Optimal control via positivity certificates

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MASTER 2 OPTIMIZATION: Paris-Orsay

The generalized problem of moments (GPM)

The GPM is the convex optimization problem:

$$\inf_{\mu \in M(\mathbf{K})} \left\{ \int_{\mathbf{K}} f \, d\mu \;:\; \underbrace{\int_{\mathbf{K}} h_j \, d\mu}_{\text{generalized moment}} \geq b_j, \quad j = 1, \ldots, p \right\}$$

where

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☞ Its DUAL is the infinite-dimensional LINEAR PROGRAM

$$\sup_{\lambda \geq 0} \left\{ \sum_{j=1}^{p} \lambda_j \, b_j : \quad f(\mathbf{x}) - \sum_{j=1}^{p} \lambda_j \, h_j(\mathbf{x}) \, \geq \, 0, \quad \forall \mathbf{x} \in \mathbf{K} \right\}$$

In both cases one minimizes a LINEAR FUNCTIONAL under:

- K-MOMENT constraints for the primal.
- POSITIVITY-ON-K constraints for the dual.

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We even consider the more general GPM

$$\min_{\mu_i \in M(\mathbf{K}_i)} \{ \sum_{i \in I} \int_{\mathbf{K}_i} f_i \, d\mu_i \; : \quad \underbrace{\sum_{i \in I} \int_{\mathbf{K}_i} h_{ji} \, d\mu_i \geq b_j}_{\text{moment constraints}}, \quad j \in J \}$$

where for all $i \in I$,

- $\mathbf{K}_i \subseteq \mathbb{R}^{n_i}$, and
- $M(\mathbf{K}_i)$ is the space of finite Borel measures on $\mathbf{K}_i$.

The index set I is FINITE whereas J may be COUNTABLE.

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$$\sup_{\lambda \geq 0} \left\{ \sum_{j \in J} \lambda_j \, b_j : \quad \underbrace{f_i(\mathbf{x}) - \sum_{j \in J} \lambda_j \, h_{ji}(\mathbf{x}) \, \geq \, 0, \quad \forall \mathbf{x} \in \mathbf{K}_i,}_{\text{positivity constraint}} \quad i \in I \right\}$$

Again one minimizes a LINEAR FUNCTIONAL under:

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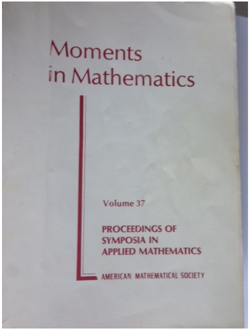

☞ The GPM has great modelling power, in various fields.
Global Optimization (continuous, discrete), Control (Robust and optimal control), Nonlinear Equations, Probability and Statistics, Performance Evaluation (in e.g. Mathematical finance, Markov chains), Inverse Problems (cristallography, tomography), Numerical multivariate Integration, etc ...

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Optimal Control as an instance of the GPM

Deterministic Optimal Control

$$
\begin{aligned}
j^* := \quad & \min_{\mathbf{u}} \int_0^T h(s, x(s), u(s))\, ds + H(x(T)) \\
\dot{x}(s) \quad = \quad & f(s,\, x(s), u(s)), \quad s \in [0, T] \\
(x(s), u(s)) \quad \in \quad & X \times U, \quad s \in [0, T) \\
x(T) \quad \in \quad & X_T,
\end{aligned}
$$

and with initial condition $x(0) = x_0 \in X$, and
- $X, X_T \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are basic semi-algebraic sets.
- $h, f \in \mathbb{R}[t, x, u]$, $H \in \mathbb{R}[x]$

- Consider an admissible trajectory $(s, x(s), u(s))$

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \in [0, T),$$

and let $g : [0, T] \times X \to \mathbb{R}$ be differentiable.

Define $v : [0, T] \to \mathbb{R}$ to be the function:

$$t \mapsto v(t) \; := \; g(t, x(t)), \qquad \forall t \in [0, T].$$

Then if v is continuously differentiable

$$v(T) - v(0) \; = \; \int_0^T v'(t)\, dt.$$

Observe that for all $t \in [0, T]$

$$v'(t) \; = \; \frac{\partial g(s, x(s))}{\partial t} + \frac{\partial g(s, x(s))}{\partial x} f(s, x(s), u(s)).$$

Therefore

$$v(T) = v(0) + \int_0^T v'(t)\,dt$$

is equivalent to:

$$\begin{aligned} g(T, x(T)) &= g(0, x_0) \\ &+ \int_0^T \frac{\partial g(s, x(s))}{\partial t} + \frac{\partial g(s, x(s))}{\partial x} f(s, x(s), u(s))\,ds \end{aligned}$$

