centered at the Santaló point of K (where the Santaló point of K is defined as in appendix A.3, Theorem A.1).*

Proof. existence. By applying an affine transformation in \mathbb{R}^{n+1} , we may assume that the Santaló point of K lies at the origin, and that

$$K \subset \{(x, 0) : x \in \mathbb{R}^n\}.$$

Write $K_1 \subset \mathbb{R}^n$ for the interior of the set $\{(x, 0) : x \in \mathbb{R}^n\}$. Then K_1 is an open, convex set whose Santaló point lies at the origin. According to Corollary A.2, $K_1^\circ \subset \mathbb{R}^n$ is a compact, convex set containing 0 in its interior such that the barycenter of

K_1° lies at the origin. We write $L \subset \mathbb{R}^n$ for the interior of K_1° . According to Corollary 3.1, there exists a proper, convex function $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\overline{\text{dom}(\psi)} = \bar{L}$ such that

$$M = \text{Graph}_L(\psi)$$

is affinely-spherical with center at the origin. Moreover, $\nabla\psi(L) = \mathbb{R}^n$ and $\psi(0) < 0$. Denote

$$\tilde{K} = \text{Epigraph}(\psi)^\circ.$$

According to Corollary A.1, the hypersurface M^* is also affinely-spherical with center at the origin. Furthermore, Lemma 3.4 shows that $\tilde{K} \subset \mathbb{R}^{n+1}$ is an $(n+1)$ -dimensional, compact convex set and

$$M^* = (\partial\tilde{K}) \cap \mathcal{H}^- \quad \text{while} \quad (\partial\tilde{K}) \setminus \mathcal{H}^- = L^\circ \times \{0\} = K.$$

Consequently, $M^* \subset \mathcal{H}^-$ does not intersect the hyperplane ∂H^- that contains K , while $\partial\tilde{K} = M^* \cup K$. Thus, the hypersurface M^* is an affine hemisphere with anchor K , which is centered at the Santaló point of K .

uniqueness. Suppose that M is an affine hemisphere with anchor K , and let \tilde{K} be as in Definition 3.3. By applying an affine transformation in \mathbb{R}^{n+1} , we may assume that M is affinely-spherical with center at the origin, and that

$$K \subset \{(x, 0) : x \in \mathbb{R}^n\} \quad \text{while} \quad \tilde{K} \subset \overline{\mathcal{H}^-}. \quad (66)$$

Definition 3.3 implies that the origin belongs to the relative interior of the n -dimensional, compact, convex set K . Hence there exists a bounded, open, convex set $L \subset \mathbb{R}^n$ containing the origin such that $K = L^\circ \times \{0\}$. From (66) and Definition 3.3, we conclude that $M = (\partial\tilde{K}) \cap \mathcal{H}^- \subset \mathcal{H}^-$. Lemma 3.5 shows that $M^* = \text{Graph}_L(\psi)$ for a certain convex function $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying the requirements of Corollary 3.1-(b). Corollary 3.1 now implies that the barycenter of L lies at the origin, and hence the affine hemisphere M is centered at the Santaló point of K . Moreover, Corollary 3.1 also shows that the convex function ψ is uniquely determined up to transformation. Thus, M is uniquely determined up to transformation. □

Lemma 3.5. *Let $L \subset \mathbb{R}^n$ be a bounded, open, convex set containing the origin. Let $M \subset \mathcal{H}^-$ be an affine hemisphere with anchor $L^\circ \times \{0\} \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ and center at the origin. Then M^* is well-defined, and there exists a function ψ as in Corollary 3.1 such that $M^* = \text{Graph}_L(\psi)$.*

