Moments and Positive Polynomials for Optimization II: LP- VERSUS SDP-relaxations

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Recall the Global Optimization problem **P**:

$$f^* := \min\{\ f(x) \mid \ g_j(x) \geq 0,\ j = 1, \ldots, m\},$$

where f and g_j are all POLYNOMIALS, and let

$$\mathbf{K} := \{\ x \in \mathbb{R}^n \mid \ g_j(x) \geq 0, \quad j = 1, \ldots, m\}$$

be the feasible set (a compact basic semi-algebraic set)

Putinar Positivstellensatz

Assumption 1:

For some $M > 0$, the quadratic polynomial $M - \|X\|^2$ belongs to the quadratic module $Q(g_1, \ldots, g_m)$

Theorem (Putinar-Jacobi-Prestel)

Let $\mathbf{K}$ *be compact and Assumption 1 hold. Then*

$$[\,\mathbf{f} \in \mathbb{R}[\mathbf{X}] \quad \textit{and} \quad \mathbf{f} > \mathbf{0} \textit{ on } \mathbf{K}\,] \Rightarrow \mathbf{f} \in \mathbf{Q}(\mathbf{g_1}, \ldots, \mathbf{g_m}), \quad \textit{i.e.,}$$

$$f(x) = \sigma_0(x) + \sum_{j=1}^{m} \sigma_j(x)\, g_j(x), \qquad \forall x \in \mathbb{R}^n$$

for some s.o.s. polynomials $\{\sigma_j\}_{j=0}^{m}$.

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for some s.o.s. polynomials $\{\sigma_j\}_{j=0}^{m}$.

• **If one fixes an a priori bound on the degree of the s.o.s. polynomials** $\{\sigma_j\}$, checking $f \in Q(g_1, \ldots, g_m)$ reduces to solving a SDP!!

• **Moreover, Assumption 1 holds true if e.g. :**
- all the g_j's are linear (hence $\mathbf{K}$ is a polytope), or if
- the set $\{ x \mid g_j(x) \geq 0\}$ is compact for some $j \in \{1, \ldots, m\}$.

• **If** $x \in \mathbf{K} \Rightarrow \|x\| \leq M$ **for some (known)** M, then it suffices to add the redundant quadratic constraint $M^2 - \|X\|^2 \geq 0$, in the definition of $\mathbf{K}$.

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Krivine-Handelman-Vasilescu Positivstellensatz

Assumption I:

With $g_0 = 1$, the family $\{g_0, \ldots, g_m\}$ generates the algebra $\mathbb{R}[x]$, that is, $\mathbb{R}[x_1, \ldots, x_n] = \mathbb{R}[g_0, \ldots, g_m]$.

Assumption II:

Recall that $\mathbf{K}$ is compact. Hence we also assume with no loss of generality (but possibly after scaling) that for every $j = 1, \ldots, m$:

$$0 \leq g_j(x) \leq 1 \qquad \forall x \in \mathbf{K}.$$

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Notation: for $\alpha, \beta \in \mathbb{N}^m$, let

$$\begin{aligned} g(x)^\alpha &= g_1(x)^{\alpha_1} \cdots g_m(x)^{\alpha_m} \\ (1-g(x))^\beta &= (1-g_1(x))^{\beta_1} \cdots (1-g_m(x))^{\beta_m} \end{aligned}$$

Theorem (Krivine,Vasilescu Positivstellensatz)

Let Assumption I and Assumption II hold:

*If $f \in \mathbb{R}[x_1, \ldots, x_m]$ is POSITIVE on **K** then*

$$f(x) = \sum_{\alpha,\beta \in \mathbb{N}^m} c_{\alpha\beta}\, g(x)^\alpha\, (1-g(x))^\beta, \quad \forall x \in \mathbb{R}^n,$$

for finitely many positive coefficients $(c_{\alpha\beta})$.

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Testing whether

$$f(x) = \sum_{\alpha,\beta\in\mathbb{N}^m} c_{\alpha\beta}\, g(x)^\alpha (1-g(x))^\beta, \quad \forall x \in \mathbb{R}^n,$$

for finitely many positive coefficients $(c_{\alpha\beta})$ and with $\sum_i \alpha_i + \beta_i \leq d$

... reduces to solving a LP!.