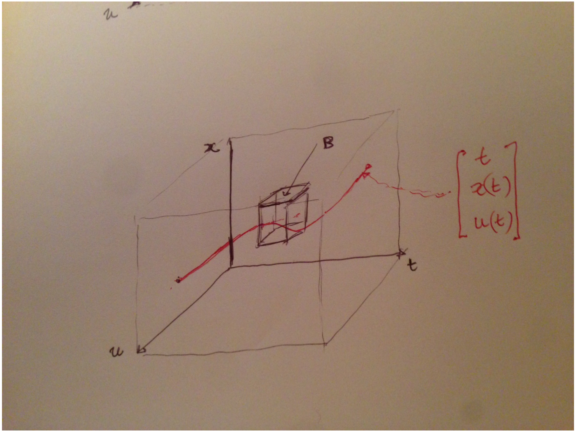
Next, with $\mathbf{u} = \{u(t),\ 0 \leq t < T\}$ the admissible control,

introduce the probability measure $\nu^{\mathbf{u}}$ on $\mathbb{R}^n$, and the measure $\mu^{\mathbf{u}}$ on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$, defined by

$$\begin{aligned} \nu^{\mathbf{u}}(B) &:= I_B[x(T)], \quad B \in \mathcal{B}(\mathbb{R}^n) \\ \mu^{\mathbf{u}}(A \times B \times C) &:= \int_{[0,T]\cap A} I_{B\times C}[(x(s), u(s))]\, ds, \end{aligned}$$

for all hyper-rectangles (A, B, C).

- $\mu^{\mathbf{u}}$ is called the *occupation measure* of the state-action (deterministic) process $(s, x(s), u(s))$ *up to* time T, whereas
- $\nu^{\mathbf{u}}$ is the *occupation measure* of the state $x(T)$ *at* time T.



$\mu^{\mathbf{u}}(B)$ is the time spent by the trajectory $\begin{bmatrix} t \\ x(t) \\ u(t) \end{bmatrix}$ in the set B.

Then for every continuous function $h : X \times U \times [0,T] \to \mathbb{R}$:

$$\int_{X \times U \times [0,T]} h(x,u,t)\, d\mu^{\mathbf{u}}(x,u,t) \;=\; \int_0^T h(x(t),u(t),t)\, dt.$$

and for every continuous function $p : X_T \to \mathbb{R}$

$$\int_{X_T} p(x)\, d\nu^{\mathbf{u}}(x) \;=\; p(x(T)).$$

With this newly introduced occupation measures $\nu^{\mathbf{u}}$ and $\mu^{\mathbf{u}}$,

the *time* integration

$$g(T,x(T)) = g(0,x_0) + \int_0^T \frac{\partial g(s,x(s))}{\partial t} + \frac{\partial g(s,x(s))}{\partial x}\, f(s,x(s),u(s))\,ds$$

is equivalent to the *spatial* integration

$$\int_{X_T} g_T(x)\,d\nu^{\mathbf{u}}(x) = g(0,x_0) + \int_{[0,T]\times X\times U} \left[\frac{\partial g(t,x)}{\partial t} + \frac{\partial g(t,x)}{\partial x}\, f(t,x,u)\right] d\mu^{\mathbf{u}}(t,x,u)$$

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with $g_T(x) := g(T,x)$ for all x.

• Similarly, the cost

$$\int_0^T h(s,\, x(s), u(s))\, ds + H(x(T))$$

of this trajectory reads as

$$\int_{[0,T]\times X\times U} h(s,x,u)\, d\mu^{\mathbf{u}}(s,x,u) + \int_{X_T} H(x)\, d\nu^{\mathbf{u}}(x)$$

So stating that a trajectory $(s, x(s), u(s))$ is admissible is equivalent to stating that

$$x(s) \in X, \quad x(T) \in X_T, \quad u(s) \in U$$

AND

EITHER

$$\dot{x}(s) = f(s, x(s), u(s)), \quad s \in [0, T]; \quad x(0) = x_0,$$

OR

$$g(T, x(T)) = g(0, x_0) + \int_0^T \frac{\partial g(s, x(s))}{\partial t} + \frac{\partial g(s, x(s))}{\partial x} f(s, x(s), u(s))\, ds$$

for SUFFICIENTLY MANY TEST FUNCTIONS $g \in G$.

OR the occupation measures (ν^u, μ^u) satisfy

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The so-called weak formulation of the optimal control problem is the infinite-dimensional LP

$$\begin{cases} \rho^* = \min_{\mu,\nu} \int H\, d\nu + \int h\, d\mu \\ \text{s.t.} \quad \int g_T\, d\nu - \int \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f \right) d\mu = g(0, x_0), \quad \forall g \in G \\ \\ \mu: \quad \text{measure supported on } [0, T] \times X \times U \\ \nu: \quad \text{prob. measure supported on } X_T \end{cases}$$

over some space of MEASURES (μ, ν) with appropriate SUPPORTS $[0, T] \times X \times U$ and X_T.

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This LP is a RELAXATION of the original problem BECAUSE ...

if to every FEASIBLE TRAJECTORY $(s, x(s), us)$ correspond the two OCCUPATION MEASURES μ^u and ν^u which are admissible for this LP ...

On the other hand, NOT EVERY FEASIBLE SOLUTION (μ, ν) of this LP are occupation measures associated with some FEASIBLE TRAJECTORY $(s, x(s), u(s))$ of the initial optimal control problem!