Proof. The hypersurface $M \subset \mathcal{H}^-$ is an affine hemisphere with anchor $L^\circ \times \{0\}$ which is centered at the origin. Let \tilde{K} be as in Definition 3.3. Denote $B = \tilde{K} \cap \mathcal{H}^-$ which is a convex, relatively-closed subset of \mathcal{H}^- with $\overline{B} = \tilde{K}$. The convex set B is bounded from below in \mathcal{H}^- since \tilde{K} is compact. Moreover, by Definition 3.3, the set

$$M = \left(\partial \tilde{K} \right) \cap \mathcal{H}^- = (\partial B) \cap \mathcal{H}^- \quad (67)$$

is a smooth, connected, locally strongly-convex hypersurface. Additionally, it follows from Definition 3.3 that

$$(\partial B) \cap \mathcal{H}^- = \left(\partial \tilde{K} \right) \cap \mathcal{H}^- = K = L^\circ \times \{0\}. \quad (68)$$

Thus the relatively-closed, convex set $B \subset \mathcal{H}^-$ satisfies all of the requirements of Lemma A.2. Therefore, there exists a proper, convex function $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$\text{Epigraph}(\psi)^\circ = \overline{B} = \tilde{K} \quad (69)$$

and $\psi(0) < 0$, $\overline{\text{dom}(\psi)} = \overline{L}$ while ψ is smooth and strongly-convex in L with $\nabla\psi(L) = \mathbb{R}^n$. Thanks to (67) and (69), Lemma 3.4 shows that

$$\text{Graph}_L(\psi) = M^*.$$

Since M is affinely-spherical with center at the origin, according to Corollary A.1, $\text{Graph}_L(\psi)$ is also affinely-spherical with center at the origin. Thus, the function ψ satisfies all of the conditions of Corollary 3.1.

□

A Appendix: Useful facts

A.1 Some results on polarity transforms related to Legendre transforms

The aim of this section is to summarize a few results relating to convexity considered by Artstein-Avidan and Milman [3] and by Rockafellar ([20], section 15). We denote

$$\mathcal{H}^+ := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\} \quad \text{and} \quad \mathcal{H}^- := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t < 0\}. \quad (\text{A.70})$$

Consider the fractional-linear transformations $I^+ : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ and $I^- : \mathcal{H}^- \rightarrow \mathcal{H}^+$ defined via

$$I^+(x, t) := \left(\frac{x}{t}, -\frac{1}{t} \right) \quad \text{and} \quad I^-(y, s) := \left(-\frac{y}{s}, -\frac{1}{s} \right). \quad (\text{A.71})$$

A subset $V \subset \mathcal{H}^\pm$ is a relative half-space if $V = A \cap \mathcal{H}^\pm$ where A is set of points $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ such that $\langle x, \theta \rangle + bt + c \geq 0$ for some $\theta \in \mathbb{R}^n$ and $b, c \in \mathbb{R}$. Note that a relative half-space $V \subset \mathcal{H}^\pm$ is a relatively-closed subset of \mathcal{H}^\pm . A relative half-space $V \subset \mathcal{H}^\pm$ is called proper if both V and $\mathcal{H}^\pm \setminus V$ are non-empty.

Lemma A.1. (a) *The maps I^+ and I^- transform relative half-spaces to relative half-spaces.*

(b) *The maps I^+ and I^- transform relatively-closed, convex sets to relatively-closed, convex sets.*

(c) *The two diffeomorphisms I^\pm transform smooth, connected, locally strongly-convex hypersurfaces to smooth, connected, locally strongly-convex hypersurfaces.*

For a convex set $S \in \mathbb{R}^n$, the polar body of S is defined by

$$S^\circ := \{x \in \mathbb{R}^n : \langle x, s \rangle \leq 1 \text{ for all } s \in S\} = \{x \in \mathbb{R}^n : \sup_{s \in S} \langle x, s \rangle \leq 1\} \quad (\text{A.72})$$

For $S \in \mathbb{R}^n$ and $g : S \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$\text{Epigraph}_S(g) := \{(x, t) \in S \times \mathbb{R} : g(x) \leq t\} \subset \mathbb{R}^{n+1} \quad (\text{A.73})$$

When $S = \mathbb{R}^n$, we denote $\text{Epigraph}(g) = \text{Epigraph}_{\mathbb{R}^n}(g)$.