Indeed, recall that $f(x) = \sum_\gamma f_\gamma x^\gamma$. So expand

$$\sum_{\alpha,\beta\in\mathbb{N}^m} c_{\alpha\beta}\, g(x)^\alpha (1-g(x))^\beta = \sum_{\gamma\in\mathbb{N}^n} \theta_\gamma(c)\, x^\gamma$$

and state that

$$f_\gamma = \theta_\gamma(c), \quad \forall \gamma \in \mathbb{N}^n_{2d}; \quad c \geq 0. \quad \rightarrow \text{ a linear system!}$$

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DUAL side: The K-moment problem

Let $\{X^\alpha\}$ be a canonical basis for $\mathbb{R}[X]$, and let $y := \{y_\alpha\}$ be a given sequence indexed in that basis.

Recall the K-moment problem

Given $\mathbf{K} \subset \mathbb{R}^n$, **does there exist a measure** μ on $\mathbf{K}$, such that

$$y_\alpha = \int_{\mathbf{K}} X^\alpha \, d\mu, \qquad \forall \alpha \in \mathbb{N}^n \quad ?$$

(where $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$).

Given $y = \{y_\alpha\}$, let $L_y : \mathbb{R}[X] \to \mathbb{R}$, be the linear functional

$$f \left(= \sum_\alpha f_\alpha X^\alpha\right) \quad \mapsto \quad L_y(f) := \sum_{\alpha \in \mathbb{N}^n} f_\alpha y_\alpha.$$

Moment matrix $M_d(y)$

with rows and columns also indexed in the basis $\{X^\alpha\}$.

$$M_d(y)(\alpha, \beta) := L_y(X^{\alpha+\beta}) = y_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{N}^n, \quad |\alpha|, |\beta| \leq d.$$

For instance in $\mathbb{R}^2$:

$$M_1(y) = \begin{array}{c} \begin{array}{ccc} 1 & X_1 & X_2 \end{array} \\ \left[\begin{array}{c|cc} y_{00} & y_{10} & y_{01} \\ \hline y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{array}\right] \end{array}$$

Importantly ...

$$M_d(y) \succeq 0 \quad \Longleftrightarrow \quad L_y(h^2) \geq 0, \qquad \forall h \in \mathbb{R}[X]_d$$

Localizing matrix

The “Localizing matrix" $M_d(\theta y)$ w.r.t. a polynomial $\theta \in \mathbb{R}[X]$

with $X \mapsto \theta(X) = \sum_\gamma \theta_\gamma X^\gamma$, has its rows and columns also indexed in the basis $\{X^\alpha\}$ of $\mathbb{R}[X]_d$, and with entries:

$$\begin{aligned} M_d(\theta\, y)(\alpha,\beta) &= L_y(\theta\, X^{\alpha+\beta}) \\ &= \sum_{\gamma\in\mathbb{N}^n} \theta_\gamma\, y_{\alpha+\beta+\gamma}, \qquad \left\{ \begin{array}{l} \alpha,\beta \in \mathbb{N}^n \\ |\alpha|, |\beta| \leq d. \end{array} \right. \end{aligned}$$

For instance, in $\mathbb{R}^2$, and with $X \mapsto \theta(X) := 1 - X_1^2 - X_2^2$,

$$M_1(\theta\, y) = \left[\begin{array}{ccc} \overbrace{y_{00} - y_{20} - y_{02},}^{1} & \overbrace{y_{10} - y_{30} - y_{12},}^{X_1} & \overbrace{y_{01} - y_{21} - y_{03}}^{X_2} \\ y_{10} - y_{30} - y_{12}, & y_{20} - y_{40} - y_{22}, & y_{11} - y_{21} - y_{12} \\ y_{01} - y_{21} - y_{03}, & y_{11} - y_{21} - y_{12}, & y_{02} - y_{22} - y_{04} \end{array}\right].$$

Importantly . . .

$$M_d(\theta\, y) \succeq 0 \quad \Longleftrightarrow \quad L_y(h^2\, \theta) \,\geq 0, \qquad \forall h \in \mathbb{R}[X]_d$$

Putinar's dual conditions

Again $\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^{\mathbf{n}} \mid \mathbf{g_j(x)} \geq \mathbf{0},\ \mathbf{j} = \mathbf{1}, \ldots, \mathbf{m}\}$.