Therefore necessarily $\rho^* \leq j^*$.

QUESTION: When do we have $\rho^* = j^*$?

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Theorem (R. Vinter)

If X, X_T, U are compact, $f(s, x, U)$ is convex for all $(s, x) \in [0, T] \times X$, and h satisfies some technical condition, then $\rho^* = j^*$.

However, in general ...
WE DON'T KNOW HOW TO SOLVE (OR APPROXIMATE) THIS LP!

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if f, h, H are polynomials and X and U are compact basic semi-algebraic sets then the Moment-SOS approach can be implemented!

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The dual of the infinite-LP formulated on measures has a nice interpretation in terms of the Hamilton-Jacobi-Bellman (HJB) equation: Let

$$
\begin{aligned}
\phi^*(t,x) := & \quad \min_{\mathbf{u}} \quad \int_t^T h(s,x(s),u(s))\,ds + H(x(T)) \\
\dot{x}(s) &= f(s,\,x(s),u(s)), \quad s \in [0,T] \\
(x(s),u(s)) &\in X \times U, \quad s \in [0,T) \\
x(T) &\in X_T \\
x(t) &= x,
\end{aligned}
$$

that is, $\phi^*(t,x)$ is the Optimal Value of the OCP on the interval $[t,T]$, with initial condition $x(t) = x$ at time t.
$\rightarrow$ ϕ^* is called the Optimal Value Function.

Under some conditions, the Optimal Value Function ϕ^* is the unique solution (in the sense of VISCOSITY) of the HJB OPTIMALITY EQUATION:

$$\begin{cases} \frac{\partial \phi^*(t,x)}{\partial t} + \min_{u \in U} \left\{ \frac{\partial \phi^*(t,x)}{\partial x} f(t,x,u) + h(t,x,u) \right\} = 0, \\ \text{for all } (t,x) \in [0,T] \times X \\ \\ \phi^*(T,x) = H(x), \quad \forall x \in X_T \end{cases}$$

The dual of the infinite-LP formulated on measures is the LP on functions:

$$
\left\{
\begin{array}{rl}
\rho^* = & \sup_\phi \phi(0, x_0) \\
\text{s.t.} & \frac{\partial\phi(t,x)}{\partial t} + \frac{\partial\phi(t,x)}{\partial x} f(t,x,u) + h(t,x,u) \geq 0, \\
 & \text{for all } (t,x,u) \in [0,T] \times X \times U \\
 & \phi(T,x) \leq H(x), \quad \forall x \in X_T \\
 & \phi \in C^1([0,T) \times X).
\end{array}
\right.
$$

and any feasible solution ϕ is called a SUB-SOLUTION of the HJB OPTIMALITY-EQUATION

The moment-SOS approach

As the polynomials are dense in $C^1([0,T] \times X)$

replace

$\phi \in C^1([0,T] \times X)$ with $\phi \in \mathbb{R}[t,x]$

and use Putinar's positivity certificate

for the positivity constraints

$$I. \quad \frac{\partial \phi(t,x)}{\partial t} + \frac{\partial \phi(t,x)}{\partial x} f(t,x,u) + h(t,x,u) \geq 0$$

$\forall\, (t,x,u) \in [0,T] \times X \times U$.

and

$$II. \quad H(x) - \phi(T,x) \geq 0, \quad \forall x \in X_T$$

That is, $\phi \in \mathbb{R}[t,x]_{2d}$, and

$$\frac{\partial\phi(t,x)}{\partial t} + \frac{\partial\phi(t,x)}{\partial x}\, f(t,x,u) + h(t,x,u)$$

$$= \underbrace{\sigma_0(t,x,u)}_{SOS} + \sum_{j=1}^{m} \underbrace{\sigma_j(t,x,u)}_{SOS}\, g_j(t,x,u),$$

with $\deg(\sigma_0) <= 2d$, $\deg(\sigma_j g_j) <= 2d$, and where:

$$[0,T] \times X \times U \;=\; \{\,(t,x,u) : \; g_j(t,x,u) \;\geq\; 0,\; j = 1,\ldots,m\,\}.$$

Similarly

$$H(x) - \phi(T,x) \,=\, \underbrace{\psi_0(x)}_{SOS} + \sum_{k=1}^{s} \underbrace{\psi_k(x)}_{SOS}\, h_k(x),$$

with $\deg(\psi_0) <= 2d$, $\deg(\psi_k h_k) <= 2d$, and where :

$$X_T \,=\, \{\, x : \; h_k(x) \,\geq\, 0,\; k = 1, \ldots, s \,\}.$$

This yields a hierarchy of Semidefinite Programs $(\mathbf{Q}_d)_{d\in\mathbb{N}}$ with optimal value ρ_d, where d is the degree allowed for ϕ and for the SOS weights (σ_j) in Putinar's certificate, and

$$\lim_{d\to\infty} \rho_d = \rho^* = j^*.$$

☞ Lass, Henrion, Prieur and Trélat, Nonlinear optimal control via occupation measures and LMI-relaxations, SIAM J. Control & Optimization 47 (2008)