Proposition A.1. *Let $\varphi : \mathbb{R}^n \rightarrow (0, +\infty]$ be a proper, convex function and denote $\psi = \varphi^*$. Then,*

$$I^+(\text{Epigraph}(\varphi)) = \text{Epigraph}(\psi)^\circ \cap \mathcal{H}^-.$$

Moreover, if $\psi(0) < +\infty$ then $\text{Epigraph}(\psi)^\circ \setminus \mathcal{H}^- = \{(x, 0) : x \in \text{dom}(\psi)^\circ\}$.

We say that a subset $A \subset \mathcal{H}^\pm$ is bounded from below if there exists $(x_0, t_0) \in \mathcal{H}^\pm$ such that $t > t_0$ for all $(x, t) \in A$. We write $\overline{A} \subset \mathbb{R}^{n+1}$ and $\partial A \subset \mathbb{R}^{n+1}$ for the usual closure and boundary of the set A , viewed as a subset of \mathbb{R}^{n+1} .

Lemma A.2. *Let $L \subset \mathbb{R}^n$ be a bounded, open, convex set containing the origin. Let $B \subset \mathcal{H}^-$ be a relatively-closed, convex set that is bounded from below. Assume that the set $(\partial B) \cap \mathcal{H}^-$ is a smooth, connected, locally strongly-convex hypersurface, while $(\partial B) \setminus \mathcal{H}^- = \{(x, 0) : x \in L^\circ\}$.*

Then there exists a proper, convex function $\psi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with $\overline{\text{dom}(\psi)} = \overline{L}$, that is smooth and strongly-convex in L , with $\nabla \psi(L) = \mathbb{R}^n$, $\psi(0) < 0$ and $\overline{B} = \text{Epigraph}(\psi)^\circ$. Moreover, $I^-(B) = \text{Epigraph}(\varphi)$ where $\varphi = \psi^$.*

A.2 Some results in differential geometry

Proposition A.2. (see [19], section II.5) *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly-convex hypersurface. For $y \in M$ write $K_y > 0$ for the Gauss curvature of M at the point y and denote*

$$\rho_y = \langle y, N_y \rangle$$

where $N_y \in \mathbb{R}^{n+1}$ is the Euclidean unit normal to M at the point y , pointing to the concave side of M . Then M is affinely-spherical with center at the origin if and only if there exists $C \in \mathbb{R} \setminus \{0\}$ such that $\rho_y^{n+2}/K_y = C$ for all $y \in M$.

We define the cone measure on a smooth hypersurface $M \subset \mathbb{R}^{n+1}$ to be the measure μ_M supported on M whose density with respect to the surface area measure on M is the function $y \mapsto |\rho_y| / (n + 1)$.

Proposition A.3. *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly-convex hypersurface. Then M is affinely-spherical with center at the origin if and only if the following holds: The polarity map $\nu : M \rightarrow M^*$ is well-defined, and it pushes forward the cone measure μ_M to a measure proportional to the cone measure μ_{M^*} .*

Corollary A.1. *Let $M \subset \mathbb{R}^{n+1}$ be an affinely-spherical hypersurface with center at the origin. Then the polar hypersurface M^* is well-defined, and it is again affinely-spherical with center at the origin.*

A.3 Blaschke-Santaló inequality

Theorem A.1. (see [17, 2]) *Let $K \subset \mathbb{R}^n$ be an n -dimensional, non-empty, bounded, convex set. Then, there exists a unique point z in the interior of K such that*

$$\text{Vol}_n((K - z)^\circ) = \inf_{x \in \text{int}(K)} (\text{Vol}_n((K - x)^\circ)),$$

and for this point we have

$$\text{Vol}_n(K - z) \cdot \text{Vol}_n((K - z)^\circ) \leq (\text{Vol}_n(\overline{B}))^2$$

where $\overline{B} = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and equality holds if and only if K is an ellipsoid.

The unique point z in Theorem A.1 is usually called the Santaló point of K and is characterized by the following property.

Theorem A.2. *The polar body $(K - z)^\circ$ has its barycenter at the origin if and only if z is the Santaló of K .*

Corollary A.2. *The Santaló point of K lies at the origin if and only if the barycenter of K° lies at the origin.*

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