Assumption 1: For some $M > 0$, the quadratic polynomial

$M - \|X\|^2$ is in the quadratic module $Q(g_1, \ldots, g_m)$

Theorem (Putinar: dual side)

Let $\mathbf{K}$ be compact, and Assumption 1 hold.

Then a sequence $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, has a representing measure μ on $\mathbf{K}$ if and only if

$$(**) \quad L_y(f^2) \geq 0; \quad L_y(f^2\, g_j) \geq 0, \quad \forall j = 1, \ldots, m; \quad \forall\, f \in \mathbb{R}[X].$$

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Checking whether (**) holds for all $f \in \mathbb{R}[X]$ with degree $\leq d$

reduces to checking whether $M_d(y) \succeq 0$ and $M_d(g_j\, y) \succeq 0$, for all $j = 1, \ldots, m$!

$\rightarrow$ $m + 1$ LMI conditions to verify!

Krivine-Vasilescu: dual side

Theorem

Let **K** *be compact, and Assumption I and II hold.*

Then the sequence $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, *has a representing measure* μ *on* **K** *if and only if*

$$L_y(g^\alpha (1-g)^\beta) \geq 0, \qquad \forall \alpha, \beta \in \mathbb{N}^m.$$

LP-relaxations

With $f \in \mathbb{R}[x]$, consider the hierarchy of LP-relaxations

$$\left\{ \begin{array}{rl} \rho_d^* = \max\limits_{\lambda, c_{\alpha\beta}} & \lambda \\ & f - \lambda = \displaystyle\sum_{\alpha,\beta \in \mathbb{N}^m} c_{\alpha\beta}\, g^\alpha\, (1-g)^\beta \\ & c_{\alpha\beta} \geq 0, \qquad \forall\, |\alpha+\beta| \leq 2d \end{array} \right.$$

and of course $\rho_d = \rho_d^*$ for all d.

Theorem

Assume that **K** *is compact and Assumption I and II hold. Then the LP-relaxations CONVERGE, that is,*

$$\rho_d \uparrow f^* \quad \text{as } d \to \infty.$$

• The SHERALI-ADAMS RLT's hierarchy is exactly this type of LP-relaxations.

• Its convergence for 0/1 programs was proved with ah-hoc arguments.

• In fact, the rationale behind such convergence if Krivine-Vasilescu Positivstellensatz.

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Some remarks on LP-relaxations

1. Notice the presence of binomial coefficients in both primal and dual LP-relaxations ... which yields numericall ill-conditioning for relatively large d.

2. Let $x^* \in \mathbf{K}$ be a global minimizer, and for $x \in \mathbf{K}$, let $J(x)$ be the set of active constraints, i.e., $g_j(x) = 0$ or ...$1 - g_k(x) = 0$.

Then FINITE convergence CANNOT occur if there exists nonoptimal $x \in \mathbf{K}$ with $J(x) \supseteq J(x^*)$!

$\rightarrow$ And so ... not possible for CONVEX problems in general!

For instance, if $\mathbf{K}$ is a Polytope then FINITE convergence is possible only if every global minimizer is a vertex of $\mathbf{K}$!

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LP- and SDP-Relaxations with their dual

Primal LP-relaxation	Primal SDP-relaxation
$\min_y \; L_y(f)$ $L_y(g^\alpha (1-g)^\beta) \geq 0$ $\forall\, \alpha, \beta \in \mathbb{N}^m, \quad \lvert\alpha+\beta\rvert \leq 2d$	$\min_y \; L_y(f)$ $L_y(h^2\, g_j) \geq 0,\ j = 1, \ldots, m$ $\forall h,\ \deg(h\, g_j) \leq 2d,\ j \leq m$
Dual LP-relaxation	**Dual SDP-relaxation**
$\max_{\lambda, \{c_{\alpha\beta}\}} \; \lambda$ $f - \lambda = \sum_{\alpha,\beta \in \mathbb{N}^m} c_{\alpha\beta}\, g^\alpha (1-g)^\beta$ $c_{\alpha\beta} \geq 0;\ \lvert\alpha+\beta\rvert \leq 2d$	$\max_{\lambda, \{\sigma_j\}} \; \lambda$ $f - \lambda = \sum_{j=0}^{m} \sigma_j\, g_j$ $\deg(\sigma_j\, g_j) \leq 2d,\ j \leq m$